Large deviations for triangles in scale-free random graphs

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Abstract

We provide large deviations estimates for the upper tail of the number of triangles in scale-free inhomogeneous random graphs where the degrees have power law tails with index $-\alpha, \alpha \in (1, 2)$. We show that upper tail probabilities for triangles undergo a phase transition. For $\alpha < 4/3$, the upper tail is caused by many vertices of degree of order n, and this probability is semi-exponential. In this regime, additional triangles consist of two hubs. For $\alpha > 4/3$ on the other hand, the upper tail is caused by one hub of a specific degree, and this probability decays polynomially in n, leading to additional triangles with one hub. In the intermediate case $\alpha = 4/3$, we show polynomial decay of the tail probability caused by multiple but finitely many hubs. In this case, the additional triangles contain either a single hub or two hubs. Our proofs are partly based on various concentration inequalities. In particular, we tailor concentration bounds for empirical processes to make them well-suited for analyzing heavy-tailed phenomena in nonlinear settings.

1 Introduction and main results

Many real-world networks were found to have degree distributions that can be approximated by a power-law distribution, where the fraction of vertices of degree k scales as a power law with infinite variance [38]. Therefore, random graph models that serve as benchmark for these real-world networks therefore often focus on networks with power-law degree distributions. These random graphs are constructed to have similar degree distributions as real-world networks, but other graph properties are not prescribed by the model. The behavior of network properties of random graph models with a prescribed degree sequence has therefore been an object of intensive study [15, 19, 20, 26, 22, 41].

In this paper, we focus on the property of triangle counts. Triangle counts measure the tendency of two neighbors of a vertex to be connected as well, allowing to analyze the network's clustering properties. While many real-world networks were found to be highly clustered, many random graph models are locally tree-like, and therefore only contain few triangles in the large-network limit. In power-law random graphs however, the random graphs may still possess a polynomial number of triangles, and the average clustering coefficient vanishes extremely slowly in the network size [21]. Motivated by this slow decay of the average clustering coefficient, we focus on the question: How unlikely is it that a power-law random graph contains a large number of triangles?

The tail probability for triangle counts in Erdős-Rényi random graphs has been studied extensively since [24, 25, 28], and a matching upper and lower bound were finally provided in [9, 14] when the random graphs are not too sparse, that is, when the average degree of each vertex tends to infinity in the network size; see also [4]. For sparser Erdős-Rényi graphs with finite average vertex degrees, [8] showed that the probability of observing many triangles is extremely small, and that the unlikely event that a large number of triangles is present is driven by a localized almost clique structure.

Significantly fewer results exist on large deviations for random graphs with heavy-tailed degrees. Most existing work assumes light-tailed degree distributions, or a finite second moment [2, 8, 16], as this allows to write the connection probability in a product form of the weights of the two involved nodes, while an infinite second moment makes this impossible. Under an infinite second moment, the connection probability depends on both vertex weights in a way that cannot be split into their individual contributions, creating so-called degree-degree correlations [35, 40]. Other work focuses on a regime where the average degree grows [31], which is not applicable when the tail of the degree distribution behaves like a power law with index $-\alpha, \alpha \in (1, 2)$. Results on large-deviations analysis for power-law random graphs are so far restricted to the Pagerank functional [11, 32], and edge counts [27, 37].

In this paper, we derive tail asymptotics for the probability that the number of triangles is larger than average for the sparse, power-law case with $\alpha \in (1, 2)$. Interestingly, we show that there is a phase transition in the degree exponent α. When $\alpha < 4/3$, the probability that the number of triangles is larger than expected decays semi-exponential in the network size n. For $\alpha \geq 4/3$ on the other hand, this probability decays polynomially in n. Furthermore, in contrast to the non-power-law case, deviations of the triangle counts are caused by the presence of one or more hubs of specific degree.

1.1 Model description

To give a precise description of our main results, we now provide a model description. We consider the rank-1 inhomogeneous random graph (or hidden variable model). This model constructs simple graphs with soft constraints on the degree sequence [7, 12]. The graph consists of n vertices with non-negative weights $W_i, i = 1, ..., n$. These weights are an i.i.d. sample from the continuous heavy-tailed distribution $F(x)$,

$$\bar{F}(x) := \mathbb{P}(W > x) = x^{-\alpha} L(x), \quad x \geq 0 \tag{1.1}$$

for some slowly varying function $L(x)$, $\alpha \in (1, 2)$.

We denote $\mu = \mathbb{E}[W_i]$. Then, every pair of vertices with weights (h, h') is connected with probability $p(h, h')$. In this paper, we take

$$p(h, h') = \min\left(\frac{hh'}{\mu n}, 1\right), \tag{1.2}$$

which is the Chung-Lu version of the rank-1 inhomogeneous random graph [12]. This connection probability ensures that the degree of a vertex with weight h will be close to h [7].

We are interested in the number of triangles $\triangle_n$ contained in a sample of the rank-1 homogeneous random graph. Denote

$$f_n(u, v, w) = \min\left\{\frac{uv}{\mu n}, 1\right\} \min\left\{\frac{vw}{\mu n}, 1\right\} \min\left\{\frac{uw}{\mu n}, 1\right\}. \tag{1.3}$$

The next result describes the growth rate of $m_n = \mathbb{E}[\triangle_n]$, extending previous work [23, 36] on the pure power law case.

Lemma 1.1. *Let $\alpha \in (1, 2)$. $m_n = (1 + o(1))Hn^3(\bar{F}(\sqrt{n}))^3$, with*

$$H = \frac{\alpha^3}{6} \int_{u=0}^{\infty} \int_{v=0}^{\infty} \int_{w=0}^{\infty} f_1(u, v, w) u^{-\alpha-1} v^{-\alpha-1} w^{-\alpha-1} du dv dw. \tag{1.4}$$

In particular, m_n is regularly varying with index $3 - \frac{3}{2}\alpha$.

1.2 Main results and discussion

We are interested in the event that $\triangle_n$ deviates from its mean m_n by a factor $a > 0$. It turns out that there is a qualitative difference determined by the question whether m_n is increasing to ∞ faster than n or not, which, due to Lemma 1.1, is determined by whether the parameter α is bigger than or smaller than $4/3$.

(a) $\alpha < 4/3$ or more than n triangles: exponential deviations, caused by many hubs of weight of $O(n)$ that form many triangles with one node of constant weight.

(b) $\alpha > 4/3$: polynomial deviations, caused by one hub of weight of $O(n^{(\alpha+2\theta)/(4(\alpha-1))})$ that forms triangles with vertices of lower degrees. $\theta = 0$ in Theorem 1.3.

Figure 1: Illustration of the events that cause polynomial and exponential deviations.

The case $\alpha > 4/3$: single hub As we will see, an unusually large number of triangles will be caused by one or more hubs. To determine how large a hub needs to be, we define for $a > 0$,

$$c_a(n) := \inf\Big\{c : n^2 \frac{1}{2}\int_0^\infty \int_0^\infty f_n(x,y,c)dF(x)dF(y) \geq m_n a\Big\}. \tag{1.5}$$

Intuitively, $c_a(n)$ is the smallest size of a hub needed to create am_n additional triangles. The next lemma estimates its order of magnitude. Let $f(n) \sim g(n)$ denote $f(n)/g(n) \to 1$.

Lemma 1.2. *$c_a(n)$ is regularly varying of index $\beta = \frac{1}{4}\frac{\alpha}{\alpha-1}$. In particular, there exists a slowly varying function L^* such that*

$$c_a(n) \sim L^*(n) n^\beta a^{\frac{1}{2}\frac{1}{\alpha-1}} \tag{1.6}$$

for every $a > 0$.

The index β equals 1 at $\alpha = 4/3$ and decreases in α. β equals $1/2$ when $\alpha = 2$. One can show using extreme-value theory that the typical value of the largest weight in the random graph is regularly varying with index $1/\alpha$. For all $\alpha \in (1,2)$ we have $\beta > 1/\alpha$. A hub of size $c_a(n)$ is therefore a rare event, and our first main theorem confirms that it is the most likely rare event leading to $(1+a)m_n$ triangles when $\alpha > 4/3$.

Theorem 1.3. *Let $\alpha > 4/3$ and $a > 0$. As $n \to \infty$,*

$$\mathbb{P}(\triangle_n > (1+a)m_n) = (1+o(1))n\mathbb{P}(W > c_a(n)). \tag{1.7}$$

By applying a recent concentration result of Chatterjee [9], we show in Section 4 that it suffices to investigate the asymptotics of $\mathbb{P}(G_n > (1+a)m_n)$, with

$$G_n = \mathbb{E}[\triangle_n \mid W_1, ..., W_n]. \tag{1.8}$$

To analyze $\mathbb{P}(G_n > (1+a)m_n)$, we extend ideas from heavy-tailed large deviations theory (see e.g. [13, 18, 29, 34, 42]) and formalize the intuition that a single big hub is needed. A major step is to show that the event $\{G_n > (1+a)m_n\}$ is much more unlikely when all nodes have weight smaller than $\varepsilon c_a(n)$ for some suitable $\varepsilon > 0$. However, existing heavy-tailed large deviation tools focus on essentially linear processes and are less suitable to apply to functionals of random graphs which are essentially nonlinear as in (1.8).

For this reason we develop a different approach: we write the number of triangles as a functional of the empirical distributions of the weights as in (2.1), and we derive a novel concentration result (Proposition 2.2), building on a classical concentration result for weighted empirical processes [39]. For a more precise statement we refer to (2.18). This approach seems promising for other nonlinear functionals of heavy-tailed random variables, like U-statistics, or other observables of random graphs. Examples of recent work on nonlinear large deviations in a light-tailed setting are [4, 10].

The case $\alpha < 4/3$: many hubs For $\alpha < 4/3$, the probability of a larger than average number of triangles is semi-exponential instead:

Theorem 1.4. *Suppose that $\mathbb{P}(W \geq 1) = 1$ and $\mathbb{P}(W > x) \sim Cx^{-\alpha}$ for $x \geq 1$, $\alpha \in (1, 4/3)$. For any fixed $a > 0$,*

$$\lim_{n\to\infty} \frac{\log \mathbb{P}\left(\triangle_n > n^{(3-\alpha 3/2)}(C^3 H + a)\right)}{n^{1-\alpha 3/4}\log(n)} = -\sqrt{2a}\frac{\alpha}{4}. \tag{1.9}$$

In this setting, a large number of triangles is caused by a sublinear (but polynomially growing in n) number of vertices of weight μn. If $\mathbb{P}(W \geq 1) = 1$, a vertex of weight μn connects to all other vertices with probability one. Thus, in pairs of two, these hubs form triangles with all other vertices, see Figure 1a.

The square root of a can intuitively be explained by the fact that B vertices of weight μn create $nB(B-1)/2 \approx nB^2/2$ triangles. Indeed, each of the $B^2/2$ pairs of vertices of weight at least μn creates triangles with each of the n other vertices. Thus, with $B = \sqrt{2a}n^{1-\alpha 3/4}$ create at least $an^{(3-\alpha 3/2)}$ triangles.

To prove an asymptotic upper bound, we will split all triples of nodes into three sets based on the triangle weights and then construct an upper bound on the triangle counts with edges in these weight groups one by one. Here we will use similar concentration bounds as developed in the proof of Theorem 1.3 to get rid of the random weights, as long as these weights are sufficiently small. For triangles containing larger weights, we will use properties of the function f_n instead to deal with the multiple sources of randomness.

The phenomenon of having a number of big hubs that is growing with n is non-standard in the context of heavy tails; a related example appears in an exit problem for the sample average of a random walk, where the number of big values required to avoid escaping a convex set for n time units is logarithmically increasing with n [5].

We believe that it is possible to extend Theorem 1.4 to allow for non-trivial slowly varying functions. In this setting, the denominator of the scaling will likely also include a term with $L(n)$, the slowly varying function evaluated at n, due to the fact that the main contribution is from vertices of weight n. However, in the current proof we distinguish different types of triangles at different scales of the weights. Proving such a statement with slowly varying functions then entails showing that the contribution of the slowly varying functions at other scales that will appear in the probabilities of these non-dominating triangles are small compared to the contribution of $L(n)$, which becomes rather technical, especially when $L(n)$ oscillates.

Behavior at the boundary $\alpha = 4/3$: multiple hubs The above two theorems show a stark contrast in the way additional triangles are generated: if $\alpha > 4/3$ they consist of two regular nodes and one hub, and if $\alpha < 4/3$, they consist one regular node, and two hubs. At the boundary $\alpha = 4/3$ both may occur. When $\alpha = 4/3$, $c_a(n)$ defined in (1.5) is regularly varying of index 1, as is m_n. To avoid technical complications with slowly varying functions that can arise on the boundary (for example, m_n/n could be oscillating between 0 and ∞), we assume that $\mathbb{P}(W > n) = (1 + o(1))Cn^{-\alpha}$, in which case $m_n \sim C^3 Hn$.

Depending on the value of a, a single big value of the weights W_i may not be enough to create $(1+a)m_n$ triangles, for which we now provide some intuition. If there are l hubs with a weight of infinite size and n regular nodes, each hub forms a triangle with any of the μn edges, leading to $l\mu n$ additional triangles consisting of a single hub and two regular nodes. In addition, each of the $l(l-1)/2$ pairs of hubs form n triangles with the regular nodes. Therefore, if we wish to exceed the number of triangles with a factor am_n we need $k(a)$ hubs where $k(a)$ is defined as

$$k(a) := \inf\{l : l\mu + l(l-1)/2 > aC^3 H\}. \tag{1.10}$$

To derive a precise result, we need to take into account that hubs have weight of $O(n)$ rather than

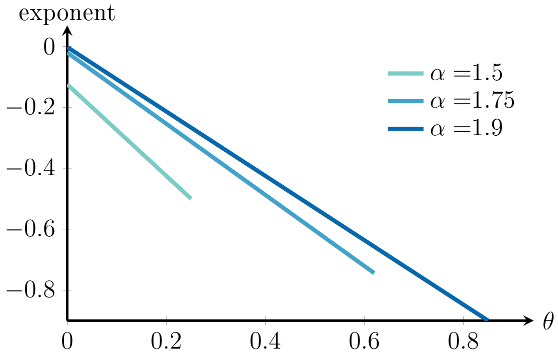


Figure 2: The exponent of Theorem 1.6 plotted against θ for several values of α.

∞. To this end, we define

$$K_l(z_1,...,z_l) = \frac{1}{2\mu}\sum_{i=1}^{l}(\frac{z_i}{\mu}\mathbb{E}\left[W^2 I(W\leq \mu/z_i)\right] + \mathbb{E}\left[WI(W>\mu/z_i)\right])^2 + \sum_{i=1,j>i}^{l}\mathbb{E}\left[\min\{\frac{z_i}{\mu}W,1\},\min\{\frac{z_j}{\mu}W,1\}\right], \tag{1.11}$$

where I denotes the indicator function. This can be interpreted as the expected number of additional triangles caused by l hubs of size $z_i n, i=1,...,l$. We can now formulate our main theorem for $\alpha = 4/3$. Let, for $b>0$, $X_i^b, i\geq 1$, be an i.i.d. sequence such that $\mathbb{P}(X_i^b > x) = (x/b)^{-\alpha}, x\geq b$. Set $\eta(a)$ as the smallest number η for which $((k(a)-1)\mu + K_1(\eta)\geq C^3H(1+a)$. Note that $\eta(a)>0$ if $(k(a)-1)\mu + (k(a)-1)(k(a)-2)/2 < aC^3H$.

Theorem 1.5. *Suppose that $P(W>x)\sim Cx^{-4/3}$ and suppose that $(k(a)-1)\mu+(k(a)-1)(k(a)-2)/2 < aC^3H$. Then*

$$\mathbb{P}\left(\triangle_n > (1+a)m_n\right) \sim \mathbb{P}(K_{k(a)}(X_1^{\eta(a)},\dots,X_{k(a)}^{\eta(a)}) \geq C^3Ha)(n\mathbb{P}\left(W>\eta(a)n\right))^{k(a)}. \tag{1.12}$$

We prove Theorem 1.5 along similar lines as Theorem 1.3, namely by first showing the analogous result for G_n. The technical condition $(k(a)-1)\mu + (k(a)-1)(k(a)-2)/2 < aC^3H$ is needed to be able to pin down the number of hubs that is required for G_n and $\triangle_n$ to be large.

Larger deviations We now show that larger deviations of an order of magnitude n^θ from the mean again induce a phase transition. First, the probability that a factor of n^θ more triangles than average are present decays regularly varying in n up until $\theta = (3/2)\alpha - 2$:

Theorem 1.6. *Let $\alpha > 4/3$ and $\theta\in(0,\frac{3}{2}\alpha-2)$. Then $c_{n^\theta}(n)\sim L^*(n)n^{\beta+\frac{\theta}{2}\frac{1}{\alpha-1}} = o(n)$ for some slowly varying L^* and*

$$\mathbb{P}\left(\triangle_n > m_n(1+n^\theta)\right) \sim n\mathbb{P}\left(W > c_{n^\theta}(n)\right). \tag{1.13}$$

In particular, $\mathbb{P}\left(\triangle_n > m_n(1+n^\theta)\right)$ is regularly varying with exponent $1-\alpha\beta-\alpha\frac{\theta}{2}\frac{1}{\alpha-1}$.

The proof of Theorem 1.6 follows the same steps as the proof of Theorem 1.3, but is at several points slightly more technical. For readability we provide these additional technical details separately in Appendix D.

Note that Theorem 1.6 applies up to $\theta = (3/2)\alpha - 2$. Thus, the higher α, the larger the factor of deviations that can still be computed with this theorem, as also shown in Figure 2. Furthermore, Figure 2 illustrates that for larger values of α, a deviation of n^θ is more likely than for smaller values of α. That is, more degree inhomogeneity makes deviations of the triangle counts more unlikely. At first sight, this may be in contrast with the intuition that degree inhomogeneity makes it more likely for extreme values of the weight sequence to appear, and therefore could make a deviation of the triangle counts more likely. However, the average number of triangles is also larger for low values of α, so that a lower number of triangles is required for higher values of α to get the same deviating factor than for lower values of α.

Combining the upper bound $(3/2)\alpha - 2$ for θ with the fact that $\mathbb{E}[\triangle_n] = O(n^{3-3/2\alpha})$ shows that the theorem applies until deviations of order n, as $n^{3-3/2\alpha}n^{(3/2)\alpha-2} = n$. Intuitively, this is because the inhomogeneous random graph has on average μn edges. A single high-degree vertex can therefore only create μn triangles. However, the maximum possible number of triangles in a graph on n vertices scales as n^3.

We complement Theorem 1.6 by studying the cases where $\alpha < 4/3$ or $\alpha > 4/3$ and deviations of more than a factor of $n^{(3/2)\alpha-2}$ from average:

Theorem 1.7. *Suppose that $\mathbb{P}(W \geq 1) = 1$ and $\mathbb{P}(W > x) \sim Cx^{-\alpha}$ for $x \geq 1$. For $\gamma \in (\max(1, 3 - 3/2\alpha), 3)$ and $\alpha > 1$ and $a > 0$,*

$$\lim_{n\to\infty} \frac{\log \mathbb{P}(\triangle_n > an^\gamma)}{n^{(\gamma-1)/2}\log(n)} = \sqrt{2a}\Big(\frac{3-\gamma}{2} - \alpha\Big) < 0. \tag{1.14}$$

The proof of this theorem is similar to the proof of Theorem 1.4, and can be found in Appendix C. The logarithmic asymptotics of Theorems 1.6 and 1.7 match at the boundary where $\gamma = 1$ or $\theta = (3/2)\alpha - 2$. We expect that a similar theorem as Theorem 1.5 holds in this case.

There is a sharp phase transition for $\alpha > 4/3$ between deviations up to a factor of $n^{(3/2)\alpha-2}$ (Theorem 1.6) and larger deviations (Theorem 1.7). Similarly to the smaller deviations, this phase transition happens when the single hub scaling equals n. Indeed, for Theorem 1.6, the driving event is one hub of magnitude $n^{(\alpha+2\theta)/(4(\alpha-1))}$. For $\theta = (3/2)\alpha - 2$, this means that one hub of order n is necessary. A hub weight of μn already makes all connection probabilities equal to one, so that increasing the hub weight further will not increase the number of triangles. Thus, to create even more triangles, a larger number of hubs is necessary, explaining the phase transition between Theorems 1.6 and Theorem 1.7.

Interestingly, in the regime of Theorem 1.7, a lower value of α makes the probability of n^θ more triangles than expected more likely than a higher value of α, contrary to the regime of Theorem 1.6.

1.3 Organization of the paper

In Section 2, we analyze the behavior of G_n in the case $\alpha > 4/3$, after having first developed a concentration bound (Proposition 2.2). The tail behavior of G_n for the boundary case $\alpha = 4/3$ is analyzed in Section 3. Section 4 completes the proofs of Theorem 1.3 and 1.5. The case with many hubs, in particular Theorem 1.4 is proven in Section 5.

We collect several proofs with more standard and/or repetitive arguments in the appendices. The proof of the lemmas presented so far, as well as proofs of various auxiliary results in the subsequent three sections are given in Appendix A. Auxiliary results for Section 5 are proven in Appendix B. Finally, Appendix C and D contain the necessary additional details which are needed to complete the proofs of Theorem 1.7 and Theorem 1.6.

2 Nonlinear heavy-tailed large deviations

Let F_n be the empirical distribution function associated with the weights $W_1, \ldots, W_n$ and observe that

$$G_n = n^3 \int_0^\infty \int_x^\infty \int_y^\infty f_n(x, y, z)dF_n(z)dF_n(y)dF_n(x), \tag{2.1}$$

with f_n as in (1.3). As a convention, integration regions are always of the form (x, ∞), (y, ∞), etc. to avoid double counting. This section proves the following theorem.

Theorem 2.1. *If $\alpha \in (4/3, 2)$, then*

$$\mathbb{P}\left(G_n > m_n(1+a)\right) = (1+o(1))n\mathbb{P}\left(W > c_a(n)\right). \tag{2.2}$$

This theorem serves as a major stepping stone towards the proof of Theorem 1.3, which will be completed in Section 4. The proof of Theorem 2.1 consists of the following steps. Let

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(W_i \le x), \qquad x \ge 0. \tag{2.3}$$

1. Building on concentration results for weighted empirical processes, dating back to [39], we construct an event of high probability on which we can bound F_n with a suitable function F_n^*, which is essentially a mixture of F, and two large (in n) atoms.
2. To effectively use F_n^*, we develop some estimates for the expected number of triangles generated by one or two large hubs.
3. We combine both previous steps in constructing a sharp bound for G_n which holds with high probability when there is no hub of size bigger than $\varepsilon c_a(n)$, for some $\varepsilon > 0$.
4. Using this sharp upper bound for G_n, we complete the proof of Theorem 2.1.

These four steps are worked out in the next four subsections.

2.1 Concentration of weighted empirical distribution functions

In this subsection, we construct a convenient upper bound for $\bar{F}_n(x) = 1 - F_n(x)$. Set uniform random variables $U_i = F(W_i), i \ge 1$. Set F_n^U as the empirical distribution function for uniform random variables. Define

$$h(x) = x(\log x - 1) + 1. \tag{2.4}$$

Then, Lemma 1 of [39] states, for $\lambda \ge 1, y \in (0,1]$,

$$\mathbb{P}\left(\sup_{t\in[y,1]} |F_n^U(t)/t| \ge \lambda\right) \le e^{-nyh(\lambda)}. \tag{2.5}$$

Now, let

$$a(n) = \bar{F}^{-1}(1/n)n^{-\delta},$$

for some $\delta > 0$. By taking $y = \bar{F}(a(n))$ and $\lambda = A + 1$ in (2.5) we get

$$\mathbb{P}\left(\sup_{x<a(n)} |\frac{\bar{F}_n(x)}{\bar{F}(x)} - 1| > A\right) \le e^{-n\bar{F}(a(n))h(A+1)}. \tag{2.6}$$

Let $\bar{F}(x) = 1 - F(x)$ and

$$b(n) = \bar{F}^{-1}(1/n)n^{\delta}.$$

Observe that (2.6) implies

$$\mathbb{P}\left(\sup_{x\in[a(n),b(n)]} \bar{F}_n(x) > (1+A)\bar{F}(a(n))\right) \le e^{-n\bar{F}(a(n))h(A+1)}, \tag{2.7}$$

as $\bar{F}_n(x)$ is non-increasing. Finally, we investigate the range $[b(n), \infty)$. Let $c > 0$ and note that

$$\mathbb{P}\left(\sup_{x\in[b(n),\infty)} \bar{F}_n(x) > c/n\right) \le P(\bar{F}_n(b(n)) > c/n) = \mathbb{P}\left(\sum_{i=1}^{n} I(U_i < \bar{F}(b(n))) > c\right).$$

We can use Lemma 2.3 from [37] to bound this and get

$$\mathbb{P}\left(\sup_{x\in[b(n),\infty)} \bar{F}_n(x) > c/n\right) \leq e^{n\bar{F}(b(n))}(n\bar{F}(b(n)))^{\lceil c\rceil}. \tag{2.8}$$

Note that the bounds (2.6), (2.7), (2.8) hold for any continuous distribution F with support on $[0,\infty)$. If $\bar{F}(x)$ is regularly varying with index $-\alpha < -1$, then $a(n)$ is regularly varying with index $1/\alpha - \delta$ and $b(n)$ is regularly varying with index $1/\alpha + \delta$. This makes the upper bounds in (2.6) and (2.7) go to 0 at a faster rate than polynomial.

If $\alpha > 1$ and $\delta > 0$ is such that $1/\alpha + \delta < 1$, the upper bound in (2.8) is regularly varying with index $-\lceil c\rceil(1 - 1/\alpha + \delta)$, which can be made to go to 0 at any desired polynomial rate by picking c appropriately large. We summarize our findings in the following proposition.

Proposition 2.2. *For a fixed $\delta \in (0, 1/\alpha - 1)$, $c > 0$, and $A > 0$ define the event $E_n(A, c, \delta)$ by*

$$\left\{\sup_{x<a(n)} \frac{\bar{F}_n(x)}{\bar{F}(x)} \leq 1 + A, \sup_{x\in[a(n),b(n)]} \bar{F}_n(x) \leq (1 + A)\bar{F}(a(n)), \sup_{x\in[b(n),\infty)} \bar{F}_n(x) \leq c/n\right\}. \tag{2.9}$$

Let $\beta > 0$. Then $\mathbb{P}(E_n(A, c, \delta)^c) = o(n^{-\beta})$ if $\lceil c\rceil > \beta(1 - 1/\alpha + \delta)^{-1}$.

Application to sample averages. To illustrate the use of Proposition 2.2, consider the sample mean $\tau_n = \int_0^\infty x dF_n(x)$; in particular the probability

$$\mathbb{P}(\tau_n > (1 + a)\mu). \tag{2.10}$$

A classical result dating back to Nagaev [30] is that the most likely way this event occurs is by a single big jump of size $a\mu n$ and that this probability is regularly varying with index $-(\alpha - 1)$. A critical step in the proof is to show that a jump of size at least εn is really necessary. In our notation, this entails showing that for every $\beta < \infty$ there exists an $\varepsilon > 0$ such that

$$\mathbb{P}\left(\int_0^{\varepsilon n} x dF_n(x) > \mu + a\right) = o(n^{-\beta}). \tag{2.11}$$

A version of this result (which we actually need in Section 4), can be found in [33], and can also be proven with Bennett's inequality. We now show how (2.11) follows from Proposition 2.2. On the set $E_n(A, \delta, c)$, the following upper bound holds for $\bar{F}_n$ on $x \in [0, \varepsilon n]$:

$$\begin{aligned}\bar{F}_n(x) \leq{}& (1 + A)\bar{F}(x)I(x < a(n)) + (1 + A)\bar{F}(a(n))I(a(n) \leq x < b(n)) \\ &+ \frac{c}{n} I(x \in [b(n), \varepsilon n]).\end{aligned} \tag{2.12}$$

Informally, the random variable with distribution function F_n truncated at εx is stochastically bounded by a random variable with distribution tail $(1 + A)\bar{F}(x)$ on $[0, a(n))$, an atom of size $\bar{F}(a(n)) - c/n$ at $b(n)$, and an atom of size c/n at εn. Since we integrate against a non-decreasing function, we see that, on $E_n(A, \delta, c)$,

$$\int_0^{\varepsilon n} x dF_n(x) \leq (1 + A)\int_0^{a(n)} x dF(x) + (1 + A)b(n)\bar{F}(a(n)) + (c/n)(\varepsilon n). \tag{2.13}$$

This is smaller than $\mu+c\varepsilon+o(1)$, since $b(n)\bar{F}(a(n))$ is regularly varying of index $1/\alpha+\delta-\alpha(1\alpha-\delta) < 0$. The desired result (2.11) now follows by choosing ε small enough so that $c\varepsilon < a$.

Below, we apply this bounding technique to the nonlinear functional (2.1). We believe the technique can be applied to other nonlinear functionals of F_n, like U-statistics, and other observables of inhomogeneous random graphs, such as clustering coefficients, degree correlations or general subgraph counts.

2.2 Estimating the number of triangles generated by large hubs

To successfully apply Proposition 2.2 to the particular nonlinear functional (2.1), we need several auxiliary estimates. In particular, the following two lemmas will be convenient in the estimation of various single and double integrals appearing in our upper bound of G_n, obtained after applying Proposition 2.2 to (2.1). These integrals approximate with high probability the number of additional triangles caused by one or two hubs. Their proofs can be found in Appendix A.

Lemma 2.3. *There exists a constant K_1 such that the following holds. Let $1 > \alpha_c \geq \alpha_b > 1/2$. Let $b(n)$ be regularly varying of index α_b and let $c(n)$ be regularly varying of index α_c with either $\alpha_c > \alpha_b$ or $c(n) = b(n)$. Then*

$$S_{b,c}(n) := \int_0^\infty f_n(x, b(n), c(n))dF(x) \sim K_1 \frac{b(n)}{c(n)} \bar{F}(n/c(n)). \tag{2.14}$$

In particular, $S_{b,c}(n)$ is regularly varying of index $-[\alpha(1-\alpha_c) + \alpha_c - \alpha_b]$.

Lemma 2.4. *There exists a constant K_2 such that the following holds. Let $1 > \alpha_b > 1/2$. Let $b(n)$ be regularly varying of index α_b. Then*

$$S_b(n) := \int_0^\infty \int_x^\infty f_n(x, y, b(n))dF(y)dF(x) \sim K_2 \frac{n}{(b(n))^2} \bar{F}(n/b(n))^2. \tag{2.15}$$

In particular, $S_b(n)$ is regularly varying of index $-[2(\alpha - 1)(1 - \alpha_b) + 1]$.

2.3 G_n cannot be large without a big hub

In this section, we establish a key result, namely that the following nonlinear analogue of (2.11) holds. Define $L_n(z)$ as the number of W_i for which $W_i > z$.

Proposition 2.5. *There exists an $\varepsilon > 0$ such that*

$$\mathbb{P}(G_n > (1+a)m_n; L_n(\varepsilon c_a(n)) = 0) = o(n\mathbb{P}(W_1 > c_a(n))). \tag{2.16}$$

Proof. On the set $E_n^{\varepsilon,c}$, the following upper bound holds for $\bar{F}_n$ on $x \in [0, \varepsilon c_a(n)]$:

$$\bar{F}_n(x) \leq \bar{F}_n^*(x), \tag{2.17}$$

with $F_n^*(x)$ defined by

$$(1+A)\bar{F}(x)I(x < a(n)) + (1+A)\bar{F}(a(n))I(a(n) \leq x < b(n)) + \frac{c}{n}I(x \in [b(n), \varepsilon c_a(n)]). \tag{2.18}$$

Again, on the set $E_n^{\varepsilon,c}$, the random variable with distribution function F_n truncated at εx is stochastically bounded by a random variable with distribution tail $(1+A)\bar{F}(x)$ on $[0, a(n))$, an atom of size $\bar{F}(a(n)) - c/n$ at $b(n)$, and an atom of size c/n at $\varepsilon c_a(n)$. We have that $b(n)/c_a(n) \to 0$ [for δ small enough, to be specified later]. Since f_n is nondecreasing in each coordinate, we see that, on the set $E_n(A, \delta, c) \cap \{L_n(\varepsilon c_a(n)) = 0\}$,

$$G_n \leq n^3 \int_0^\infty \int_x^\infty \int_y^\infty f_n(x, y, z)dF_n^*(z)dF_n^*(y)dF_n^*(x). \tag{2.19}$$

The next step is to evaluate the integral on the RHS, where we need to keep track of the value of each of the 3 coordinates: they may be of (s)mall ($< a(n)$), (m)edium ($b(n)$) or (l)arge ($\varepsilon c_a(n)$) value. There are 10 different combinations: (s,s,s), (s,s,m), (s,s,l), (s,m,m), (s,l,l), (s,m,l), (m,m,m), (m,m,l), (m,l,l), (l,l,l). Thus, on the set $E_n(A, c, \delta) \cap \{L_n(\varepsilon c_a(n)) = 0\}$,

$$\frac{G_n}{1+A} \leq m_n + n^3 \bar{F}(a(n)) \int_0^\infty \int_x^\infty f_n(x, y, b(n))dF(y)dF(x)$$

$$
\begin{aligned}
&+ n^3 \frac{c}{n} \int_0^\infty \int_x^\infty f_n(x, y, \varepsilon c(n)) dF(y) dF(x) \\
&+ n^3 (\bar{F}(a(n)))^2 \int_0^\infty f_n(x, b(n), b(n)) dF(x) \\
&+ n^3 (c/n)^2 \int_0^\infty f_n(x, \varepsilon c_a(n), \varepsilon c_a(n)) dF(x) \\
&+ n^3 \bar{F}(a(n)) \frac{c}{n} \int_0^\infty f_n(x, b(n), \varepsilon c_a(n)) dF(x) \\
&+ n^3 (\bar{F}(a(n)))^3 f_n(b(n), b(n), b(n)) + n^3 (\bar{F}(a(n)))^2 (c/n) f_n(\varepsilon c(n), b(n), b(n)) \\
&+ n^3 (\bar{F}(a(n)))(c/n)^2 f_n(\varepsilon c(n), \varepsilon c(n), b(n)) + n^3 (c/n)^3 f_n(\varepsilon c(n), \varepsilon c(n), \varepsilon c(n)). \qquad (2.20)
\end{aligned}
$$

Apart from the main term m_n, we need to bound 9 terms in total. In what follows, we often use that $a(n)$ is regularly varying of index $1/\alpha - \delta$, that $\bar{F}(a_n)$ is regularly varying of index $\alpha\delta - 1$, that $b(n)$ is regularly varying of index $1/\alpha + \delta$, and that $c(n) = c_a(n)$ is regularly varying of index $\alpha_c = \frac{1}{4}\frac{\alpha}{\alpha-1}$. The last 4 terms are all bounded by at most $O(n^{3\alpha\delta})$ since $f_n \leq 1$, and are therefore $o(m_n)$. We now examine Terms 2–6 in more detail.

Term 2: Lemma 2.4 with $\alpha_b = 1/\alpha + \delta$ yields that $n^3 \bar{F}(a(n)) \int f_n(x, y, b(n)) dF(x) dF(y)$ is regularly varying of index $5 - 2(\alpha + 1/\alpha) + \delta(3\alpha - 2)$. For δ small enough, this can be made strictly smaller than $3 - \alpha 3/2$, using the fact that $\alpha + 1/\alpha > 2$ when $\alpha \in (1, 2)$. Thus, we can conclude that, for δ sufficiently small,

$$
n^3 \bar{F}(a(n)) \int_0^\infty \int_x^\infty f_n(x, y, b(n)) dF(y) dF(x) = o(m_n). \qquad (2.21)
$$

Term 3: invoking Lemma 2.4 with $\varepsilon c_a(n)$, and using definition (1.5) we obtain that

$$
n^3 \frac{c}{n} \int_0^\infty f_n(x, y, \varepsilon c_a(n)) dF(x) dF(y) \sim c \varepsilon^{(\alpha-1)2} a m_n, \qquad (2.22)
$$

for every $\varepsilon > 0$.

Term 4: invoking Lemma 2.3 with both sequences equal to $b(n)$, and $\alpha_b = \alpha_c = 1/\alpha + \delta$, it follows that $n^3 (\bar{F}(a(n)))^2 \int f_n(x, b(n), b(n)) dF(x)$ is regularly varying of index $2 - \alpha + \delta 3\alpha$, which is smaller than $3 - \alpha 3/2$ for a suitable choice of δ, as $2 - \alpha < 3 - \alpha 3/2$ for $\alpha < 2$. Thus, we can conclude that, for δ sufficiently small,

$$
n^3 (\bar{F}(a(n)))^2 \int_0^\infty f_n(x, b(n), b(n)) dF(x) = o(m_n). \qquad (2.23)
$$

Term 5: invoking Lemma 2.3 with both sequences equal to $\varepsilon c_a(n)$ and $\alpha_b = \alpha_c = \frac{1}{4}\frac{\alpha}{\alpha-1}$ it follows that $n^3 (c/n)^2 \int_0^\infty f_n(x, \varepsilon c_a(n), \varepsilon c_a(n)) dF(x)$ is regularly varying of index $1 - \alpha + \alpha^2/(4(\alpha - 1)$. This is strictly smaller than $3 - \alpha 3/2$: the inequality $1 - \alpha + \alpha^2/(4(\alpha - 1) < 3 - \alpha 3/2$ can be rewritten into $(\alpha - 4/3)(\alpha + 2) < 0$ which is true due to our assumption $\alpha \in (4/3, 2)$. Thus, we can conclude that, for δ sufficiently small,

$$
n^3 (c/n)^2 \int_0^\infty f_n(x, \varepsilon c_a(n), \varepsilon c_a(n)) dF(x) = o(m_n). \qquad (2.24)
$$

Term 6: invoking Lemma 2.3 with sequences $b(n)$ and $\varepsilon c_a(n)$, such that $\alpha_b = 1/\alpha + \varepsilon$ and $\alpha_c = \frac{1}{4}\frac{\alpha}{\alpha-1}$, it follows that this term behaves like behaves like Term 5, times an additional factor which is regularly varying of index $\delta(1 + \alpha) - (\frac{1}{4}\frac{\alpha}{\alpha-1} - \frac{1}{\alpha})$. As this factor converges to 0 for sufficiently small δ, we conclude also that

$$
n^3 \bar{F}(a(n)) \frac{c}{n} \int_0^\infty f_n(x, b(n), \varepsilon c_a(n)) dF(x) = o(m_n). \qquad (2.25)
$$

Concluding, we see that, for every $A > 0$, on the set $E_n(A, \delta, c) \cap \{L_n(\varepsilon c_a(n)) = 0\}$,

$$G_n \leq (1 + A)m_n(1 + 3c\varepsilon^{(\alpha-1)2}a)(1 + o(1)) \tag{2.26}$$

which is strictly smaller than $1 + a$ for A, ε sufficiently small. We conclude that

$$\mathbb{P}\left(G_n > (1 + a)m_n; E_n(A, \delta, c) \cap \{L_n(\varepsilon c_a(n)) = 0\}\right) = 0$$

for sufficiently large n, so that

$$\begin{aligned} &\mathbb{P}\left(G_n > (1 + a)m_n; \{L_n(\varepsilon c_a(n)) = 0\}\right) \\ &\quad \leq \mathbb{P}\left(G_n > (1 + a)m_n; \{L_n(\varepsilon c_a(n)) = 0\}; E_n(A, \delta, c)\right) + \mathbb{P}\left(E_n(A, \delta, c)^c\right), \end{aligned} \tag{2.27}$$

which can be made to go to 0 at any polynomial rate by a suitable choice of c and ε, using Proposition 2.2. □

2.4 Proof of Theorem 2.1

Using a simple bound for binomial distributions (e.g. [37, Lemma 2.3]) one can show that

$$\mathbb{P}\left(G_n > (1 + a)m_n; L_n(\varepsilon c_a(n)) \geq 2\right) \leq \mathbb{P}\left(L_n(\varepsilon c_a(n)) \geq 2\right) = o(n\mathbb{P}\left(X_1 > c_a(n)\right)). \tag{2.28}$$

In view of this estimate, Proposition 2.5 and symmetry of G_n as a function of the weights, it suffices to show that

$$P(G_n > (1 + a)m_n, W_n > \varepsilon c_a(n) > W_i, i < n) = (1 + o(1))\mathbb{P}\left(W > c_a(n)\right). \tag{2.29}$$

Write $D_n = \{W_n > \varepsilon c_a(n) > W_i, i < n\}$ and

$$\mathbb{P}\left(G_n > (1 + a)m_n, D_n\right) = \mathbb{P}\left(G_n > (1 + a)m_n \mid D_n\right)\mathbb{P}\left(D_n\right). \tag{2.30}$$

Next, we condition on the value of $W_n/c_a(n)$ given D_n:

$$\begin{aligned} &\mathbb{P}\left(G_n > (1 + a)m_n \mid D_n\right) \\ &= \int_{y=\varepsilon}^{\infty} \mathbb{P}\left(G_n > (1 + a)m_n \mid D_n; W_n = yc_a(n)\right) d\mathbb{P}\left(W_n/c_a(n) \leq y \mid D_n\right). \end{aligned} \tag{2.31}$$

We now state the following proposition, providing a version of the weak law of large numbers for G_n, which will be proven later on.

Proposition 2.6. *As $n \to \infty$,*

$$\mathbb{P}\left(G_n > (1 + a)m_n \mid D_n; W_n = yc_a(n)\right) \to \begin{cases} 0 & y < 1, \\ 1 & y > 1 \end{cases} \tag{2.32}$$

Applying Proposition 2.6 to (2.31), we see that

$$\mathbb{P}\left(G_n > (1 + a)m_n \mid D_n\right) \sim \mathbb{P}\left(W_1 > c_a(n) \mid D_n\right), \tag{2.33}$$

which together with (2.30) implies (2.29), proving Theorem 2.1.

Proof of Proposition 2.6. We first prove (2.32) for $y > 1$. For $W_i^z = W_i I(W_i < z)$, define the truncated mean $m_n(z) = E[g_n(X_1^z, ..., X_n^z)]$, and observe that (e.g. by inspecting the proof of Lemma 1.1), for $a(n) = \bar{F}^{-1}(n)n^{-\delta}$,

$$m_n(a(n)) = (1 + o(1))m_n \tag{2.34}$$

if $\delta > 0$ is small enough. Now, observe that, for any constant $A > 0$, with large probability, (2.6) implies that $\bar{F}_n(x) \geq (1-A)\bar{F}(x)$ on $x \in [0, a(n)]$. On the intersection of this event with D_n and $W_n = yc_a(n)$ we can write, since $a(n) \geq \varepsilon c_a(n)$ for n large enough,

$$\begin{aligned} G_{n+1}(W_1^{a(n)}, ..., W_n^{a(n)}, yc_n(a)) &\geq n^3 \int_0^{a(n)} \int_u^{a(n)} \int_v^{a(n)} f_n(u,v,w) dF_n(w) dF_n(v) dF_n(u) \\ &\quad + n^2 \int_0^{a(n)} \int_u^{a(n)} f_n(u,v,yc_a(z)) dF_n(v) dF_n(u) \\ &\geq (1-A)m_n(a(n)) + (1-A)m_n a y^{3-\alpha 3/2}(1+o(1)), \end{aligned} \tag{2.35}$$

where we applied also Lemma 2.4 with $yc_a(n)$, the defining property of $c_a(n)$, and regular variation. Since $y > 1$, there exists an $A > 0$ such that the RHS is larger than $(1+a)m_n$ with high probability, in view of (2.34), which proves (2.32) for $y > 1$.

We proceed with $y < 1$. Fix $y \in (0,1)$ and let $\varepsilon \in (0, y)$. Recall the definition (2.18) of F_n^*. For any $A > 0$ we have with high probability on D_n

$$\begin{aligned} G_{n+1}(W_1^{\varepsilon c_a(a)}, ..., W_n^{\varepsilon c_a(a)}, yc_n(a)) \leq &\text{ RHS of (2.20)} \\ &+ n^2 \int_0^{\varepsilon c_a(n)} \int_u^{\varepsilon c_a(n)} f_n(u,v,yc_a(z)) dF_n^*(v) dF_n^*(u). \end{aligned} \tag{2.36}$$

As we have proven, the RHS of (2.20) is close to m_n by taking ε sufficiently small. It therefore suffices to show that the second term is strictly smaller than am_n for n sufficientlly large. The second term can be analyzed in a way similar to (2.20). In particular, we can upper bound it with

$$\begin{aligned} &O(n^{2\alpha\delta}) + n^2(1+A)S_{yc_a(n)}(n) + n^2(1+A)\frac{c}{n}S_{\varepsilon c_a(n), yc_a(n)} \\ &\quad + n^2(1+A)\bar{F}(a(n))S_{b(n), yc_a(n)}(n). \end{aligned}$$

Call the three main terms on the RHS of this expression Term A,B,C. Term B behaves similar to Term 5 in the RHS of (2.20), and Term C behaves like Term 6 in the RHS of (2.20). In particular, both terms are $o(m_n)$. Finally, Term A behaves like $(1+A)m_n a y^{3-\alpha 3/2}$, in view of Lemma 2.4 with $yc_a(n)$, the defining property of $c_a(n)$, and regular variation. Since $y < 1$, there exists an $A > 0$ such that the last display is strictly smaller than am_n for n sufficiently large, proving (2.32) for $y < 1$. □

3 The boundary case $\alpha = 4/3$

Recall that, for $b > 0$, $X_i^b, i \geq 1$, is an i.i.d. sequence such that $\mathbb{P}(X_i^b > x) = (x/b)^{-\alpha}, x \geq b$. Set $\eta(a)$ as the smallest number η for which $((k(a)-1)\mu + K_1(\eta) \geq C^3 H(1+a)$. Note that $\eta(a) > 0$ when $(k(a)-1)\mu + (k(a)-1)(k(a)-2)/2 < aC^3 H$.

The goal of this section is to prove the following theorem, which serves as a major stepping stone towards Theorem 1.5.

Theorem 3.1. *Suppose that $P(W > x) \sim Cx^{-4/3}$ and suppose that $(k(a)-1)\mu + (k(a)-1)(k(a)-2)/2 < aC^3 H$. Then*

$$\mathbb{P}(G_n > (1+a)m_n) \sim \mathbb{P}(K_{k(a)}(X_1^{\eta(a)}, \ldots, X_{k(a)}^{\eta(a)}) \geq C^3 Ha)(n\mathbb{P}(W > \eta(a)n))^{k(a)}. \tag{3.1}$$

A key step towards proving Theorem 3.1 is to show that $k(a)$ hubs are really needed. Its proof is a refinement of (and builds on) the arguments developed in the case $\alpha > 4/3$.

Proposition 3.2. *For every $\beta > 0$ there exists an $\varepsilon > 0$ such that*

$$\mathbb{P}(G_n > m_n(1+a), L_n(\varepsilon n) < k(a)) = o(n^{-\beta}). \tag{3.2}$$

Proof. Fix $l < k(a)$ and observe that $G_n \leq G_{n+l}$. Note that

$$\mathbb{P}\left(G_{n+l} > m_n(1+a); L_n(\varepsilon n) = l\right) = \binom{n+l}{l}\mathbb{P}\left(G_{n+l} > m_n(1+a); W_i < \varepsilon n \text{ iff } i \leq n\right). \quad (3.3)$$

On the event $\{W_i < \varepsilon n \text{ iff } i \leq n\}$ we can write

$$\begin{aligned} G_{n+l} = G_n + n^2 \sum_{i=1}^{l} \int_0^{\varepsilon n} \int_x^{\varepsilon n} f_n(x, y, W_{n+i}) dF_n(y) dF_n(x) \\ + n \sum_{i=1, j>i}^{l} \int_0^{\varepsilon n} f_n(x, W_{n+i}, W_{n+j}) dF_n(x). \end{aligned} \quad (3.4)$$

We wish to use Proposition 2.5 with $\varepsilon c_a(n)$ replaced by εn to show that G_n is sufficiently close to m_n with high probability, but in the analysis of Term 5 in (2.20) we assumed that $\alpha > 4/3$. For $\alpha = 4/3$, this term behaves as

$$n^3(c/n)^2 \int_0^\infty f_n(x, \varepsilon n, \varepsilon n) dF(x) \sim nc^2 \int_0^\infty \min\{x, \varepsilon/\mu\}^2 dF(x) \leq \varepsilon^2 nc^2/\mu$$

which is negligible as $\varepsilon \downarrow 0$. In addition, Term 3 in (2.20) behaves like $nC_1(\varepsilon)$ when $\alpha = 4/3$ and $c_a(n)$ is replaced by εn which is negligible since $C_1(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$. Based on this, and repeating the arguments in the other terms in (2.20) for $\alpha = 4/3$ and $c_a(n)$ replaced with εn, we obtain that for every $\beta < \infty$ and $\eta > 0$ there exists a $c > 0$ and $\varepsilon > 0$ such that

$$\mathbb{P}\left(G_n > m_n(1+\eta); W_i < \varepsilon n, i \leq n\right) = o(n^{-\beta}). \quad (3.5)$$

We now analyze the second term in (3.4). By bounding W_{n+i} with ∞ we get

$$n^2 \int_0^\infty \int_x^\infty f_n(x, y, W_{n+i}) dF_n(y) dF_n(x) \leq \frac{n}{\mu}\Big(\int_0^{\varepsilon n} x dF_n(x)\Big)^2 \leq n\mu. \quad (3.6)$$

Combining this bound with the estimate (2.11) implies that for every $\beta < 0$ and $\eta > 0$ there exists an $\varepsilon > 0$ such that

$$\mathbb{P}\left(n^2 \sum_{i=1}^{l} \int_0^{\varepsilon n} \int_x^{\varepsilon n} f_n(x, y, W_{n+i}) dF_n(x) dF_n(y) > n(l\mu + \eta); W_i < \varepsilon n, i \leq n\right) = o(n^{-\beta}). \quad (3.7)$$

The proof is now finished by bounding the last term in (3.4) by $nl(l-1)/2$, and combining it with (3.5), (3.7), and (3.3), noting that $l\mu + l(l-1)/2 < aC^3H$ since $l < k(a)$. □

Using a simple tail bound for binomial distributions we obtain

Proposition 3.3.

$$\mathbb{P}\left(G_n > m_n(1+a), L_n(\varepsilon n) > k(a)\right) = o(n^{k(a)}\mathbb{P}\left(W > n\right)^{k(a)}). \quad (3.8)$$

Define

$$\begin{aligned} G_{n,l}(z_1, ..., z_l) = n^2 \sum_{i=1}^{l} \int_0^\infty \int_x^\infty f_n(x, y, nz_i) dF_n(x) dF_n(y) \\ + n \sum_{i=1, j>i}^{l} \int_0^\infty f_n(x, nz_i, nz_j) dF_n(x). \end{aligned} \quad (3.9)$$

Our next proposition is the final main ingredient of the proof of Theorem 3.1.

Proposition 3.4. *The following convergence holds in probability:*

$$\frac{1}{n}G_{n,l}(z_1,...,z_l) \to K_l(z_1,...,z_l). \tag{3.10}$$

Proof. Using definition (1.3), write the ith term of the first part of $G_{n,l}(z_1,...,z_l)/n$ as

$$\int_0^\infty \int_x^\infty \min\{xy/\mu, n\} \min\{xz_i/\mu, 1\} \min\{yz_i/\mu, 1\} dF_n(y) dF_n(x).$$

We proceed by analyzing upper and lower bounds. For the upper bound, use $\min\{xy/\mu, n\} \le xy/\mu$ to get the upper bound $(\int_0^\infty x \min\{xz_i/\mu, 1\} dF_n(x))^2/\mu$. Furthermore,

$$\int_0^\infty x \min\{xz_i/\mu, 1\} dF_n(x) = \int_0^{\mu/z_i} ((z_i/\mu)x^2 - x) dF_n(x) + \int_0^\infty x dF_n(x).$$

The first integral converges as $F_n \to F$, and the second integral converges due to the weak law of large numbers. The limit equals $(z_i/\mu)\mathbb{E}\left[W^2 I(W \le \mu/z_i)\right] + \mathbb{E}\left[WI(W > \mu/z_i)\right]$. This leads to the desired upper bound. A lower bound follows by bounding $\min\{xy/\mu, n\}$ from below by $\min\{xy/\mu, K\}$, using that $F_n \to F$ and Fatou's lemma, and then take K arbitrarily large. Finally, note that each term in the second part of $G_{n,l}(z_1,...,z_l)/n$ converges to its desired limit using the bounded convergence theorem. □

Proof of Theorem 3.1. Abbreviate $k = k(a)$. Using straightforward combinatorial arguments, it suffices to show that

$$\begin{aligned}&\mathbb{P}\left(G_{n,l}(W_{n+1},...,W_{n+k}) > m_n a, W_i < \varepsilon n \text{ iff } i \le n\right) \\ &\quad \sim \mathbb{P}(K_{k(a)}(X_1^{\eta(a)}, \ldots, X_{k(a)}^{\eta(a)}) \ge C^3 Ha)(\mathbb{P}\left(W > \eta(a)n\right))^{k(a)}.\end{aligned} \tag{3.11}$$

We now write the probability on the LHS of the last display as

$$\begin{aligned}&\int_{(\varepsilon,\infty)^k} \mathbb{P}\left(G_{n,l}(z_1, \ldots, z_k) > m_n a; W_i < \varepsilon n, i \le n\right) d\prod_{i=1}^k \mathbb{P}\left(\frac{W_i}{n} \le z_i \mid W_i > \varepsilon n\right) \\ &\times \mathbb{P}\left(W_1 > \varepsilon n\right)^k.\end{aligned} \tag{3.12}$$

Now $\mathbb{P}\left(W_i/n \le z_i \mid W_i > \varepsilon n\right)$ converges to the continuous distribution $\mathbb{P}\left(X_i^\varepsilon \le x_i\right)$. Recalling that $m_n \sim nC^3H$, and applying proposition 3.4, we obtain that the integral in the last display converges to $\mathbb{P}\left(K_{k(a)}(X_1^\varepsilon, \ldots, X_{k(a)}^\varepsilon) \ge C^3 Ha\right)$.

Because $(k(a)-1)\mu + (k(a)-1)(k(a)-2)/2 < aC^3H$, $K_{k(a)}(\varepsilon, \infty, \ldots, \infty) = 0$ for all $\varepsilon < \eta(a)$. Since $K_{k(a)}$ is symmetric, a similar property holds for the other coordinates. Therefore, if $\varepsilon < \eta(a)$,

$$\mathbb{P}(K_{k(a)}(X_1^\varepsilon, \ldots, X_{k(a)}^\varepsilon) \ge a) = (\eta/\varepsilon)^{-k(a)\alpha}\mathbb{P}(C(X_1^{\eta(a)}, \ldots, X_{k(a)}^{\eta(a)}) \ge a). \tag{3.13}$$

Furthermore, by regular variation,

$$\mathbb{P}(W_1 > \varepsilon n)^{k(a)} \sim (\eta(a)/\varepsilon)^{k(a)\alpha}\mathbb{P}(W_1 > \eta(a)n)^{k(a)}. \tag{3.14}$$

Putting everything together, we conclude that (3.11) holds. □

4 Completing the proofs of Theorem 1.3 and 1.5

In this section we use the precise tail asymptotics for G_n, $\alpha > 4/3$ in Theorem 2.1 and for $\alpha = 4/3$ in Theorem 3.1, to complete the proofs of Theorem 1.3 and 1.5. Our argument will be based on the identity $G_n = E[\triangle_n \mid W_1, ..., W_n]$, an argument showing that $\triangle_n$ and G_n are close, also in the rare event context we consider. Our proof is based on asymptotic upper and lower bounds, which are facilitated by two auxiliary lemmas. The first lemma is helpful for an asymptotic upper bound.

Lemma 4.1. *For any $\zeta > 0$ there exists some $\varepsilon > 0$, such that*

$$\mathbb{P}\left(\triangle_n \geq (1+\zeta)G_n\right) \leq \exp(-n^{\varepsilon}). \tag{4.1}$$

We prove this lemma in the next subsection, using a recent concentration bound from [9]. A crucial argument in the lower bound is to show that large hubs generate sufficiently many additional triangles. This is covered by the next lemma.

Lemma 4.2. *Let $\triangle_n(\delta, a)$ be the number of triangles containing nodes i such that $W_i \leq n^{1/2+\delta}$ for $i < n$ and node n with $W_n = c_a(n)$. The following convergence holds in probability for $\alpha \geq 4/3$, and $\delta > 0$ such that $1/2 + \delta < 1/\alpha$:*

$$\triangle_n(\delta, a)/m_n \to 1 + a. \tag{4.2}$$

Let $\triangle_{n,l}(\delta, z_1, ..., z_l)$ be the number of triangles containing nodes i such that $W_i \leq n^{1/2+\delta}$ for $i > l$ and $W_{n-i} = nz_i, i = 1, ..., l$. The following convergence holds in probability as $n \to \infty$ for $\alpha = 4/3$, and $\delta > 0$ such that $1/2 + \delta < 3/4$:

$$\triangle_{n,l}(\delta, z_1, ..., z_l)/n \to C^3H + C_l(z_1, ..., z_l). \tag{4.3}$$

Proof of Theorem 1.3 and Theorem 1.5. The proofs of both theorems are similar. We first prove an asymptotic upper bound. Write for $\zeta > 0$,

$$\begin{aligned}\mathbb{P}\left(\triangle_n > (1+a)m_n\right) &\leq \mathbb{P}\left(\triangle_n \geq (1+\zeta)G_n\right) + \mathbb{P}\left(G_n/(1+\zeta) \geq \triangle_n \geq (1+a)m_n\right) \\ &\leq e^{-n^{\varepsilon}} + \mathbb{P}\left(G_n > (1+(a-\zeta)(1-\zeta))m_n\right).\end{aligned}$$

Consequently,

$$\limsup_{n\to\infty} \frac{\mathbb{P}\left(\triangle_n > (1+a)m_n\right)}{\mathbb{P}\left(G_n > (1+a)m_n\right)} \leq \limsup_{n\to\infty} \frac{\mathbb{P}\left(G_n > (1+(a-\zeta)(1-\zeta))m_n\right)}{\mathbb{P}\left(G_n > (1+a)m_n\right)}$$

The RHS of this expression converges to 1 as $\zeta \downarrow 0$: for $\alpha > 4/3$ this follows from Theorem 2.1, and for $\alpha = 4/3$ this follows from Theorem 3.1, in particular from (3.1): under our assumptions, $k(\cdot)$ is continuous at a, and so are $\eta(a)$ and $\mathbb{P}(K_{k(a)}(X_1^{\eta(a)}, \ldots, X_{k(a)}^{\eta(a)}) \geq C^3Ha)$.

We proceed with the proof of an asymptotic lower bound. First, consider $\alpha > 4/3$, and write, for sufficiently large n

$$\mathbb{P}\left(\triangle_n > (1+a)m_n\right) \geq n\mathbb{P}\left(W_n > c_{a+\zeta}(n)\right)\mathbb{P}\left(\triangle_n(\delta, a+\zeta) > (1+a)m_n\right).$$

By Lemma 4.2, $\mathbb{P}\left(\triangle_n(\delta, a+\eta) > (1+a)m_n\right) \to 1$ for sufficiently small h, and we obtain

$$\liminf_{n\to\infty} \frac{\mathbb{P}\left(\triangle_n > (1+a)m_n\right)}{\mathbb{P}\left(G_n > (1+a)m_n\right)} \geq \liminf_{n\to\infty} \frac{\mathbb{P}\left(W > c_{a+\zeta}(n)\right)}{\mathbb{P}\left(W > c_a(n)\right)}. \tag{4.4}$$

The RHS converges to 1 as $\zeta \downarrow 0$ due to the properties of $c_a(n)$ in Lemma 1.2.

Next, consider $\alpha = 4/3$, and write, for $\zeta > 0$ and sufficiently large n

$$\begin{aligned}\mathbb{P}\left(\triangle_n > (1+a)m_n\right) \geq n^{k(a)}&\mathbb{P}\left(K_{k(a)}(W_1/n, ..., W_{k(a)}/n) > (a+\zeta)m_n\right) \\ &\cdot \mathbb{P}\left(\triangle_n > (1+a)m_n \mid K_{k(a)}(W_1/n, ..., W_{k(a)}/n) > (a+\zeta)m_n\right).\end{aligned}$$

By conditioning on $(W_1/n, ..., W_{k(a)}/n)$ and applying the second part of Lemma 4.2, we see that $\mathbb{P}\left(\triangle_n > (1+a)m_n \mid K_{k(a)}(W_1/n, ..., W_{k(a)}/n) > (a+\zeta)m_n\right) \to 1$. Since, for sufficiently small $\zeta > 0$, $k(a) = k(a+\zeta)$, we have that

$$\begin{aligned}&\mathbb{P}\left(K_{k(a)}(W_1/n, ..., W_{k(a)}/n) > (a+\zeta)m_n\right) \\ &\sim \mathbb{P}(K_{k(a)}(X_1^{\eta(a+\zeta)}, \ldots, X_{k(a)}^{\eta(a+\zeta)}) \geq C^3H(a+\zeta))(\mathbb{P}\left(W > \eta(a+\zeta)n\right))^{k(a)}.\end{aligned} \tag{4.5}$$

Consequently, using Theorem 3.1,

$$\liminf_{n\to\infty} \frac{\mathbb{P}\left(\triangle_n > (1+a)m_n\right)}{\mathbb{P}\left(G_n > (1+a)m_n\right)}$$
$$\geq \liminf_{n\to\infty} \frac{\mathbb{P}(K_{k(a)}(X_1^{\eta(a+\zeta)}, \ldots, X_{k(a)}^{\eta(a+\zeta)}) \geq C^3 H(a+\zeta))(\mathbb{P}\left(W > \eta(a+\zeta)n\right))^{k(a)}}{\mathbb{P}(C_{k(a)}(X_1^{\eta(a)}, \ldots, X_{k(a)}^{\eta(a)}) \geq C^3 Ha)(\mathbb{P}\left(W > \eta(a)n\right))^{k(a)}}. \tag{4.6}$$

Since $\eta(\cdot)$ is continuous at a, the RHS converges to 1 as $\zeta \downarrow 0$. □

4.1 Proof of Lemma 4.1

The number of triangles $\triangle_n$ equals the sum of the indicators that i, j, k forms a triangle over all i, j, k. In the proofs, we will often make use of a recent concentration bound from [9] to deal with the dependencies of the presences of different triangles, which we state here for completeness:

Lemma 4.3 (Theorem 3.1 from [9])**.** *Let F be a finite set and $(X_i)_{i\in F}$, $(X_i')_{i\in F}$ $(X_{i(j)})_{i,j\in F}$ be collections of nonnegative random variables with finite moment generating functions, defined on the same probability space, and satisfying the following conditions:*

a) For all i, $X_i \leq X_i'$.

b) For all i, the random variables X_i' and $\sum_{j\in F} X_{j(i)}$ are independent.

c) For all i, $\sum_{j\in F} X_{j(i)} \leq \sum_{j\in F} X_j$.

d) There is a constant a such that for all i, when $X_i > 0$, we have

$$\sum_{j\in F} X_j \leq a + \sum_{j\in F} X_{j(i)}. \tag{4.7}$$

Let $\lambda = \sum_{j\in F} \mathbb{E}\left[X_j'\right]$. Then for any $t \geq \lambda$,

$$\mathbb{P}\left(\sum_{i\in F} X_i \geq t\right) \leq \exp\Big(-\frac{t}{a}\log\Big(\frac{t}{\lambda} - 1 + \frac{\lambda}{t}\Big)\Big). \tag{4.8}$$

We will apply this lemma to the number of triangles, where $X_{ijk(uvw)}$ deals with the dependence of the events that the triangle ijk is present and the event that the triangle uvw is present. To carry out this idea, we need several additional supporting results.

The second preparatory lemma shows that edges between vertices of low weights appear in relatively few triangles:

Lemma 4.4. *Suppose that $1 < \alpha < 2$. Then, when $K > n^{(2-\alpha)/2+\varepsilon}$ for some $\varepsilon > 0$,*

$$\mathbb{P}\left(\{i,j\} \text{ in } \geq K \text{ triangles} \mid W_iW_j < \mu n\right) \leq \exp(-c_1 K), \tag{4.9}$$

for some $c_1 > 0$ and n sufficiently large.

The proof of this lemma is based on bounding the number of triangles for low-weight vertices from above with a binomial random variable with the right probability, and can be found in Appendix A.

Define the event $E_w = \{W_1 = w_1, ..., W_n = w_n\}$ and set $g_n = \sum_{i<j<k} f_n(w_i, w_j, w_k)$ as the expected number of triangles conditional on specific values of the weights. Let $\triangle_n(w)$ be the number of triangles on the event E_w. Finally, set for $\zeta > 0$,

$$J(\eta) = (1+\zeta)\log(\zeta + 1/(1+\zeta))/3. \tag{4.10}$$

Lemma 4.5. *There exists an $\varepsilon > 0$ such that, for $\zeta > 0$, and all $w = (w_1, ..., w_n)$:*

$$\mathbb{P}\left(\triangle_n(w) > (1+\zeta)g_n\right) \le e^{-J(\zeta)g_n/n^{2-\alpha}} + e^{-n^{\varepsilon}}. \tag{4.11}$$

Proof of Lemma 4.5. By Lemma 4.4, when $w_i w_j < \mu n$, and choosing $K = n^{2-\alpha}$,

$$\mathbb{P}\left(\{i,j\} \text{ in } \ge n^{2-\alpha} \text{ triangles}\right) \le \exp(-K_1 n^{2-\alpha}), \tag{4.12}$$

for some $K_1 > 0$. This indicates that with probability at least $1-n^2\exp(-K_1 n^{2-\alpha}) \ge 1-\exp(-n^{\varepsilon})$, all edges between vertices of weights $w_i w_j < \mu n$ are in at most $n^{2-\alpha}$ triangles. We now work on this event, which we call $\mathcal{E}'$.

We set $X_{ijk} = X'_{ijk}$ as the indicator that a triangle is present between vertices i, j, k. Furthermore, we set $X_{ijk(uvw)} = X_{ijk}$ when $|\{i,j,k,u,v,w\}| \ge 5$. When $|\{i,j,k,u,v,w\}| = 4$, we set $X_{ijk(uvw)} = X_{ijk}$ when the overlap of ijk and uvw occurs at an edge with $w_u w_v > \mu n$, and we set $X_{ijk(uvw)} = 0$ otherwise or when $\{i,j,k\} = \{u,v,w\}$. Then, $\sum_{i,j,k} X_{ijk(uvw)}$ and X_{uvw} are independent. Indeed, when two triangles do not overlap at an edge, their presence is independent conditionally on the weights, as the edge indicators are independent conditionally on the weights. When the edge overlap occurs at an edge that is present with probability one, the presence of the two triangles is still independent conditionally on the weights. In all other cases, $X_{ijk(uvw)} = 0$, which is also independent of X_{uvw}.

On the event $\mathcal{E}'$,

$$\triangle_n = \sum_{i,j,k} X_{ijk} \le \sum_{ijk} X_{ijk(uvw)} + 3n^{2-\alpha}. \tag{4.13}$$

Lemma 4.3 with $a = 3n^{2-\alpha}$, $t = (1+\zeta)g_n$ and $\lambda = g_n$ concludes the assertion. □

Proof of Lemma 4.1. Recall that $F_n(x)$ denotes the empirical weight distribution. Now by (4.16), the event $\mathcal{E}_1 = \{\sup_{x<\sqrt{\mu n}} |\frac{\bar{F}_n(x)}{\bar{F}(x)} - 1| \le \eta\}$ happens with probability of at least $1 - e^{-n^{\varepsilon}}$ for some $\varepsilon > 0$. Fix some $a < 1$. On the event $\mathcal{E}_1$,

$$\begin{aligned} G_n &\ge n^3(1-\eta)^3 \int_{a\sqrt{\mu n}}^{\sqrt{\mu n}} \int_{a\sqrt{\mu n}}^{\sqrt{\mu n}} \int_{a\sqrt{\mu n}}^{\sqrt{\mu n}} f_n(u,v,w) dF_n(u) dF_n(v) dF_n(w) \\ &= (1-\eta)^3 \Big(\int_{a\sqrt{\mu n}}^{\sqrt{\mu n}} w^2 dF_n(w) \Big)^3 \ge \varepsilon_2 n^{3-3/2\alpha} L(\sqrt{n})^3 \end{aligned} \tag{4.14}$$

for some $\varepsilon_2 > 0$. Now, write

$$\mathbb{P}\left(\triangle_n > (1+\zeta)G_n\right) \le \mathbb{P}\left(\mathcal{E}_1^c\right) + \mathbb{P}\left(\triangle_n > (1+\zeta)G_n; \mathcal{E}_1\right). \tag{4.15}$$

The first term is asymptotically small. The second term can be bounded by applying Lemma 4.5, using the lower bound (4.14) for G_n on $\mathcal{E}_1$ to conclude the assertion. □

4.2 Proof of Lemma 4.2

To prove (4.2), we wish to apply Chebyshev's inequality, which requires appropriate estimates for the first two moments of $\triangle_n(\delta, a)$. Note first that

$$\begin{aligned} \mathbb{E}\left[\triangle_n(\delta,a)\right] = (n-1)^3 &\int_{0<x<y<z<n^{1/2+\delta}} f_{n-1}(x,y,z) dF(x) dF(y) dF(z) \\ &+ (n-1)^2 \int_{0<x<y<n^{1/2+\delta}} f_{n-1}(x,y,c_a(n)) dF(x) dF(y). \end{aligned}$$

Using this expression, and applying similar ideas as in the proofs of Lemma 1.1 and Lemma 2.4, along with the definition of $c_a(n)$ it follows that $\mathbb{E}\left[\triangle_n(\delta,a)\right] \sim (1+a)m_n$.

We proceed by analyzing the second moment of $\triangle_n(\delta, a)$ by using the concentration bound Lemma 4.5, as well as a concentration bound for F_n. In particular, it follows from (2.6) that for sufficiently small $\delta > 0$ there exists an $\varepsilon > 0$ such that

$$\mathbb{P}\left(\sup_{x<n^{1/2+\delta}} |\frac{\bar{F}_n(x)}{\bar{F}(x)} - 1| > \zeta\right) \le e^{-n^{\varepsilon}}. \tag{4.16}$$

On the event $E_n(\zeta) = \{\sup_{x<n^{1/2+\delta}} |\frac{\bar{F}_n(x)}{\bar{F}(x)} - 1| \le \zeta\}$, $G_{n-1}/n^{2-\alpha} > n^{\varepsilon}$ for some $\varepsilon > 0$ for n sufficiently large (see also the detailed computation in the proof of Lemma 4.5). In addition, using the expression

$$\begin{aligned}\triangle_n(\delta, a) = (n-1)^3 &\int_{0<x<y<z<n^{1/2+\delta}} f_{n-1}(x, y, z) dF_{n-1}(x) dF_{n-1}(y) dF_{n-1}(z) \\ &+ (n-1)^2 \int_{0<x<y<n^{1/2+\delta}} f_{n-1}(x, y, c_a(n)) dF_{n-1}(x) dF_{n-1}(y),\end{aligned}$$

we see that $\triangle_n(\delta, a) < (1+\zeta)^2(1+a)m_n$ with high probability, and therefore

$$\mathbb{E}\left[\triangle_n(\delta, a)^2\right] \le n^6(2e^{-n^{\varepsilon}} + e^{-J(\zeta)n^{\varepsilon}}) + (1+\zeta)^4(1+a)^2 m_n^2. \tag{4.17}$$

Next, fix $\delta > 0$. Using Chebyshev's inequality and the previous bounds we obtain for every $\zeta > 0$,

$$\limsup_{n\to\infty} \mathbb{P}\left(|\triangle_n(\delta, a)/\mathbb{E}\left[\triangle_n(\delta, a)\right] - 1| > \delta\right) \le \frac{1}{\delta^2}((1+\zeta)^4 - 1)).$$

The proof of (4.2) is now completed by letting $\zeta \downarrow 0$.

We now turn to the proof of (4.3) Note that

$$\begin{aligned}\mathbb{E}\left[\triangle_{n,l}(\delta, z_1, ..., z_l)\right] = (n-l)^3 &\int_{0<x<y<z<n^{1/2+\delta}} f_{n-l}(x, y, z) dF(x) dF(y) dF(z) \\ &+ (n-l)^2 \sum_{i=1}^{l} \int_{0<x<y<n^{1/2+\delta}} f_{n-l}(x, y, (n-l)z_i) dF(x) dF(y) \\ &+ (n-l) \sum_{i=1, j>i}^{l} \int_0^{n^{1/2+\delta}} f_n(x, (n-l)z_i, (n-l)z_j) dF(x).\end{aligned}$$

Observe that the second and third term in this expression are very related to the quantities analyzed in Proposition 3.4. Using similar ideas as in the proof of Proposition 3.4 with F_n replaced by F, it follows from the above expression that

$$\mathbb{E}\left[\triangle_{n,l}(\delta, z_1, ..., z_l)\right]/n \to C_l(z_1, ..., z_l). \tag{4.18}$$

We proceed by analyzing the second moment of $\triangle_{n,l}(\delta, z_1, ..., z_l)$, using concentration bounds. As before, note that on the event $E_n(\zeta) = \{\sup_{x<n^{1/2+\delta}} |\frac{\bar{F}_n(x)}{\bar{F}(x)} - 1| \le \zeta\}$, $G_{n-l}/n^{2-\alpha} > n^{\varepsilon}$ for some $\varepsilon > 0$ for n sufficiently large.

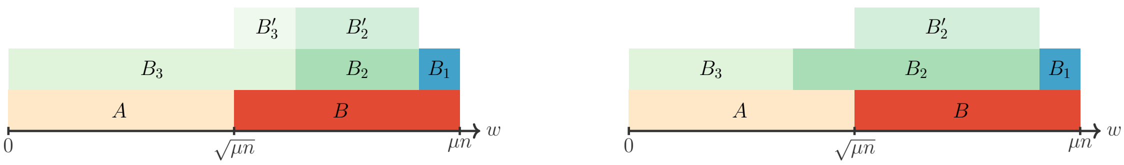
Modify the definition of $G_{n-l,l}$ in Proposition 3.4 to $G_{n-l,l}^{n^{1/2+\delta}}$ to truncate all integrals at $n^{1/2+\delta}$ rather than ∞, i.e.

$$\begin{aligned}G_{n,l}^{n^{1/2+\delta}}(z_1, ..., z_l) = n^2 &\sum_{i=1}^{l} \int_{0<x<y<n^{1/2+\delta}} f_n(x, y, nz_i) dF_n(x) dF_n(y) \\ &+ n \sum_{i=1, j>i}^{l} \int_0^{n^{1/2+\delta}} f_n(x, nz_i, nz_j) dF_n(x).\end{aligned} \tag{4.19}$$

Again using Lemma 4.5, we obtain that, with high probability,

$$\triangle_{n,l}(\delta, z_1, ..., z_l) \le (1+\zeta)(G_{n-l} + G_{n,l}^{n^{1/2+\delta}}(z_1, ..., z_l)), \tag{4.20}$$

which is in turn bounded by $(1+\zeta)^2 n(C^3 H + C_l(z_1, ..., z_l))$ with high probability. The proof is now completed by using Chebyshev's inequality, and letting $\zeta \downarrow 0$ as before.



(a) Case 1: $A \subseteq B_3$

(b) Case 2: $B_3 \subseteq A$.

Figure 3: Illustration of the sets A, B and B_1, B_2, B_3

5 Many dominating hubs

To prove Theorems 1.4 and 1.7, we distinguish several types of vertices, based on the vertex weights. Specifically, fix $\zeta > 2$ and $\varepsilon > 0$, then

- Type A: $W_i \leq \sqrt{\mu n}$,
- Type B_3: $W_i \leq D^{1/\alpha} n^{(3-\gamma)/(2\alpha)} \log(n^{(\gamma-3)/2+\alpha})^{-1/\alpha}$,
- Type B_2: $W_i \in [D^{1/\alpha} n^{(3-\gamma)/(2\alpha)} \log(n^{(\gamma-3)/2+\alpha})^{-1/\alpha}, n/\log(n)^{\zeta/\alpha}]$,
- Type B_1: $W_i \geq n/\log(n)^{\zeta/\alpha}$,

where we set

$$D = \frac{\sqrt{2a}}{h(1+\varepsilon)}, \tag{5.1}$$

with $h(\cdot)$ as in (2.4). Now depending on γ and α, vertices of type A and B_3 or type A and B_2 may overlap, see Figure 3. In the proof of Theorem 1.4, we will therefore sometimes split up the sets B_3 or B_2 at $\sqrt{\mu n}$ to avoid this overlap. We denote the number of triangles between one vertex from B_i one from B_j and one from B_k by $\triangle_{B_i,B_j,B_k}$. Similarly, we denote the number of edges with one end in B_i and one end in B_j by E_{B_i,B_j}. The main strategy of the proof of Theorem 1.7 is to split up the triangles into different types, and bounds their contributions. The main contribution is from triangles of type $B_1B_1B_i$ for $i = 1, 2, 3$, that is, triangles with two high-degree vertices and one other vertex. We show that all other types of triangles appear less often with high enough probability. Here we will use that the empirical weight distribution of B_3 vertices is close to its mean with sufficiently high probability, to get rid of the randomness caused by the weight sampling. On B_2, we get rid of the random weights by using the fact that the probability that a triangle appears is non-decreasing in the weights, so that we may assume that all B_2 vertices have weights $n/\log(n)^{\zeta/\alpha}$.

Before we prove Theorem 1.4, we first provide several lemmas. We begin by recalling a variation of Theorem A.1.4 in [1], to bound tail probabilities of sums of independent Bernoulli random variables.

Lemma 5.1. *Let $B_i, i \geq 1$ be a sequence of independent Bernoulli random variables with $p_i = \mathbb{P}(B_i = 1) = 1 - \mathbb{P}(B_i = 0)$. Set $m_n = \sum_{i=1}^n p_i$. For every $b > 0$ we have*

$$\mathbb{P}(\sum_{i=1}^n B_i > (1+b)m_n) \leq e^{-m_n I_B(b)}, \qquad \mathbb{P}(\sum_{i=1}^n B_i < (1-b)m_n) \leq e^{-m_n I_B(-b)}, \tag{5.2}$$

with $I_B(b) = (1+b)\log(1+b) - b$.

We next provide an elementary lemma that bounds the probability that polynomially vertices have at least a given weight:

Lemma 5.2. *Suppose that $\mathbb{P}(W > x) \sim Cx^{-\alpha}$. Then, for $\gamma > 1 - \alpha\beta$ and $d, u > 0$,*

$$\mathbb{P}\left(un^\gamma \text{ vertices of weight } > dn^\beta\right) \leq \exp\Big(-un^\gamma \log(n^{\gamma-1+\alpha\beta})\Big)(1+o(1)), \tag{5.3}$$

and

$$\mathbb{P}\left(un^{\gamma} \textit{ vertices of weight } > dn^{\beta}\right) \geq \frac{1}{\sqrt{2n}} \exp\Big(- un^{\gamma}\log(n^{\gamma-1+\alpha\beta})\Big)(1+o(1)). \tag{5.4}$$

Proof. By (1.1), the probability that a vertex has weight at least dn^{β} is given by $Cd^{-\alpha}n^{-\alpha\beta}(1 + o(1))$, and is independent for each vertex. Thus, by Lemma 5.1,

$$\begin{aligned}
&\mathbb{P}\left(n^{\gamma} \text{ vertices of weight } > cn^{\beta}\right) \\
&\leq \exp\Big(- un^{\gamma}\log\Big(\frac{un^{\gamma}}{Cn^{1-\alpha\beta}d^{-\alpha}}\Big) + n^{\gamma} - Cd^{-\alpha}n^{-\alpha\beta}\Big)(1+o(1)) \\
&\leq \exp\Big(- n^{\gamma}\log(n^{\gamma-1+\alpha\beta})\Big)(1+o(1)).
\end{aligned} \tag{5.5}$$

The second inequality follows similarly, using [3, Lemma 4.7.2] instead to get the lower bound. □

We now provide bounds on the number of vertices in B_1 and B_2, $N(B_1)$ and $N(B_2)$ (we will usually upper bound the number of B_3 vertices by the total number of vertices n).

Lemma 5.3. *Suppose that* $\mathbb{P}(W > x) \sim Cx^{-\alpha}$. *For* $K \gg n^{(\gamma-1)/2}\log(n^{(\gamma-3)/2+\alpha})$,

$$\mathbb{P}(N(B_1 \cup B_2) > K) \leq \exp\Big(- K\log\Big(\frac{K}{D^{-\alpha}n^{(\gamma-1)/2}\log(n^{(\gamma-3)/2+\alpha})}\Big)\Big)(1+o(1)), \tag{5.6}$$

and for $K \gg n^{1-\alpha}\log(n)^{\zeta}$,

$$\mathbb{P}(N(B_1) > K) \leq \exp\Big(- K\log\Big(\frac{K}{n^{1-\alpha}\log(n)^{\zeta}}\Big)\Big)(1+o(1)). \tag{5.7}$$

Proof. As the number of vertices of weight at least $x \gg 1$, N_x is binomial with parameters n and $Cx^{-\alpha}(1 + o(1))$, Lemma 5.1 gives that

$$\begin{aligned}
\mathbb{P}(N_x > K) &\leq \exp\Big(- K\log\Big(\frac{K}{Cnx^{-\alpha}}\Big) + K - 1\Big)(1+o(1)) \\
&\leq \exp\Big(- K\Big(\log\Big(\frac{K}{Cnx^{-\alpha}}\Big) - 1\Big)\Big)(1+o(1)).
\end{aligned} \tag{5.8}$$

Plugging in the lower weight bounds for $B_1 \cup B_2$ and for B_1, $n^{(3-\gamma)/(2\alpha)}\log(n^{(\gamma-3)/2+\alpha})^{-1/\alpha}$ and $n/\log(n)^{\zeta/\alpha}$ respectively and noticing that under the given constraints on K the -1 in the exponent is of lower order of magnitude, gives the result. □

5.1 Bounding specific triangle and edge types

We now turn to investigating the number of edges and triangles between vertices of the groups A, B_1, B_2, B_3, that will later be used in the upper bound for the total number of triangles. We first provide some lemmas that bound the number of edges and triangles between B_3 vertices:

Lemma 5.4. *Suppose that* $\mathbb{P}(W > x) \sim Cx^{-\alpha}$. *Let* B_3 *be the set of vertices with weights at most*

$$Q_n = D^{-1/\alpha}n^{(3-\gamma)/(2\alpha)}\log(n^{(\gamma-3)/2+\alpha})^{-1/\alpha}. \tag{5.9}$$

Then, for $\gamma \in (1, 3)$

$$\mathbb{P}(\triangle_{B_3,B_3,B_3} > Kn^{\gamma}) \leq n\exp\Big(- \sqrt{2}an^{(\gamma-1)/2}\log(n^{(\gamma-3)/2+\alpha})\Big) \tag{5.10}$$

when $Kn^{\gamma} > (1 + \varepsilon)^3C^3Hn^{3-3/2\alpha}$. *Furthermore, for* $K > (1 + \varepsilon)^2\mu n$,

$$\mathbb{P}(E_{B_3,B_3} > K) \leq \exp(-Dn^{(\gamma-1)/2}\log(n^{(\gamma-3)/2+\alpha})h(1 + \varepsilon)). \tag{5.11}$$

Proof. By (2.6), for all $\varepsilon > 0$ and with Q_n as in (5.9),

$$\mathbb{P}(\mathcal{E}_1) := \mathbb{P}\left(\sup_{x\in[1,Q_n]} \frac{|1-F_n(x)|}{\bar{F}(x)} \leq (1+\varepsilon)\right) \geq 1 - \exp\big(-Dh(1+\varepsilon)n^{(\gamma-1)/2}\log(n^{(\gamma-3)/2+\alpha})\big)(1+o(1)). \tag{5.12}$$

As the number of edges and triangles are non-decreasing in the weights, on $\mathcal{E}_1$,

$$\mathbb{P}(\triangle_{B_3,B_3,B_3} \geq K) \leq \mathbb{P}\left(\triangle_{B_3,B_3,B_3} \geq K \mid \bar{F}_n(x) = (1+\varepsilon)\bar{F}(x) \text{ for } x\in[1,Q_n]\right), \tag{5.13}$$

and the same holds for edge counts.

Note that $\bar{F}_n(x) = (1+\varepsilon)\bar{F}(x)$ for $x\in[1,Q_n]$ may not always be possible for all values of n, as $\bar{F}_n(x)$ needs to be a multiple of $1/n$ for all x, but adding this restriction would only make our bound tighter. Given the weight distribution $\bar{F}_n(x) = (1+\varepsilon)\bar{F}(x)$, the edge count is a sum of independent Bernoulli random variables with mean at most $(1+\varepsilon)\mu n$. Thus, on $\mathcal{E}_1$, Lemma 5.1 yields that for $K > (1+\varepsilon)^2\mu n$,

$$\begin{aligned}\mathbb{P}(E_{B_3,B_3} \geq K) &\leq \mathbb{P}\left(E_{B_3,B_3} \geq K \mid \bar{F}_n(x) = (1+\varepsilon)\bar{F}(x) \text{ for } x\in[1,Q_n]\right)\\ &\leq \exp\Big(-(1+\varepsilon)\mu n((1+\varepsilon)\log(1+\varepsilon)-\varepsilon)\Big)\\ &\leq \exp(-\sqrt{2a}n^{(\gamma-1)/2}\log(n^{(\gamma-3)/2+\alpha})),\end{aligned} \tag{5.14}$$

for $\gamma < 3$ and n sufficiently large. To obtain an upper bound for the triangle counts, we apply Lemma 4.3. Let $X_{ijk} = X'_{ijk}$ denote the indicator that i, j, k forms a triangle. Now given the weight distribution, X_{ijk} and X_{uvw} are independent as long as $|\{i,j,k,u,v,w\}| \geq 5$. Thus, we define $X_{ijk(uvw)} = X_{ijk}$ when $|\{i,j,k,u,v,w\}| \geq 5$ and $X_{ijk(uvw)} = 0$ otherwise. Then, for all uvw,

$$\sum_{i,j,k,\in B_3} X_{ijk} \leq d_{u,B_3} + d_{v,B_3} + d_{w,B_3} + \sum_{i,j,k,\in B_3} X_{ijk(uvw)} \tag{5.15}$$

where d_{u,B_3} denotes the degree of vertex u to other B_3 vertices. As $d_{u,B_3} \leq n$ for all u, on $\mathcal{E}'$, by (5.15) and Lemma 4.3,

$$\begin{aligned}\mathbb{P}(\triangle_{B_3,B_3,B_3} \geq Kn^\gamma) &\leq \exp\Big(-\frac{Kn^\gamma}{3n}\log\Big(\frac{Kn^\gamma}{(1+\varepsilon)^3Hn^{3-3/2\alpha}} - 1 + \frac{(1+\varepsilon)^3Hn^{3-3/2\alpha}}{Kn^\gamma}\Big)\Big)\\ &\leq \exp(-Dh(1+\varepsilon)n^{(\gamma-1)/2}\log(n^{(\gamma-3)/2+\alpha}))\end{aligned} \tag{5.16}$$

for $\gamma \geq 1$, as on $\mathcal{E}_1$, the mean number of triangles is bounded by $(1+\varepsilon)^3Hn^{3-3/2\alpha}(1+o(1))$. Plugging in the value of D then proves the lemma. □

We now investigate vertices and triangles from B_2 vertices. The proof of this Lemma follows a similar structure as the proof of Lemma 5.4, and can be found in Appendix B.

Lemma 5.5. *Suppose that $\mathbb{P}(W > x) \sim Cx^{-\alpha}$ and that $d_u \leq M$ for all $u \in A$. Then,*

$$\mathbb{P}\Big(E_{A,B_2} > n^{(\gamma+1)/2}\Big) \leq \exp\left(-\frac{n^{(\gamma+1)/2}}{M}\log\Big(\frac{n^{(\gamma-1)/2}}{K_1N(B_2)\log(n)^{-\zeta}}\Big)\right), \tag{5.17}$$

for some $K_1 > 0$, where $N(B_2)$ denotes the number of B_2 vertices. Furthermore,

$$\mathbb{P}(\triangle_{A,B_2,B_2} > n^\gamma) \leq \exp\Big(-\frac{n^\gamma}{M^2}\log\Big(\frac{n^{\gamma-1}}{3K_1N(B_2)^2\log(n)^{-\zeta}}\Big)\Big). \tag{5.18}$$

To prove Theorem 1.7, we finally state a lemma that bounds the degree of type A vertices. Its proof can also be found in Appendix B.

Lemma 5.6. *Suppose that $\mathbb{P}(W > x) \sim Cx^{-\alpha}$. Let $\mathcal{F}$ denote the event that all vertices of weight at most $\sqrt{\mu n}$ have degrees at most*

$$M = \begin{cases} 4\sqrt{\mu n} & \gamma < 2 \\ 4n^{\gamma/3} & \gamma \in [2, 3). \end{cases}$$

Then, for all $D > 0$,

$$\mathbb{P}(\bar{\mathcal{F}}) \leq \exp\Big(- Dn^{(\gamma-1)/2} \log(n^{(\gamma-3)/2+\alpha})\Big). \tag{5.19}$$

5.2 Proof of Theorem 1.4

Proof of Theorem 1.4. To prove this theorem, we lower bound the number of triangles with triangles between two weight n vertices and one other vertex, and we upper bound by considering triangles between vertices of weights in classes B_1, B_2, B_3, and split into all possible cases, as different types of triangles can be upper bounded by different terms.

Lower bound. As a lower bound of $\triangle_n$, we compute the number of triangles with at least one vertex of weight $> \mu n$. Note that as the minimal weight is at least 1, vertices of weight μn connect to all other vertices with probability 1. The probability that at least $\sqrt{2a}n^{1-\alpha 3/4}$ vertices of weights at least μn are present can be bounded by Lemma 5.2 as

$$\mathbb{P}\Big(\sqrt{2a}n^{1-\alpha 3/4} \text{ vertices of weight } > \mu n\Big) \geq \frac{1}{\sqrt{2n}} \exp\Big(- \sqrt{2a}n^{1-\alpha 3/4} \log\big(n^{\alpha/4}\big)\Big). \tag{5.20}$$

Now, $\sqrt{2a}n^{1-\alpha 3/4}$ vertices of weight at least μn generate $n(\sqrt{2a}n^{1-\alpha 3/4})^2/2 = an^{3-3/2\alpha}$ triangles. Indeed, every pair of 2 vertices of weight at least μn forms a triangle with any of the other n vertices, creating $an^{3-3/2\alpha}$ triangles. Furthermore, the vertices of weight at most $n^{1/\alpha}$ generate $C^3Hn^{3-3/2\alpha}(1 + o(1))$ triangles with high probability [21]. Therefore,

$$\mathbb{P}\Big(\triangle_n > n^{3-3/2\alpha}(C^3H + a)\Big) \geq \frac{1}{\sqrt{2n}} \exp\Big(- \sqrt{2a}n^{1-\alpha 3/4} \log\big(n^{\alpha/4}\big)\Big)(1 + o(1)). \tag{5.21}$$

Thus,

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}\left(\triangle_n > n^{3-3/2\alpha}(C^3H + a)\right)}{n^{1-\alpha 3/4}\log(n)} \geq \sqrt{2a}\Big(1 - \alpha - 1 - \alpha 3/4\Big) = \sqrt{2a}\frac{-\alpha}{4}. \tag{5.22}$$

Upper bounds. When $\gamma = 3 - 3/2\alpha$, B_3 vertices have weights up to $Dn^{3/4}\log(n^{\alpha/4})^{1/\alpha}$. We denote $B_3' = B_3 \cap B$. Then, $B_3' \subseteq B$ and $A \subseteq B_3$, as illustrated in Figure 3a. Therefore, we distinguish the types of triangles $B_3B_3B_3$, $B_3B_3B_i$, $B_1B_1B_i$, $B_3'B_iB_j$, $\mathrm{A}B_iB_j$, $B_iB_jB_k$ for $i, j, k \in \{1, 2\}$ and bound the number of these triangles one by one.

$B_3B_3B_3$ triangles. For $B_3B_3B_3$ triangles, we use Lemma 5.4 with $D = \sqrt{2a}/h(1 + \varepsilon)$ to obtain that for $\delta > \varepsilon$,

$$\mathbb{P}\Big(\triangle_{B_3B_3B_3} > (C^3H + a)n^{3-3/2\alpha}\Big) \leq \exp\Big(- \sqrt{2a}n^{1-\alpha 3/4} \log\big(n^{\alpha/4}\big)\Big). \tag{5.23}$$

$B_3B_3B_i$ triangles for $i \in \{1, 2\}$. We bound the number of $B_3B_3B_i$ for $i = 1, 2$ by the number of B_i vertices times the number of edges between B_3 vertices. By (5.6) with $\gamma = 3 - 3/2\alpha$ and $D = \sqrt{2a}/h(1 + \varepsilon)$,

$$\mathbb{P}(N(B_1 \cup B_2) > K) \leq \exp\Big(- K \log\Big(\frac{K}{\sqrt{2a}h(1 + \varepsilon)^{-1}n^{1-\alpha 3/4}\log(n^{\alpha/4})}\Big)\Big). \tag{5.24}$$

By (5.11) with $D = \sqrt{2a}$

$$\mathbb{P}\Big(E_{B_3,B_3} > n^{1-3\alpha/8}\Big) \leq \exp(-\sqrt{2a}n^{(\gamma-1)/2}\log(\sqrt{2a}n^{(\gamma-3)/2+\alpha})). \tag{5.25}$$

On the complement of this event, by (5.24),

$$\begin{aligned}\mathbb{P}\Big(\triangle_{B_3B_3B_i} > \varepsilon n^{3-3/2\alpha}\Big) &\leq \mathbb{P}\Big(N(B_1\cup B_2) > \varepsilon n^{2-9\alpha/8}\Big)\\ &\leq \exp\Big(-\varepsilon n^{2-9\alpha/8}\log(\delta^{-\alpha/2}n^{2-9\alpha/8-1-\alpha 3/4}))\log(n^{\alpha/4})^{-1}\Big)\\ &\leq \exp(-\sqrt{2a}n^{(\gamma-1)/2}\log(n^{(\gamma-3)/2+\alpha}))\end{aligned} \tag{5.26}$$

for n sufficiently large and $\alpha < 4/3$.

$B_iB_1B_1$ triangles for $i \in \{1,2,3\}$. We now bound the number of $B_iB_1B_1$ type triangles $i \in \{1,2,3\}$ by n times the number of pairs of two B_1 vertices. This is the number of ways to choose 2 type B_1 vertices, and one other vertex. Thus, by (5.7),

$$\begin{aligned}\mathbb{P}\Big(\triangle_{B_iB_1B_1} > an^{3-3/2\alpha}\Big) &\leq \mathbb{P}\Big(N(B_1) > \sqrt{2a}n^{1-\alpha 3/4}\Big)\\ &\leq \exp\Big(-\sqrt{2a}n^{1-\alpha 3/4}\log\Big(\frac{\sqrt{2a}n^{1-\alpha 3/4}}{n^{1-\alpha}\log(n)^{\zeta}}\Big)\Big).\end{aligned} \tag{5.27}$$

Thus, for fixed δ,

$$\lim_{n\to\infty}\frac{\log\Big(\mathbb{P}\left(\triangle_{AB_1B_1} > an^{3-3/2\alpha}\right)\Big)}{n^{1-\alpha 3/4}\log(n)} = -(1-\frac{3\alpha}{4}) + (1-\alpha) = \frac{-\alpha}{4}. \tag{5.28}$$

$B_iB_jB_k$ triangles for $i,j,k \in \{1,2\}$. Any triple of vertices in $B_1 \cup B_2$ have weights at least $\sqrt{\mu n}$ and thus form a triangle with probability one. Thus, an upper bound for these triangles is the number of vertices in $B_2 \cup B_1$ to the power three.

$$\begin{aligned}&\mathbb{P}\Big(\triangle_{B_iB_jB_k} > \varepsilon n^{(2-\alpha)3/2}\Big) \leq \mathbb{P}\Big(N(B_2\cup B_1)^3 > \varepsilon n^{(2-\alpha)3/2}\Big)\\ &= \mathbb{P}\Big(N(B_2\cup B_1) > \varepsilon^{1/3}n^{(2-\alpha)/2}\Big).\end{aligned} \tag{5.29}$$

Now by (5.24)

$$\begin{aligned}&\mathbb{P}\Big(N(B_2\cup B_1) > \varepsilon^{1/3}n^{(2-\alpha)/2}\Big)\\ &\leq \exp\Big(-\varepsilon^{1/3}n^{(2-\alpha)/2}\log\Big(\frac{\varepsilon^{1/3}n^{(2-\alpha)/2}}{\sqrt{2a}h(1+\varepsilon)^{-1}n^{1-\alpha 3/4}\log(n^{\alpha/4})}\Big)\Big)\\ &\leq \exp\Big(-\sqrt{2a}n^{1-\alpha 3/4}\log(n^{\alpha/4})\Big),\end{aligned} \tag{5.30}$$

when $\zeta > 1$ and n is sufficiently large, since $(2-\alpha)/2 > 1-\alpha 3/4$ for $1 < \alpha < 4/3$. Thus, for $\varepsilon, \delta > 0$, and n sufficiently large,

$$\mathbb{P}\Big(\triangle_{B_iB_jB_k} > \varepsilon n^{3-3/2\alpha}\Big) \leq \exp\Big(-\sqrt{2a}n^{1-\alpha 3/4}\log\left(n^{\alpha/4}\right)\Big) \tag{5.31}$$

for $i,j,k \in \{1,2\}$.

$B_3'B_iB_j$ triangles $i,j \in \{1,2\}$. Again, any triple of vertices in $B_3'B_iB_j$ have weights at least $\sqrt{\mu n}$ and thus form a triangle with probability one. Thus, an upper bound for these triangles is the number of vertices of weight at least $\sqrt{\mu n}$ squared times the number of vertices in $B_2 \cup B_1$.

$$\mathbb{P}\Big(\triangle_{B_3'B_iB_j} > \varepsilon n^{(2-\alpha)3/2}\Big) \leq \mathbb{P}\Big(N(B_2\cup B_1)N(\geq\sqrt{\mu n})^2 > \varepsilon n^{(2-\alpha)/2}\Big). \tag{5.32}$$

Now by (5.24)

$$\mathbb{P}\Big(N(B_2\cup B_1) > n^{1-\alpha 3/4}\log(n)^{\zeta}\Big) \leq \exp\Big(-\sqrt{2a}n^{1-\alpha 3/4}\log(n^{\alpha/4})\Big), \tag{5.33}$$

when $\zeta > 1$ and n is sufficiently large. On the complement of this event, Lemma 5.2 yields

$$\begin{aligned}\mathbb{P}\Big(\triangle_{B_3' B_i B_j} > \varepsilon n^{(2-\alpha)3/2}\Big) &\le \mathbb{P}\Big(N(B)^2 > \varepsilon n^{1-3\alpha/8}\log(n)^{-\zeta}\Big)\\ &\le \exp\Big(-\varepsilon n^{1-3\alpha/8}\log(n)^{-\zeta}\log(\frac{\varepsilon n^{1-3\alpha/8}\log(n)^{-\zeta}}{n^{(2-\alpha)/2}})\Big)\\ &\le \exp\Big(-\sqrt{2a}n^{1-\alpha 3/4}\log\big(n^{\alpha/4}\big)\Big), \end{aligned} \tag{5.34}$$

since $1-3/8\alpha > 1-\alpha 3/4$ for $1<\alpha<4/3$. Thus, for $\varepsilon, \delta > 0$, and n sufficiently large,

$$\mathbb{P}\Big(\triangle_{B_3' B_i B_j} > \varepsilon n^{3-3/2\alpha}\Big) \le 2\exp\Big(-\sqrt{2a}n^{1-\alpha 3/4}\log\big(n^{\alpha/4}\big)\Big). \tag{5.35}$$

AB_2B_2 triangles. We now consider AB_2B_2 triangles. By Lemma 5.5, for all $\varepsilon > 0$

$$\begin{aligned}\mathbb{P}\Big(\triangle_{AB_2B_2} > \varepsilon n^{3-3/2\alpha}\Big) &\le \exp\Big(-\varepsilon\frac{n^{3-3/2\alpha}}{K^2 n}\log\Big(\frac{\varepsilon n^{3-3/2\alpha}}{3K_1 N(B_2)^2 n\log(n)^{-\zeta}}\Big)\Big)\\ &\le \exp\Big(-\sqrt{2a}n^{2-3/2\alpha}\log(n^{\alpha/4})\Big), \end{aligned} \tag{5.36}$$

for n sufficiently large when $N(B_2) < Cn^{1-\alpha 3/4}\log(n)^{\zeta/2}$ for some $C > 0$. Now by (5.33)

$$\mathbb{P}\Big(N(B_2) > n^{1-\alpha 3/4}\log(n)^{\zeta/2}\Big) \le \exp\Big(-\sqrt{2a}n^{1-\alpha 3/4}\log(n^{\alpha/4})\Big), \tag{5.37}$$

for $\zeta > 2$.

AB_2B_1 triangles. We now bound the number of AB_2B_1 triangles by $N(B_1)$ times the number of AB_2 edges. By Lemma 5.5 with $M = \max(\sqrt{n}, n^{\gamma/3})$,

$$\mathbb{P}\Big(E_{A,B_2} > n^{1-3\alpha/8}\Big) \le \exp\Big(-\sqrt{2a}n^{1-\alpha 3/4}\log(n^{\alpha/4})\Big) \tag{5.38}$$

as long as $N(B_2) \le n^{1-\alpha 3/4}\log(n)^{\zeta}$, which happens with probability (5.33). Thus, by (5.7),

$$\begin{aligned}\mathbb{P}\Big(\triangle_{AB_1B_2} > \varepsilon n^{3-3/2\alpha}\Big) &\le \mathbb{P}\Big(N(B_1)E_{AB_2} > \varepsilon n^{3-3/2\alpha}\Big)\\ &\le \mathbb{P}\Big(N(B_1) > \varepsilon n^{2-9\alpha/8}\Big)\\ &\le \exp\Big(-\sqrt{2a}n^{1-\alpha 3/4}\log(n^{\alpha/4})\Big), \end{aligned} \tag{5.39}$$

for $\alpha < 4/3$ and $\varepsilon > 0$.

Thus, (5.23), (5.26), (5.31), (5.35) (5.36) and (5.39) yield that

$$\mathbb{P}\left(\sum_{\{i,j,k\}\neq\{3,3,3\},\{i,1,1\}} \triangle_{B_iB_jB_k} > \varepsilon n^{3-3/2\alpha}\right) \le K\exp\Big(-\sqrt{2a}n^{1-\alpha 3/4}\log(n^{\alpha/4})\Big), \tag{5.40}$$

for any $\varepsilon > 0$ and some $K > 0$. Combining this with (5.23) and (5.27)yields that for $\delta > \varepsilon$,

$$\begin{aligned}\mathbb{P}\Big(\triangle_n > (H+a)n^{3-3/2\alpha}\Big) &\le \mathbb{P}\Big(\triangle_{B_3B_3B_3} + \triangle_{B_1B_1B_i} > (C^3H + a - \varepsilon)n^{3-3/2\alpha}\Big)\\ &\quad + K\exp\Big(-\sqrt{2a}n^{1-\alpha 3/4}\log(n^{\alpha/4})\Big)\\ &\le (K+1)\exp\Big(-\sqrt{2a}n^{1-\alpha 3/4}\log(n^{\alpha/4})\Big)\\ &\quad + \mathbb{P}\Big(\triangle_{B_iB_1B_1} > (C^3H + a - 2\varepsilon)n^{3-3/2\alpha}\Big). \end{aligned} \tag{5.41}$$

Thus,

$$\limsup_{n\to\infty}\frac{\log\Big(\mathbb{P}\big(\triangle_n > (C^3H+a)n^{3-3/2\alpha}\big)\Big)}{n^{1-\alpha 3/4}\log(n)} \le \sqrt{2a-2\varepsilon}\Big(\frac{-\alpha}{4}\Big). \tag{5.42}$$

Letting $\varepsilon \downarrow 0$ yields the result. □

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A Additional proofs for Sections 1–4

Proof of Lemma 1.1. Note that f_1 is absolutely continous, so there exists a function h such that $f_1(x,y,z) = \int_0^x \int_0^y \int_0^z h(u,v,w)dudvdw$. Now, write

$$\begin{aligned}\frac{6m_n}{n^3\bar{F}(\sqrt{n})^3} &= \frac{1}{\bar{F}(\sqrt{n})^3}\int_{x=0}^{\infty}\int_{y=0}^{\infty}\int_{z=0}^{\infty} f_n(x,y,z)dF(x)dF(y)dF(z)\\ &= \frac{1}{\bar{F}(\sqrt{n})^3}\int_{x=0}^{\infty}\int_{y=0}^{\infty}\int_{z=0}^{\infty} f_n(\sqrt{n}x,\sqrt{n}y,\sqrt{n}z)dF(x)dF(y)dF(z)\\ &= \int_{u=0}^{\infty}\int_{v=0}^{\infty}\int_{w=0}^{\infty} f_1(u,v,w)d\hat{F}_n(u)d\hat{F}_n(v)d\hat{F}_n(w)\\ &= \int_{u=0}^{\infty}\int_{v=0}^{\infty}\int_{w=0}^{\infty} h(u,v,w)\frac{\bar{F}(u\sqrt{n})}{\bar{F}(\sqrt{n})}dudvdw,\end{aligned}$$

where we made the transformation $u = x/\sqrt{n}$ (and similar for (v,w)) in the third step, with $\hat{F}_n(u) = F(\sqrt{n}u)/\bar{F}(\sqrt{n})$. To show this integral converges we use the Potter bounds, which imply that for each $\delta > 0$ there exists a constant M such that $\frac{\bar{F}(u\sqrt{n})}{\bar{F}(\sqrt{n})} \leq Mu^{-\alpha-\delta}, u < 1$ and $\frac{\bar{F}(u\sqrt{n})}{\bar{F}(\sqrt{n})} \leq Mu^{-\alpha+\delta}, u > 1$. Define the function $d(u) = Mu^{-\alpha-\delta}I(u \leq 1) + Mu^{-\alpha+\delta}I(u > 1)$. Since for $\alpha \in (1,2)$, the integral

$$\int_{v=0}^{\infty}\int_{w=0}^{\infty} h(u,v,w)d(u)d(v)d(w)dudvdw$$

converges for δ sufficiently small, we can use dominated converge to conclude

$$\begin{aligned}\frac{m_n}{n^3\bar{F}(\sqrt{n})^3} &\to \frac{1}{6}\int_{u=0}^{\infty}\int_{v=0}^{\infty}\int_{w=0}^{\infty} h(u,v,w)u^{-\alpha}v^{-\alpha}w^{-\alpha}dudvdw\\ &= \frac{\alpha^3}{6}\int_{u=0}^{\infty}\int_{v=0}^{\infty}\int_{w=0}^{\infty} f_1(u,v,w)u^{-\alpha-1}v^{-\alpha-1}w^{-\alpha-1}dudvdw.\end{aligned}$$

□

Proof of Lemma 1.2. Using Lemma 2.4 we know that, for a sequence $b(n)$ regularly varying of index α_b,

$$n^2\int_0^{\infty}\int_0^{\infty} f_n(x,y,b(n))dF(x)dF(y) \sim K_2\frac{n^3}{(b(n))^2}\bar{F}(n/b(n))^2. \tag{A.1}$$

Now, choose b such that $K_2 \frac{n^3}{(b(n))^2} \bar{F}(n/b(n))^2 = am_n$. Dividing both sides of this equation with n, and noting that $y\bar{F}(y)$ has an asymptotic inverse $h(y)$ which is regularly varying of index $-1/(\alpha-1)$ (cf. [6], Section 1.7), we see that $b(n) \sim n/h(\sqrt{a\mu n/(nK_2)})$. Since m_n is regularly varying of index $3-\alpha 3/2$, b is regularly varying of index $\beta > 1/2$. The proof is now completed by observing that $c_a(n) \sim b(n)$. The asymptotic form of $c_a(n)$ and the dependence on a now follows straightforwardly. □

Proof of Lemma 2.3. For large enough n, $b(n)c(n) \gg n$ so that we can write

$$\int_0^\infty f_n(x, b(n), c(n))dF(x) = \int_0^\infty \min\{xb(n)/(\mu n), 1\} \min\{xc(n)/(\mu n), 1\} dF(x).$$

The RHS can be decomposed as follows:

$$\frac{b(n)c(n)}{n^2\mu^2} \int_0^{\mu n/c(n)} x^2 dF(x) + \frac{b(n)}{\mu n} \int_{\mu n/c(n)}^{\mu n/b(n)} x dF(x) + \bar{F}(\mu n/b(n)).$$

Call these terms I,II, III. We can use Karamata's theorem to estimate Term I:

$$\frac{b(n)c(n)}{n^2\mu^2} \int_0^{\mu n/c(n)} x^2 dF(x) \sim L(\mu n/c(n)) \frac{1}{2-\alpha} \frac{b(n)c(n)}{n^2\mu^2} (\mu n/c(n))^{2-\alpha}.$$

This is regularly varying with index $-[\alpha(1-\alpha_c) + \alpha_c - \alpha_b]$. Term III is regularly varying of index $-\alpha(1-\alpha_b)$ and is therefore of small order of term I if $\alpha_c > \alpha_b$, and behaves like Term I if $c(n) = b(n)$. Term II behaves like for some constant K_{II},

$$\frac{b(n)}{\mu n} \int_{\mu n/c(n)}^\infty x dF(x) \sim K_{II} \frac{b(n)}{c(n)} \bar{F}(n/c(n)),$$

which is regularly varying of $-[\alpha(1-\alpha_c) + \alpha_c - \alpha_b]$, like Term I. Inspecting the slowly varying parts, it can be shown that the asymptotic behavior of Term I and Term II is the same up to a constant. □

Proof of Lemma 2.4. We write

$$\int_0^\infty \int_x^\infty f_n(x, y, b(n))dF(y)dF(x) = (1/2) \int_0^\infty \int_0^\infty f_n(x, y, b(n))dF(x)dF(y)$$

and we split the integral $\int_0^\infty \int_0^\infty f_n(x, y, b(n))dF(x)dF(y)$ in two terms, where $xy < \mu n$ (Term I), and $xy \geq \mu n$ (Term II).

$$I = \frac{1}{\mu n} \int_{xy<\mu n} xy \min\{xb(n)/(\mu n), 1\} \min\{yb(n)/(\mu n), 1\} dF(x)dF(y).$$

We break up Term I into 3 more regions: a region where both x, y are smaller than $\mu n/b(n)$ (Term Ii), a region where both x, y are larger than $\mu n/b(n)$ (Term Iii), and the region where one is smaller, and one larger (Term Iiii). For n large enough, $\mu n/b(n)$ is much smaller than $\sqrt{\mu n}$, in which case Ii equals

$$\frac{b(n)^2}{(\mu n)^3} \left(\int_{x=0}^{\mu n/b(n)} x^2 dF(x)\right)^2 \sim K \frac{b(n)^2}{(\mu n)^3} \left((\mu n/b(n))^2 \bar{F}(\mu n/b(n))\right)^2 = \frac{K\mu n}{b(n)^2} \bar{F}(\mu n/b(n))^2. \quad \text{(A.2)}$$

This term is regularly varying of index $-[(2\alpha_b - 1) + 2\alpha(1-\alpha_b)]$.

For Term Iii, we can lower and upper bound the region by respectively including constraints $x < \sqrt{\mu n}$, $y < \sqrt{\mu n}$, and by removing the constraint $xy < \mu n$. In both cases, we end up with the square of an integral with the same asymptotic behavior, so we get

$$Iii \sim \frac{1}{\mu n} \left(\int_{x=\mu n/b(n)}^\infty x dF(x)\right)^2 \sim \frac{K}{n} (n/b(n))^2 \bar{F}(n/b(n))^2. \quad \text{(A.3)}$$

This term is regularly varying of index $-[2(\alpha-1)(1-\alpha_b)+1]$, like Term Ii. Term Iiii can be written as

$$\frac{2b(n)}{(\mu n)^2}\int_{xy<\mu n, x<\mu n/b(n)<y} x^2 y dF(x)dF(y). \tag{A.4}$$

Again, the constraint $xy<\mu n$ is asymptotically irrelevant, leading to the behavior

$$\frac{2b(n)}{(\mu n)^2}\left(\int_0^{\mu n/b(n)} x^2 dF(x)\right)\left(\int_{\mu n/b(n)}^{\infty} y dF(y)\right) \sim \frac{Kb(n)}{n^2}(n/b(n))^3 \bar{F}(n/b(n))^2. \tag{A.5}$$

Once more, this term is regularly varying of index $-[2(\alpha-1)(1-\alpha_b)+1]$. Note that $[2(\alpha-1)(1-\alpha_b)+1]<\alpha$ if $\alpha_b>1/2$.

We now turn to Term II. By bounding the two minima by 1, we see that

$$II=\int_{xy>\mu n}\min\{xb(n)/(\mu n),1\}\min\{yb(n)/(\mu n),1\}dF(x)dF(y)\leq \mathbb{P}\left(XY>\mu n\right),$$

with X, Y iid and regularly varying of index $-\alpha$. Due to [17], the product is also regularly varying of index $-\alpha$. Since $\alpha>[2(\alpha-1)(1-\alpha_b)+1]$, Term II is asymptotically negligible. ☐

Proof of Lemma 4.4. For any given edge between vertices i and j of weights $w_i>w_j$ such that $w_iw_j<\mu n$, the number of triangles it is involved in is a binomial random variable with $n-2$ trials and probability at most

$$\begin{aligned}
&\int_1^{\infty}\min\Big(\frac{w_iw_k}{\mu n},1\Big)\min\Big(\frac{w_jw_k}{\mu n},1\Big)dF(w_k)\\
&\leq \int_1^{\infty} h(w_i,w_j,w_k)w_k^{-\alpha}L(w_k)dw_k\\
&\leq \int_1^{\infty} h(w_i,w_j,w_k)w_k^{-\alpha+\delta}dw_k\\
&=\int_1^{\infty}\min\Big(\frac{w_iw_k}{\mu n},1\Big)\min\Big(\frac{w_jw_k}{\mu n},1\Big)w_k^{-\alpha-1+\delta}dw_k,
\end{aligned} \tag{A.6}$$

where $h(w_i,w_j,w_k)$ is a function such that $\int_1^{w_k} h(w_i,w_j,x)dx=\min(\frac{w_iw_k}{\mu n},1)\min(\frac{w_jw_k}{\mu n},1)$, and where in the second inequality we have used the Potter bound on the slowly varying function $L(x)$. Now

$$\begin{aligned}
&\int_1^{\infty} w_k^{-\alpha-1+\delta}\min\Big(\frac{w_iw_k}{\mu n},1\Big)\min\Big(\frac{w_jw_k}{\mu n},1\Big)dw_k\\
&=\int_1^{\mu n/w_i} w_k^{1-\alpha+\delta}\frac{w_iw_j}{(\mu n)^2}dw_k+\int_{\mu n/w_i}^{\mu n/w_j} w_k^{-\alpha+\delta}\frac{w_j}{\mu n}dw_k+\int_{\mu n/w_j}^{\infty} w_k^{-\alpha-1+\delta}dw_k\\
&=\left(\frac{\left(\frac{\mu n}{w_i}\right)^{2-\alpha+\delta}}{2-\alpha+\delta}\frac{w_iw_j}{(\mu n)^2}+\frac{\left(\frac{\mu n}{w_i}\right)^{1-\alpha+\delta}}{\alpha-\delta-1}\frac{w_j}{\mu n}+\frac{1}{\alpha-\delta}\Big(\frac{\mu n}{w_j}\Big)^{-\alpha+\delta}\right)(1+o(1))\\
&=(\mu n)^{-\alpha+\delta}\Big(\frac{w_i^{\alpha-1}w_j}{(2-\alpha+\delta)(\alpha-\delta-1)}+\frac{w_j^{\alpha-\delta}}{\alpha-\delta}\Big)(1+o(1)).
\end{aligned} \tag{A.7}$$

Now when $w_i\geq w_j$ and $w_iw_j\leq \mu n$, this expression is maximized by $w_i=w_j=\sqrt{\mu n}$ for $1<\alpha<2$. Thus, the number of triangles $\{i,j\}$ is involved in is a binomial random variable with probability at most

$$C_1(\mu n)^{-\alpha+\delta}\sqrt{\mu n}^{-\alpha+\delta}(1+o(1))\leq C_2 n^{(-\alpha+\delta)/2}, \tag{A.8}$$

for some $C_1, C_2>0$ and n sufficiently large, where we have used that $w_iw_j\leq \mu n$ and that $w_i>w_j$. Thus, the average number of triangles involving $\{i,j\}$ is at most $C_2n^{(2-\alpha+\delta)/2}$. When choosing δ sufficiently small such that $C_2 n^{(2-\alpha)/2+\varepsilon} \gg n^{(2-\alpha+\delta)/2}$, Lemma 5.1 yields that for $K > C_2 n^{(2-\alpha)/2+\varepsilon}$,

$$\mathbb{P}\left(\{i,j\} \text{ in } \geq K \text{ triangles}\right) \leq \exp(-c_1 K)(1+o(1)), \tag{A.9}$$

for some $c_1 > 0$. □

B Additional proofs for Section 5

Proof of Lemma 5.5. We first bound the number of AB_2 edges. As the connection probability is increasing in the vertex weights, the number of AB_2 edges and A, B_2, B_2 triangles is stochastically dominated by the number of such edges and triangles respectively when all B_2 vertices have the upper bound of their weights of $n/\log(n)^{\zeta/\alpha}$. We therefore assume that all B_2 vertices have weights $n/\log(n)^{\zeta/\alpha}$.

Now $\mathbb{P}(W > x) \sim Cx^{-\alpha}$ implies that

$$\mathbb{P}(W > x) \leq \tilde{C}x^{-\alpha}, \quad x \geq 1 \tag{B.1}$$

for some $\hat{C} > 0$. By (B.1), the average number of AB_2 edges can then be upper bounded by

$$N(B_2)n\hat{C}\left(\int_1^{\log(n)^{\frac{\zeta}{\alpha}}} x^{-\alpha-1}\frac{x}{\log(n)^{\frac{\zeta}{\alpha}}}dx + \int_{\log(n)^{\frac{\zeta}{\alpha}}}^{\sqrt{n}} x^{-\alpha-1}dx\right) \leq K_1 N(B_2)n\log(n)^{-\zeta}, \tag{B.2}$$

for some $K_1 > 0$. Now, let X_{ij} denote the indicator that i,j forms an edge and $i \in A, j \in B_2$. Define $X_{ij(uv)} = X_{ij}$ when $i \neq u$ and 0 otherwise. Because $d_u \leq M$,

$$\sum_{i\in A, j\in B_2} X_{ij} \leq M + \sum_{i\in A, j\in B_2} X_{ij(uv)}. \tag{B.3}$$

Thus, by Lemma 4.3

$$\begin{aligned}\mathbb{P}\left(E_{A,B_2} > n^{(\gamma+1)/2}\right) &= \mathbb{P}\left(\sum_{i\in A, j\in B_2} X_{ij} > n^{(\gamma+1)/2}\right)\\ &\leq \exp\left(-\frac{n^{(\gamma+1)/2}}{M}\log\left(\frac{n^{(\gamma-1)/2}}{N(B_2)\log(n)^{-\zeta}}\right)\right),\end{aligned} \tag{B.4}$$

which proves the first part of the lemma.

Similarly, conditionally on the number of type B_2 vertices, the average number of AB_2B_2 triangles can be bounded by

$$\begin{aligned}&N(B_2)^2 n\hat{C}\left(\int_1^{\log(n)^{\frac{\zeta}{\alpha}}} x^{-\alpha-1}\left(\frac{x}{\log(n)^{\frac{\zeta}{\alpha}}}\right)^2 dx + \int_{\log(n)^{\frac{\zeta}{\alpha}}}^{\sqrt{n}} x^{-\alpha-1}dx\right)\\ &\leq K_2 N(B_2)^2 n\log(n)^{-\zeta},\end{aligned} \tag{B.5}$$

for some $K_2 > 0$.

Now, for all $i \in A$ and $j,k \in B_2$, let X_{ijk} denote the indicator that i,j,k forms a triangle. Let $X_{ijk(uvw)} = X_{ijk}$ when $i \neq u$, and 0 otherwise. Then, $X_{ijk(uvw)} \leq X_{ijk}$, and as edges between j and k vertices have weights $n/\log(n)^{\zeta/\alpha}$ and therefore are present with probability one, $\sum_{ijk} X_{ijk(uvw)}$ is independent from X_{uvw}. Furthermore,

$$\sum_{i,j,k} X_{ijk} \leq d_u^2 + \sum_{i,j,k} X_{ijk(uvw)} \leq M^2 + \sum_{i,j,k} X_{ijk(uvw)}. \tag{B.6}$$

Therefore, Lemma 4.3 yields

$$\mathbb{P}\left(\triangle_{A,B_2,B_2} > n^{\gamma}\right) \leq \exp\Big(- \frac{n^{\gamma}}{M^2} \log\Big(\frac{n^{\gamma-1}}{3K_1 N(B_2)^2 \log(n)^{-\zeta}})) \Big). \tag{B.7}$$

□

Proof of Lemma 5.6. As the degree of a vertex of lower weights is stochastically dominated by the degree of a vertex with higher weight, we assume that vertex i has the maximal weight $\sqrt{\mu n}$. Given the weights, the degree of a vertex with weight $\sqrt{\mu n}$ is a sum of independent Bernoulli random variables with mean $\sum_i \min(W_i/\sqrt{\mu n}, 1)$. We now compute the probability of the event that this mean is large. First of all,

$$\mathrm{Var}\left(W \mathbb{1}_{\{W \leq \sqrt{\mu n}\}}\right) \leq \mathbb{E}\left[W^2 \mathbb{1}_{\{W \leq \sqrt{\mu n}\}}\right] \leq \hat{C} \int_1^{\sqrt{\mu n}} x^{1-\alpha} dx = \hat{C}(\mu n)^{(2-\alpha)/2}(1+o(1)). \tag{B.8}$$

Thus, by Bernsteins' inequality,

$$\begin{aligned}\mathbb{P}\left(|\sum_i W_i \mathbb{1}_{\{W_i \leq \sqrt{\mu n}\}} - \mu n| > \mu n \right) &\leq \exp\Big(- \frac{(\mu n)^2}{2n\hat{C}(\mu n)^{(2-\alpha)/2} + 2(\mu n)^{3/2}} \Big) \\ &\leq \exp(-(\mu n)^{1/2}) \\ &\leq \exp(-n^{(\gamma-1)/2}),\end{aligned} \tag{B.9}$$

for all $\alpha \in (1,2)$ and $\gamma \leq 2$. For $\gamma > 2$, and n sufficiently large, Bernstein's inequality yields

$$\begin{aligned}\mathbb{P}\left(|\sum_i W_i \mathbb{1}_{\{W_i \leq \sqrt{\mu n}\}} - \mu n| > n^{1/2+\gamma/3} \right) &\leq \exp(- \frac{n^{1+2\gamma/3}}{(1+o(1))2(\mu n)^{3/2}}) \\ &\leq \exp\Big(- n^{(\gamma-1)/2} \Big).\end{aligned} \tag{B.10}$$

Thus,

$$\mathbb{P}\left(\sum_i \frac{W_i}{\sqrt{\mu n}} \mathbb{1}_{\{W_i \leq \sqrt{\mu n}\}} > 2\sqrt{\mu n} + n^{\gamma/3} \right) \leq \exp\Big(- n^{(\gamma-1)/2} \Big). \tag{B.11}$$

Thus, the probability that the mean degree of a vertex with weight $W_i = \sqrt{\mu n}$ is at most $2\sqrt{\mu n}$ or $n^{\gamma/3}$ is at most $\exp(-n^{(\gamma-1)/2})$. On this event, the degree of a vertex of weight $\sqrt{\mu n}$ is a sum of indicators with mean at most $2\sqrt{\mu n}$ or $n^{\gamma/3}$.

We use Lemma 5.1 and the union bound to show that for $\gamma < 2$

$$\begin{aligned}\mathbb{P}\left(\bar{\mathcal{F}}\right) &\leq \sum_{i=1}^n \mathbb{P}\left(d_i > 4\sqrt{\mu n} \mid W_i \leq \sqrt{\mu n}\right) \\ &\leq n\mathbb{P}\left(d_i > 4\sqrt{\mu n} \mid W_i = \sqrt{\mu n}\right) \leq n \exp(-2\sqrt{\mu n}(2\log(2) - 1)).\end{aligned} \tag{B.12}$$

Similarly, for $\gamma > 2$,

$$\mathbb{P}\left(\bar{\mathcal{F}}\right) \leq \sum_{i=1}^n \mathbb{P}\left(d_i > 4n^{\gamma/3} \mid W_i = \sqrt{\mu n}\right) \leq n \exp(-2n^{\gamma/3}(2\log(2) - 1)). \tag{B.13}$$

Thus, as $\gamma/3 > (\gamma-1)/2$ for $\gamma \in (1,3)$,

$$\mathbb{P}\left(\bar{\mathcal{F}}\right) \leq \exp\Big(- Dn^{(\gamma-1)/2} \log(n^{(\gamma-3)/2+\alpha}) \Big). \tag{B.14}$$

□

C Proof of Theorem 1.7: many dominating hubs

Proof of Theorem 1.7. Lower bound. As a lower bound, we compute the number of triangles with at least one vertex of weight $> \mu n$. By Lemma 5.2, the probability that at least n^λ vertices of weights at least μn are present can be bounded by

$$\mathbb{P}\left(n^\lambda \text{ vertices of weight } > \mu n\right) \geq \frac{1}{\sqrt{2n}} \exp(-n^\lambda \log(n^{\lambda+\alpha-1}))(1+o(1)). \tag{C.1}$$

Now, n^λ vertices of weight at least μn generate $n^{1+2\lambda}/2$ triangles. Indeed, every pair of 2 hubs forms a triangle with any of the other n vertices, so $n^{1+2\lambda}/2$ in total. Therefore, $\sqrt{2a}n^{(\gamma-1)/2}$ vertices of weight at least n create an^γ triangles. Thus,

$$\mathbb{P}\left(\triangle_n > an^\gamma\right) \geq \frac{1}{\sqrt{2n}} \exp\big(-\sqrt{2a}n^{(\gamma-1)/2} \log\big(\frac{\gamma-3}{2}+\alpha\big)\big)(1+o(1)). \tag{C.2}$$

Upper bound. We now distinguish and bound different types of triangles.

BBB: all weights larger than $\sqrt{\mu n}$. Let $X_{u,v,w}$ denote the event that u,v,w forms a triangle and that $w_u, w_v, w_w > \sqrt{\mu n}$. Then,

$$\sum_{u,v,w} X_{u,v,w} \leq \sum_{u,v,w} \mathbb{1}_{\{w_u,w_v,w_w>\sqrt{\mu n}\}} = \Big(\sum_u \mathbb{1}_{\{w_u>\sqrt{\mu n}\}}\Big)^3. \tag{C.3}$$

By Lemma 5.2, when $\gamma > 3 - 3/2\alpha$,

$$\begin{aligned}\mathbb{P}\left(\sum_{u,v,w} X_{u,v,w} > \varepsilon n^\gamma\right) &\leq \mathbb{P}\left(\varepsilon^{1/3} n^{\gamma/3} \text{ vertices of weight } > \sqrt{\mu n}\right)\\ &\leq \exp\Big(-\varepsilon^{1/3} n^{\gamma/3} \log(n^{\gamma/3-(2-\alpha)/2})\Big),\end{aligned} \tag{C.4}$$

by Lemma 5.2. Now for fixed $\varepsilon > 0$ and $\gamma > 1$ therefore

$$\mathbb{P}\left(\triangle_{BBB}\right) \leq \exp\Big(-\sqrt{2a}n^{(\gamma-1)/2} \log(n^{(\gamma-3)/2+\alpha})\Big), \tag{C.5}$$

for n sufficiently large.

AAA: all weights smaller than $\sqrt{\mu n}$. By Lemma 5.6, we may work on the event $\mathcal{F}$, so that all degrees are bounded by $M = 4\max(\sqrt{n}, n^{\gamma/3})$. We aim to design two sets of indicators that deal with the dependencies between the presences of different triangles, so that we can use Lemma 4.3. Let Y_{uvw} denote the indicator that u,v,w forms a triangle and that u,v,w have degree at most $\sqrt{\mu n}$. Now $Y_{uvw} \leq X_{uvw}$. We now define a set of indicators $Y_{xyz(uvw)}$.

When $|\{x,y,z,u,v,w\}| = 6$, we set $Y_{xyz(uvw)} = Y_{xyz}$, and otherwise, we set $Y_{xyz(uvw)} = 0$. Now $\sum_{x,y,z} Y_{xyz(uvw)}$ is independent of X_{uvw}, as none of the entries of the summation depend on the edges uv, uw, vw.

Furthermore, $\sum_{x,y,z} Y_{xyz(uvw)} \leq \sum_{x,y,z} Y_{x,y,z}$. Finally,

$$\sum_{x,y,z} Y_{x,y,z} \leq 3M^2 + \sum_{x,y,z} Y_{xyz(uvw)}, \tag{C.6}$$

as at most M^2 triangles involve vertex u since its maximal degree is M. Thus, by Lemma 4.3,

$$\mathbb{P}\left(\sum_{u,v,w} Y_{uvw} > \varepsilon n^\gamma\right) \leq \exp\left(-\frac{\varepsilon n^\gamma}{M^2} \log(n^{\gamma-\frac{(2-\alpha)}{2}})\right). \tag{C.7}$$

Again, for $\gamma > 1$ and n sufficiently large this indicates that

$$\mathbb{P}\left(\triangle_{AAA}\right) \leq \exp\Big(-\sqrt{2a}n^{(\gamma-1)/2} \log(n^{(\gamma-3)/2+\alpha})\Big), \tag{C.8}$$

as $\gamma - 2\gamma/3 > (\gamma-1)/2$ for $\gamma \in (1,3)$.

ABB triangles.

To bound the number of these triangles, we split the B vertices into B_1 vertices, $B_2' = B \cap B_2$ and $B_3' = B \cap B_3$. We first investigate the number of $AB_3'B_3'$ triangles. When $B \cap B_3 = \emptyset$, we are done. Otherwise, $A \subseteq B_3$ and by (5.10), for $\gamma > 3 - 3/2\alpha$ and $D = \sqrt{2a}/h(1+\varepsilon)$,

$$\mathbb{P}\left(\triangle_{A,B_3',B_3'} > \varepsilon n^{\gamma}\right) \leq \mathbb{P}\left(\triangle_{B_3,B_3,B_3} > \varepsilon n^{\gamma}\right) \leq \exp(-\sqrt{2a}n^{(\gamma-1)/2}\log(n^{(\gamma-3)/2+\alpha})). \tag{C.9}$$

We now bound the number of AB_1B_1 type triangles by n times the number of B_1 vertices squared. This is the number of ways to choose 2 type B_1 vertices, and one other vertex. Thus, by (5.7),

$$\begin{aligned}\mathbb{P}\left(\triangle_{AB_1B_1} > an^{\gamma}\right) &\leq \mathbb{P}\Big(N(B_1) > \sqrt{2a}n^{(\gamma-1)/2}\Big)\\ &\leq \exp\Big(-\sqrt{2a}n^{(\gamma-1)/2}\log(n^{(\gamma-3)/2+\alpha}\log(n)^{-\zeta})\Big).\end{aligned} \tag{C.10}$$

Consequently,

$$\lim_{n\to\infty}\frac{\log\Big(\mathbb{P}\left(\triangle_{AB_1B_1} > an^{\gamma}\right)\Big)}{n^{(\gamma-1)/2}\log(n)} = \sqrt{2a}\Big(\frac{\gamma-3}{2}+\alpha\Big). \tag{C.11}$$

We bound the number of $AB_3'B_i'$ for $i=1,2$ by the number of B_i' vertices times the number of edges between AB_3' vertices. Again, when B_3' is empty, we are done. Otherwise, $A \subseteq B_3$, so that by (5.11)

$$\mathbb{P}\Big(E_{A,B_3'} > n^{(\gamma+1)/4}\Big) \leq \mathbb{P}\Big(E_{B_3,B_3} > n^{(\gamma+1)/4}\Big) \leq \exp(-\sqrt{2a}n^{(\gamma-1)/2}\log(n^{(\gamma-3)/2+\alpha})). \tag{C.12}$$

On this event, by (5.6), and for n sufficiently large,

$$\begin{aligned}\mathbb{P}\left(\triangle_{AB_3'B_i'} > \varepsilon n^{\gamma}\right) &\leq \mathbb{P}\Big(N(B_1\cup B_2) > \varepsilon n^{(3\gamma-1)/4}\Big)\\ &\leq \exp\Big(-\varepsilon n^{(3\gamma-1)/4}\log(n^{(\gamma+1)/4}\log(n^{(\gamma-3)/2+\alpha})^{-1})\Big)\\ &\leq \exp\Big(-\sqrt{2a}n^{(\gamma-1)/2}\log(n^{(\gamma-3)/2+\alpha}\log(n)^{-\alpha})\Big).\end{aligned} \tag{C.13}$$

We now bound the number of $AB_2'B_1$ triangles by $N(B_1)$ times the number of AB_2 edges. By Lemma 5.5 with $M = \max(\sqrt{n}, n^{\gamma/3})$, and for n sufficiently large

$$\begin{aligned}\mathbb{P}\left(E_{A,B_2'} > \frac{\varepsilon}{\sqrt{2a}}n^{(\gamma+1)/2}\right) &\leq \mathbb{P}\left(E_{A,B_2} > \frac{\varepsilon}{\sqrt{2a}}n^{(\gamma+1)/2}\right)\\ &\leq \exp(-\sqrt{2a}n^{(\gamma-1)/2}\log(n^{(\gamma-3)/2+\alpha}))\end{aligned} \tag{C.14}$$

as long as $N(B_2) \leq n^{(\gamma-1)/2}\log(n)^{\zeta}$. By (5.6) this happens with probability

$$\begin{aligned}\mathbb{P}\Big(N(B_2) > n^{(\gamma-1)/2}\log(n)^{\zeta}\Big) &\leq \exp\Big(-n^{(\gamma-1)/2}\log(n)^{\zeta}\log\Big(\log(n)^{\zeta}\log(n^{(\gamma-3)/2+\alpha})^{-1}\Big)\Big)\\ &\leq \exp\Big(-\sqrt{2a}n^{(\gamma-1)/2}\log(n^{(\gamma-3)/2+\alpha})\Big),\end{aligned} \tag{C.15}$$

when $\zeta > 1$ and n is sufficiently large. Thus, on the event of (C.14), by (5.7),

$$\begin{aligned}\mathbb{P}\left(\triangle_{AB_1B_2'} > \varepsilon n^{\gamma}\right) &\leq \mathbb{P}\left(N(B_1)E_{AB_2'} > \varepsilon n^{\gamma}\right)\\ &\leq \mathbb{P}\Big(N(B_1) > \sqrt{2a}n^{(\gamma-1)/2}\Big)\\ &\leq \exp\Big(-\sqrt{2a}n^{(\gamma-1)/2}\log(n^{(\gamma-3)/2+\alpha}\log(n)^{-\zeta})\Big).\end{aligned} \tag{C.16}$$

We now consider A$B_2'B_2'$ triangles. By Lemma 5.5,

$$\begin{aligned}\mathbb{P}\left(\triangle_{A,B_2',B_2'} > n^\gamma\right) &\leq \exp\Big(-\frac{n^\gamma}{M^2}\log(n^\gamma/(3K_1N(B_2')^2 n\log(n)^{-\zeta}))\Big)\\ &\leq \exp\Big(-\sqrt{2a}n^{(\gamma-1)/2}\log(n^{(\gamma-3)/2+\alpha})\Big),\end{aligned} \tag{C.17}$$

as long as $N(B_2) \leq n^{(\gamma-1)/2}\log(n)^\zeta$ and $\gamma > 1$, which happens with probability (C.15).

Thus, (C.11), (C.9), (C.13), (C.16) and (C.17) yield that

$$\lim_{n\to\infty} \frac{\log\Big(\mathbb{P}\left(\triangle_{ABB} > n^\gamma\right)\Big)}{n^{(\gamma-1)/2}\log(n)} = \sqrt{2a}\Big(\frac{\gamma-3}{2}+\alpha\Big). \tag{C.18}$$

AAB triangles. When bounding the number of AAB triangles, we work on the event $\mathcal{F}$ from Lemma 5.6, so that all type-A vertices have degrees at most $M = \max(4\sqrt{\mu n}, 4n^{\gamma/3})$. Again, we split the B vertices into B_1', B_2' and B_3'.

We now bound the number of AAB_i' triangles with $i \in B_1' \cap B_2'$. To do so, we first show that the event $\mathcal{E}$ that there are at most $n^{(\gamma+1)/2}$ edges between type A vertices happens with high probability. Let X_{ij}' denote the indicator that an edge is present between i and j, let X_{ij} be the indicator that $i, j \in A$ and that $\{i, j\}$ is an edge. Finally, we define $X_{ij(uv)} = X_{ij}$ when i, j, u, v are all distinct, and set $X_{ij(uv)} = 0$ when i or j overlaps with u or v. Then $\sum_{ij} X_{ij(uv)}$ is independent from X_{uv}, and furthermore

$$E_{A,A} := \sum_{ij} X_{ij} \leq d_u + d_v + \sum_{ij} X_{ij(uv)} \leq 2M + \sum_{ij} X_{ij(uv)} \tag{C.19}$$

Then, Lemma 4.3 shows that

$$\begin{aligned}\mathbb{P}\Big(E_{A,A} > n^{(3\gamma-1)/4}\Big) &\leq \exp\Big(-\frac{n^{(3\gamma-1)/4}}{2n^{\gamma/3}}\log\Big(\frac{n^{(3\gamma-1)/2}}{3\mu n}\Big)\Big)\\ &< \exp\Big(-\sqrt{2a}n^{(\gamma-1)/2}\log(n^{(\gamma-3)/2+\alpha})\Big),\end{aligned} \tag{C.20}$$

for $\gamma \in [2, 3)$. Furthermore,

$$\begin{aligned}\mathbb{P}(\mathcal{E}) := \mathbb{P}(E_{A,A} > 4\mu n) &\leq \exp\Big(-\frac{4\mu n}{8K\sqrt{\mu n}}\log\Big(\frac{4\mu n}{3\mu n}\Big)\Big)\\ &< \exp\Big(-\sqrt{2a}n^{(\gamma-1)/2}\log(n^{(\gamma-3)/2+\alpha})\Big),\end{aligned} \tag{C.21}$$

for $\gamma \in [1, 2)$ and n sufficiently large.

We now condition on the event $\mathcal{E}$. On this event, we can bound the number of AAB_i triangles for $i = 1, 2$ by $4\mu n$ times the number of type B_i vertices for $\gamma \in [1, 2)$. Thus, on $\mathcal{E}$, by Lemma 5.2

$$\begin{aligned}\mathbb{P}\left(\triangle_{AAB_i} > \varepsilon n^\gamma\right) \leq \mathbb{P}\left(N(B_2' \cup B_1) > \varepsilon n^{\gamma-1}/(4\mu)\right) &\leq \exp\Big(-\frac{\varepsilon n^{\gamma-1}}{4\mu}\log(n^{\gamma-1-(\gamma-1)/2})\Big)\\ &\leq \exp\Big(-\sqrt{2a}n^{(\gamma-1)/2}\log(n^{(\gamma-1)/2-(2-\alpha)/2})\Big),\end{aligned} \tag{C.22}$$

for $\gamma \in [1, 2)$ and n sufficiently large. Similarly, for $\gamma \in [2, 3)$, by Lemma 5.2

$$\begin{aligned}\mathbb{P}\left(\triangle_{AAB_i} > \varepsilon n^\gamma\right) &\leq \mathbb{P}\Big(N(B_2' \cup B_1) > \varepsilon n^{(\gamma+1)/4}\Big)\\ &\leq \exp(-\varepsilon n^{(\gamma+1)/4}\log(n^{(\gamma+1)/4-(\gamma-1)/2}))\\ &\leq \exp\Big(-\sqrt{2a}n^{(\gamma-1)/2}\log(n^{(\gamma-1)/2-(2-\alpha)/2})\Big),\end{aligned} \tag{C.23}$$

for n sufficiently large.

Finally, we bound the number of AAB_3' triangles. When B_3' is empty, we are done. Otherwise, $A \subseteq B_3$, and any AAB_3' triangle is also an $B_3B_3B_3$ triangle, whose number can be bounded by (C.9).

To conclude, (C.5), (C.5), (C.18), (C.22) and (C.23) with a limit of $\varepsilon \downarrow 0$ yield that

$$\log\left(\mathbb{P}\left(\triangle_n > an^{\gamma}\right)\right) \leq -\sqrt{2a}n^{(\gamma-1)/2}\log(n^{(\gamma-3)/2+\alpha})(1+o(1)). \tag{C.24}$$

Combining this with (C.2) gives

$$\lim_{n\to\infty} \frac{\log \mathbb{P}\left(\triangle_n > n^{\gamma}\right)}{n^{(\gamma-1)/2}\log(n)} = \sqrt{2a}\Big(\frac{\gamma-3}{2}+\alpha\Big). \tag{C.25}$$

□

D Proof of Theorem 1.6

The proof follows the same steps as the proof of Theorem 1.3. To avoid unnecessary repetitions, we restrict ourselves to indicating which steps require nontrivial modifications. First of all, the asymptotic expansion $c_{n^\theta}(n) \sim L^*(n)n^{\beta+\frac{\theta}{2}\frac{1}{\alpha-1}} = o(n)$ follows from a straightforward modification of the proof of Lemma 1.2, and the $o(n)$ behavior follows from the inequality $\theta < \frac{3}{2}\alpha - 2$.

The first major step is to prove that the analogue of Theorem 2.1 holds, with a replaced by n^{γ}. To establish this, we first show how to modify the proof of Proposition 2.5, in particular how to manage the ten-term bound (2.20). The three terms that require modification are the third, fifth and sixth term. For the third term, we can again invoke Lemma 2.4 to conclude that this term behaves like $c\varepsilon^{(\alpha-1)/2}n^{\theta}\mu_n$. For the fifth term, apply Lemma 2.3 with $\alpha_b = \alpha_c = \beta + \frac{\theta}{2}\frac{1}{\alpha-1}$, to conclude that this term is regularly varying with index $1-\alpha+\alpha^2/(4(\alpha-1))+\alpha\theta/(2(\alpha-1))$. To show that the fifth term is of small order in n, it suffices to show that

$$1-\alpha+\alpha^2/(4(\alpha-1))+\alpha\theta/(2(\alpha-1)) < 3-\alpha 3/2+\gamma,$$

which is equivalent to

$$3\alpha^2 - 2\alpha + \theta(4-2\alpha) < 8(\alpha-1).$$

This follows from the inequality $\theta < \alpha 3/2 - 2$. As before, the sixth term is of smaller order than the fifth term, and therefore is also negligible as n grows large. This leads to the conclusion that

$$\mathbb{P}\left(G_n > (1+n^{\theta})m_n; L_n(\varepsilon c_{n^\theta}(n)) = 0\right) = o(n\mathbb{P}\left(W_1 > c_{n^\theta}(n)\right)), \tag{D.1}$$

which is the desired extension of Proposition 2.5. The statement and the proof of Proposition 2.6 extends straightforwardly to handle the case where a is replaced by n^{θ}. This readily leads to the conclusion that

$$\mathbb{P}\left(G_n > m_n(1+n^{\theta})\right) = (1+o(1))n\mathbb{P}\left(W > c_{n^\theta}(n)\right). \tag{D.2}$$

To derive the same estimate for $\triangle_n$ we use the same steps as in Section 4: the proof of the asymptotic upper bound again follows from Lemma 4.1. To derive an asymptotic lower bound, we need to modify the first part Lemma 4.2. In particular, we need that

$$\triangle_n(\delta, n^{\theta})/(m_n(1+n^{\theta})) \to 1 \tag{D.3}$$

in probability. The proof of this statement follows by simply following the same steps as the first part of Lemma 4.2, with a replaced by n^{θ}. With the appropriate analogues of Lemma's 4.1 and 4.2 in place, the proof of Theorem 1.6 follows straightforwardly.