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Scalable global state synchronization of discrete-time double integrator multi-agent systems with input saturation via linear protocol

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Abstract: This paper studies scalable global state synchronization of discrete-time double integrator multi-agent systems in presence of input saturation based on localized information exchange. A *scale-free* collaborative linear dynamic protocols design methodology is developed for discrete-time multi-agent systems with both full and partial-state couplings. And the protocol design methodology does not need any knowledge of the directed network topology and the spectrum of the associated Laplacian matrix. Meanwhile, the protocols are parametric based on a parameter set in which the designed protocols can guarantee the global synchronization result. Furthermore, the proposed protocol is scalable and achieves synchronization for any arbitrary number of agents.

Key Words: Discrete-time double integrator multi-agent systems, Global state synchronization, Scale-free linear protocol

1 Introduction

In recent years, the synchronization or consensus problem of multi-agent system (MAS) has attracted much more attention, due to its wide potential for applications in several areas such as automotive vehicle control, satellites/robots formation, sensor networks, and so on. See for instance the books [1, 2, 11, 23, 27, 28, 39] and references therein.

At present, most work in synchronization for MAS focused on state synchronization of continuous-time and discrete-time homogeneous networks. State synchronization based on diffusive *full-state coupling* (it means that all states are communicated over the network) has been studied where the agent dynamics progress from single- and double-integrator (e.g. [6, 9, 12, 24, 25, 26, 35]) to more general dynamics (e.g. [34, 38, 41]). State synchronization based on diffusive *partial-state coupling* (i.e., only part of the states are communicated over the network) has also been considered, including static design ([3, 20, 21]), dynamic design ([10, 18, 30, 33, 36, 37]), and the design with additional communication ([4, 14, 29]).

On the other hand, it is worth to note that actuator saturation is pretty common and indeed is ubiquitous in engineering applications. Some researchers have tried to establish (semi) global state and output synchronization results for both continuous- and discrete-time MAS in the presence of input saturation. From the existing literature for a linear system subject to actuator saturation, we have the following conclusion [28]:

1) A linear protocol is used if we consider synchronization in the semi-global framework (i.e. initial conditions of agents are in a priori given compact set).
2) Synchronization in the global sense (i.e., when initial conditions of agents are anywhere) in general requires a nonlinear protocol.
3) Synchronization in the presence of actuator saturation requires eigenvalues of agents to be in the closed left half plane for continuous-time systems and in the closed unit disc for discrete-time systems, that is the agents are at most weakly unstable.

The semi-global synchronization has been studied in [32] via full-state coupling. For partial state coupling, we have [31, 42] which are based on the extra communication. Meanwhile, the result without the extra communication is developed in [43]. Then, the static controllers via partial state coupling is designed in [17] by passifying the original agent model.

On the other hand, global synchronization for full-state coupling has been studied by [22] (continuous-time) and [40] (discrete-time) for neutrally stable and double-integrator agents. The global framework has only been studied for static protocols under the assumption that the agents are neutrally stable and the network is detailed balanced or undirected. Partial-state coupling has been studied in [5] using an adaptive approach but the observer requires extra communication. The result dealing with networks that are not detailed balanced are based on [13] which intrinsically requires the agents to be single integrators. Recently, we introduce a scale-free linear collaborative protocols for global regulated state synchronization of continuous and discrete-time homogeneous MAS, see [19] and [16]. This *scale-free* protocol means the design is independent of the information about the associated communication graph or the size of the network, i.e., the number of agents.

In this paper, we focus on scalable linear protocol design for global state synchronization of discrete-time double-integrator MAS in presence of input saturation. The contributions of this paper are stated as follows:

- A class of parametric linear protocol is established based on a parameter set in which the designed parametric protocol makes all states of MAS synchronized.
- Meanwhile, the linear protocol design is scale-free and do not need any information about communication network. In other words, the proposed protocols work for any MAS with any communication graph with arbitrary number of agents as long as the communication graph has a path among each agent.

Notations and definitions

Given a matrix $A \in \mathbb{R}^{m\times n}$, A^{T} denotes its conjugate transpose and $\|A\|$ is the induced 2-norm. A square matrix A is said to be Schur stable if all its eigenvalues are in the closed unit disk. $A\otimes B$ depicts the Kronecker product between A and B. I_n denotes the n-dimensional identity matrix and 0_n denotes $n\times n$ zero matrix; sometimes we drop the subscript if the dimension is clear from the context. A matrix $D = [d_{ij}]_{N\times N}$ is called a row stochastic matrix if (a) $d_{ij} > 0$ for any i, j and (b) $\sum_j^N d_{ij} = 1$ for $i = 1, \cdots, N$. A row stochastic matrix D has at least one eigenvalue at 1 with right eigenvector $\mathbf{1}$.

A *weighted graph* $\mathcal{G}$ is defined by a triple $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, \dots, N\}$ is a node set, $\mathcal{E}$ is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the weighting matrix. Each pair in $\mathcal{E}$ is called an *edge*, where $a_{ij} > 0$ denotes an edge $(j, i) \in \mathcal{E}$ from node j to node i with weight a_{ij}. Moreover, $a_{ij} = 0$ if there is no edge from node j to node i. We assume there are no self-loops, i.e. we have $a_{ii} = 0$. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1, \dots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, \dots, k-1$. A *directed tree* with root r is a subgraph of the graph $\mathcal{G}$ in which there exists a unique path from node r to each node in this subgraph. A *directed spanning tree* is a directed tree containing all the nodes of the graph. A directed graph may contain many directed spanning trees, and thus there may be several choices for the root agent. The set of all possible root agents for a graph $\mathcal{G}$ is denoted by π_g.

The *weighted in-degree* of node i is given by $d_{\text{in}}(i) = \sum_{j=1}^N a_{ij}$. For a weighted graph $\mathcal{G}$, the matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^N a_{ik}, & i = j, \\ -a_{ij}, & i \neq j, \end{cases}$$

is called the *Laplacian matrix* associated with the graph $\mathcal{G}$. The Laplacian matrix L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $\mathbf{1}$ [7].

2 Problem formulation

Consider a MAS consisting of N identical discrete-time double integrator with input saturation:

$$\begin{cases} x_i(k+1) = Ax_i(k) + B\sigma(u_i(k)), \\ y_i(k) = Cx_i(k) \end{cases} \tag{1}$$

where $x_i(k) \in \mathbb{R}^{2n}$, $y_i(k) \in \mathbb{R}^n$ and $u_i(k) \in \mathbb{R}^n$ are the state, output, and the input of agent $i = 1, \dots, N$, respectively. And

$$A = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}, B = \begin{pmatrix} 0 \\ I \end{pmatrix}, C = \begin{pmatrix} I & 0 \end{pmatrix}$$

Meanwhile,

$$\sigma(v) = \begin{pmatrix} \text{sat}(v_1) \\ \text{sat}(v_2) \\ \vdots \\ \text{sat}(v_m) \end{pmatrix} \quad \text{where} \quad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \in \mathbb{R}^m$$

with $\text{sat}(w)$ is the standard saturation function,

$$\text{sat}(w) = \text{sgn}(w) \min(1, |w|).$$

The network provides agent i with the following information,

$$\zeta_i(k) = \sum_{j=1}^N a_{ij}(y_i(k) - y_j(k)), \tag{2}$$

where $a_{ij} \geqslant 0$ and $a_{ii} = 0$. This communication topology of the network can be described by a weighted graph $\mathcal{G}$ associated with (2), with the a_{ij} being the coefficients of the weighting matrix $\mathcal{A}$. In terms of the coefficients of the associated Laplacian matrix L, ζ_i can be rewritten as

$$\zeta_i(k) = \sum_{j=1}^N \ell_{ij} y_j(k). \tag{3}$$

We refer to this as *partial-state coupling* since only part of the states are communicated over the network. When $C = I$, it means all states are communicated over the network, we call it *full-state coupling*. Then, the original agents are expressed as

$$x_i(k+1) = Ax_i(k) + B\sigma(u_i(k)) \tag{4}$$

and ζ_i is rewritten as

$$\zeta_i(k) = \sum_{j=1}^{N} \ell_{ij} x_j(k). \tag{5}$$

We need the following definition to explicitly state our problem formulation.

Definition 1 *We define the following set. $\mathbb{G}^N$ denotes the set of directed graphs of N agents which contains a directed spanning tree. Moreover, for any $\mathcal{G} \in \mathbb{G}^N$, we denote the root set of the $\mathcal{G}$ by π_g.*

Remark 1 *When the undirected or strongly connected graph is considered, it is obvious that the set π_g will include all nodes of networks.*

We consider the **state synchronization** problem under the graph set $\mathbb{G}^N$ satisfying Definition 1. Here, its objective is that the agents achieve state synchronization, that is

$$\lim_{k\to\infty} (x_i(k) - x_j(k)) = 0. \tag{6}$$

for all $i, j \in 1, ..., N$.

Meanwhile, we introduce an additional information exchange among each agent and its neighbors. In particular, each agent $i = 1, \ldots, N$ has access to additional information, denoted by $\hat{\zeta}_i$, of the form

$$\hat{\zeta}_i(k) = \sum_{j=1}^{N} a_{ij}(\xi_i(k) - \xi_j(k)) \tag{7}$$

where $\xi_j \in \mathbb{R}^n$ is a variable produced internally by agent j and to be defined in next sections.

Then, we formulate the problem for global state synchronization of a MAS via linear protocols based on additional information exchange (7).

Problem 1 *Consider a MAS described by* (1) *and* (2). *Let the set $\mathbb{G}^N$ denote all graphs satisfy Definition 1.*

*The **scalable global state synchronization problem with additional information exchange via linear dynamic protocol** is to find a linear dynamic protocol, using only the knowledge of agent model (A, B, C), of the form*

$$\begin{cases} x_{c,i}(k+1) = A_{c,i}x_{c,i}(k) + B_{c,i}\sigma(u_i(k)) \\ \qquad\qquad\qquad +C_{c,i}\zeta_i(k) + D_{c,i}\hat{\zeta}_i(k), \\ u_i(k) = \quad K_{c,i}x_{c,i}(k) \end{cases} \tag{8}$$

where $\hat{\zeta}_i$ is defined in (7) *with $\xi_i = H_{c,i}x_{c,i}$, and $x_{c,i} \in \mathbb{R}^{n_c}$, such that state synchronization* (6) *is achieved for any N and any graph $\mathcal{G} \in \mathbb{G}^N$, and for all initial conditions of the agents $x_i(0) \in \mathbb{R}^n$, and all initial conditions of the protocols $x_{c,i}(0) \in \mathbb{R}^{n_c}$.*

3 Protocol design

3.1 Full-state coupling

Let $\mathcal{G}$ be any graph belongs to $\mathbb{G}^N$, and we choose agent θ where θ is any node in the root set π_g. Then, we propose the following protocol.

Linear Protocol 1: Full-state coupling

$$\begin{cases} \chi_i(k+1) = A\chi_i(k) + B\sigma(u_i(k)) \\ \qquad\qquad\qquad +\frac{1}{1+D_{in}(i)}\left[A\zeta_i(k) - A\hat{\zeta}_i(k)\right] \\ u_i(k) \quad = K\chi_i(k), \quad i = \{1,\ldots,N\}\setminus\theta \\ u_\theta(k) \quad \equiv 0, \end{cases} \tag{9}$$

where $D_{\rm in}(i)$ is the upper bound of $d_{in}(i) = \sum_{j=1}^{N} a_{ij}$. Then, we still choose matrix $K = -\begin{pmatrix} k_1 I & k_2 I\end{pmatrix}$, where $k_1 \in (0,1)$ and $k_2 > 0$ satisfy the following condition

$$(1 + k_1 - k_2)^2 < 1 - k_1. \tag{10}$$

$\hat{\zeta}_i(k)$ and $\zeta_i(k)$ are defined by (7) and (2), respectively. And the agents communicate $\xi_i(k)$ which is chosen as $\xi_i(k) = \chi_i(k)$.

The condition (10) can be shown as Fig. 1, where the zone encircled by parabola and line $(0,0)$ to $(0,2)$.

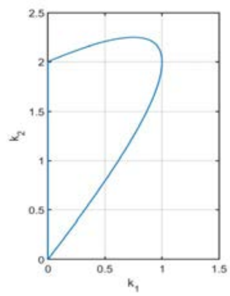

Figure 1: Solvable zone of k_1, k_2 for synchronization

Theorem 1 *Consider a MAS described by* (4) *and* (5). *Let the set $\mathbb{G}^N$ denote all graphs satisfy Definition 1.*

Then, the scalable global state synchronization problem with additional information exchange as stated in Problem 1 is solvable. In particular, for any given $k_1 \in (0,1)$ and $k_2 > 0$ satisfying (10)*, the linear dynamic protocol* (9) *solves the global state synchronization problem for any N and any graph $\mathcal{G} \in \mathbb{G}^N$.*

To obtain this theorem we need the following lemma.

Lemma 1 *For all $u, v \in \mathbb{R}^n$, we have*

$$(\sigma(v) - \sigma(u))^{\rm T}(u - \sigma(u)) \leqslant 0. \tag{11}$$

Proof: Note that we have:

$$(\sigma(v) - \sigma(u))^{\rm T}(u - \sigma(u)) = \sum_{i=1}^{n}(\sigma(v_i) - \sigma(u_i))(u_i - \sigma(u_i)) \tag{12}$$

when $u = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix}^{\mathrm{T}}$ and $v = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}^{\mathrm{T}}$. Next note that if $u_i \geqslant 1$ we have $\sigma(v_i) - \sigma(u_i) = \sigma(v_i) - 1 \leqslant 0$ and $u_i - \sigma(u_i) = u_i - 1 \geqslant 0$ and hence:

$$(\sigma(v_i) - \sigma(u_i))(u_i - \sigma(u_i)) \leqslant 0 \tag{13}$$

On the other hand if $u_i \leqslant -1$ we have $\sigma(v_i) - \sigma(u_i) = \sigma(v_i) + 1 \geqslant 0$ and $u_i - \sigma(u_i) = u_i + 1 \leqslant 0$ and (13) is still satisfied. Finally, if $|u_i| \leqslant 1$ then $u_i - \sigma(u_i) = 0$ and (13) is also satisfied.

Since (13) is satisfied for all i and using (12) we find (11) holds for all u and v. ■

The proof of Theorem 1: Since we have $u_\theta(k) \equiv 0$, we obtain $\sigma(u_\theta(k)) = 0$. The model of agent θ is rewritten as $x_\theta(k+1) = Ax_\theta(k)$.

Then, let $\bar{x}(k) = x_i(k) - x_\theta(k)$, we have

$$\begin{cases} \bar{x}_i(k+1) = A\bar{x}_i(k) + B\sigma(u_i(k)) \\ \chi_i(k+1) = A\chi_i(k) + B\sigma(u_i(k)) \\ \qquad + \frac{1}{1+D_{in}(i)} \sum_{j=1}^{N-1} \ell_{ij} A\left[\bar{x}_i(k) - \chi_i(k)\right] \\ u_i(k) = -\begin{pmatrix} k_1 I & k_2 I \end{pmatrix} \chi_i(k) \end{cases} \tag{14}$$

for $i = \{1, \ldots, N\} \setminus \theta$. Then by defining $2(N-1)n$-dimensional vectors

$$\bar{x}(k) = \begin{pmatrix} \bar{x}_1(k) \\ \vdots \\ \bar{x}_{N-1}(k) \end{pmatrix}, \chi(k) = \begin{pmatrix} \chi_1(k) \\ \vdots \\ \chi_N(k) \end{pmatrix},$$

$$u(k) = \begin{pmatrix} u_1(k) \\ \vdots \\ u_N(k) \end{pmatrix}, \sigma(u(k)) = \begin{pmatrix} \sigma(u_1(k)) \\ \vdots \\ \sigma(u_N(k)) \end{pmatrix}$$

where $\chi_\theta(k)$, $u_\theta(k)$, and $\sigma(u_\theta(k))$ are not included. We have the following closed-loop system

$$\begin{cases} \bar{x}(k+1) = (I_{N-1} \otimes A)\bar{x}(k) + (I_{N-1} \otimes B)\sigma(u(k)) \\ \chi(k+1) = (I_{N-1} \otimes A)\chi(k) + (I_{N-1} \otimes B)\sigma(u(k)) \\ \qquad + ((I_{N-1} - \bar{D}) \otimes A)(\bar{x}(k) - \chi(k)) \\ u(k) = -(I_{N-1} \otimes \begin{pmatrix} k_1 I & k_2 I \end{pmatrix})\chi(k) \end{cases}$$

where $\bar{D} = I_{N-1} - (I_{N-1} + D_{d,\text{in}})^{-1}\hat{L}$, $D_{d,\text{in}} = \text{diag}\{D_{in}(1), D_{in}(2), \cdots, D_{in}(N)\}$ without $D_{in}(\theta)$, and $\hat{L}$ is the matrix obtained from L by deleting the θth row and the θth column. Meanwhile, according to [8, Lemma 1], we have the real part of all eigenvalues of $\hat{L}$ are greater than zero. Thus, it implies all eigenvalues' absolute value of $\bar{D} \in \mathbb{R}^{(N-1)\times(N-1)}$ are less than 1.

Let $e(k) = \bar{x}(k) - \chi(k)$, we have

$$\begin{cases} \bar{x}(k+1) = (I_{N-1} \otimes A)\bar{x}(k) + (I_{N-1} \otimes B)\sigma(u(k)) \\ e(k+1) = (\bar{D} \otimes A)e(k) \\ u(k) = -(I_{N-1} \otimes \begin{pmatrix} k_1 I & k_2 I \end{pmatrix})(\bar{x}(k) - e(k)) \end{cases} \tag{15}$$

Then, let $\bar{x}(k) = \begin{pmatrix} \bar{x}_1^{\mathrm{T}}(k) & \bar{x}_2^{\mathrm{T}}(k) \end{pmatrix}^{\mathrm{T}}$, we have

$$\begin{cases} \bar{x}_1(k+1) = \bar{x}_1(k) + \bar{x}_2(k) \\ \bar{x}_2(k+1) = \bar{x}_2(k) + \sigma(u(k)) \\ e(k+1) = (\bar{D} \otimes A)e(k) \\ u(k) = -k_1\bar{x}_1(k) - k_2\bar{x}_2(k) \\ \qquad + \left(I_{N-1} \otimes \begin{pmatrix} k_1 I & k_2 I \end{pmatrix}\right) e(k) \end{cases} \tag{16}$$

The eigenvalues of $\bar{D} \otimes A$ are of the form $\lambda_i\mu_j$, with λ_i and μ_j eigenvalues of $\bar{D}$ and A, respectively. Since $|\lambda_i| < 1$ and $\mu_j \equiv 1$, we find $\bar{D} \otimes A$ is asymptotically stable. Therefore we find that:

$$\lim_{k\to\infty} e_i(k) \to 0. \tag{17}$$

Thus, we just need to prove the stability of (16). Namely, we have $\bar{x}(k) \to 0$ as $k \to \infty$ with $e_i \to 0$, which will obtain the synchronization result.

To prove the synchronization result, we consider the following weighting Lyapunov function

$$V(k) = (1-h)V_1(k) + hV_2(k) \tag{18}$$

where $h \in (0,1)$, $V_2(k) = e^{\mathrm{T}}(k)P_D e(k)$, $P_D > 0$ satisfies

$$(\bar{D} \otimes A)^{\mathrm{T}} P_D (\bar{D} \otimes A) - P_D \leqslant -2I_{2(N-1)n}, \tag{19}$$

$$V_1(k) = \begin{pmatrix} \sigma(u(k)) \\ \bar{x}_2(k) \end{pmatrix}^{\mathrm{T}} \left[\begin{pmatrix} 1 & k_1 \\ k_1 & k_1 \end{pmatrix} \otimes I_{(N-1)n}\right] \begin{pmatrix} \sigma(u(k)) \\ \bar{x}_2(k) \end{pmatrix} + 2\sigma(u(k))^{\mathrm{T}}(u(k) - \sigma(u(k)))$$

Here, we obtain $V_1(k)$ and $V_2(k)$ are positive, i.e. $V_1(k) > 0$ except for $(u(k), \bar{x}_2(k)) = 0$ when $V_1(k) = 0$ and $V_2(k) > 0$ except for $e(k) = 0$ when $V_2(k) = 0$. Then, we have

$$\begin{aligned} \Delta V_1(k) =& V_1(k+1) - V_1(k) \\ \leqslant& 2(1 + k_1 - k_2)\sigma(u(k+1))^{\mathrm{T}}\sigma(u(k)) \\ &- \sigma(u(k+1))^{\mathrm{T}}\sigma(u(k+1)) \\ &- (1 - k_1)\sigma(u(k))^{\mathrm{T}}\sigma(u(k)) \\ &+ 2\sigma(u(k+1))^{\mathrm{T}}(I_{N-1} \otimes \begin{pmatrix} k_1 I & k_2 I \end{pmatrix}\Psi)e(k) \end{aligned}$$

since $(\sigma(u(k+1)) - \sigma(u(k)))^{\mathrm{T}}(u(k) - \sigma(u(k))) \leqslant 0$ based on Lemma 1, where $\Psi = \bar{D} \otimes A - I_{2(N-1)n}$. Meanwhile, for $V_2(k)$ we have

$$\Delta V_2(k) = V_2(k+1) - V_2(k) \leqslant -2e^{\mathrm{T}}(k)e(k)$$

based on condition (19). Thus, one can obtain

$$\begin{aligned} &\Delta V(k) \leqslant (1-h)\Delta V_1(k) + h\Delta V_2(k) \\ &\leqslant (1-h) \begin{pmatrix} \sigma(u(k+1)) \\ \sigma(u(k)) \end{pmatrix}^{\mathrm{T}} (\Phi \otimes I_{(N-1)n}) \begin{pmatrix} \sigma(u(k+1)) \\ \sigma(u(k)) \end{pmatrix} \\ &\quad - h\|e(k)\|^2 \end{aligned}$$

where $\Phi = \begin{pmatrix} -1 + \frac{\|\Psi\|^2(1-h)(k_1^2+k_2^2)}{h} & 1 + k_1 - k_2 \\ 1 + k_1 - k_2 & -(1-k_1) \end{pmatrix}$.

Obviously we just need to prove $\Phi < 0$. Without loss of generality, there exists an $\varepsilon > 0$ such that

$$(1 + k_1 - k_2)^2 = (1-\varepsilon)(1-k_1). \tag{20}$$

By using Schur Compliment, we have $\Phi < 0$ is equivalent to $-1 + \frac{\|\Psi\|^2(1-h)(k_1^2+k_2^2)}{h} + \frac{(1+k_1-k_2)^2}{1-k_1} < 0$. From condition (20), we can obtain $-1 + \frac{\|\Psi\|^2(1-h)(k_1^2+k_2^2)}{h} + \frac{(1+k_1-k_2)^2}{1-k_1} < \frac{\|\Psi\|^2(1-h)(k_1^2+k_2^2)}{h} - \varepsilon$. For h sufficiently close to 1, one can obtain $\frac{\|\Psi\|^2(1-h)(k_1^2+k_2^2)}{h} < \varepsilon$. It means that we obtain $\Phi < 0$.

Thus, we have $\Delta V(k) < 0$ for $\begin{pmatrix}\sigma(u(k+1))\\ \sigma(u(k))\end{pmatrix} \neq 0$, $\bar{x}(k) \to 0$ as $k \to \infty$.

Furthermore, when $\Delta V(k) = 0$, we obtain $u(k+1) = u(k) = 0$ and $e(k) = 0$. It is easy to obtain $\bar{x}_1(k) = \bar{x}_2(k) = 0$ at $\Delta V(k) = 0$.

Thus, the invariance set $\{(\bar{x}(k), e(k)) : \Delta V(\bar{x}(k), e(k)) = 0\}$ contains no trajectory of the system except the trivial trajectory $(\bar{x}(k), e(k)) = (0, 0)$. (15) is globally asymptotically stable based on LaSalle's invariance principle. Finally, we obtain the global state synchronization result. ∎

3.2 Partial-state coupling

Let $\mathcal{G}$ be any graph belongs to $\mathbb{G}^N$, and also we choose agent θ where θ is any node in the root set π_g. Then, we propose the following linear protocol.

Linear protocol 2: Partial-state coupling

$$\begin{cases} \hat{x}_i(k+1) = (A - FC)\hat{x}_i(k) \\ \qquad\qquad + \frac{1}{1+D_{in}(i)}\left[B\hat{\zeta}_{i2}(k) + F\zeta_i(k)\right] \\ \chi_i(k+1) = A\chi_i(k) + Bu_i(k) \\ \qquad\qquad + A\hat{x}_i(k) - \frac{1}{1+D_{in}(i)}A\hat{\zeta}_{i1}(k) \\ u_i(k) \quad = K\chi_i(k), \quad i = \{1, \ldots, N\} \setminus \theta \\ u_\theta(k) \quad \equiv 0, \end{cases} \tag{21}$$

where $D_{\text{in}}(i)$ is the upper bound of $d_{in}(i) = \sum_{j=1}^N a_{ij}$. Then, we choose matrix $K = -\begin{pmatrix}k_1 I & k_2 I\end{pmatrix}$, where $k_1 \in (0,1), k_2 > 0$ satisfy condition (10). In this protocol, the agents communicate $\xi_i(k) = \begin{pmatrix}\xi_{i1}^{\mathrm{T}}(k) & \xi_{i2}^{\mathrm{T}}(k)\end{pmatrix}^{\mathrm{T}} = \begin{pmatrix}\chi_i^{\mathrm{T}}(k) & \sigma(u_i(k))^{\mathrm{T}}\end{pmatrix}^{\mathrm{T}}$, i.e. each agent has access to additional information $\hat{\zeta}_i(k) = \begin{pmatrix}\hat{\zeta}_{i1}^{\mathrm{T}}(k) & \hat{\zeta}_{i2}^{\mathrm{T}}(k)\end{pmatrix}$, where $\begin{cases}\hat{\zeta}_{i1}(k) = \sum_{j=1}^N a_{ij}(\chi_i(k) - \chi_j(k)) \\ \hat{\zeta}_{i2}(k) = \sum_{j=1}^N a_{ij}(\sigma(u_i(k)) - \sigma(u_j(k)))\end{cases}$, while $\bar{\zeta}_i(k)$ is defined via (2).

Theorem 2 *Consider a MAS described by* (1) *and* (2)*. Let the set* $\mathbb{G}^N$ *denote all graphs satisfy Definition 1.*

Then, the scalable global state synchronization problem with additional information exchange as stated in Problem 1 is solvable. In particular, for any given $k_1 \in (0,1)$ *and* $k_2 > 0$ *satisfying* (10)*, the linear dynamic protocol* (21) *solves the global state synchronization problem for any N and any graph* $\mathcal{G} \in \mathbb{G}^N$.

The proof of Theorem 2: Similar to Theorem 1, by defining $\bar{x}_i(k) = x_i(k) - x_\theta(k)$, $e(k) = \bar{x}(k) - \chi(k)$, and $\bar{e}(k) = [(I_{N-1} - \bar{D}) \otimes I]\bar{x}(k) - \hat{x}(k)$, we have the matrix expression of closed-loop system

$$\begin{cases} \bar{x}_1(k+1) = \bar{x}_1(k) + \bar{x}_2(k) \\ \bar{x}_2(k+1) = \bar{x}_2(k) + \sigma(u(k)) \\ e(k+1) = (\bar{D} \otimes A)e(k) + \bar{e}(k) \\ \bar{e}(k+1) = [I_{N-1} \otimes (A - FC)]\bar{e}(k) \\ u(k) = -\left(I_{N-1} \otimes \begin{pmatrix}k_1 I & k_2 I\end{pmatrix}\right)\chi(k) \end{cases}$$

Since the eigenvalues of $A - FC$ and $\bar{D} \otimes A$ are in open unit disk, we just need to prove the stability of $\bar{x}_1(k)$ and $\bar{x}_2(k)$. Similar to the proof of Theorem 1, the state synchronization result can be obtained. ∎

Since the space limitation, we put the part of numerical example to the completed version, see [15].

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