# Finite element hybridization of port-Hamiltonian systems 

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#### Abstract

In this contribution, we extend the hybridization framework for the Hodge Laplacian [Awanou et al., Hybridization and postprocessing in finite element exterior calculus, 2023] to port-Hamiltonian systems. To this aim, a general dual field continuous Galerkin discretization is introduced, in which one variable is approximated via conforming finite element spaces, whereas the second is completely local. This scheme retains a discrete power balance and discrete conservation laws and is directly amenable to hybridization. The hybrid formulation is equivalent to the continuous Galerkin formulation and to a power preserving interconnection of port-Hamiltonian systems, thus providing a system theoretic interpretation of finite element assembly. The hybrid system can be efficiently solved using a static condensation procedure in discrete time. The size reduction achieved thanks to the hybridization is greater than the one obtained for the Hodge Laplacian as one field is completely discarded. Numerical experiments on the 3D wave and Maxwell equations show the equivalence of the continuous and hybrid formulation and the computational gain achieved by the latter.


Keywords: Port-Hamiltonian systems, Finite element exterior calculus, Hybridization, Dual Field

## 1. Introduction

The port-Hamiltonian $(\mathrm{pH})$ framework is a novel modelling paradigm that structurally describes energy routing within a system or a network. Distributed pH systems were first introduced in the seminal work [1] and comprehensively reviewed in [2]. In order to account for boundary interaction of systems of coupled conservation laws, the framework relies on the concept of StokesDirac structure. This geometrical structure represents a particular instance of Dirac manifolds [3]. The Stokes theorem allows defining appropriate boundary variables to account for the power exchange through the boundary of the spatial domain. This geometrical structure ignores the actual boundary conditions of the problem at hand and captures all admissible boundary flows. Because of the fact that Dirac manifolds are composable, port-Hamiltonian systems are closed under interconnection, thus well suited for modeling complex multi-physical systems as an interconnection of simpler subsystems.

The preservation of the pH structure at the discrete level is an important active area of research. Many numerical algorithms have been proposed to capture the underlying conservation laws and provide a systematic way to handle boundary conditions, so that the resulting discrete pH system defines a Dirac structure. One challenging task in deriving discrete representations of pH systems is the construction of a discrete isomorphic Hodge star [4, 5]. Several approaches have been used to achieve this purpose. For instance, the discrete exterior calculus framework [6] can be used to formulate pH systems in a purely discrete manner, as shown in [7]. In this

[^0]framework dual topological meshes (based on the Delaunay-Voronoi duality) are used to define an isomorphic discrete Hodge star. The main drawback of dual meshes resides in the introduction of ghost points that do not lie inside the physical domain, thus making it difficult to interconnect systems. A different approach is pursued in [8], where a Galerkin formulation based on Whitney forms is detailed. This method does not require dual meshes and instead constructs the Hodge star via a tunable projector (like it is done in the construction of numerical fluxed in discontinuous method). The best choice of the parameters associated with the projector depends strongly on the application at hand. Another interesting and very recent contribution describes a discontinuous Galerkin framework for systematic discretization of port-Hamiltonian systems [9]. Therein, the duality between polynomial families, employed to define degrees of freedom for finite element differential forms, is used to construct a discrete Stokes-Dirac structure. The Hodge operator is subsequently embedded in the inner product. In our previous work [10, we have proposed a strategy based on finite element exterior calculus [11]. We have shown that port-Hamiltonian systems admit a primal dual representation, that is constructed by introducing an adjoint system, which embeds the Hodge operators into the codifferential. The primal dual structure describes the original unknowns of the problems together with their Hodge duals and thus produces a dual field representation of the variables. The dual field formulation leads to a discrete power balance and conservation laws, thus retaining the main properties of port-Hamiltonian systems. This discretization strategy was originally introduced to approximate the Navier-Stokes equations in a way that preserves mass and energy and helicity [12]. After time discretization the resulting scheme is linear and hence extremely efficient.

The computational efficiency of continuous finite element discretization schemes, such as our dual field method, can be boosted using hybridization [13. Hybridization leads to numerous theoretical and practical benefits. First of all the Lagrange multiplier functions correspond to weak boundary traces of solution components, thus providing additional information about the solution. An important computational advantage is given by the so called static condensation [14]: degrees of freedom for discontinuous function spaces can be locally eliminated using the Schur complement. The resulting system of equations only involves degrees of freedom for the globally coupled boundary Lagrange multipliers. This leads to a smaller system to be solved than the original problem. Via local postprocessing the quality of the approximated solution can be improved. Very recently, finite element exterior calculus hybridization and post-processing have been detailed for the Hodge Laplacian [15]. This framework represents the basis for the present contribution.

In this work, we present a hybridization scheme for a dual-field continuous Galerkin formulation of port-Hamiltonian systems. The properties of our original dual field method [10], in particular the discrete power balance and conservation laws, are left untouched by this scheme. A major distinction to our formulation in [10] is that we starting by demanding less regularity for the variable that does not undergo the exterior derivative. This variable is discretized by using broken finite element differential forms in order to verity a local discrete subcomplex property. Following the framework described in [15, the continuous Galerkin scheme is then hybridized by introducing a broken (and therefore local) multiplier, capturing the information related to the normal trace, and an unbroken global Lagrange, representing the tangential trace of the regular variable. The hybrid formulation is completely equivalent to the continuous Galerkin discretization and thus retains all its properties. Furthermore, an interesting connection between discretization and system theory is established, as the local problems can be formulated as port-Hamiltonian differential algebraic systems (or port-Hamiltonian descriptor systems [16]) and the hybridization scheme can be reinterpreted as interconnection of the local problems. The time discretization is obtained by using the implicit midpoint method, that allows to preserve the symplectic structure of the problem. The resulting system is a saddle point problem that can be efficiently solved via a static condensation procedure. This leads to a considerable reduction of the system to be solved as only the global trace variable of the continuous field need to be solved for. Local variables can be subsequently computed in parallel, as they are completely uncoupled. Because of the peculiar
structure of port-Hamiltonian systems, the size reduction obtained by means of the hybridization is even more important than the one obtained for the Hodge Laplacian in [15], as the $L^{2}$ variable, being of local nature, is completely discarded from the global problem.

The paper is organised as follows: in Sec. 2 a brief discussion of the $L^{2}$ theory of differential forms and the notation is presented. In Sec. 3 we introduce the modified continuous Galerkin formulation by making use of broken finite elements for differential forms. The hybridization of this formulation is presented in Sec. 4, where it is shown that the hybrid version is completely equivalent to the continuous formulation. In Sec. 5, the algebraic realization of the weak formulation is detailed. The reinterpretation of finite element assembly as interconnection of port-Hamiltonian descriptor system is also given. A time discrete system is obtained using the implicit midpoint method. Via static condensation the local variable are then eliminated. Section 6 presents numerical experiments for the wave and Maxwell equations in 3D. The equivalence of the continuous and hybrid formulation is verified numerically and the convergence rate of the variables is assessed.

## 2. Preliminaries

In this section we will introduce some preliminary concepts and their notations which are needed for the treatment ahead.

### 2.1. Smooth differential forms

Let $M \subset \mathbb{R}^{n}$ be a bounded Lipschitz Riemannian manifold with boundary $\partial M$ and metric $g$. The space of smooth differential forms on $M$ (i.e. the space of smooth sections of the $k$ th exterior power of the cotangent bundle $T^{*} M$ ) is denoted by $\Omega^{k}(M)$. For the sake of additional clarity, the degree of differential forms will always be indicated. We assume that the reader is familiar with the basic operators and results of exterior calculus

- wedge product $\wedge: \Omega^{k}(M) \times \Omega^{l}(M) \rightarrow \Omega^{k+l}(M)$;
- the exterior derivative d : $\Omega^{k}(M) \rightarrow \Omega^{k+1}(M)$;
- the trace operator $\operatorname{tr}:=\iota^{*}$, i.e. the pull back of the inclusion map $\iota: \partial M \rightarrow M$;
- the Stokes theorem: for $\omega^{n-1} \in \Omega^{n-1}$ it holds $\int_{M} \mathrm{~d} \omega^{n-1}=\int_{\partial M} \operatorname{tr} \omega^{n-1}$;

Differential forms possess a natural duality product which is often not discussed in numerical methods.

Definition 1 (Duality product). Given a smooth manifold $M$ of dimension n, the duality product is denoted by

$$
\begin{equation*}
\left(\alpha^{k} \mid \beta^{n-k}\right)_{M}:=\int_{M} \alpha^{k} \wedge \beta^{n-k}, \quad \alpha^{k} \in \Omega^{k}(M), \quad \beta^{n-k} \in \Omega^{n-k}(M), \quad k=0, \ldots, n \tag{1}
\end{equation*}
$$

The duality product is also defined on the boundary $\partial M$ (whose orientation is inherited from the one of the manifold)

$$
\begin{equation*}
\left\langle\operatorname{tr} \alpha^{k} \mid \operatorname{tr} \beta^{n-k-1}\right\rangle_{\partial M}:=\int_{\partial M} \operatorname{tr} \alpha^{k} \wedge \operatorname{tr} \beta^{n-k-1}, \quad \alpha^{k} \in \Omega^{k}(M), \quad \beta^{n-k-1} \in \Omega^{n-k-1}(M) \tag{2}
\end{equation*}
$$

where $k=0, \ldots, n-1$.
Combing together the Leibniz rule and the Stokes theorem, one has the integration by parts formula
$(\mathrm{d} \alpha \mid \beta)_{M}+(-1)^{k}(\alpha \mid \mathrm{d} \beta)_{M}=\langle\operatorname{tr} \alpha \mid \operatorname{tr} \beta\rangle_{\partial M}, \quad \alpha \in \Omega^{k}(M), \quad \beta \in \Omega^{n-k-1}(M), \quad k=0, \ldots, n-1$.
This duality pairing is fundamental in port-Hamiltonian systems as it defines the power flow.

## 2.2. $L^{2}$ theory of differential forms

Given a coordinate chart $\xi_{i}: M \rightarrow \mathbb{R} \quad i=1, \ldots, n$, we can represent locally a point $p$ in the manifold $M$ with the tuple $\xi(p):=\left(\xi_{1}(p), \ldots, \xi_{n}(p)\right)$. Then, the local representation of a form $\alpha^{k} \in \Omega^{k}(M)$ reads

$$
\begin{equation*}
\alpha^{k}(p)=\sum_{I} \alpha_{I}(\xi(p)) \mathrm{d} \xi^{i_{1}} \wedge \cdots \wedge \mathrm{~d} \xi^{i_{k}} \tag{4}
\end{equation*}
$$

where the multi-index $I:=i_{1}, \ldots, i_{k}, 1 \leq i_{1} \leq \cdots \leq i_{k} \leq n$ has been introduced. Thanks to the metric structure $g$ of the Riemannian manifold (that establishes an inner product of vectors) and the duality between vectors and differential forms, the space of $k$-forms can be equipped with a pointwise inner product

$$
\begin{equation*}
\left(\alpha^{k}, \beta^{k}\right):=g^{i_{1} j_{1}} \ldots g^{i_{k} j_{k}} \alpha_{i_{1} \ldots i_{k}} \beta_{j_{1} \ldots j_{k}}, \quad \forall \alpha^{k}, \beta^{k} \in \Omega^{k}(M) \tag{5}
\end{equation*}
$$

where $g^{k l}:=\left(\mathrm{d} \xi^{k}, \mathrm{~d} \xi^{l}\right) \in C^{\infty}(M)$ are the components of the inverse metric tensor.
Once an orientation is given to the Riemannian manifold, the Hodge star operator $\star$ can be properly defined. The Hodge maps inner-oriented (or true) forms, measuring intensities, to outeroriented (or pseudo) forms [17, 18, measuring quantities, and vice versa. This distinction is of fundamental importance and allows defining integral quantities that are orientation independent (like mass, energy, etc.). In this paper outer-oriented forms are denoted by means of a hat, i.e. $\widehat{\alpha}^{k} \in \widehat{\Omega}^{k}(M)$ where $\widehat{\Omega}^{k}(M)$ is the space of outer oriented (or pseudo) forms. This distinction is often disregarded in the mathematical literature but is well known in the physics community. This distinction is important also in numerical methods 19 and is fundamental to our proposed dual-field scheme.

Definition 2 (Hodge-丸 operator). The Hodge- $\star$ operator, defined for an $n$-dimensional Riemannian manifold $M$, is the operator $\star: \Omega^{k}(M) \rightarrow \widehat{\Omega}^{n-k}(M)$ such that

$$
\alpha^{k} \wedge \star \beta^{k}=\left(\alpha^{k}, \beta^{k}\right) \operatorname{vol}, \quad \alpha^{k}, \beta^{k} \in \Omega^{k}(M)
$$

where the inner product is defined in (5). The standard volume form in local coordinates is given by [18, Page 362]

$$
\begin{equation*}
\operatorname{vol}=\sqrt{\left|\operatorname{det}\left(g_{i j}\right)\right|} \epsilon_{1 \ldots n} \mathrm{~d} \xi^{1} \wedge \cdots \wedge \mathrm{~d} \xi^{n} \tag{6}
\end{equation*}
$$

where $g_{i j}$ are the components of the metric tensor in the chosen chart $\xi$ and $\epsilon_{1 \ldots n}$ is the Levi-Civita symbol. Notice the often forgotten importance of the Levi-Civita symbol, that indicates that the volume form is indeed a pseudo form $\mathrm{vol}=\star 1$, as the total volume of a portion of space cannot be negative.

We now introduce the $L^{2}$ inner product of forms.
Definition 3 ( $L^{2}$ inner product). Given a smooth manifold $M$ of dimension $n$, the $L^{2}$ inner product is defined by

$$
\left(\alpha^{k}, \beta^{k}\right)_{M}:=\int_{M}\left(\alpha^{k}, \beta^{k}\right) \mathrm{vol}=\int_{M} \alpha^{k} \wedge \star \beta^{k}, \quad \alpha, \beta \in \Omega^{k}(M)
$$

As in vector calculus, the $L^{2}$ Hilbert space of differential forms is the completion of the space of smooth forms $\Omega^{k}(M)$ in the norm induced by the $L^{2}$ inner product.

Taking $d$ in the sense of distributions allows it to be extended to a closed, densely defined operator with domain

$$
H \Omega^{k}(M):=\left\{\omega^{k} \in L^{2} \Omega^{k}(M) \mid \mathrm{d} \omega^{k} \in L^{2} \Omega^{k+1}(M)\right\}, \quad k=0, \ldots, n-1
$$

The space $H \Omega^{k}(M)$ is again an Hilbert space with graph inner product $\left(v^{k}, \omega^{k}\right)_{H \Omega^{k}(M)}=\left(v^{k}, \omega^{k}\right)_{M}+$ $\left(\mathrm{d} v^{k}, \mathrm{~d} \omega^{k}\right)_{M}$. These spaces, connected by the operator d , form the de Rham domain complex, as illustrated in Fig. 1 .

The formal adjoint of the exterior derivative is the codifferential operator.

$$
H \Omega^{0}(M) \xrightarrow{\mathrm{d}} \ldots \xrightarrow{\mathrm{~d}} H \Omega^{k}(M) \xrightarrow{\mathrm{d}} \ldots \xrightarrow{\mathrm{~d}} H \Omega^{n}(M)
$$

Figure 1: The domain de Rham complex.

Definition 4 (Codifferential). The co-differential map $\mathrm{d}^{*}: \Omega^{k}(M) \rightarrow \Omega^{k-1}(M)$ is defined by

$$
\begin{equation*}
\mathrm{d}^{*}:=(-1)^{k} \star^{-1} \mathrm{~d} \star . \tag{7}
\end{equation*}
$$

The codifferential verifies the following integration by parts formula

$$
\begin{equation*}
\left(\alpha^{k}, \mathrm{~d}^{*} \beta^{k+1}\right)_{M}=\left(\mathrm{d} \alpha^{k}, \beta^{k+1}\right)_{M}-\left\langle\operatorname{tr} \alpha^{k} \mid \operatorname{tr} \star \beta^{k+1}\right\rangle_{\partial M}, \quad \alpha \in \Omega^{k}(M), \beta \in \Omega^{k+1}(M) \tag{8}
\end{equation*}
$$

The integration by parts formula (8) can be rewritten using the normal trace.
Definition 5 (Normal trace). Denote the Hodge star on the boundary $\partial M$ by $\star_{\partial}$, constructed using the pullback of the metric at the boundary and the associated volume form (6). Given $\omega^{k} \in \Omega^{k}(M)$, its normal trace is defined by

$$
\operatorname{tr}_{\boldsymbol{n}} \omega^{k}=\star_{\partial}^{-1} \operatorname{tr} \star \omega^{k} \in \Omega^{k-1}(\partial \Omega)
$$

Using the definition of normal trace, the duality pairing in Eq. (8) is converted into an inner product over the boundary

$$
\begin{equation*}
\left\langle\operatorname{tr} \alpha^{k} \mid \operatorname{tr} \star \beta^{k+1}\right\rangle_{\partial M}=\left\langle\operatorname{tr} \alpha^{k}, \operatorname{tr}_{\boldsymbol{n}} \beta^{k+1}\right\rangle_{\partial M}:=\int_{\partial M} \operatorname{tr} \alpha^{k} \wedge \star \partial \operatorname{tr}_{\boldsymbol{n}} \beta^{k+1} \tag{9}
\end{equation*}
$$

The integration by parts formula (8) is then rewritten as follows

$$
\begin{equation*}
\left\langle\operatorname{tr} \alpha^{k}, \operatorname{tr}_{\boldsymbol{n}} \beta^{k+1}\right\rangle_{\partial M}=\left(\mathrm{d} \alpha^{k}, \beta^{k+1}\right)_{M}-\left(\alpha^{k}, \mathrm{~d}^{*} \beta^{k+1}\right)_{M} \tag{10}
\end{equation*}
$$

For the reader convenience, Table 1 resumes the notation used for inner and duality products over the domain $M$ and the boundary $\partial M$.

|  | Inner product | Dual Product |
| :---: | :---: | :---: |
| Domain $M$ | $(\alpha, \beta)_{M}=\int_{M} \alpha \wedge \star \beta$ | $(\alpha \mid \beta)_{M}=\int_{M} \alpha \wedge \beta$ |
| Boundary $\partial M$ | $\langle\alpha, \beta\rangle_{\partial M}=\int_{\partial M} \alpha \wedge \star \partial \beta$ | $\langle\alpha \mid \beta\rangle_{\partial M}=\int_{\partial M} \alpha \wedge \beta$ |

Table 1: Notation for inner and duality product in the domain $M$ and its boundary. Inner and dual products on the domain $M$ are denoted using round brackets ( ), whereas on the boundary $\partial M$ they are denoted using angle brackets $\rangle$. The forms $\alpha$ or $\beta$ have appropriate degree according to the definitions.

Analogously to the exterior derivative, the codifferential d* may be also extended to a closed, densely defined operator with domain

$$
H^{*} \Omega^{k}(M):=\left\{\omega^{k} \in L^{2} \Omega^{k}(M): \mathrm{d}^{*} \omega^{k} \in L^{2} \Omega^{k-1}(M)\right\}=\star H \Omega^{n-k}(M)
$$

which is a Hilbert space with the graph inner product. In 20], it has been shown that the integration by parts formula (10) can be extended to $\alpha^{k} \in H \Omega^{k}(M)$ and $\beta^{k+1} \in H^{*} \Omega^{k+1}(M)$ and to manifolds with Lipschitz boundary. The spaces where the trace variables live are denoted by $\operatorname{tr} \alpha^{k} \in H \Omega^{k, \boldsymbol{t}}(\partial M)$ and $\operatorname{tr}_{\boldsymbol{n}} \beta^{k+1} \in H^{*} \Omega^{k, \boldsymbol{n}}(\partial M)$, which are subspaces of the fractional Sobolev space ${ }^{1} H^{-1 / 2} \widehat{\Omega}^{k}(\partial M)$. Thus, $\left\langle\operatorname{tr} \alpha^{k}, \operatorname{tr}_{\boldsymbol{n}} \beta^{k+1}\right\rangle_{\partial M}$ represents a duality pairing ${ }^{2}$ extending to the $L^{2}$

[^1]inner product on $\partial M$. The exact definitions of $H \Omega^{k, \boldsymbol{t}}(\partial M), H^{*} \Omega^{k, \boldsymbol{n}}(\partial M)$ are involved. However, these spaces can be easily characterized since they are isomorphic to the following quotient spaces (cf. [22, Thms. 5 and 7])
\[

$$
\begin{aligned}
H \Omega^{k, \boldsymbol{t}}(\partial M) & \cong H \Omega^{k}(M) / \stackrel{\circ}{H} \Omega^{k}(M) \\
H^{*} \Omega^{k, \boldsymbol{n}}(\partial M) & \cong H^{*} \Omega^{k+1}(M) / \stackrel{\circ}{H}^{*} \Omega^{k+1}(M)
\end{aligned}
$$
\]

where the subspaces $\stackrel{\circ}{H} \Omega^{k}(M) \subset H \Omega^{k}(M)$ and $\stackrel{\circ}{H}^{*} \Omega^{k}(M) \subset H^{*} \Omega^{k}(M)$ are defined as the closure of compactly supported smooth function in the $H \Omega^{k}(M)$ and $H^{*} \Omega^{k}(M)$ norm respectively. Therefore $H \Omega^{k, \boldsymbol{t}}(\partial M)$ and $H^{*} \Omega^{k, n}(\partial M)$ can be treated as quotient spaces equipped with the quotient norm. This equivalent characterization allows deducing that $\stackrel{\circ}{H} \Omega^{k}(M)$ and $\stackrel{\circ}{H}^{*} \Omega^{k}(M)$ are the spaces of function with vanishing tangential and normal trace respectively (see [15, Lemma 2.3 and Rmk. $2]$ for details)

$$
\begin{aligned}
\stackrel{\circ}{H} \Omega^{k}(M) & =\left\{\omega^{k} \in H \Omega^{k}(M)\left|\operatorname{tr} \omega^{k}\right|_{\partial M}=0\right\} \\
\stackrel{\circ}{H}^{*} \Omega^{k}(M) & =\left\{\omega^{k} \in H^{*} \Omega^{k}(M)\left|\operatorname{tr}_{n} \omega^{k}\right|_{\partial M}=0\right\}
\end{aligned}
$$

Mixed boundary conditions will be considered in this work. We denote with $\Gamma_{1}$ and $\Gamma_{2}$ two open subsets of the boundary that verify $\bar{\Gamma}_{1} \cup \bar{\Gamma}_{2}=\partial M$ and $\Gamma_{1} \cap \Gamma_{2}=\emptyset$. Sobolev spaces with homogeneous boundary conditions are defined as

$$
H \Omega^{k}\left(M, \Gamma_{i}\right):=\left\{\omega^{k} \in H \Omega^{k}(M)\left|\operatorname{tr} \omega^{k}\right|_{\Gamma_{i}}=0\right\}, \quad i=1,2
$$

For an analysis of these spaces at the continuous level see [23, 24]. For results concerning finite element exterior calculus and mixed boundary conditions consult [25].

### 2.3. Port-Hamiltonian systems

Port-Hamiltonian systems encoding conservation laws are associated to the geometrical StokesDirac structure. Consider the state trajectories $\widehat{\alpha}^{p}(t):\left[0, T_{\text {end }}\right] \rightarrow \widehat{\Omega}^{p}(M), \beta^{q}(t):\left[0, T_{\text {end }}\right] \rightarrow$ $\Omega^{q}(M)$ with $p+q=n+1$ and the following quadratic Hamiltonian functional

$$
\begin{equation*}
H\left(\widehat{\alpha}^{p}, \beta^{q}\right)=\int_{M} \mathcal{H}\left(\widehat{\alpha}^{p}, \beta^{q}\right), \quad \mathcal{H}\left(\widehat{\alpha}^{p}, \beta^{q}\right)=\widehat{\alpha}^{p} \wedge \star \widehat{\alpha}^{p}+\beta^{q} \wedge \star \beta^{q} \tag{11}
\end{equation*}
$$

with Hamiltonian density function $\mathcal{H}: \widehat{\Omega}^{p}(M) \times \Omega^{q}(M) \rightarrow \Omega^{n}(M)$.
Remark 1. For simplicity we do not consider material properties in this work. These material properties are embedded in the Hodge operator that induces a weighted inner product.

A fundamental notion is the variational derivative of the Hamiltonian functional [1, 26].
Definition 6 (Variational derivative). The variational derivatives of the Hamiltonian $\delta_{\widehat{\alpha}} H^{n-p} \in$ $\Omega^{n-p}(M), \delta_{\beta} H^{n-q} \in \widehat{\Omega}^{n-q}(M)$ are defined implicitly by

$$
\begin{array}{cl}
\left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0} H\left(\widehat{\alpha}^{p}+\varepsilon \delta \widehat{\alpha}^{p}, \beta^{q}\right)=\left(\delta_{\widehat{\alpha}} H^{n-p} \mid \delta \widehat{\alpha}^{p}\right)_{M}, & \forall \delta \widehat{\alpha}^{p} \in \widehat{\Omega}^{p}(M), \\
\left.\frac{d}{d \varepsilon}\right|_{\varepsilon=0} H\left(\widehat{\alpha}^{p}, \beta^{q}+\varepsilon \delta \beta^{q}\right)=\left(\delta_{\beta} H^{n-q} \mid \delta \beta^{q}\right)_{M}, & \forall \delta \beta^{q} \in \Omega^{q}(M)
\end{array}
$$

Because of the assumed quadratic Hamiltonian in (11), the variational derivatives of the Hamiltonian, computed according to Definition 6, can be seen to be

$$
\begin{equation*}
\delta_{\widehat{\alpha}} H^{n-p}=(-1)^{p(n-p)} \star \widehat{\alpha}^{p}, \quad \delta_{\beta} H^{n-q}=(-1)^{q(n-q)} \star \beta^{q} . \tag{12}
\end{equation*}
$$

The variational derivatives of the Hamiltonian are also called the co-energy variables in the literature. Consider then the following equations which can be seen to represent a system of two conservation laws with canonical inter-domain coupling [1]

$$
\binom{\partial_{t} \widehat{\alpha}^{p}}{\partial_{t} \beta^{q}}=\left[\begin{array}{cc}
0 & (-1)^{p} \mathrm{~d}  \tag{13}\\
-(-1)^{p(n-p)} \mathrm{d} & 0
\end{array}\right]\binom{\delta_{\widehat{\alpha}} H^{q-1}}{\delta_{\beta} H^{p-1}}
$$

and since $p+q=n+1$, it holds $n-p=q-1$, and $n-q=p-1$. The initial conditions are given by $\widehat{\alpha}^{p}(0)=\widehat{\alpha}_{0}^{p}$ and $\beta^{q}(0)=\beta_{0}^{q}$.
Remark 2. To simplify the computations, the canonical definition of port-Hamiltonian system [1] has been modified, introducing a factor by $(-1)^{p(n-p)}$ in the first state variable.

Boundary conditions. Each boundary subpartition $\Gamma_{1}, \Gamma_{2}$ is associated with one boundary condition. In particular, the values of $\delta_{\widehat{\alpha}} H^{n-p}$ (resp. $\delta_{\beta} H^{n-q}$ ) are imposed on $\Gamma_{1}$ (resp. $\Gamma_{2}$ ). Since in this paper we consider boundary controlled systems, the boundary conditions are assigned by means of the inputs

$$
\begin{align*}
\left.(-1)^{p(n-p)} \operatorname{tr} \delta_{\widehat{\alpha}} H^{n-p}\right|_{\Gamma_{1}} & =u_{1}^{q-1} \in \Omega^{q-1}\left(\Gamma_{1}\right), \\
\left.(-1)^{p} \operatorname{tr} \widehat{\beta}^{p-1}\right|_{\Gamma_{2}} & =\widehat{u}_{2}^{p-1} \in \widehat{\Omega}^{p-1}\left(\Gamma_{2}\right) . \tag{14}
\end{align*}
$$

The outputs are defined in such a way that they are power conjugated to the inputs

$$
\begin{align*}
y_{1}^{q-1}:= & \left.(-1)^{p(n-p)} \operatorname{tr} \delta_{\widehat{\alpha}} H^{n-p}\right|_{\Gamma_{2}} \in \Omega^{q-1}\left(\Gamma_{2}\right),  \tag{15}\\
& \widehat{y}_{2}^{p-1}:=\left.(-1)^{p} \operatorname{tr} \widehat{\beta}^{p-1}\right|_{\Gamma_{1}} \in \widehat{\Omega}^{p-1}\left(\Gamma_{1}\right) .
\end{align*}
$$

Along trajectories of the pH system (13), the rate of change of the Hamiltonian (11) is expressed as

$$
\begin{align*}
\dot{H} & =\left(\delta_{\widehat{\alpha}} H^{q-1} \mid \partial_{t} \widehat{\alpha}^{p}\right)_{M}+\left(\delta_{\beta} H^{p-1} \mid \partial_{t} \beta^{q}\right)_{M}, \\
& =\left(\delta_{\widehat{\alpha}} H^{q-1} \mid(-1)^{p} \mathrm{~d} \delta_{\beta} H^{p-1}\right)_{M}-\left(\delta_{\beta} H^{p-1} \mid(-1)^{p(n-p)} \mathrm{d} \delta_{\widehat{\alpha}} H^{q-1}\right)_{M}, \\
& =(-1)^{p}\left\{\left(\mathrm{~d} \delta_{\beta} H^{p-1} \mid(-1)^{p(n-p)} \delta_{\widehat{\alpha}} H^{q-1}\right)_{M}+(-1)^{p-1}\left(\delta_{\beta} H^{p-1} \mid(-1)^{p(n-p)} \mathrm{d} \delta_{\widehat{\alpha}} H^{q-1}\right)_{M}\right\},  \tag{16}\\
& =\left\langle(-1)^{p} \operatorname{tr} \delta_{\beta} H^{p-1} \mid(-1)^{p(n-p)} \operatorname{tr} \delta_{\widehat{\alpha}} H^{q-1}\right\rangle_{\partial M}, \\
& =\left\langle\widehat{y}_{2}^{p-1} \mid u_{1}^{q-1}\right\rangle_{\Gamma_{1}}+\left\langle\widehat{u}_{2}^{p-1} \mid y_{1}^{q-1}\right\rangle_{\Gamma_{2}},
\end{align*}
$$

where the last equality descends from Stokes theorem and the additive property of the integration and the definition of the boundary variables given in (14), (15). 16) characterizes the total power balance of the system such that the change of energy within the spatial domain $M$ is equal to the supplied power through its boundary $\partial M$. Notice that since $p+q=n+1$, the boundary variables are dual forms on the boundary $(n-p)+(n-q)=2 n-(n+1)=n-1$.

### 2.4. The primal-dual structure of port-Hamiltonian systems

By combining the canonical port-Hamiltonian system and its adjoint, two different formulations are obtained [10]. Each of these two formulation is ruled by a formally skew-adjoint operator. These two formulations incorporate mixed boundary control and observation. The variables of one system corresponds to the Hodge dual of those belonging to the second one. For this reason, we refer to these systems as the primal and the dual formulation.

For notational simplicity, let us introduce the following variables

$$
\begin{equation*}
\binom{\alpha^{q-1}}{\widehat{\beta}^{p-1}}:=\binom{(-1)^{p(n-p)} \delta_{\widehat{\alpha}} H^{q-1}}{\delta_{\beta} H^{p-1}} . \tag{17}
\end{equation*}
$$

Primal system of outer oriented forms. Find $\widehat{\alpha}^{p}:(0, T] \rightarrow \widehat{\Omega}^{p}(M)$ and $\widehat{\beta}^{p-1}:(0, T] \rightarrow \widehat{\Omega}^{p-1}(M)$ such that

$$
\frac{\partial}{\partial t}\binom{\widehat{\alpha}^{p}}{\widehat{\beta}^{p-1}}=(-1)^{p}\left[\begin{array}{cc}
0 & \mathrm{~d}  \tag{18}\\
-\left.\mathrm{d}^{*} \star \widehat{\alpha}^{p}\right|_{\Gamma_{1}} & =u_{1}^{q-1} \in\binom{\widehat{\alpha}^{p}}{\widehat{\beta}^{p-1}}, \\
\left.(-1)^{p} \operatorname{tr} \widehat{\beta}^{p-1}\right|_{\Gamma_{2}} & =\widehat{u}_{2}^{p-1} \in \widehat{\Omega}^{p-1}\left(\Gamma_{2}\right),
\end{array}\right.
$$

with initial condition $\widehat{\alpha}^{p}(0)=\widehat{\alpha}_{0}^{p}, \widehat{\beta}^{p-1}(0)=\widehat{\beta}_{0}^{p-1}$. The collocated outputs are given by

$$
\begin{array}{ll}
y_{1}^{q-1}:=\left.\operatorname{tr} \star \widehat{\alpha}^{p}\right|_{\Gamma_{2}} & \in \Omega^{q-1}\left(\Gamma_{2}\right),  \tag{19}\\
\widehat{y}_{2}^{p-1}:=\left.(-1)^{p} \operatorname{tr} \widehat{\beta}^{p-1}\right|_{\Gamma_{1}} & \in \widehat{\Omega}^{p-1}\left(\Gamma_{1}\right) .
\end{array}
$$

Dual system of inner oriented forms. Find $\alpha^{q-1}:(0, T] \rightarrow \Omega^{q-1}(M)$ and $\beta^{q}:(0, T] \rightarrow \Omega^{q}(M)$ such that

$$
\frac{\partial}{\partial t}\binom{\alpha^{q-1}}{\beta^{q}}=\left[\begin{array}{cc}
0 & \mathrm{~d}^{*}  \tag{20}\\
-\mathrm{d} & 0
\end{array}\right]\binom{\left.\alpha^{q-1} \alpha^{q-1}\right|_{\Gamma_{1}}=u_{1}^{q-1} \in \Omega^{q-1}\left(\Gamma_{1}\right)}{\beta^{q}},\left.\quad(-1)^{p+q(n-q)} \operatorname{tr} \star \beta^{q}\right|_{\Gamma_{2}}=\widehat{u}_{2}^{p-1} \in \widehat{\Omega}^{p-1}\left(\Gamma_{2}\right), ~
$$

with initial condition $\alpha^{q-1}(0)=\alpha_{0}^{q-1}, \beta^{q}(0)=\beta_{0}^{q}$. The outputs are given by

$$
\begin{array}{ll}
y_{1}^{q-1}:=\left.\operatorname{tr} \alpha^{q-1}\right|_{\Gamma_{2}} & \in \Omega^{q-1}\left(\Gamma_{2}\right), \\
y_{2}^{p-1}:=\left.(-1)^{p+q(n-q)} \operatorname{tr} \star \beta^{q}\right|_{\Gamma_{1}} & \in \widehat{\Omega}^{p-1}\left(\Gamma_{1}\right) . \tag{21}
\end{array}
$$

## 3. Continuous Galerkin discretization of port-Hamiltonian systems

In our previous work [10], it has been shown that port-Hamiltonian systems present an intrinsic primal-dual weak formulation. By interpreting the codifferential in 18 and 20 weakly using integration by parts, the two boundary inputs (14) appear in the weak formulation naturally. This weak formulation uses Sobolev spaces that appear in the standard de Rham complex. In our current formulation, a major distinction with respect to [10] is that the variables that do not undergo differentiation will be taken to be in a particular subspace of $L^{2}$, namely a broken Sobolev spaces, as presented here after. In what follows we introduce this more generic construction, together with the associated continuous Galerkin discretization.

### 3.1. Broken Sobolev spaces

Let $\mathcal{T}_{h}$ denote a regular mesh corresponding to the spatial manifold $M$ which can be thought of as a domain decomposition. We denote single elements of $\mathcal{T}_{h}$ by $T$. The broken Sobolev spaces are defined as follows

$$
\begin{equation*}
H \Omega^{k}\left(\mathcal{T}_{h}\right):=\prod_{T \in \mathcal{T}_{h}} H \Omega^{k}(T), \quad H^{*} \Omega^{k}\left(\mathcal{T}_{h}\right):=\prod_{T \in \mathcal{T}_{h}} H^{*} \Omega^{k}(T) \tag{22}
\end{equation*}
$$

and analogously for the space of outer oriented forms. For what concerned $L^{2}$ spaces, their broken version is isomorphic to the unbroken one

$$
L^{2} \Omega^{k}\left(\mathcal{T}_{h}\right):=\prod_{T \in \mathcal{T}_{h}} L^{2} \Omega^{k}(T) \cong L^{2} \Omega^{k}(M)
$$

Since these spaces are product spaces, they inherit the inner product of the factors

$$
\begin{equation*}
(\cdot, \cdot)_{\mathcal{T}_{h}}:=\sum_{T \in \mathcal{T}_{h}}(\cdot, \cdot)_{T}, \quad(\cdot, \cdot)_{H \Omega^{k}\left(\mathcal{T}_{h}\right)}:=\sum_{T \in \mathcal{T}_{h}}(\cdot, \cdot)_{H \Omega^{k}(T)}, \quad(\cdot, \cdot)_{H^{*} \Omega^{k}\left(\mathcal{T}_{h}\right)}:=\sum_{T \in \mathcal{T}_{h}}(\cdot, \cdot)_{H^{*} \Omega^{k}(T)} \tag{23}
\end{equation*}
$$

Since $H \Omega^{k}(T) \subset L^{2} \Omega^{k}(T)$, the broken Sobolev spaces are $L^{2}$ subspaces $H \Omega^{k}\left(\mathcal{T}_{h}\right) \subset L^{2} \Omega^{k}(M)$. Sobolev spaces are naturally included in their broken versions

$$
H \Omega^{k}(M) \hookrightarrow H \Omega^{k}\left(\mathcal{T}_{h}\right), \quad H^{*} \Omega^{k}(M) \hookrightarrow H^{*} \Omega^{k}\left(\mathcal{T}_{h}\right)
$$

by restriction over each cell.

### 3.2. Weak primal and dual port-Hamiltonian system

The aforementioned broken spaces will replace the Sobolev space used in 10 for the variable that does not undergo differentiation, i.e. $\widehat{\alpha}^{p}$ in the primal system and $\beta^{q}$ for the dual. It will be shown in Sec. 3.4 that such a modification still leads to a discrete power balance.

To simplify the notation, the spaces for outer oriented forms will be denoted from this point further as

$$
\begin{equation*}
\widehat{W}^{k}:=H \widehat{\Omega}^{k}\left(\mathcal{T}_{h}\right), \quad \widehat{V}^{k}:=H \widehat{\Omega}^{k}(M), \quad \widehat{V}^{k}\left(\Gamma_{2}\right):=H \widehat{\Omega}^{k}\left(M, \Gamma_{2}\right) \tag{24}
\end{equation*}
$$

whereas for inner oriented forms

$$
\begin{equation*}
W^{k}:=H \Omega^{k}\left(\mathcal{T}_{h}\right), \quad V^{k}:=H \Omega^{k}(M), \quad V^{k}\left(\Gamma_{1}\right):=H \Omega^{k}\left(M, \Gamma_{1}\right) \tag{25}
\end{equation*}
$$

Note that $W$ and $\widehat{W}$ represent broken Sobolev spaces while $V$ and $\widehat{V}$ are unbroken ones. For what concerns the control inputs, let us introduce the control spaces

$$
\begin{aligned}
& U_{1}:=H^{-1 / 2} \Omega^{q-1}\left(\Gamma_{1}\right)=\left\{\left.\operatorname{tr} \omega^{k}\right|_{\Gamma_{1}}, \omega^{q-1} \in H \Omega^{q-1}(M)\right\} \\
& \widehat{U}_{2}:=H^{-1 / 2} \widehat{\Omega}^{p-1}\left(\Gamma_{2}\right)=\left\{\left.\operatorname{tr} \widehat{\omega}^{p-1}\right|_{\Gamma_{2}}, \widehat{\omega}^{p-1} \in H \Omega^{p-1}(M)\right\}
\end{aligned}
$$

Weak primal system. The weak formulation for the primal system (18) reads: find $\widehat{\alpha}^{p} \in \widehat{W}^{p}, \widehat{\beta}^{p-1} \in \widehat{V}^{p-1}$ such that $\left.(-1)^{p} \operatorname{tr} \widehat{\beta}^{p-1}\right|_{\Gamma_{2}}=\widehat{u}_{2}^{p-1} \in \widehat{U}_{2}$ and

$$
\begin{align*}
\left(\widehat{v}^{p}, \partial_{t} \widehat{\alpha}^{p}\right)_{M} & =(-1)^{p}\left(v^{p}, \mathrm{~d} \widehat{\beta}^{p-1}\right)_{M}, & \forall \widehat{v}^{p} \in \widehat{W}^{p}  \tag{26a}\\
\left(\widehat{v}^{p-1}, \partial_{t} \widehat{\beta}^{p-1}\right)_{M} & =(-1)^{p}\left\{-\left(\mathrm{d} \widehat{v}^{p-1}, \widehat{\alpha}^{p}\right)_{M}+\left\langle\operatorname{tr} \widehat{v}^{p-1} \mid u_{1}^{q-1}\right\rangle_{\Gamma_{1}}\right\}, & \forall \widehat{v}^{p-1} \in \widehat{V}^{p-1}\left(\Gamma_{2}\right) . \tag{26b}
\end{align*}
$$

By integration by parts, the control variable $u_{1}^{q-1} \in U_{1}$ is naturally included above.
Weak dual system. For the dual system (20), the following weak formulation is obtained: find $\alpha^{q-1} \in V^{q-1}, \beta_{2}^{q} \in W^{q}$ such that $\left.\operatorname{tr} \alpha^{q-1}\right|_{\Gamma_{1}}=u_{1}^{q-1} \in U_{1}$, and

$$
\begin{array}{rlrl}
\left(v^{q-1}, \partial_{t} \alpha^{q-1}\right)_{M} & =\left(\mathrm{d} v^{q-1}, \beta^{q}\right)_{M}+(-1)^{(p-1)(q-1)}\left\langle\operatorname{tr} v^{q-1} \mid \widehat{u}_{2}^{p-1}\right\rangle_{\Gamma_{2}}, & \forall v^{q-1} \in V^{q-1}\left(\Gamma_{1}\right), \\
& \left(v^{q}, \partial_{t} \beta^{q}\right)_{M} & =-\left(v^{q}, \mathrm{~d} \alpha^{q-1}\right)_{M}, & \forall v^{q} \in W^{q}, \tag{27b}
\end{array}
$$

Again, the control input $\widehat{u}_{2}^{p-1} \in U_{2}$ is naturally included in this formulation.
Remark 3 (The dual system on the adjoint complex). Instead of writing the dual system on the $H \Omega(M)$ complex, it can be equivalently rewritten on the adjoint complex $H^{*} \Omega(M)$ [15]. In this case the forms have the same degree as in the primal system and only the codifferential appears in the formulation.

### 3.3. Dual field continuous Galerkin discretization

A discrete continuous Galerkin formulation can be obtained by considering spaces of finite element differential forms constituting a subcomplex of the de Rham complex $V_{h}^{k} \subset H \Omega^{k}(M)$ (for instance one can choose the trimmed polynomial family $V_{h}^{k}=\mathcal{P}^{-} \Omega^{k}\left(\mathcal{T}_{h}\right)$ [11] or the mimetic polynomial spaces [17, 27]). The test functions are taken in the corresponding discrete space with boundary conditions

$$
\widehat{V}_{h}^{k}\left(\Gamma_{2}\right):=\left\{\widehat{\omega}_{h}^{k} \in \widehat{V}_{h}^{k}\left|\operatorname{tr} \widehat{\omega}_{h}^{k}\right| \Gamma_{2}=0\right\}, \quad V_{h}^{k}\left(\Gamma_{1}\right):=\left\{\omega_{h}^{k} \in V_{h}^{k}\left|\operatorname{tr} \omega_{h}^{k}\right| \Gamma_{1}=0\right\}
$$

The less regular variables, i.e. those belonging to $H \Omega^{k}\left(\mathcal{T}_{h}\right)$, are naturally discretized using broken spaces of differential forms. For each $T \in \mathcal{T}_{h}$, let $W_{h}^{k}(T) \subset H \Omega^{k}(T)$ be a finite subcomplex and

$$
\begin{equation*}
W_{h}^{k}:=\prod_{T \in \mathcal{T}_{h}} W_{h}^{k}(T) \tag{28}
\end{equation*}
$$

Then $W_{h}^{k} \subset H \Omega^{k}\left(\mathcal{T}_{h}\right)$ corresponds to the discrete space associated with variables belonging to $H \Omega^{k}\left(\mathcal{T}_{h}\right)$. In particular, conforming spaces $V_{h}^{k}$ are naturally included in their broken counterpart $V_{h}^{k} \hookrightarrow W_{h}^{k}$. This property is crucial as it leads to a discrete power balance, as shown in Sec. 3.4.

The discrete control spaces correspond to the restriction at the boundary of the $V_{h}^{k}$ spaces

$$
\begin{equation*}
U_{1, h}=\left.\operatorname{tr} V_{h}^{q-1}\right|_{\Gamma_{1}}, \quad \widehat{U}_{2, h}=\left.\operatorname{tr} \widehat{V}_{h}^{p-1}\right|_{\Gamma_{2}} \tag{29}
\end{equation*}
$$

Discrete primal system. The discrete formulation for the primal system reads: find $\widehat{\alpha}_{h}^{p} \in \widehat{W}_{h}^{p}, \widehat{\beta}_{h}^{p-1} \in \widehat{V}_{h}^{p-1}$ such that $\left.(-1)^{p} \operatorname{tr} \widehat{\beta}_{h}^{p-1}\right|_{\Gamma_{2}}=\widehat{u}_{2, h}^{p-1} \in U_{2, h}$ and

$$
\begin{array}{rlrl}
\left(\widehat{v}_{h}^{p}, \partial_{t} \widehat{\alpha}_{h}^{p}\right)_{M} & =(-1)^{p}\left(\widehat{v}_{h}^{p}, \mathrm{~d} \widehat{\beta}_{h}^{p-1}\right)_{M}, & \forall \widehat{v}_{h}^{p} \in \widehat{W}_{h}^{p}, \\
\left(\widehat{v}_{h}^{p-1}, \partial_{t} \widehat{\beta}_{h}^{p-1}\right)_{M} & =(-1)^{p}\left\{-\left(\mathrm{d} \widehat{v}_{h}^{p-1}, \widehat{\alpha}_{h}^{p}\right)_{M}+\left\langle\operatorname{tr} \widehat{v}_{h}^{p-1} \mid u_{1, h}^{q-1}\right\rangle_{\Gamma_{1}}\right\} & & \forall \widehat{v}^{p-1} \in \widehat{V}_{h}^{p-1}\left(\Gamma_{2}\right), \tag{30b}
\end{array}
$$

where $u_{1, h}^{q-1} \in U_{1, h}$.
Discrete dual system. The discrete dual port-Hamiltonian system is given by:
find $\alpha_{h}^{q-1} \in V_{h}^{q-1}, \beta_{h}^{q} \in W_{h}^{q}$ such that $\left.\operatorname{tr} \alpha_{h}^{q-1}\right|_{\Gamma_{1}}=u_{1, h}^{q-1} \in U_{1, h}$ and

$$
\begin{align*}
& \left(v_{h}^{q-1}, \partial_{t} \alpha_{h}^{q-1}\right)_{M}=\left(\mathrm{d} \widehat{v}_{h}^{q-1}, \beta_{h}^{q}\right)_{M}+(-1)^{(p-1)(q-1)}\left\langle\operatorname{tr} v_{h}^{q-1} \mid \widehat{u}_{2, h}^{p-1}\right\rangle_{\Gamma_{2}}, \quad \forall v_{h}^{q-1} \in V_{h}^{q-1}\left(\Gamma_{1}\right),  \tag{31a}\\
& \left(v_{h}^{q}, \partial_{t} \beta_{h}^{q}\right)_{M}=-\left(v_{h}^{q}, \mathrm{~d} \alpha_{h}^{q-1}\right)_{M}, \quad \forall v_{h}^{q} \in W_{h}^{q}, \tag{31b}
\end{align*}
$$

where $\widehat{u}_{2, h}^{p-1} \in \widehat{U}_{2, h}$.

### 3.4. Discrete power balance

The dual field continuous Galerkin discretization is capable of retaining the following discrete power balance (that characterizes the Dirac structure underlying port-Hamiltonian systems).

Proposition 1. The primal-dual discrete port-Hamiltonian systems (30), 31) encode the following discrete power balance

$$
(-1)^{p(n-p)}\left(\alpha_{h}^{q-1} \mid \partial_{t} \widehat{\alpha}_{h}^{p}\right)_{M}+\left(\widehat{\beta}_{h}^{p-1} \mid \partial_{t} \beta_{h}^{q}\right)_{M}=\left\langle\widehat{y}_{2, h}^{p-1} \mid u_{1, h}^{q-1}\right\rangle_{\Gamma_{1}}+\left\langle\widehat{u}_{2, h}^{p-1} \mid y_{1, h}^{q-1}\right\rangle_{\Gamma_{2}}
$$

Proof. The proof generalizes Pr. 5 in [10]. Consider (30a), 31b). Since the conforming discrete spaces $V_{h}^{k}$ form a de Rham subcomplex and conforming spaces are naturally included in their broken counterpart, it holds

$$
\mathrm{d} \widehat{V}_{h}^{p-1} \subset \widehat{V}_{h}^{p} \hookrightarrow \widehat{W}_{h}^{p}, \quad \mathrm{~d} V_{h}^{q-1} \subset V_{h}^{q} \hookrightarrow W_{h}^{q}
$$

This implies that two discrete conservation laws hold

$$
\partial_{t} \widehat{\alpha}_{h}^{p}=(-1)^{p} \mathrm{~d} \widehat{\beta}_{h}^{p-1}, \quad \partial_{t} \beta_{h}^{q}=-\mathrm{d} \alpha_{h}^{q-1}
$$

Take the duality product of the first equation with $(-1)^{p(n-p)} \alpha_{h}^{q-1}$ and with $\widehat{\beta}^{p-1}$ for the second. Summing the two it is obtained and following the same computation as in 16

$$
\begin{aligned}
(-1)^{p(n-p)}\left(\alpha_{h}^{q-1} \mid \partial_{t} \widehat{\alpha}_{h}^{p}\right)_{M}+\left(\widehat{\beta}_{h}^{p-1} \mid \partial_{t} \beta_{h}^{q}\right)_{M} & =\sum_{T \in \mathcal{T}_{h}}(-1)^{p}\left(\mathrm{~d} \widehat{\beta}_{h}^{p-1} \mid \alpha_{h}^{q-1}\right)_{T}-\left(\widehat{\beta}_{h}^{p-1} \mid \mathrm{d} \alpha_{h}^{q-1}\right)_{T}, \\
& =\sum_{T \in \mathcal{T}_{h}}(-1)^{p}\left(\operatorname{tr} \widehat{\beta}_{h}^{p-1} \mid \operatorname{tr} \alpha_{h}^{q-1}\right)_{\partial T} \\
& =\left\langle\widehat{y}_{2, h}^{p-1} \mid u_{1, h}^{q-1}\right\rangle_{\Gamma_{1}}+\left\langle\widehat{u}_{2, h}^{p-1} \mid y_{1, h}^{q-1}\right\rangle_{\Gamma_{2}} .
\end{aligned}
$$

The final equality is obtained using the continuity property of differential forms (cf. Appendix A in [10]).

## 4. Hybridization of continuous Galerkin schemes

For simplicity, we will assume in what follows that $\mathcal{T}_{h}$ is a regular mesh. However, the hybridization strategy can be generalized to non-conforming meshes.

### 4.1. Decomposition of Hilbert complexes

For convenience, we recall here some concepts related to decomposition of Hilbert complex, following closely [15].

Given broken version of Sobolev spaces, broken versions of the differential operators are defined element-wise:

- d : $H \Omega^{k}\left(\mathcal{T}_{h}\right) \rightarrow H \Omega^{k+1}\left(\mathcal{T}_{h}\right)$ is $\left.\mathrm{d}\right|_{H \Omega^{k}(T)}$ on each $T \in \mathcal{T}_{h}$;
- $\mathrm{d}^{*}: H^{*} \Omega^{k}\left(\mathcal{T}_{h}\right) \rightarrow H^{*} \Omega^{k-1}\left(\mathcal{T}_{h}\right)$ is $\left.\mathrm{d}^{*}\right|_{H^{*} \Omega^{k}(T)}$ on each $T \in \mathcal{T}_{h}$.

The broken Hilbert complexes corresponds to the $H \Omega, H^{*} \Omega$ on the disjoint union $\bigsqcup_{T \in \mathcal{T}_{h}} T$. Tangential and normal traces on the mesh disjoint facets $\partial \mathcal{T}_{h}:=\bigsqcup_{T \in \mathcal{T}_{h}} \partial T$ are defined by considering traces on the boundary $\partial T$ of each cell $T$. Defining $\langle\cdot, \cdot\rangle_{\partial \mathcal{T}_{h}}:=\sum_{T \in \mathcal{T}_{h}}\langle\cdot, \cdot\rangle_{\partial T}$, the following integration by parts formula is readily obtained by summing the contributions of all cells $T \in \mathcal{T}_{h}$

$$
(\mathrm{d} \omega, \sigma)_{\mathcal{T}_{h}}-\left(\omega, \mathrm{d}^{*} \sigma\right)_{\mathcal{T}_{h}}=\left\langle\operatorname{tr} \omega, \operatorname{tr}_{\boldsymbol{n}} \sigma\right\rangle_{\partial \mathcal{T}_{h}}, \quad \forall \omega \in H \Omega^{k-1}\left(\mathcal{T}_{h}\right), \quad \sigma \in H^{*} \Omega^{k}\left(\mathcal{T}_{h}\right)
$$

and analogously for the spaces of outer oriented differential forms. Any edge shared by two elements $e=\partial T^{+} \cup \partial T^{-}$appears twice in the disjoint union $\partial \mathcal{T}_{h}$ : once for $\partial T^{+}$, and once for $\partial T^{-}$. Traces of broken differential forms are double valued, as no continuity is imposed at interfaces.

The following proposition, corresponding to Pr. 3.1. in [15], gives a characterization of unbroken Sobolev spaces as single-valued broken spaces.

Proposition 2. Let $\mathcal{T}_{h}$ be a Lipschitz decomposition of $M$ then

$$
\begin{aligned}
& H \Omega^{k}(M)=\left\{\omega \in H \Omega^{k}\left(\mathcal{T}_{h}\right):\left\langle\operatorname{tr} \omega, \operatorname{tr}_{\boldsymbol{n}} \tau\right\rangle_{\partial \mathcal{T}_{h}}=0, \quad \forall \tau \in \stackrel{\circ}{H}^{*} \Omega^{k+1}(M)\right\}, \\
& \stackrel{\circ}{H} \Omega^{k}(M)=\left\{\omega \in H \Omega^{k}\left(\mathcal{T}_{h}\right):\left\langle\operatorname{tr} \omega, \operatorname{tr}_{\boldsymbol{n}} \tau\right\rangle_{\partial \mathcal{T}_{h}}=0, \quad \forall \tau \in H^{*} \Omega^{k+1}(M)\right\}, \\
& H^{*} \Omega^{k}(M)=\left\{\omega \in H^{*} \Omega^{k}\left(\mathcal{T}_{h}\right):\left\langle\operatorname{tr} \sigma, \operatorname{tr}_{n} \omega\right\rangle_{\partial \mathcal{T}_{h}}=0, \quad \forall \sigma \in \stackrel{\circ}{H} \Omega^{k-1}(M)\right\}, \\
& \stackrel{\circ}{H}^{*} \Omega^{k}(M)=\left\{\omega \in H^{*} \Omega^{k}\left(\mathcal{T}_{h}\right):\left\langle\operatorname{tr} \sigma, \operatorname{tr}_{\boldsymbol{n}} \omega\right\rangle_{\partial \mathcal{T}_{h}}=0, \quad \forall \sigma \in H \Omega^{k-1}(M)\right\} .
\end{aligned}
$$

### 4.2. Decomposition of the primal and dual port-Hamiltonian systems

The primal system. The decomposition of the primal system is based on the fact that the weak formulation 26 is equivalent to the following local problem: find $\widehat{\alpha}^{p} \in H \widehat{\Omega}^{p}(T), \widehat{\beta}^{p-1} \in H \widehat{\Omega}^{p-1}(T)$ such that

$$
\begin{align*}
\left(\widehat{v}^{p}, \partial_{t} \widehat{\alpha}^{p}\right)_{T} & =(-1)^{p}\left(v^{p}, \mathrm{~d} \widehat{\beta}^{p-1}\right)_{T}, & \forall \widehat{v}^{p} \in H \widehat{\Omega}^{p}(T), \\
\left(\widehat{v}^{p-1}, \partial_{t} \widehat{\beta}^{p-1}\right)_{T} & =-(-1)^{p}\left(\mathrm{~d} \widehat{v}^{p-1}, \widehat{\alpha}^{p}\right)_{T}, & \forall \widehat{v}^{p-1} \in H{ }^{\circ} \widehat{\Omega}^{p-1}(T), \tag{32}
\end{align*}
$$

with essential boundary conditions $\operatorname{tr} \widehat{\beta}^{p-1}=\widehat{\beta}^{p-1, t}$. The tangential trace $\widehat{\beta}^{p-1, t}$ is now considered as an independent variable. Contrarily to [15 which focuses only on static problems, we do not account for harmonic forms as the problem is dynamical.

The following broken and unbroken facet spaces will be used

$$
\begin{aligned}
\widehat{W}^{k, \boldsymbol{n}} & :=\left\{\operatorname{tr}_{\boldsymbol{n}} \widehat{\omega}: \widehat{\omega} \in H^{*} \widehat{\Omega}^{k+1}\left(\mathcal{T}_{h}\right)\right\}, \\
\widehat{V}^{k, \boldsymbol{t}} & :=\left\{\operatorname{tr} \widehat{\omega}: \widehat{\omega} \in H \widehat{\Omega}^{k}(\Omega)\right\}, \\
\widehat{V}^{k, \boldsymbol{t}}\left(\Gamma_{2}\right) & :=\left\{\operatorname{tr} \widehat{\omega}: \widehat{\omega} \in H \widehat{\Omega}^{k}\left(\Omega, \Gamma_{2}\right)\right\} .
\end{aligned}
$$

The spaces $\widehat{V}^{k, t}, \widehat{V}^{k, t}\left(\Gamma_{2}\right)$ consists of single valued traces from the unbroken space, whereas the space $\widehat{W}^{k, n}$ contain broken forms. Consider the variational problem: find

- Local variables: $\widehat{\alpha}^{p} \in \widehat{W}^{p}, \quad \widehat{\beta}^{p-1} \in \widehat{W}^{p-1}, \quad \widehat{\alpha}^{p-1, n} \in \widehat{W}^{p-1, n}$;
- Global variables $\widehat{\beta}^{p-1, \boldsymbol{t}} \in \widehat{V}^{p-1, \boldsymbol{t}}$;
such that $\left.\widehat{\beta}^{p-1, t}\right|_{\Gamma_{2}}=\widehat{u}_{2}^{p-1}$ and

$$
\begin{align*}
\left(\widehat{v}^{p}, \partial_{t} \widehat{\alpha}^{p}\right)_{\mathcal{T}_{h}} & =(-1)^{p}\left(v^{p}, \mathrm{~d} \widehat{\beta}^{p-1}\right)_{\mathcal{T}_{h}}, & & \forall \widehat{v}^{p} \in \widehat{W}^{p}, \\
\left(\widehat{v}^{p-1}, \partial_{t} \widehat{\beta}^{p-1}\right)_{\mathcal{T}_{h}} & =(-1)^{p}\left\{-\left(\mathrm{d} \widehat{v}^{p-1}, \widehat{\alpha}^{p}\right)_{\mathcal{T}_{h}}+\left\langle\operatorname{tr} \widehat{v}^{p-1}, \widehat{\alpha}^{p-1, \boldsymbol{n}}\right\rangle_{\partial \mathcal{T}_{h}}\right\}, & & \forall \widehat{v}^{p-1} \in \widehat{W}^{p-1},  \tag{0}\\
0 & =-(-1)^{p}\left\langle\widehat{v}^{p-1, \boldsymbol{n}}, \operatorname{tr} \widehat{\beta}^{p-1}-\widehat{\beta}^{p-1, \boldsymbol{t}}\right\rangle_{\partial \mathcal{T}_{h}}, & & \forall \widehat{v}^{p-1, \boldsymbol{n}} \in \widehat{W}^{p-1, \boldsymbol{n}},  \tag{33c}\\
0 & =(-1)^{p}\left\{-\left\langle\widehat{v}^{p-1, t}, \widehat{\alpha}^{p-1, \boldsymbol{n}}\right\rangle_{\partial \mathcal{T}_{h}}+\left\langle\widehat{v}^{p-1, \boldsymbol{t}} \mid u_{1}^{q-1}\right\rangle_{\Gamma_{1}}\right\}, & & \forall \widehat{v}^{p-1, \boldsymbol{t}} \in \widehat{V}^{p-1, t}\left(\Gamma_{2}\right) . \tag{33~d}
\end{align*}
$$

The dual system. Analogously to the primal system, the decomposition is based on the fact that the weak formulation 27 ) is equivalent to the following local problem: find $\alpha^{q-1} \in H \Omega^{q-1}(T)$, $\beta_{2}^{q} \in H \Omega^{q}(T)$

$$
\begin{align*}
\left(v^{q-1}, \partial_{t} \alpha^{q-1}\right)_{T} & =\left(\mathrm{d} v^{q-1}, \beta^{q}\right)_{T}, & & \forall v^{q-1} \in \stackrel{\circ}{H} \Omega^{q-1}(T), \\
\left(v^{q}, \partial_{t} \beta^{q}\right)_{T} & =-\left(v^{q}, \mathrm{~d} \alpha^{q-1}\right)_{T}, & & \forall v^{q} \in H \Omega^{q}(T), \tag{34}
\end{align*}
$$

with essential boundary conditions $\operatorname{tr} \alpha^{q-1}=\alpha^{q-1, t}$. Inner oriented broken and unbroken facet spaces will be used for the discretization

$$
\begin{aligned}
W^{k, \boldsymbol{n}} & :=\left\{\operatorname{tr}_{\boldsymbol{n}} \omega: \omega \in H^{*} \Omega^{k+1}\left(\mathcal{T}_{h}\right)\right\}, \\
V^{k, t} & :=\left\{\operatorname{tr} \omega: \omega \in H \Omega^{k}(\Omega)\right\}, \\
V^{k, t}\left(\Gamma_{1}\right) & :=\left\{\operatorname{tr} \omega: \omega \in H \Omega^{k}\left(\Omega, \Gamma_{1}\right)\right\} .
\end{aligned}
$$

Consider the variational problem: find

- Local variables: $\alpha^{q-1} \in W^{q-1}, \quad \beta_{2}^{q} \in W^{q}, \quad \beta^{q-1, \boldsymbol{n}} \in W^{q-1, \boldsymbol{n}}$;
- Global variables $\alpha^{q-1, t} \in V^{q-1, t}$;
such that $\left.\alpha^{q-1, t}\right|_{\Gamma_{1}}=u_{1}^{q-1}$ and

$$
\begin{align*}
\left(v^{q-1}, \partial_{t} \alpha^{q-1}\right)_{\mathcal{T}_{h}} & =\left(\mathrm{d} v^{q-1}, \beta^{q}\right)_{\mathcal{T}_{h}}-\left\langle\operatorname{tr} v^{q-1}, \beta^{q-1, \boldsymbol{n}}\right\rangle_{\partial \mathcal{T}_{h}}, & & \forall v^{q-1} \in W^{q-1}, \\
\left(v^{q}, \partial_{t} \beta^{q}\right)_{\mathcal{T}_{h}} & =-\left(v^{q}, \mathrm{~d} \alpha^{q-1}\right)_{\mathcal{T}_{h}}, & & \forall v^{q} \in W^{q},  \tag{35b}\\
0 & =\left\langle v^{q-1, \boldsymbol{n}}, \operatorname{tr} \alpha^{q-1}-\alpha^{q-1, t}\right\rangle_{\partial \mathcal{T}_{h}}, & & \forall v^{q-1, \boldsymbol{n}} \in W^{q-1, \boldsymbol{n}},  \tag{35c}\\
0 & =\left\langle v^{q-1, t}, \beta^{q-1, n}\right\rangle_{\partial \mathcal{T}_{h}}+(-1)^{(p-1)(q-1)}\left\langle v^{q-1, t} \mid \widehat{u}_{2}^{p-1}\right\rangle_{\Gamma_{2}}, & & \forall v^{q-1, t} \in V^{q-1, t}\left(\Gamma_{1}\right) . \tag{35~d}
\end{align*}
$$

Equivalence of the continuous and hybrid formulation. The following proposition shows that this problem is equivalent to the weak formulation (26). The proof is analogous to one of theorem 3.2. in [15] for the Hodge Laplacian, with the exception of additional boundary terms.

Theorem 1. For the primal system, the following are equivalent:

- $\widehat{\alpha}^{p}, \widehat{\beta}^{p-1}, \widehat{\alpha}^{p-1, n}, \widehat{\beta}^{p-1, \boldsymbol{t}}$ is a solution to (33);
- $\widehat{\alpha}^{p}, \widehat{\beta}^{p-1}$ is a solution to 26). Moreover, $\widehat{\beta}^{p-1, t}=\operatorname{tr} \widehat{\beta}^{p-1}$ and $\widehat{\alpha}^{p-1, \boldsymbol{n}}=\operatorname{tr}_{\boldsymbol{n}} \widehat{\alpha}^{p}$ on $\partial \mathcal{T}_{h}$;

Analogously, for the dual system, the following are equivalent:

- $\alpha^{q-1}, \beta^{q}, \beta^{q-1, \boldsymbol{n}}, \alpha^{q-1, \boldsymbol{t}}$ is a solution to (35);
- $\alpha^{q-1}, \beta^{q}$ is a solution to (27) and it holds $\alpha^{q-1, \boldsymbol{t}}=\operatorname{tr} \alpha^{q-1}, \operatorname{tr}_{\boldsymbol{n}} \beta^{q}=\beta^{q-1, \boldsymbol{n}}$.

Proof. The proof will be given only for the primal system. The one for the dual system is completely analogous. Assume a solution to 33 is given. Then from the variational formulation, it follows that $\widehat{\beta}^{p-1, \boldsymbol{t}}=\operatorname{tr} \widehat{\beta}^{p-1}$ and $\widehat{\alpha}^{p-1, \boldsymbol{n}}=\operatorname{tr}_{\boldsymbol{n}} \widehat{\alpha}^{p}$ on $\partial \mathcal{T}_{h}$. Since $\widehat{\beta}^{p-1, \boldsymbol{t}}=\operatorname{tr} \widehat{\beta}^{p-1}$, by Pr. 2 this means that $\widehat{\beta}^{p-1} \in \widehat{H} \Omega^{p-1}(M)$. This immediately implies 26a). Taking the test function $\widehat{v}^{p-1} \in H \widehat{\Omega}^{p-1}\left(M, \Gamma_{2}\right)$ in 33 b$)$, the boundary term on $\partial \mathcal{T}_{h}$ is replaced by the contribution on the $\Gamma_{1}$ boundary, thus leading to 26 b .

For the converse direction, given a solution $\widehat{\alpha}^{p}$, $\widehat{\beta}^{p-1}$ to $(26)$, then (33a), (33b), (33c) hold. Choosing $\widehat{v}^{p-1} \in H \Omega^{p-1}\left(M, \Gamma_{2}\right)$ in 33 b and combining it with 26b gives

$$
-\left\langle\operatorname{tr} \widehat{v}^{p-1}, \widehat{\alpha}^{p-1, n}\right\rangle_{\partial \mathcal{T}_{h}}+\left\langle\operatorname{tr} \widehat{v}^{p-1} \mid u_{1}^{q-1}\right\rangle_{\Gamma_{1}}, \quad \text { for } \widehat{v}^{p-1} \in H \Omega^{p-1}\left(M, \Gamma_{2}\right)
$$

This implies that 33 d holds.

### 4.3. Discrete hybridized port-Hamiltonian systems

Let $W_{h}^{k} \subset H \Omega^{k}\left(\mathcal{T}_{h}\right)$ be defined as in 28. Conforming finite element spaces for the Sobolev spaces are given by $V_{h}^{k}=V^{k} \cup W_{h}^{k}$ where $V^{k}=H \Omega^{k}(M)$. We recall from [15] the broken and unbroken space of tangential traces finite element differential forms are then defined as follows

$$
\begin{aligned}
W_{h}^{k, \boldsymbol{t}} & :=\left\{\operatorname{tr} \omega_{h}^{k}: \omega_{h}^{k} \in W_{h}^{k}\right\}, \\
V_{h}^{k, t} & :=\left\{\operatorname{tr} \omega_{h}^{k}: \omega_{h}^{k} \in V_{h}^{k}\right\}=V^{k} \cap W_{h}^{k, t} \\
V_{h}^{k, \boldsymbol{t}}\left(\Gamma_{1}\right) & :=\left\{\operatorname{tr} \omega_{h}^{k}: \omega_{h}^{k} \in V_{h}^{k}\left(\Gamma_{1}\right)\right\}=V^{k}\left(\Gamma_{1}\right) \cap W_{h}^{k, t} .
\end{aligned}
$$

For what concerns the space $W^{k, n}$, its discrete version is taken to be the dual space of $W_{h}^{k, t}$, i.e. $W_{h}^{k, n}=\left(W_{h}^{k, t}\right)^{*}$.

Remark 4 (Construction of the $W_{h}^{k, n}$ space as a dual of an $L^{2}$ space). Duality is employed to identify $W_{h}^{k, n}$ with the degrees of freedom of $W_{h}^{k, t}$. Since discrete tangential traces are piecewise polynomial, they are in $L^{2}\left(\partial \mathcal{T}_{h}\right)$ and so $W_{h}^{k, n}=W_{h}^{k, t}$ where $\langle\cdot, \cdot\rangle_{\partial \mathcal{T}_{h}}$ is the $L^{2}$ inner product.

Primal discrete system. Consider the variational problem: find

- Local variables: $\widehat{\alpha}_{h}^{p} \in \widehat{W}_{h}^{p}, \quad \widehat{\beta}_{h}^{p-1} \in \widehat{W}_{h}^{p-1}, \quad \widehat{\alpha}_{h}^{p-1, \boldsymbol{n}} \in \widehat{W}_{h}^{p-1, \boldsymbol{n}} ;$
- Global variables $\widehat{\beta}_{h}^{p-1, t} \in \widehat{V}_{h}^{p-1, t}$;
such that $\left.\widehat{\beta}_{h}^{p-1, t}\right|_{\Gamma_{2}}=\widehat{u}_{2, h}^{p-1}$ and

$$
\begin{align*}
\left(\widehat{v}_{h}^{p}, \partial_{t} \widehat{\alpha}_{h}^{p}\right)_{\mathcal{T}_{h}} & =(-1)^{p}\left(v_{h}^{p}, \mathrm{~d} \widehat{\beta}_{h}^{p-1}\right)_{\mathcal{T}_{h}}, & & \forall \widehat{v}_{h}^{p} \in \widehat{W}_{h}^{p},  \tag{36a}\\
\left(\widehat{v}_{h}^{p-1}, \partial_{t} \widehat{\beta}_{h}^{p-1}\right)_{\mathcal{T}_{h}} & =(-1)^{p}\left\{-\left(\mathrm{d} \widehat{v}_{h}^{p-1}, \widehat{\alpha}_{h}^{p}\right)_{\mathcal{T}_{h}}+\left\langle\operatorname{tr} \widehat{v}_{h}^{p-1}, \widehat{\alpha}_{h}^{p-1, \boldsymbol{n}}\right\rangle_{\partial \mathcal{T}_{h}}\right\}, & & \forall \widehat{v}_{h}^{p-1} \in \widehat{W}_{h}^{p-1},  \tag{36b}\\
0 & =-(-1)^{p}\left\langle\widehat{v}_{h}^{p-1, \boldsymbol{n}}, \operatorname{tr} \widehat{\beta}_{h}^{p-1}-\widehat{\beta}_{h}^{p-1, \boldsymbol{t}}\right\rangle_{\partial \mathcal{T}_{h}}, & & \forall \widehat{v}_{h}^{p-1, \boldsymbol{n}} \in \widehat{W}_{h}^{p-1, \boldsymbol{n}},  \tag{36c}\\
0 & =(-1)^{p}\left\{-\left\langle\widehat{v}_{h}^{p-1, \boldsymbol{t}}, \widehat{\alpha}_{h}^{p-1, \boldsymbol{n}}\right\rangle_{\partial \mathcal{T}_{h}}+\left\langle\widehat{v}_{h}^{p-1, \boldsymbol{t}} \mid u_{1, h}^{q-1}\right\rangle_{\Gamma_{1}}\right\}, & & \forall \widehat{v}_{h}^{p-1, \boldsymbol{t}} \in \widehat{V}_{h}^{p-1, \boldsymbol{t}}\left(\Gamma_{2}\right) . \tag{36d}
\end{align*}
$$

Dual discrete system. Consider the variational problem: find

- Local variables: $\alpha_{h}^{q-1} \in W_{h}^{q-1}, \quad \beta_{h}^{q} \in W_{h}^{q}, \quad \beta_{h}^{q-1, \boldsymbol{n}} \in W_{h}^{q-1, \boldsymbol{n}} ;$
- Global variables $\alpha_{h}^{q-1, t} \in V_{h}^{q-1, t}$;
such that $\left.\alpha_{h}^{q-1, t}\right|_{\Gamma_{1}}=u_{1, h}^{q-1}$ and

$$
\begin{align*}
\left(v_{h}^{q-1}, \partial_{t} \alpha_{h}^{q-1}\right)_{\mathcal{T}_{h}} & =\left(\mathrm{d} v_{h}^{q-1}, \beta_{h}^{q}\right) \mathcal{T}_{h}-\left\langle\operatorname{tr} v_{h}^{q-1}, \beta_{h}^{q-1, n}\right\rangle_{\partial \mathcal{T}_{h},}, & & \forall v_{h}^{q-1} \in W_{h}^{q-1},(37 \mathrm{a}) \\
\left(v_{t} \beta_{h}^{q}\right) \mathcal{T}_{h} & =-\left(v_{h}^{q}, \mathrm{~d} \alpha_{h}^{q-1}\right)_{\mathcal{T}_{h}}, & & \forall v_{h}^{q} \in W_{h}^{q},(37 \mathrm{~b}) \\
0 & =\left\langle v_{h}^{q-1, n}, \operatorname{tr} \alpha_{h}^{q-1}-\alpha_{h}^{q-1, t}\right\rangle_{\partial \mathcal{T}_{h}}, & & \forall v_{h}^{q-1, n} \in W_{h}^{q-1, n},  \tag{37c}\\
0 & =\left\langle v_{h}^{q-1, t}, \beta_{h}^{q-1, n}\right\rangle_{\partial \mathcal{T}_{h}}+(-1)^{(p-1)(q-1)}\left\langle v_{h}^{q-1, t} \mid \widehat{u}_{2, h}^{p-1}\right\rangle_{\Gamma_{2}}, & & \forall v_{h}^{q-1, t} \in V_{h}^{q-1, t}\left(\Gamma_{1}\right) . \tag{37d}
\end{align*}
$$

Equivalence of the discrete continuous and hybrid formulation. The mixed and hybrid primal discrete formulations are equivalent. The only difference with respect to the weak formulations (33) and (35) is that $\widehat{\alpha}_{h}^{p-1, n}$ (resp. $\beta_{h}^{q-1, n}$ ) are not exactly equal to the normal trace of $\widehat{\alpha}_{h}^{p}$ (resp. $\beta_{h}^{q}$ ), but only weakly [15].

Theorem 2. For the primal system, the following are equivalent:

- $\widehat{\alpha}_{h}^{p}, \widehat{\beta}_{h}^{p-1}, \widehat{\alpha}_{h}^{p-1, n}, \widehat{\beta}_{h}^{p-1, t}$ is a solution to (36);
- $\widehat{\alpha}_{h}^{p}, \widehat{\beta}_{h}^{p-1}$ is a solution to (30). Moreover, $\widehat{\beta}_{h}^{p-1, t}=\operatorname{tr} \widehat{\beta}_{h}^{p-1}$ and $\widehat{\alpha}_{h}^{p-1, n}$ is uniquely determined by (36b).

For the dual system, the following are equivalent:

- $\alpha_{h}^{q-1}, \beta_{h}^{q}, \beta_{h}^{q-1, n}, \alpha_{h}^{q-1, t}$ is a solution to (37);
- $\alpha_{h}^{q-1}, \beta_{h}^{q}$ is a solution to (31). Moreover, $\alpha_{h}^{q-1, t}=\operatorname{tr} \alpha_{h}^{q-1}$ and $\beta_{h}^{q-1, n}$ is uniquely determined by (37a).
Proof. The proof is the same as in Thm. 1. Uniqueness of $\widehat{\alpha}_{h}^{p-1, n}, \beta_{h}^{q-1, n}$ follows from the fact that the broken space of normal traces is in duality with the broken space of tangential traces.

Remark 5. Since the primal and dual discrete hybrid systems are equivalent to the continuous Galerkin formulations, they satisfy a discrete power balance.

## 5. Algebraic realization of the discrete systems and static condensation

Given a finite element basis, the algebraic realization of the different terms in the weak formulation can be computed. In Appendix A the matrix realization of the local and global weak formulations is described.

Both the primal discrete system (36) and dual discrete system (37) have the following common structure

$$
\left[\begin{array}{cc}
\mathbf{E}_{l} & \mathbf{0}  \tag{38}\\
\mathbf{0} & \mathbf{0}
\end{array}\right]\binom{\dot{\mathbf{x}}_{l}}{\dot{\mathbf{x}}_{g}}=\left[\begin{array}{cc}
\mathbf{J}_{l} & \mathbf{C}_{l g} \\
-\mathbf{C}_{l g}^{\top} & \mathbf{0}
\end{array}\right]\binom{\mathbf{x}_{l}}{\mathbf{x}_{g}}+\left[\begin{array}{cc}
\mathbf{B}_{l} & \mathbf{0} \\
\mathbf{0} & \mathbf{B}_{g}
\end{array}\right]\binom{\mathbf{u}_{l}}{\mathbf{u}_{g}},
$$

where the subscript $l$ denotes the local variables and the subscript $g$ denotes the global variable. Matrix $\mathbf{E}_{l}$ is symmetric and positive semi-definite, while $\mathbf{J}_{l}$ is skew-symmetric $\mathbf{J}_{l}=-\mathbf{J}_{l}^{\top}$. This structure is a particular instance of a port-Hamiltonian descriptor system of index 2, or pHDAE [16].

Next we present the exact expression of the primal and dual algebraic systems. We shall use in what follows the notation $[\mathbf{x}]_{R}$ to indicatethe vector containing the rows of vector $\mathbf{x}$ associated with the index set $R$ only. In particular, we will denote the index sets corresponding to the degrees of freedom of the interior of the domain by $I$, and the ones associated to the subpartitions $\Gamma_{1}$ and $\Gamma_{2}$ by the same letters $\Gamma_{1}, \Gamma_{2}$ (with a slight abuse of notation, since these symbols are also used for the corresponding continuous stes).

The primal system. For what concerns the primal system, the state and input variables are the following

$$
\mathbf{x}_{l}=\left(\begin{array}{c}
\widehat{\boldsymbol{\alpha}}^{p} \\
\widehat{\boldsymbol{\beta}}^{p-1} \\
\widehat{\boldsymbol{\alpha}}^{p-1, \boldsymbol{n}}
\end{array}\right), \quad \mathbf{x}_{g}=\left[\widehat{\boldsymbol{\beta}}^{p-1, \boldsymbol{t}}\right]_{I \cup \Gamma_{1}}, \quad\binom{\mathbf{u}_{l}}{\mathbf{u}_{g}}=\binom{\widehat{\mathbf{u}}_{2}^{p-1}}{\mathbf{u}_{1}^{q-1}} .
$$

The specific structure of the matrices appearing in (38) is given by

$$
\left.\begin{array}{rl}
\mathbf{E}_{l} & =\left[\begin{array}{ccc}
\mathbf{M}_{\mathcal{T}_{h}}^{p} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{M}_{\mathcal{T}_{h}}^{p-1} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}\right], \quad \mathbf{J}_{l}=(-1)^{p}\left[\begin{array}{cc}
\mathbf{0} & \mathbf{D}_{\mathcal{T}_{h}}^{p-1}
\end{array}\right. \\
\left(\mathbf{D}_{\mathcal{T}_{h}}^{p-1}\right)^{\top} & \mathbf{0} \\
\mathbf{0} & \begin{array}{c}
\mathbf{0} \\
-\mathbf{M}_{\partial \mathcal{T}_{h}}^{p-1} \mathbf{T}_{\partial \mathcal{T}_{h}}^{p-1}
\end{array} \\
\left(\mathbf{T}_{\partial \mathcal{T}_{h}}^{p-1}\right)^{\top} \mathbf{M}_{\partial \mathcal{T}_{h}}^{p-1} \\
\mathbf{0}
\end{array}\right],
$$

The dual system. For the dual system, the state and input variables are the following

$$
\mathbf{x}_{l}=\left(\begin{array}{c}
\boldsymbol{\alpha}^{q-1} \\
\boldsymbol{\beta}^{q} \\
\boldsymbol{\beta}^{q-1, \boldsymbol{n}}
\end{array}\right), \quad \mathbf{x}_{g}=\left[\boldsymbol{\alpha}^{q-1, \boldsymbol{t}}\right]_{I \cup \Gamma_{2}}, \quad\binom{\mathbf{u}_{l}}{\mathbf{u}_{g}}=\binom{\mathbf{u}_{1}^{q-1}}{\widehat{\mathbf{u}}_{2}^{p-1}}
$$

The matrices exhibit the following structure

$$
\begin{aligned}
\mathbf{E}_{l} & =\left[\begin{array}{ccc}
\mathbf{M}_{\mathcal{T}_{h}}^{q-1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{M}_{\mathcal{T}_{h}}^{q} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}\right], \quad \mathbf{J}_{l}=\left[\begin{array}{ccc}
\mathbf{0} & \left(\mathbf{D}_{\mathcal{T}_{h}}^{q-1}\right)^{\top} & -\left(\mathbf{T}_{\partial \mathcal{T}_{h}}^{q-1}\right)^{\top} \mathbf{M}_{\partial \mathcal{T}_{h}}^{q-1} \\
-\left(\mathbf{D}_{\mathcal{T}_{h}}^{q-1}\right) & \mathbf{0} & \mathbf{0} \\
\mathbf{M}_{\partial \mathcal{T}_{h}}^{q-1} \mathbf{T}_{\partial \mathcal{T}_{h}}^{q-1} & \mathbf{0} & \mathbf{0}
\end{array}\right], \\
\mathbf{C}_{l g} & =-\left[\begin{array}{c}
\mathbf{0} \\
\mathbf{0} \\
\left(\mathbf{\Xi}_{\mathcal{F}_{h} / \Gamma_{1}}^{q-1}\right)^{\top} \mathbf{M}_{\mathcal{F}_{h} / \Gamma_{1}}^{q-1}
\end{array}\right], \quad \mathbf{B}_{l}=-\left[\begin{array}{c}
\mathbf{0} \\
\mathbf{0} \\
\left(\mathbf{T}_{\Gamma_{1}}^{p-1}\right)^{\top} \mathbf{M}_{\Gamma_{1}}^{p-1}
\end{array}\right], \\
\mathbf{B}_{g} & =(-1)^{(p-1)(q-1)}\left(\mathbf{T}_{\Gamma_{2}}^{q-1, t}\right)^{\top} \mathbf{\Psi}_{\Gamma_{2}}^{p-1} .
\end{aligned}
$$

### 5.1. Time discretization and static condensation

The time discretization leads to a linear saddle point system that can be efficiently solved via static condensation [14]. Since we are interested in Hamiltonian conservative systems, conservation of energy is of utmost importance. Implicit Runge-Kutta methods based on Gauss Legendre collocation points can be used to this aim 28. These methods are also the only collocation schemes that lead to an exact discrete energy balance in the linear case [29]. The implicit midpoint method is here used to illustrate the time discretization. Other schemes and in particular multistep schemes can be used as well.

Consider a total simulation time $T_{\text {end }}$ and a equidistant splitting given by the time step $\Delta t=$ $T_{\text {end }} / N_{t}$, where $N_{t}$ is the total number of simulation instants. The evaluation of a generic variable $\mathbf{x}$ at the time instant $t^{n}=n \Delta t$ is denoted by $\mathbf{x}^{n}$. Consider once again system 38 . The application of the implicit midpoint scheme leads to the following algebraic system

$$
\left[\begin{array}{cc}
\mathbf{E}_{l}-\frac{1}{2 \Delta t} \mathbf{J}_{l} & -\frac{1}{2 \Delta t} \mathbf{C}_{l g} \\
\frac{1}{2 \Delta t} \mathbf{C}_{l g}^{\top} & \mathbf{0}
\end{array}\right]\binom{\mathbf{x}_{l}^{n+1}}{\mathbf{x}_{g}^{n+1}}=\left[\begin{array}{cc}
\mathbf{E}_{l}+\frac{1}{2 \Delta t} \mathbf{J}_{l} & \frac{1}{2 \Delta t} \mathbf{C}_{l g} \\
-\frac{1}{2 \Delta t} \mathbf{C}_{l g}^{\top} & \mathbf{0}
\end{array}\right]\binom{\mathbf{x}_{l}^{n}}{\mathbf{x}_{g}^{n}}+\left[\begin{array}{cc}
\mathbf{B}_{l} & \mathbf{0} \\
\mathbf{0} & \mathbf{B}_{g}
\end{array}\right]\binom{\mathbf{u}_{l}^{n+1 / 2}}{\mathbf{u}_{g}^{n+1 / 2}}
$$

This system can be rewritten as the following saddle point problem

$$
\left[\begin{array}{cc}
\mathbf{A} & -\mathbf{C}  \tag{39}\\
\mathbf{C}^{\top} & \mathbf{0}
\end{array}\right]\binom{\mathbf{x}_{l}}{\mathbf{x}_{g}}=\binom{\mathbf{b}_{l}}{\mathbf{b}_{g}}
$$

The application of a Schur complement leads to the following system for the global variables

$$
\mathbf{C}^{\top} \mathbf{A}^{-1} \mathbf{C} \mathbf{x}_{g}=\mathbf{b}_{g}-\mathbf{C}^{\top} \mathbf{A}^{-1} \mathbf{b}_{l}
$$

The matrix $\mathbf{A}$ is block diagonal and contains local information. Each block can be inverted numerically or even analytically. Furthermore by Corollary 4.3. in 30], the Schur complement $\mathbf{C}^{\top} \mathbf{A}^{-1} \mathbf{C}$ has a positive semidefinite symmetric part, like the original matrix. This property can be exploited to design efficient preconditioning strategies 30. Once the global variable has been computed, the local unknown is then computed by solving the following system

$$
\mathbf{x}_{l}=\mathbf{A}^{-1} \mathbf{b}_{l}+\mathbf{A}^{-1} \mathbf{C} \mathbf{x}_{g}
$$

This system is block diagonal and can be solved in parallel.
Remark 6 (Well-posedness of the problem). The matrix $\mathbf{A}$ is invertible as it is given by a saddle point matrix whose off diagonal block are full rank and $\mathbf{C}$ matrix is full rank. These properties are a consequence of the fact that $\langle\cdot, \cdot\rangle_{\partial \mathcal{T}_{h}}$ is simply an $L^{2}$ inner product at the discrete level. Therefore, the algebraic system (39) is well posed.

A major advantage when using hybridization strategies is the reduced size of the system of equations to be solved. For simplicity consider the case in which no essential boundary conditions are present (this depends on which of the two system, primal or dual, is considered). The continuous Galerkin discretization leads to a total size of $\operatorname{dim} W_{h}^{k}+\operatorname{dim} V_{h}^{k-1}$ with $k=\{p, q\}$, whereas the hybridization strategy condenses it to $\operatorname{dim} V_{h}^{k-1, t}$. The following result is a simple application of Th. 4.4 in (15.

Theorem 3. When only natural boundary conditions apply, the reduction in size from the conforming discretization to the hybridized system is given by

$$
\begin{equation*}
\left(\operatorname{dim} W_{h}^{k}+\operatorname{dim} V_{h}^{k-1}\right)-\operatorname{dim} V_{h}^{k-1, t}=\operatorname{dim} W_{h}^{k}+\sum_{T \in \mathcal{T}_{h}} \operatorname{dim} \stackrel{\circ}{W}_{h}^{k-1}(T) \tag{40}
\end{equation*}
$$

where $\stackrel{\circ}{W}_{h}(T):=\left\{\omega_{h} \in W_{h}^{k-1},\left.\operatorname{tr} \omega_{h}\right|_{\partial T}=0\right\}$ and $k=\{p, q\}$.
Proof. The space $V_{h}^{k-1, t}$ is the image of $V_{h}^{k-1}$ under the trace map. By the rank nullity theorem their dimensions differ by the kernel

$$
\begin{align*}
\operatorname{dim} V_{h}^{k-1}-\operatorname{dim} V_{h}^{k-1, t} & =\operatorname{dim}\left\{\omega_{h} \in V_{h}^{k-1}:\left.\operatorname{tr} \omega_{h}\right|_{\partial \mathcal{T}_{h}}=0\right\} \\
& =\operatorname{dim} \prod_{T \in \mathcal{T}_{h}} \stackrel{\circ}{W}_{h}^{k-1}(T)=\sum_{T \in \mathcal{T}_{h}} \operatorname{dim} \stackrel{\circ}{W}_{h}^{k-1}(T) \tag{41}
\end{align*}
$$

### 5.2. Hybridization as transformer interconnection of port-Hamiltonian systems

In this section we give a reinterpretation of the finite element assembly as interconnection of port-Hamiltonian systems. The considerations in this section are carried out at the level of the algebraic systems arising after discretization. However they remain true if weak formulations are considered.

The local problems (32), (34) can be rewritten as port-Hamiltonian descriptor systems ( $\mathrm{pH}-$ DAE) of the following form

$$
\begin{align*}
{\left[\begin{array}{cc}
\mathbf{M}_{T} & \mathbf{0} \\
\mathbf{0} & \mathbf{0}
\end{array}\right]\binom{\dot{\mathbf{x}}_{T}}{\dot{\boldsymbol{\lambda}}_{\partial T}^{n}} } & =\left[\begin{array}{cc}
\mathbf{J}_{T} & \mathbf{G}_{\partial T} \\
-\mathbf{G}_{\partial T}^{\top} & \mathbf{0}
\end{array}\right]\binom{\mathbf{x}_{T}}{\boldsymbol{\lambda}_{\partial T}^{n}}+\left[\begin{array}{c}
\mathbf{0} \\
\mathbf{B}_{\partial T}
\end{array}\right] \mathbf{u}_{\partial T}^{t}, \\
\mathbf{y}_{\partial T}^{n} & =\left[\begin{array}{ll}
\mathbf{0} & \mathbf{B}_{\partial T}^{\top}
\end{array}\right]\binom{\mathbf{x}_{T}}{\boldsymbol{\lambda}_{\partial T}^{n}}, \tag{42}
\end{align*}
$$

where the essential boundary condition is imposed thanks to a Lagrange multiplier. The subscript $T$ and $\partial T$ refer to variables that live in a cell or its boundary respectively (cf. Appendix A.

Consider now two elements $T^{+}, T^{-}$sharing a common face $f_{i}^{ \pm}=\partial T^{+} \cap \partial T^{-}$. The input and output contribution can be split into the common face $f_{i}^{ \pm}$and the other faces $f_{e}^{ \pm}=\partial T^{ \pm} / f$ as follows

$$
\begin{align*}
{\left[\begin{array}{ccc}
\mathbf{M}_{T^{ \pm}} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}\right]\left(\begin{array}{c}
\dot{\mathbf{x}}_{T^{ \pm}} \\
\dot{\boldsymbol{\lambda}}_{f_{i}^{ \pm}}^{n} \\
\dot{\boldsymbol{\lambda}}_{f_{e}^{ \pm}}^{n}
\end{array}\right) } & =\left[\begin{array}{ccc}
\mathbf{J}_{T^{ \pm}} & \mathbf{G}_{f_{i}^{ \pm}} & \mathbf{G}_{f_{e}^{ \pm}} \\
-\mathbf{G}_{f_{i}^{ \pm}}^{\top} & \mathbf{0} & \mathbf{0} \\
-\mathbf{G}_{f_{e}^{ \pm}}^{\tau_{i}} & \mathbf{0} & \mathbf{0}
\end{array}\right]\left(\begin{array}{c}
\mathbf{x}_{T^{ \pm}} \\
\boldsymbol{\lambda}_{f_{i}^{ \pm}}^{n} \\
\boldsymbol{\lambda}_{f_{e}^{ \pm}}^{n}
\end{array}\right)+\left[\begin{array}{cc}
\mathbf{0} & \mathbf{0} \\
\mathbf{B}_{f_{i}^{ \pm}} & \mathbf{0} \\
\mathbf{0} & \mathbf{B}_{f_{e}^{ \pm}}
\end{array}\right]\binom{\mathbf{u}_{f_{i}^{ \pm}}^{t}}{\mathbf{u}_{f_{e}^{ \pm}}^{t^{ \pm}}}  \tag{43}\\
\binom{\mathbf{y}_{f_{i}^{ \pm}}^{n}}{\mathbf{y}_{f_{e}^{ \pm}}^{n}}= & {\left[\begin{array}{ccc}
\mathbf{0} & \mathbf{B}_{f_{i}^{ \pm}}^{\top} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{B}_{f_{e}^{ \pm}}^{\top}
\end{array}\right]\left(\begin{array}{c}
\mathbf{x}_{T^{ \pm}} \\
\boldsymbol{\lambda}_{f_{i}^{ \pm}}^{n} \\
\boldsymbol{\lambda}_{f_{e}^{ \pm}}^{n}
\end{array}\right) . }
\end{align*}
$$

giving rise to one system for $T^{+}$and one for $T^{-}$. The structure of this matrix system and the interconnection required to achieve the hybrid system depends on whether the primal or the dual formulation is considered.

The primal local system. The variables for the primal system are the following

$$
\mathbf{x}_{T^{ \pm}}=\binom{\widehat{\boldsymbol{\alpha}}_{T \pm \pm}^{p}}{\widehat{\boldsymbol{\beta}}_{T^{ \pm}}^{p-1}}, \quad\binom{\boldsymbol{\lambda}_{f_{i}^{ \pm}}^{\boldsymbol{n}}}{\boldsymbol{\lambda}_{f_{e}^{ \pm}}^{\boldsymbol{n}}}=\binom{\widehat{\boldsymbol{\alpha}}_{f_{i}^{ \pm}}^{p-1, \boldsymbol{n}}}{\widehat{\boldsymbol{\alpha}}_{f_{e}^{ \pm}}^{p-1, \boldsymbol{n}}},
$$

and the matrices are given by

$$
\begin{aligned}
\mathbf{M}_{T^{ \pm}} & =\left[\begin{array}{cc}
\mathbf{M}_{T^{ \pm}}^{p} & \mathbf{0} \\
\mathbf{0} & \mathbf{M}_{T^{ \pm}}^{p-1}
\end{array}\right], & \mathbf{J}_{T^{ \pm}}=(-1)^{p}\left[\begin{array}{cc}
\mathbf{0} & \mathbf{D}_{T^{ \pm}}^{p-1} \\
\left(\mathbf{D}_{T^{ \pm}}^{p-1}\right)^{\top} & \mathbf{0}
\end{array}\right] \\
\mathbf{G}_{f_{i}^{ \pm}} & =(-1)^{p}\left[\begin{array}{c}
\mathbf{0} \\
\left(\mathbf{T}_{f_{i}^{ \pm}}^{p-1}\right)^{\top} \mathbf{M}_{f_{i}^{ \pm}}^{p-1}
\end{array}\right], & \mathbf{G}_{f_{e}^{ \pm}}=(-1)^{p}\left[\begin{array}{c}
\mathbf{0} \\
\left(\mathbf{T}_{f_{i}^{ \pm}}^{p-1}\right)^{\top} \mathbf{M}_{f_{i}^{ \pm}}^{p-1}
\end{array}\right] \\
\mathbf{B}_{f_{i}^{ \pm}} & =(-1)^{p} \mathbf{M}_{f_{i}^{ \pm}}^{p-1}, & \mathbf{B}_{f_{e}^{ \pm}}=(-1)^{p} \mathbf{M}_{f_{e}^{ \pm}}^{p-1}
\end{aligned}
$$

It can be noticed that the input corresponds to the tangential trace of an outer oriented variable. More specifically

$$
\mathbf{u}_{f_{i}^{ \pm}}^{t}=\mathbf{T}_{f_{i}^{ \pm}}^{p-1} \widehat{\boldsymbol{\beta}}_{T^{ \pm}}^{p-1}, \quad \mathbf{u}_{f_{e}^{ \pm}}^{t}=\mathbf{T}_{f_{e}^{ \pm}}^{p-1} \widehat{\boldsymbol{\beta}}_{T^{ \pm}}^{p-1}
$$

The output instead corresponds to the Lagrange multiplier, that coincides with the the normal trace of an outer oriented variable (modulo a mass matrix)

$$
\mathbf{y}_{f_{i}^{ \pm}}^{\boldsymbol{n}}=(-1)^{p} \mathbf{M}_{f_{i}^{ \pm}}^{p-1} \widehat{\boldsymbol{\alpha}}_{f_{i}^{ \pm}}^{p-1, \boldsymbol{n}}, \quad \mathbf{y}_{f_{e}^{ \pm}}^{\boldsymbol{n}}=(-1)^{p} \mathbf{M}_{f_{e}^{ \pm}}^{p-1} \widehat{\boldsymbol{\alpha}}_{f_{e}^{ \pm}}^{p-1, \boldsymbol{n}}
$$

So the following transformer interconnection is used to achieved the assemble of the hybridized system of outer oriented forms

$$
\mathbf{u}_{f_{i}^{+}}^{t}=-\mathbf{u}_{f_{i}^{-}}^{t}, \quad \mathbf{y}_{f_{i}^{+}}^{n}=+\mathbf{y}_{f_{i}^{-}}^{\boldsymbol{n}}
$$

leading to the following relationship between the state variables

$$
\mathbf{T}_{f_{i}^{+}}^{p-1} \widehat{\boldsymbol{\beta}}_{T^{ \pm}}^{p-1}=-\mathbf{T}_{f_{i}^{-}}^{p-1} \widehat{\boldsymbol{\beta}}_{T^{ \pm}}^{p-1}, \quad \widehat{\boldsymbol{\alpha}}_{f_{i}^{+}}^{p-1, \boldsymbol{n}}=\widehat{\boldsymbol{\alpha}}_{f_{i}^{-}}^{p-1, \boldsymbol{n}}
$$

The equivalence of this formulation and 38 is readily obtained considering that the input corresponds to the tangential facet element

$$
\boldsymbol{\beta}^{p-1, t}=\mathbf{u}_{f_{i}^{+}}^{t}=-\mathbf{u}_{f_{i}^{-}}^{t}
$$

Once the interconnection is performed Sys. 38 with essential boundary conditions is obtained on the mesh $\mathcal{T}_{h}=T^{+} \cup T^{-}$.

The dual local system. The variables for the dual system are

$$
\mathbf{x}_{T^{ \pm}}=\binom{\boldsymbol{\alpha}_{T^{ \pm}}^{q-1}}{\boldsymbol{\beta}_{T^{ \pm}}^{q}}, \quad\binom{\boldsymbol{\lambda}_{f_{i}^{ \pm}}^{n}}{\boldsymbol{\lambda}_{f_{e}^{ \pm}}^{n}}=\binom{\boldsymbol{\beta}_{f_{i}^{ \pm}}^{q-1, \boldsymbol{n}}}{\boldsymbol{\beta}_{f_{e}^{ \pm}}^{q-1, \boldsymbol{n}}}
$$

and the matrices are given by

$$
\begin{aligned}
\mathbf{M}_{T^{ \pm}} & =\left[\begin{array}{cc}
\mathbf{M}_{T^{ \pm}}^{q-1} & \mathbf{0} \\
\mathbf{0} & \mathbf{M}_{T^{ \pm}}^{q}
\end{array}\right],
\end{aligned} \mathbf{J}_{T^{ \pm}}=\left[\begin{array}{cc}
\mathbf{0} & \left(\mathbf{D}_{T^{ \pm}}^{q-1}\right)^{\top} \\
-\mathbf{D}_{T^{ \pm}}^{q-1} & \mathbf{0}
\end{array}\right], ~\left(\begin{array}{cl}
\left.\mathbf{G}_{f_{i}^{ \pm}}^{q-1}\right)^{\top} \mathbf{M}_{f_{i}^{ \pm}}^{q-1} \\
\mathbf{0}
\end{array}\right], \quad \mathbf{G}_{f_{e}^{ \pm}}=-\left[\begin{array}{c}
\left(\mathbf{T}_{f_{i}^{ \pm}}^{q-1}\right)^{\top} \mathbf{M}_{f_{i}^{ \pm}}^{q-1} \\
\mathbf{0}
\end{array}\right], \mathbf{B}_{f_{e}^{ \pm}}=-\mathbf{M}_{f_{e}^{ \pm}}^{p-1} .
$$

In this case the input corresponds to the tangential trace of an inner oriented variable

$$
\mathbf{u}_{f_{i}^{ \pm}}^{t}=\mathbf{T}_{f_{i}^{ \pm}}^{q-1} \boldsymbol{\alpha}_{T^{ \pm}}^{q-1}, \quad \mathbf{u}_{f_{e}^{ \pm}}^{t}=\mathbf{T}_{f_{e}^{ \pm}}^{q-1} \boldsymbol{\alpha}_{T^{ \pm}}^{q-1}
$$

The output instead corresponds to the normal trace of an outer oriented variable

$$
\mathbf{y}_{f_{i}^{ \pm}}^{\boldsymbol{n}}=-\mathbf{M}_{f_{i}^{ \pm}}^{q-1} \boldsymbol{\beta}_{f_{i}^{ \pm}}^{q-1, \boldsymbol{n}}, \quad \mathbf{y}_{f_{e}^{ \pm}}^{\boldsymbol{n}}=-\mathbf{M}_{f_{e}^{ \pm}}^{q-1} \boldsymbol{\beta}_{f_{e}^{ \pm}}^{q-1, \boldsymbol{n}}
$$

So the following transformer interconnection is used to achieved the assemble of the hybridized system of inner oriented forms

$$
\mathbf{u}_{f_{i}^{+}}^{t}=+\mathbf{u}_{f_{i}^{-}}^{t}, \quad \mathbf{y}_{f_{i}^{+}}^{\boldsymbol{n}}=-\mathbf{y}_{f_{i}^{-}}^{\boldsymbol{n}}
$$

leading to the following relationship between the state variables

$$
\mathbf{T}_{f_{i}^{+}}^{q-1} \boldsymbol{\alpha}_{T^{ \pm}}^{q-1}=\mathbf{T}_{f_{i}^{-}}^{q-1} \boldsymbol{\alpha}_{T^{ \pm}}^{q-1}, \quad \boldsymbol{\beta}_{f_{i}^{+}}^{q-1, \boldsymbol{n}}=-\boldsymbol{\beta}_{f_{i}^{-}}^{q-1, \boldsymbol{n}}
$$

The equivalence of this formulation and 38 is readily obtained considering that the input corresponds to the tangential facet element

$$
\boldsymbol{\alpha}^{q-1, t}=\mathbf{u}_{f_{i}^{+}}^{t}=\mathbf{u}_{f_{i}^{-}}^{t}
$$

The presented procedure generalizes to the entire computational mesh, leading to a reinterpretation of the finite element assembly as interconnection of port-Hamiltonian systems.

## 6. Numerical experiments

In this section, the hybridization strategy is tested for the wave and Maxwell equations. The domain is the unit cube

$$
M=\left\{(x, y, z) \in[0,1]^{3}\right\}
$$

The boundary sub-partitions are selected to be

$$
\Gamma_{1}=\{(x, y, z) \mid x=0 \cup y=0 \cup z=0\}, \quad \Gamma_{2}=\{(x, y, z) \mid x=1 \cup y=1 \cup z=1\}
$$

A structured tetrahedral mesh $\mathcal{T}_{h}$ is formed by partitioning $M$ into $N_{\mathrm{el}} \times N_{\mathrm{el}} \times N_{\mathrm{el}}$ cubes, each of which is divided into six tetrahedra. The total simulation time is 1 and the time step is taken to be $\Delta t=T_{\text {end }} / 100$.

By introducing the musical isomorphism, given by the flat $b$ and the sharp operator $\sharp$, and the isomorphism $\Theta$ converting vector fields in $n-1$ forms, the commuting diagram in Fig. 2, that


Figure 2: Equivalence of vector and exterior calculus Sobolev spaces.


Figure 3: Equivalence between finite element differential forms and classical elements.
provides the link between the de Rham complex and the standard operators and Sobolev space from vector calculus, is obtained.

For the discretization, the trimmed polynomial family $\mathcal{P}_{s}^{-} \Omega^{k}\left(\mathcal{T}_{h}\right)$ is used. This family corresponds to the well known continuous Galerkin (or Lagrange) elements $\mathcal{P}_{s}^{-} \Omega^{0}\left(\mathcal{T}_{h}\right) \equiv \mathrm{CG}_{s}\left(\mathcal{T}_{h}\right)$, Nédélec of the first kind $\mathcal{P}_{s}^{-} \Omega^{1}\left(\mathcal{T}_{h}\right) \equiv \operatorname{NED}_{s}^{1}\left(\mathcal{T}_{h}\right)$, Raviart-Thomas $\mathcal{P}_{s}^{-} \Omega^{2}\left(\mathcal{T}_{h}\right) \equiv \operatorname{RT}_{s}\left(\mathcal{T}_{h}\right)$ and discontinuous Galerkin $\mathcal{P}_{s}^{-} \Omega^{3}\left(\mathcal{T}_{h}\right) \equiv \mathrm{DG}_{s-1}\left(\mathcal{T}_{h}\right)$, as illustrated in Figure 3 .

The finite element library Firedrake [31] is used for the numerical investigation. The FireDRAKE component Slate [32] is used to implement the local solvers and static condensation.

For what concerns the computation of the error, care has to be taken when considering normal trace variable $\omega_{h}^{k, n} \in W_{h}^{k, n} \equiv W_{h}^{k, t}$. Identification of $\omega_{h}^{k, n}$ with an element of $L^{2} \Omega^{k}\left(\partial \mathcal{T}_{h}\right)$ is only unique up to the annihilator $\left(W_{h}^{k, t}\right)^{\perp}[15]$. Therefore, the $L^{2}$ error has to be computed after removing the annihilator, which is equivalent to taking the following projection $P_{h} \omega_{\mathrm{ex}}^{k, n}$ of the exact solution $\omega_{\mathrm{ex}}^{k+1}$

$$
\begin{equation*}
\left\langle v_{h}^{k, \boldsymbol{t}}, P_{h} \omega_{\mathrm{ex}}^{k, \boldsymbol{n}}\right\rangle_{\partial \mathcal{T}_{h}}=\left\langle v_{h}^{k, \boldsymbol{t}}, \omega_{\mathrm{ex}}^{k, \boldsymbol{n}}\right\rangle_{\partial \mathcal{T}_{h}}, \quad \forall v_{h}^{k, \boldsymbol{t}} \in W_{h}^{k, \boldsymbol{t}} . \tag{44}
\end{equation*}
$$

The scaled $L^{2}$ norm over a cell boundary is given by $\|\|\cdot\|\|_{\partial K}:=h_{T}\|\cdot\| \partial K$ where $h_{T}$ denoted the diameter of the cell $T \in \mathcal{T}_{h}$. For the overall mesh, we use the notation $\|\|\cdot\|\|_{\partial \mathcal{T}_{h}}=\sum_{T \in \mathcal{T}_{h}} \mid\|\cdot\| \|_{\partial K}$. The error for $\omega_{h}^{k, \boldsymbol{n}}$ will then measure as

$$
\text { Error } \omega_{h}^{k, \boldsymbol{n}}=\| \| \omega_{h}^{k, \boldsymbol{n}}-P_{h} \omega_{\mathrm{ex}}^{k, \boldsymbol{n}}\| \|_{\partial \mathcal{T}_{h}}
$$

The norm $\|\|\cdot\|\|_{\partial \mathcal{T}_{h}}$ is also used to measure convergence for the tangential trace $\omega_{h}^{k, t}$.

### 6.1. The wave equation in $3 D$

The acoustic wave equation corresponds to the case $p=3$ and $q=1$. The energy variables are the pressure top-form $\widehat{p}^{3}:=\widehat{\alpha}^{3}$ and the velocity one-form $u^{1}:=\beta^{1}$. The Hamiltonian is given by

$$
\begin{equation*}
H\left(\widehat{p}^{3}, u^{1}\right)=\frac{1}{2} \int_{M} \widehat{p}^{3} \wedge \star \widehat{p}^{3}+u^{1} \wedge \star u^{1} \tag{45}
\end{equation*}
$$

with its variational derivatives given by

$$
\begin{equation*}
p^{0}:=\delta_{\widehat{p}^{3}} H=\star \widehat{p}^{3}, \quad \widehat{u}^{2}:=\delta_{u^{1}} H=\star u^{1} \tag{46}
\end{equation*}
$$

leading to the pH system

$$
\binom{\partial_{t} \widehat{p}^{3}}{\partial_{t} u^{1}}=-\left[\begin{array}{cc}
0 & \mathrm{~d}  \tag{47}\\
\mathrm{~d} & 0
\end{array}\right]\binom{\left.p^{0} p^{0}\right|_{\Gamma_{1}}=u_{1}^{0}}{\widehat{u}^{2}}, \quad \begin{aligned}
-\left.\operatorname{tr} \widehat{u}^{2}\right|_{\Gamma_{2}}=\widehat{u}_{2}^{2}
\end{aligned}
$$

The same exact solution as in 10 is here considered. Introducing the functions

$$
\begin{equation*}
g(x, y, z)=\cos (x) \sin (y) \sin (z), \quad f(t)=2 \sin (\sqrt{3} t)+3 \cos (\sqrt{3} t) \tag{48}
\end{equation*}
$$

an exact solution of 47) is given by

$$
\begin{array}{ll}
\widehat{v}_{\mathrm{ex}}^{3}=\star g \frac{d f}{d t}, & v_{\mathrm{ex}}^{0}=g \frac{d f}{d t}  \tag{49}\\
\sigma_{\mathrm{ex}}^{1}=-\mathrm{d} g f, & \widehat{\sigma}_{\mathrm{ex}}^{2}=-\star \mathrm{d} g f
\end{array}
$$

The exact solution provides the appropriate inputs to be fed into the system

$$
\begin{equation*}
u_{1}^{0}=\left.\operatorname{tr} v_{\mathrm{ex}}^{0}\right|_{\Gamma_{1}}, \quad u_{2}^{2}=-\left.\operatorname{tr} \widehat{\sigma}_{\mathrm{ex}}^{2}\right|_{\Gamma_{2}} \tag{50}
\end{equation*}
$$

The employment of the dual field discretization leads to the resolution of two systems:

- the primal system (33) of outer oriented variables $\widehat{p}^{3}, \widehat{u}^{2}, \widehat{p}^{2, n}, \widehat{u}^{2, t}$;
- the dual system (35) of inner oriented variables $p^{0}, u^{1}, u^{0, \boldsymbol{n}}, p^{0, \boldsymbol{t}}$.


### 6.1.1. Equivalence between the continuous and hybrid formulation

In this section, the equivalence of the continuous Galerkin formulation and its hybridization (cf. Theorem 22 is verified numerically. The test is performed using $N_{\text {el }}=4$ elements for each side of the cubic domain and polynomial degree $s=3$. In Figure 4 the $L^{2}$ norm of the difference of the variables $\widehat{p}^{3}, \widehat{u}^{2}, p^{0}, u^{1}$ obtained via the continuous Galerkin formulation and the hybrid version is plotted. It can be noticed that the error is zero to machine precision. In Fig. 5 the discrete power balance and the error with respect to the exact power balance are plotted. As the continuous formulation the hybrid primal-dual formulation enjoys a discrete power balance (cf. Fig. 5a). The error, reported in Fig. 5b, is only due to the interpolation. The norm of the divergence of the two forms $\widehat{E}^{2}, H^{2}$ is reported in Fig. 11 . The fields keep their solenoidal nature at the discrete level.

### 6.2. Convergence results

Having established numerically the equivalence between the continuous Galerkin formulation and its hybridized version, in this section we assess the rate of convergence for the primal and the dual formulation. The error is taken to be the $L^{2}$ norm of the error at the final time $T_{\text {end }}$.

In Fig. 6 the convergence rate of the variables is plotted against the mesh size. It can be immediately noticed that for the lowest order polynomial degree variable $\widehat{p}_{h}^{2, n}$ superconverges. This behaviour is well known for the RT and BDM elements in the static case 33. In the higher order case such superconvergent behaviour is not observed. The other variables converge with the optimal order.

For what concerns the dual system, the results are shown in Fig. 7. Variables $p_{h}^{0}, p_{h}^{0, t}$ converge with an order between $s$ and $s+1$ (the optimal order is known to be $s+1$ for homogeneous boundary condition). The degradation in the order of convergence of convergence has to be attributed to the presence of non-homogeneous boundary conditions. Variables $u_{h}^{1}, u_{h}^{0, \boldsymbol{n}}$ converge with optimal order.

In Fig. 8, the $L^{2}$ norm of the difference between the dual representation of variables. As in the dual field continuous Galerkin formulation, the dual representation of the variables converges under h-p refinement with order $h^{s}$.

The size reduction between the continuous and hybrid formulation is reported in Tables 2, 3. For the primal formulation one only solves for $40 \%$ of the degrees of freedom when third order polynomials are used. For the dual the size reduction is way more impressive (the hybrid formulation dimension is $20 \%$ of the continuous formulation in the worst case) as the broken Nédélec space is completely discarded when hybridization is used.


Figure 4: $L^{2}$ norm of the difference between the continuous Galerkin and the hybridized representation of the variables for the wave equation $\left(N_{\mathrm{el}}=4, s=3, \Delta t=\frac{1}{100}, T_{\mathrm{end}}=1\right)$.


Figure 5: Power balance (left) and error on the power flow (right) for the wave equation $\left(N_{\mathrm{el}}=4, s=3, \Delta t=\frac{1}{100}\right)$.


Figure 6: Convergence rate for the different variables in the primal formulation of the wave equation, measured at $T_{\text {end }}=1$ for $\Delta t=\frac{1}{100}$. The error is measured in the $L^{2}$ norm.

| Pol. Degree $s$ | $N_{\text {elem }}$ | $N^{\circ}$ dofs. continuous | $N^{\circ}$ dofs. hybrid | $N_{\text {hyb }} / N_{\text {cont }}$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 24 | 18 | $75 \%$ |
|  | 2 | 168 | 120 | $71 \%$ |
|  | 4 | 1248 | 864 | $69 \%$ |
|  | 8 | 9600 | 6528 | $68 \%$ |
|  | 16 | 75264 | 50688 | $67 \%$ |
| 2 | 1 | 96 | 54 | $56 \%$ |
|  | 2 | 696 | 360 | $52 \%$ |
|  | 4 | 5280 | 2592 | $49 \%$ |
|  | 8 | 41088 | 19584 | $47 \%$ |
| 3 | 1 | 240 | 108 | $45 \%$ |
|  | 2 | 1776 | 720 | $41 \%$ |
|  | 4 | 13632 | 5184 | $38 \%$ |

Table 2: Size of the primal system $\widehat{p}^{3}, \widehat{u}^{2}$ for the wave equation: continuous and hybrid formulation.


Figure 7: Convergence rate for the different variables in the dual formulation of the wave equation, measure at at $T_{\text {end }}=1$ for $\Delta t=\frac{1}{100}$. The error is measured in the $L^{2}$ norm.


Figure 8: $L^{2}$ difference of the dual representation of the solution for the wave equation at $T_{\text {end }}=1$ for $\Delta t=\frac{1}{100}$.

| Pol. Degree $s$ | $N_{\text {elem }}$ | $N^{\circ}$ dofs. continuous | $N^{\circ}$ dofs. hybrid | $N_{\text {hyb }} / N_{\text {cont }}$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 44 | 8 | $18 \%$ |
|  | 2 | 315 | 27 | $9 \%$ |
|  | 4 | 2429 | 125 | $5 \%$ |
|  | 8 | 19161 | 729 | $4 \%$ |
|  | 16 | 152369 | 4913 | $3 \%$ |
| 2 | 1 | 147 | 27 | $18 \%$ |
|  | 2 | 1085 | 125 | $12 \%$ |
|  | 4 | 8409 | 729 | $9 \%$ |
|  | 8 | 66353 | 4913 | $7 \%$ |
| 3 | 1 | 334 | 64 | $19 \%$ |
|  | 2 | 2503 | 343 | $13 \%$ |
|  | 4 | 19477 | 2197 | $11 \%$ |

Table 3: Size of the dual system $p^{0}, u^{1}$ for the wave equation: continuous and hybrid formulation.

### 6.3. The Maxwell equations in $3 D$

The Maxwell equations corresponds to the case $p=2, q=2$. The energy variables correspond to the electric displacement two form $\widehat{D}^{2}=\widehat{\alpha}^{2}$ and the magnetic field $B^{2}=\beta^{2}$. The Hamiltonian reads

$$
\begin{equation*}
H\left(\widehat{D}^{2}, B^{2}\right)=\frac{1}{2} \int_{M} \widehat{D}^{2} \wedge \star \widehat{D}^{2}+B^{2} \wedge \star B^{2}, \tag{51}
\end{equation*}
$$

The variational derivative of the Hamiltonian are given by

$$
\begin{equation*}
E^{1}:=\delta_{\widehat{D}^{2}} H=\star \widehat{D}^{2}, \quad \widehat{H}^{1}:=\delta_{B^{2}} H=\star B^{2} . \tag{52}
\end{equation*}
$$

Variables $E^{1}, \widehat{H}^{1}$ are the electric field and the magnetizing field respectively. Since the reduction of the constitutive equation is such to keep only the efforts variables and their duals, the following dynamical system is obtained.

$$
\binom{\partial_{t} \widehat{E}^{2}}{\partial_{t} H^{2}}=\left[\begin{array}{cc}
0 & \mathrm{~d}^{1}  \tag{53}\\
-\mathrm{d}^{1} & 0
\end{array}\right]\binom{E^{1}}{\widehat{H}^{1}},
$$

where $\widehat{E}^{2}=\star E^{1}, H^{2}=\star \widehat{H}^{1}$. Given the functions

$$
\boldsymbol{g}(x, y, z)=\left(\begin{array}{c}
-\cos (x) \sin (y) \sin (z)  \tag{54}\\
0 \\
\sin (x) \sin (y) \cos (z)
\end{array}\right), \quad f(t)=\frac{\sin (\sqrt{3} t)}{\sqrt{3}} .
$$

The system (53) is solved by the eigenmode

$$
\begin{array}{ll}
\widehat{E}_{\mathrm{ex}}^{2}=\star \boldsymbol{g}^{\mathrm{b}} \frac{d f}{d t}, & E_{\mathrm{ex}}^{1}=\boldsymbol{g}^{\mathrm{b}} \frac{d f}{d t},  \tag{55}\\
H_{\mathrm{ex}}^{2}=-\mathrm{d} \boldsymbol{g}^{\mathrm{b}} f, & \widehat{H}_{\mathrm{ex}}^{1}=-\star \mathrm{d} \boldsymbol{g}^{\mathrm{b}} f .
\end{array}
$$

The exact solution provides the appropriate inputs to be fed into the system

$$
\begin{equation*}
u_{1}^{1}=\left.\operatorname{tr} E_{\mathrm{ex}}^{1}\right|_{\Gamma_{1}}, \quad \widehat{u}_{2}^{1}=\operatorname{tr} \widehat{H}_{\mathrm{ex}}^{1} \mid \Gamma_{2} . \tag{56}
\end{equation*}
$$

The employment of the dual field discretization leads to the resolution of two systems:

- the primal system (33) of outer oriented variables $\widehat{E}^{2}, \widehat{H}^{1}, \widehat{E}^{1, n}, \widehat{H}^{1, t}$;
- the dual system (35) of inner oriented variables $E^{1}, H^{2}, H^{1, n}, E^{1, t}$.


### 6.3.1. Equivalence between the continuous and hybrid formulation

The equivalence between the continuous and hybrid formulation is now verified for the Maxwell equations using $N_{\mathrm{el}}=4$ elements for each side of the domain and polynomial degree $s=3$. In Figure 9 the $L^{2}$ norm of the difference of the variables $\widehat{E}^{2}, \widehat{H}^{1}, E^{1}, H^{2}$ obtained via the continuous Galerkin formulation and the hybrid version is plotted. The error is zero to machine precision. In Fig. 10 the discrete power balance and the error with respect to the exact power balance are plotted.


Figure 9: $L^{2}$ norm of the difference between the continuous Galerkin and the hybridized representation of the variables for the Maxwell equations $\left(N_{\mathrm{el}}=4, s=3, \Delta t=\frac{1}{100}, T_{\mathrm{end}}=1\right)$.

### 6.4. Convergence results

In Fig. 12 the convergence rate of the variables is plotted against the mesh size. The error is again taken to be the $L^{2}$ norm of the error at the final time $T_{\text {end }}$.

The convergence rates for the primal formulation are reported in Fig. 12 All variables converge with order $h^{s}$ in the $L^{2}$ norm. The convergence results for the dual formulation are shown in Fig. 13. The electric field $E^{1}$ and its tangential trace $E^{1, t}$ shows some superconverge at the lowest order. In the other cases, all variables converge with order $h^{s}$.

In Fig. 14 , the $L^{2}$ norm of the difference between the dual representation of variables. As in the dual field continuous Galerkin formulation, the dual representation of the variables converges under h-p refinement with order $h^{s}$.

(a) $\dot{H}_{h}=<\widehat{y}_{2, h}^{1} \mid u_{1, h}^{1}>_{\partial M}+\left\langle\widehat{u}_{2, h}^{1}\right| y_{1, h}^{1}>_{\partial M}$

(b) Error exact and interpolated boundary flow

Figure 10: Power balance (left) and error on the power flow (right) for the Maxwell equations $\left(N_{\mathrm{el}}=4, s=\right.$ $\left.3, \Delta t=\frac{1}{100}\right)$.


Figure 11: $L^{2}$ norm divergence of the two forms $\widehat{E}_{h}^{2}, H_{h}^{2}$.

The size reduction between the continuous and hybrid formulation is the same for the primal and dual system as the two formulation uses forms of the same degree and is reported in Table 4 in The hybrid system has $45 \%$ of degrees of freedom compared with the continuous formulation in the worst case. The size decreases rapidly to $30 \%$ when the number of elements and polynomial degree is increased, showing a clear computational advantage with respect to the continuous formulation presented in Sec. 3.3 .

## 7. Conclusion

In this work, the continuous dual field formulation is modified by taking the variable that does not undergo the exterior derivative to live in a broken finite element space. The resulting formulation is directly amenable to hybridization by introducing appropriate multipliers enforcing the continuity of the regular variable. The properties of the dual field scheme are left untouched and the hybrid formulation is highly advantageous since the broken variable, being local, is completely discarded from the global system, resulting in a huge computational gain obtained.


Figure 12: Convergence rate for the different variables in the primal formulation of the Maxwell equations, measured at $T_{\text {end }}=1$ for $\Delta t=\frac{1}{100}$. The error is measured in the $L^{2}$ norm.

| Pol. Degree $s$ | $N_{\text {elem }}$ | $N^{\circ}$ dofs. continuous | $N^{\circ}$ dofs. hybrid | $N_{\text {hyb }} / N_{\text {cont }}$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 43 | 19 | $44 \%$ |
|  | 2 | 290 | 98 | $38 \%$ |
|  | 4 | 2140 | 604 | $28 \%$ |
|  | 8 | 16472 | 4184 | $25 \%$ |
|  | 16 | 129328 | 31024 | $24 \%$ |
| 2 | 1 | 164 | 74 | $45 \%$ |
|  | 2 | 1156 | 436 | $37 \%$ |
|  | 4 | 8696 | 2936 | $33 \%$ |
|  | 8 | 67504 | 21424 | $32 \%$ |
| 3 | 1 | 399 | 165 | $41 \%$ |
|  | 2 | 2886 | 1014 | $35 \%$ |
|  | 4 | 21972 | 6996 | $32 \%$ |

Table 4: Size of the primal and dual system for the Maxwell equations: continuous and hybrid formulation.


Figure 13: Convergence rate for the different variables in the dual formulation of the Maxwell equation, measure at at $T_{\text {end }}=1$ for $\Delta t=\frac{1}{100}$. The error is measured in the $L^{2}$ norm.


Figure 14: $L^{2}$ difference of the dual representation of the solution for the Maxwell equation at $T_{\text {end }}=1$ for $\Delta t=\frac{1}{100}$.

The first development of this work would be the employment of algebraic dual polynomials [34] to improve the condition number of the resulting matrix system. Another important topic is the employment of higher order time integration scheme and how to apply to those the static condensation procedure. The presented framework may be extended by devising post-processing schemes. This may be achieved exploiting the primal-dual structure of the equations. Furthermore, non-conforming and hybridizable discontinuous Galerkin (HDG) methods may be devised considering a more general local problem where the exterior derivative and the codifferential are taken weakly. Another important example of port-Hamiltonian system is the Elastodynamics problem. Finite element differential forms for the de Rham complex can also be used in this case but the symmetry of the stress tensor has to be enforced weakly [35]. The dual field method may be also applied to this case, by introducing an appropriate multiplier that enforces the symmetry of the stress tensor.

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## Appendix A. Algebraic realization of the weak formulation

To discuss the hybridization we detail the algebraic computation of the different terms in the weak formulation. For a discussion of the algebraic realization arising in conforming finite element discretization, the reader can consult [10, Sec. 5.1].

A different notation is used for inner and outer oriented forms. This is important when considering the natural duality pairing of forms, as the combination of an inner and outer oriented form leads to an orientation invariant operation. For the interconnection of different elements inner and outer oriented forms undergo a different treatment. Inner oriented forms do not carry information about the normal orientation, so they keep their sign from one element to the other Outer oriented forms carry information about the ambient normal orientation, so they change sign from one element to the other. This will be addressed when introducing the jump and average operators. At the discrete level, this distinction is yet to be implemented in standard finite element libraries like Firedrake. So the same basis and degrees of freedom are used for both.

General notation. Given a computational mesh $\mathcal{T}_{h}$, the set of facets of the mesh (the codimension 1 skeleton) is the union of the cells boundary $\mathcal{F}_{h}=\bigcup_{T \in \mathcal{T}_{h}} \partial T$, whereas the disjoint union is denoted by $\partial \mathcal{T}_{h}=\bigsqcup_{T \in \mathcal{T}_{h}} \partial T$. The notation $[\mathbf{A}]_{i}^{j}$ will be used to denoted the element of matrix $\mathbf{A}$ corresponding to the $i$-th row and the $j$-th column.

Appendix A.1. Local finite element forms and their facet version
Basis expression for local forms. Let $W_{h}^{k}(T) \subset H \Omega^{k}(T)$ be a finite (local) subcomplex. For a generic inner or outer oriented $k$-discrete conforming forms one has the following finite element expansion

$$
\begin{equation*}
\mu_{h, T}^{k}=\sum_{i=1}^{N_{T}^{k}} \phi_{i, T}^{k}(\xi) \mu_{i, T}^{k}, \quad \mu_{h, T}^{k} \in W_{h}^{k}(T) \tag{A.1}
\end{equation*}
$$

where $N_{T}^{k}$ is the number of degrees of freedom for $W_{h}^{k}(T), \mu_{i, T}^{k} \in \mathbb{R}$ is the degree of freedom, and $\phi_{i, T}^{k}: M \rightarrow W_{h}^{k}(T) \subset H \Omega^{k}(T)$ is a section of $W_{h}^{k}(T)$, corresponding to a finite element basis function.

Inner product. Given two discrete local forms $\left(\nu_{h, T}^{k}, \mu_{h, T}^{k}\right) \in W_{h, T}^{k}$, the inner product reads

$$
\begin{equation*}
\left(\nu_{h, T}^{k}, \mu_{h, T}^{k}\right)_{T}=\left(\boldsymbol{\nu}_{T}^{k}\right)^{\top} \mathbf{M}_{T}^{k} \boldsymbol{\mu}_{T}^{k} \tag{A.2}
\end{equation*}
$$

where $\boldsymbol{\nu}_{T}^{k}, \boldsymbol{\mu}_{T}^{k} \in \mathbb{R}^{N_{T}^{k}}$ are the vectors collecting the degrees of freedom $\nu_{i, T}^{k}, \mu_{i, T}^{k}$ respectively and the mass matrix $\mathbf{M}_{T}^{k} \in \mathbb{R}^{N_{T}^{k} \times N_{T}^{k}}$ of order $k$ (symmetric and positive definite) is computed as

$$
\left[\mathbf{M}_{T}^{k}\right]_{i}^{j}=\left(\phi_{i, T}^{k}, \phi_{j, T}^{k}\right)_{T}
$$

Exterior derivative. The expression of the exterior derivative is here specialized for the inner product. Given a form $\nu_{h, T}^{k+1} \in W_{h}^{k+1}(T)$ and $\mu_{h, T}^{k} \in W_{h}^{k}(T)(k \leq n-1)$, the inner product of $\nu_{h, T}^{k+1}$ and $\mathrm{d} \mu_{h, T}^{k}$ is expressed by

$$
\begin{equation*}
\left(\nu_{h, T}^{k+1}, \mathrm{~d} \mu_{h, T}^{k}\right)_{T}=\left(\boldsymbol{\nu}_{T}^{k+1}\right)^{\top} \mathbf{D}_{T}^{k} \boldsymbol{\mu}_{T}^{k} \tag{A.3}
\end{equation*}
$$

where $\mathbf{D}_{T}^{k} \in \mathbb{R}^{N_{T}^{k+1} \times N_{T}^{k}}$ is computed as $\left[\mathbf{D}_{T}^{k}\right]_{i}^{j}=\left(\phi_{i, T}^{k+1}, \mathrm{~d} \phi_{j, T}^{k}\right)_{T}$.
Trace. For a discrete differential form of order $k \leq n-1$ the trace operator is also defined. When the trace of a local form is considered the simplices that do not lie on the boundary can be discarded in the expansion. Denoting the boundary of the simplicial complex on the cell by $\Delta_{j}(\partial T) \subset \Delta_{j}(T)$, the trace of a discrete $k$-form $\mu_{h, T} \in W_{h}^{k}(T)$ reads

$$
\begin{equation*}
\mu_{h, \partial T}^{k, \boldsymbol{t}}=\operatorname{tr} \mu_{h, T}^{k}=\sum_{i=1}^{N_{T}^{k}} \operatorname{tr}\left(\phi_{i, T}^{k}(\xi)\right) \mu_{i, T}^{k}=\sum_{l=1}^{N_{\partial T}^{k}} \psi_{l, \partial T}^{k, \boldsymbol{t}}(\xi) \mu_{l, \partial T}^{k, \boldsymbol{t}} \tag{A.4}
\end{equation*}
$$

where $N_{\partial T}^{k}=\sum_{j=k}^{n-1} \# \Delta_{j}(\partial T)$ is the number of degrees of freedom (with $k \leq j \leq n-1$ ) along the boundary for a polynomial differential form of order $k$. The degrees of freedom along the boundary $\mu_{l, T}^{k}$ are associated to $j$-simplices $\sigma_{j, \partial T}^{l}, k \leq j \leq n-1$ lying on the boundary. The trace matrix simply collects them

$$
\boldsymbol{\mu}_{\partial T}^{k, \boldsymbol{t}}=\mathbf{T}_{\partial T}^{k} \boldsymbol{\mu}_{T}^{k}, \quad\left[\mathbf{T}_{\partial T}^{k}\right]_{l}^{i}=\left\{\begin{array}{lll}
1, & \text { if } \sigma_{j, \partial T}^{l} \equiv \sigma_{j}^{i} \Longrightarrow \psi_{l, \partial T}^{k, \boldsymbol{t}}(\xi) \equiv \operatorname{tr}\left(\phi_{j, T}^{k}(\xi)\right) & \forall l=1, \ldots, N_{\partial T}^{k}  \tag{A.5}\\
0, & \text { otherwise }, & \forall i=1, \ldots, N_{T}^{k}
\end{array}\right.
$$

The trace can be defined analogously on every boundary subpartition $f \subset \partial T$.
Inner boundary pairing of facets elements. Facet elements are exactly does arising from the trace of local finite elements. The finite element expansion of $\mu_{h, \partial T}^{k, t} \in W_{h}^{k, t}(T)=\operatorname{tr} W_{h}^{k}(T)$ therefore reads

$$
\mu_{h, \partial T}^{k, \boldsymbol{t}}=\sum_{l=1}^{N_{\partial T}^{k}} \psi_{l, \partial T}^{k, \boldsymbol{t}}(\xi) \mu_{l, \partial T}^{k, \boldsymbol{t}}
$$

For them, the only operation allowed is the duality pairing over the facets. Following Remark 4 , since $W_{h}^{k, \boldsymbol{t}}(T) \in L^{2}(\partial T)$, the dual space $W_{h}^{k, \boldsymbol{n}}(T)$ is identified with $W_{h}^{k, \boldsymbol{t}}(T)$. So given $\nu_{h, \partial T}^{k, \boldsymbol{n}} \in$ $W_{h}^{k, \boldsymbol{n}}(T)=W_{h}^{k, \boldsymbol{t}}(T)$ its pairing with $\mu_{h, \partial T}^{k, \boldsymbol{t}}$ is the $L^{2}$ inner product over the boundary $\partial T$

$$
\left\langle\nu_{h, \partial T}^{k, \boldsymbol{n}}, \mu_{h, \partial T}^{k, \boldsymbol{t}}\right\rangle_{\partial T}=\left(\boldsymbol{\nu}_{\partial T}^{k, \boldsymbol{n}}\right)^{\top} \mathbf{M}_{\partial T}^{k} \boldsymbol{\mu}_{\partial T}^{k, \boldsymbol{t}}
$$

where the mass matrix at the boundary is computed as

$$
\left[\mathbf{M}_{\partial T}^{k}\right]_{i}^{j}=\left\langle\psi_{i, \partial T}^{k, \boldsymbol{t}}(\xi), \psi_{j, \partial T}^{k, \boldsymbol{t}}(\xi)\right\rangle_{\partial T}
$$

The mass matrix can be defined analogously on every boundary subpartition $f \subset \partial T$. If one of he two facet forms is obtained via the trace operator then

$$
\left\langle\nu_{h, \partial T}^{k, \boldsymbol{n}}, \operatorname{tr} \mu_{h, T}^{k}\right\rangle_{\partial T}=\left(\boldsymbol{\nu}_{\partial T}^{k, \boldsymbol{n}}\right)^{\top} \mathbf{M}_{\partial T}^{k} \mathbf{T}_{\partial T}^{k} \boldsymbol{\mu}_{T}^{k}
$$

Appendix A.2. Broken spaces of differential forms
Basis expression for broken forms. Since $W_{h}^{k}=\prod_{T \in \mathcal{T}} W_{h}^{k}(T)$, broken differential forms $\mu_{h}^{k} \in W_{h}^{k}$ are obtained by summing the contribution of all cells in the mesh

$$
\begin{equation*}
\mu_{h}^{k}=\sum_{T \in \mathcal{T}_{h}} \mu_{h, T}^{k}, \quad \mu_{h, T}^{k} \in W_{h}^{k}(T) \tag{A.6}
\end{equation*}
$$

All the local operation translates into block diagonal structures.

- Inner product: given two discrete broken forms $\left(\nu_{h}^{k}, \mu_{h}^{k}\right) \in W_{h}^{k}$, the inner product reads

$$
\left(\nu_{h}^{k}, \mu_{h}^{k}\right)_{\mathcal{T}_{h}}=\left(\boldsymbol{\nu}^{k}\right)^{\top} \mathbf{M}_{\mathcal{T}_{h}}^{k} \boldsymbol{\mu}^{k}
$$

where $\boldsymbol{\nu}^{k}, \boldsymbol{\mu}^{k} \in \mathbb{R}^{N_{W}^{k}}$ with $N_{W}^{k}=\sum_{T \in \mathcal{T}_{h}} N_{T}^{k}$ are the vectors collecting the degrees of freedom $\mu_{i}^{k}, \mu_{i}^{k}$ respectively. The mass matrix is given by the block diagonal matrix

$$
\mathbf{M}_{\mathcal{T}_{h}}^{k}=\operatorname{Diag}_{T \in \mathcal{T}_{h}}\left(\mathbf{M}_{T}^{k}\right)
$$

- Exterior derivative: given broken forms $\nu_{h}^{k+1} \in W_{h}^{k+1}$ and $\mu_{h, T}^{k} \in W_{h}^{k}(k \leq n-1)$, the inner product of $\nu_{h}^{k+1}$ and $\mathrm{d} \mu_{h}^{k}$ is expressed by

$$
\left(\nu_{h}^{k+1}, \mathrm{~d} \mu_{h}^{k}\right)_{\mathcal{T}_{h}}=\left(\boldsymbol{\nu}^{k+1}\right)^{\top} \mathbf{D}_{\mathcal{T}_{h}}^{k} \boldsymbol{\mu}^{k}
$$

where $\mathbf{D}_{\mathcal{T}_{h}}^{k}$ is again block diagonal $\mathbf{D}_{\mathcal{T}_{h}}^{k}=\operatorname{Diag}_{T \in \mathcal{T}_{h}}\left(\mathbf{D}_{T}^{k}\right)$.

- Trace: the trace of a broken form $\mu_{h, T}^{k} \in W_{h}^{k}(k \leq n-1)$ over $\partial \mathcal{T}_{h}$

$$
\operatorname{tr} \mu_{h}^{k}=\operatorname{tr} \sum_{T \in \mathcal{T}_{h}} \mu_{h, T}^{k}=\sum_{\partial T \in \partial \mathcal{T}_{h}} \mu_{h, \partial T}^{k, \boldsymbol{t}}=\sum_{\partial T \in \partial \mathcal{T}_{h}} \sum_{l=1}^{N_{\partial T}^{k}} \psi_{l, \partial T}^{k}(\xi) \mu_{l, \partial T}^{k, \boldsymbol{t}}
$$

gives rises to a block diagonal operator

$$
\boldsymbol{\mu}^{k, \boldsymbol{t}}=\mathbf{T}_{\partial \mathcal{T}_{h}}^{k} \boldsymbol{\mu}^{k}, \quad \mathbf{T}_{\partial \mathcal{T}_{h}}^{k}=\operatorname{Diag}_{\partial T \in \partial \mathcal{T}_{h}}\left(\mathbf{T}_{\partial T}^{k}\right)
$$

Inner boundary pairing of broken facets elements. Broken facet forms $\mu_{h}^{k, \boldsymbol{t}} \in W_{h}^{k, t}$ are expressed in a basis summing the contribution of each cell boundary

$$
\mu_{h}^{k, \boldsymbol{t}}=\sum_{\partial T \in \partial \mathcal{T}_{h}} \mu_{h, \partial T}^{k, \boldsymbol{t}}, \quad \mu_{h, \partial T}^{k, \boldsymbol{t}} \in W_{h}^{k, \boldsymbol{t}}(T)
$$

Broken facet forms can be paired together, giving rise to a block diagonal mass facets matrix

$$
\left\langle\nu_{h}^{k, \boldsymbol{n}}, \mu_{h}^{k, \boldsymbol{t}}\right\rangle_{\partial \mathcal{T}_{h}}=\left(\boldsymbol{\nu}^{k, \boldsymbol{n}}\right)^{\top} \mathbf{M}_{\partial \mathcal{T}_{h}}^{k} \boldsymbol{\mu}^{k, \boldsymbol{t}}
$$

where the mass matrix at the boundary is computed as

$$
\mathbf{M}_{\partial \mathcal{T}_{h}}^{k}=\operatorname{Diag}_{\partial T \in \partial \mathcal{T}_{h}} \mathbf{M}_{\partial T}
$$

If one of he two facet forms is obtained via the trace, then

$$
\left\langle\nu_{h}^{k, \boldsymbol{n}}, \operatorname{tr} \mu_{h}^{k}\right\rangle_{\partial \mathcal{T}_{h}}=\left(\boldsymbol{\nu}^{k, \boldsymbol{n}}\right)^{\top} \mathbf{M}_{\partial \mathcal{T}_{h}}^{k} \mathbf{T}_{\partial \mathcal{T}_{h}}^{k} \boldsymbol{\mu}^{k} .
$$

## Appendix A.3. Duality pairing of broken and unbroken facets forms

Unbroken discrete forms over the facets are obtained by making the broken version single valued, i.e. $V_{h}^{k, t}=W_{h}^{k, t} \cap V^{k}$. When an unbroken facet form $v_{h}^{k, t} \in V_{h}^{k, t}$ is paired with a broken one $\mu_{h}^{k, n} \in W_{h}^{k, n}$, the integral over the disjoint union $\partial T_{h}$ can be converted to an integral over the set of facets $\mathcal{F}_{h}$ by introducing the jump operator. As anticipated at the beginning of the appendix, inner and outer oriented forms behave differently under interconnection. When duality pairing of broken and unbroken forms is considering a jump operators appears. This jump operator sums up outer oriented forms (the sum should be zero as the normal changes sign from one element to another) and takes the difference of inner oriented forms. The normal trace of an inner oriented form is outer oriented form and vice-versa.

Normal trace of inner oriented forms. The jump operator on the normal trace of inner oriented forms sums them

$$
\begin{aligned}
& \llbracket \cdot \rrbracket: W_{h}^{k, n} \longrightarrow V_{h}^{k, n}, \\
& \llbracket \mu_{h}^{k, n} \rrbracket= \begin{cases}\mu_{h, \partial T^{+}}^{k, n}+\mu_{h, \partial T^{-}}^{k, n}, & \text { on the face } f \in \partial T^{+} \cap \partial T^{-}, \\
\mu_{h, \partial T}^{k, n}, & \text { on the face } f \in \partial T \cap \partial M .\end{cases}
\end{aligned}
$$

Consequently the duality pairing of $v_{h}^{k, t} \in V_{h}^{k, t}$ and $\mu_{h}^{k, n} \in W_{h}^{k, n}=W_{h}^{k, t}$ over $\partial \mathcal{T}_{h}$ reads

$$
\left\langle v_{h}^{k, t}, \mu_{h}^{k, n}\right\rangle_{\partial \tau_{h}}=\left\langle v_{h}^{k, t}, \llbracket \mu_{h}^{k, n} \rrbracket\right\rangle_{\mathcal{F}_{h}} .
$$

Computing the expression in a basis gives the following matrix expression

$$
\left\langle v_{h}^{k, t}, \llbracket \mu_{h}^{k, n} \rrbracket\right\rangle_{\mathcal{F}_{h}}=\mathbf{v}^{k, t} \mathbf{M}_{\mathcal{F}_{h}}^{k} \boldsymbol{\Xi}_{\mathcal{F}_{h}}^{k} \boldsymbol{\mu}^{k, n} .
$$

where $\boldsymbol{\Xi}_{\mathcal{F}_{h}}^{k}$ is the matrix expression of the jump operator.
Normal trace of outer oriented forms. The jump operator on the normal trace of outer oriented forms takes their difference

$$
\begin{aligned}
& \llbracket \cdot \rrbracket: \widehat{W}_{h}^{k, n} \longrightarrow \widehat{V}_{h}^{k, n}, \\
& \llbracket \widehat{\mu}_{h}^{k, n} \rrbracket= \begin{cases}\widehat{\mu}_{h, n}^{k, n}-\widehat{\mu}_{h, \partial T^{-}}^{k, n}, & \text { on the face } f \in \partial T^{+} \cap \partial T^{-}, \\
\widehat{\mu}_{h, \partial T}^{k, n}, & \text { on the face } f \in \partial T \cap \partial M .\end{cases}
\end{aligned}
$$

Consequently the duality pairing of $\widehat{v}_{h}^{k, t} \in \widehat{V}_{h}^{k, t}$ and $\widehat{\mu}_{h}^{k, n} \in \widehat{W}_{h}^{k, n}=\widehat{W}_{h}^{k, t}$ over $\partial \mathcal{T}_{h}$ reads

$$
\left\langle\widehat{v}_{h}^{k, t}, \widehat{\mu}_{h}^{k, n}\right\rangle_{\partial \tau_{h}}=\left\langle\widehat{v}_{h}^{k, t}, \llbracket \hat{\mu}_{h}^{k, n} \rrbracket\right\rangle_{\mathcal{F}_{h}} .
$$

Computing the expression in a basis gives the following matrix expression

$$
\left\langle\widehat{v}_{h}^{k, t}, \llbracket \hat{\mu}_{h}^{k, n} \rrbracket\right\rangle_{\mathcal{F}_{h}}=\widehat{\mathbf{v}}^{k, t} \mathbf{M}_{\mathcal{F}_{h}}^{k} \hat{\boldsymbol{\Xi}}_{\mathcal{F}_{h}}^{k} \hat{\boldsymbol{\mu}}^{k, \boldsymbol{n}} .
$$

where $\boldsymbol{\Xi}_{\mathcal{F}_{h}}^{k}$ is the matrix expression of the jump operator. The jump operator may be restricted to any facets set.

## Appendix A.4. Duality pairing of inner and outer oriented forms over the domain boundary

An unbroken facet form $v_{h}^{k, t} \in V_{h}^{k, t}$ can be restricted to the boundary of the domain $\partial M$ by taking its trace

$$
v_{h, \partial}^{k, t}=\operatorname{tr} v_{h}^{k, t}, \quad \text { or algebraically } \quad \mathbf{v}_{\partial}^{k, t}=\mathbf{T}_{\partial M}^{k, t} \mathbf{v}^{k, t} .
$$

Given an unbroken facet form $v_{h}^{k, t} \in V_{h}^{k, t}$ and an outer oriented control variables defined on the domain boundary $\left.\widehat{u}_{h}^{n-k-1} \in \operatorname{tr} \widehat{V}_{h}^{n-k-1}\right|_{\partial M}$ their duality pairing over $\partial M$ gives the following algebraic expression

$$
\left\langle\operatorname{tr} v_{h}^{k, \boldsymbol{t}} \mid \widehat{u}_{h}^{n-k-1}\right\rangle_{\partial M}=\mathbf{v}^{k, t}\left(\mathbf{T}_{\partial M}^{k, t}\right)^{\top} \boldsymbol{\Psi}_{\partial M}^{n-k-1} \widehat{\mathbf{u}}^{n-k-1} .
$$

Analogously, an outer oriented from can be paired with an outer oriented control. Furthermore the boundary integral can be restricted to the subpartitions $\Gamma_{1}, \Gamma_{2}$.


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[^1]:    ${ }^{1}$ The fractional Sobolev space $H^{-1 / 2} \Omega^{k}(\partial M)$ arise from the fact that the trace operator on $H^{1} \Omega^{k}(M)$ (the space of $k$-forms whose coefficients are in $H^{1}(M)$ ) extends to a bounded linear operator $\operatorname{tr}: H \Omega^{k}(M) \rightarrow H^{-1 / 2} \Omega^{k}(\partial M)$ [21, Theorem 6.3].
    ${ }^{2}$ It is important to underline the difference between $\langle\cdot \mid \cdot\rangle_{\partial M}$ and $\langle\cdot, \cdot\rangle_{\partial M}$. The first is given by the duality of forms and does not need to be symmetric. The second is the extension of the $L^{2}$ inner product on the boundary to distributional spaces and it is symmetric.

