# Periodic Center Manifolds and Normal Forms for DDEs in the Light of Suns and Stars 

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#### Abstract

Bifurcation theory has been very successful in the study of qualitative changes in nonlinear dynamical systems. An important tool of this theory is the existence of a center manifold near nonhyperbolic equilibria and limit cycles or homoclinic orbits. The existence has already been proven for many kinds of different systems, but not fully for limit cycles in delay differential equations (DDEs). In this paper, we prove the existence of a smooth finite-dimensional periodic center manifold near a nonhyperbolic cycle in DDEs and the existence of a special coordinate system on this manifold. This allows us to describe the local dynamics on the center manifold in terms of the standard normal forms. These results are based on the rigorous functional analytic perturbation framework for dual semigroups, also called sun-star calculus.


Keywords: delay differential equations, dual perturbation theory, sun-star calculus, center manifold theorem, normal forms, nonhyperbolic cycles

MSC: 34C20, 34C25, 34K19, 37L10

## 1 Introduction

Bifurcation theory allows us to analyze the behavior of complicated high dimensional nonlinear dynamical systems near bifurcations by reducing the system to a low dimensional invariant manifold, called the center manifold. Using normal form theory, the dynamics on the center manifold can be described by a simple canonical equation called the normal form. These bifurcations and normal forms can be categorized, and their properties can be understood in terms of certain coefficients of the normal form, see [24] for more details. Methods to compute these normal form coefficients have been implemented in software like MatCont [9] and DDE-BifTool [14, 26] to study various classes of dynamical systems.

For bifurcations of limit cycles in continuous-time dynamical systems, there are three generic codimension one bifurcations: fold (or limit point), period-doubling (or flip) and Neimark-Sacker (or torus) bifurcation. These bifurcations are well understood for ordinary differentials equations (ODEs) $[18,19,23,30]$, but for delay differential equations (DDEs) the theory is still lacking. To understand these bifurcations, one should first prove the existence of a center manifold on which one can study the dynamics near a nonhyperbolic cycle via a normal form reduction.

[^0]The first aim of this paper is to show for classical DDEs that such a center manifold exists and is sufficiently smooth. The second aim is to prove the existence of a special coordinate system on the center manifold, in which the dynamics is governed by a suitable periodic normal form. To our best knowledge, for general classical DDEs, both aims will be achieved for the first time. In an upcoming paper, we present explicit computational formulas for the critical normal form coefficients of all codimension one bifurcations of limit cycles, completely avoiding Poincaré maps. Finally, we plan to implement the obtained computational formulas into a software package like DDE-BifTool.

### 1.1 Background

Consider a delay differential equation (DDE)

$$
\begin{equation*}
\dot{x}(t)=F\left(x_{t}\right), \quad t \geq 0 \tag{1}
\end{equation*}
$$

where $x(t) \in \mathbb{R}^{n}$ and

$$
x_{t}(\theta):=x(t+\theta), \quad \theta \in[-h, 0],
$$

represents the history at time $t$ of the unknown $x$, and $0<h<\infty$ denotes the upper bound of (finite) delays. The $\mathbb{R}^{n}$-valued smooth operator $F$ is defined on the Banach space $X:=C\left([-h, 0], \mathbb{R}^{n}\right)$ consisting of $\mathbb{R}^{n}$-valued continuous functions on the compact interval $[-h, 0]$, endowed with the supremum norm.

Using the perturbation framework of dual semigroups, called sun-star calculus, developed in $[4,5$, $6,7,12$ ], the existence of a smooth finite-dimensional center manifold near a nonhyperbolic equilibrium of (1) can be rigorously established using the Lyapunov-Perron method, see [13] for the critical center manifold and [1] for the parameter-dependent center manifold. Furthermore, in [1, 13, 22], the authors derive explicit computational formulas for the normal form coefficients for all generic codimension one and two bifurcations for equilibria. These have been implemented in the MATLAB package DDE-BifTool. The question arises if this whole construction can be repeated for a nonhyperbolic periodic orbit (cycle) $\Gamma:=\left\{\gamma_{t} \in X: t \in \mathbb{R}\right\}$, where $\gamma: \mathbb{R} \rightarrow \mathbb{R}^{n}$ is a $T$-periodic solution of (1).

In this paper, we build a promising framework to generalize the described construction towards nonhyperbolic cycles in DDEs. Therefore, our first goal is to prove the existence of a smooth finitedimensional periodic center manifold in a neighborhood of $\Gamma$ using the Lyapunov-Perron method, but now in a time-dependent setting. To achieve this, we prove the existence of a smooth finite-dimensional periodic center manifold $\mathcal{W}_{\text {loc }}^{c}$ in the neighborhood of the origin of the time-dependent translated system

$$
\begin{equation*}
\dot{y}(t)=L(t) y_{t}+G\left(t, y_{t}\right) \tag{2}
\end{equation*}
$$

where $x=\gamma+y, L(t):=D F\left(\gamma_{t}\right)$ denotes the Fréchet derivative of $F$ evaluated at the point $\gamma_{t} \in X$ and $G(t, \cdot):=F\left(\gamma_{t}+\cdot\right)-F\left(\gamma_{t}\right)-L(t)$ consists of solely nonlinear terms. Note that both $L$ and $G$ are $T$-periodic in the variable $t$. Afterwards, we translate the manifold $\mathcal{W}_{\text {loc }}^{c}$, defined near the origin of (2), back towards the original cycle $\Gamma$. Hence, we obtain a smooth finite-dimensional periodic center manifold $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$ defined near the nonhyperbolic cycle $\Gamma$.

The first attempt to use a periodic center manifold for classical DDEs was made in the very interesting paper [28] by Szalai and Stépán, who heuristically applied the Lyapunov-Perron method for equilibria from [13] towards the periodic setting using sun-star calculus. However, no proof of the existence of such a center manifold was given, and in addition their results were only applicable when the period of the cycle $T$ precisely equals the delay $h$, which is a major restriction.

The existence of a finite-dimensional periodic center manifold for (2) was recently established in [2] by Church and Liu using the Lyapunov-Perron method for a specific class of delay equations, namely impulsive DDEs. These delay equations have a countable number of discontinuities in their solutions, and therefore it is in general not possible for the obtained center manifold to be smooth in the time direction. However, this smoothness will be crucial for the characterization and normal form theorems in our construction, see Section 4.3. In addition, the framework used in [2] is the formal
adjoint approach [15] and it is well known that this is an ad hoc method that can only be applied to a specific class of delay equations. Furthermore, as already remarked in [15, Section 8.2], the traditional bilinear form used in this approach is not applicable to study linear behavior of solutions near periodic orbits. Therefore, it seems difficult to derive the critical normal form coefficients for codimension one bifurcations of limit cycles using the formal adjoint approach. However, Church and Liu obtained such computational formulas, but employing the Poincaré maps [3]. When one is interested in studying numerically the local behavior of solutions in the vicinity of $\Gamma$ via the Poincare map, it is necessary to compute (higher order) derivatives of this map [23], which already does not look very promising for ODEs, let alone (impulsive) DDEs.

To overcome this problem in ODEs, the Poincaré maps have been completely avoided in [23, 30], where the authors rather worked with the results by Iooss [18, 19] on periodic normal forms. Indeed, our second goal in this paper is to generalize the results from $[18,19]$ on the existence of a special coordinate system on the center manifold, in which the system has the periodic normal form, from finite-dimensional ODEs towards infinite-dimensional DDEs, using the sun-star calculus framework. Iooss indicated in [18] that his results would be easily extendable to the infinite-dimensional setting. However, we will show in this paper some results that were truly not expected by the authors. For example, an interplay between history and periodicity for Jordan chains in Theorem 18 was a remarkable observation, since the history concept is not present in ODEs. Furthermore, the proof on the existence of this coordinate system happened to be far more involved, see Theorem 20 and especially the role of the sun-star calculus machinery in the proof.

### 1.2 Overview

The paper is organized as follows. In Section 2 we review and extend the theory of dual perturbation theory, the sun-star calculus, with time-dependent (nonlinear) perturbations, both on an abstract level as well as applied to the analysis of time-dependent (nonlinear) delay differential equations.

In Section 3 we use the theory from previous section to prove the existence of a smooth finitedimensional periodic center manifold for (2) near the origin, see Corollary 17 for the final result. Due to the dual perturbation framework, the proven results apply to a way more general class of evolution equations, as for example renewal equations [11], see Theorem 14 for the general result. Additional material on spectral decompositions can be found in Appendix A and some technical proofs on increasing smoothness and periodicity are relegated to Appendix B. To apply the general theory to classical DDEs, we also use the material presented in Appendix C.

In Section 4 we prove the existence of a special coordinate system on the center manifold near $\Gamma$ and generalize the normal form theorems from finite-dimensional ODEs [18, 19] towards infinitedimensional DDEs, see Theorem 20, Theorem 21 and Theorem 22. Examples of application of these theorems to codimension one bifurcations of limit cycles are also provided in Section 4.

## 2 Dual perturbation theory

We start by briefly recalling the general elements of (time-dependent) dual perturbation theory that are useful to study classical DDEs as dynamical systems. Standard references for this entire section are the book [13] together with the article [4] on time-dependent perturbations. All unreferenced claims relating to basic properties of time-dependent perturbations of delays equations can be found here.

### 2.1 Duality structure and time-dependent perturbations

First, we introduce the unperturbed strongly continuous semigroup $T_{0}$, which is the shift semigroup for delay equations. This is used to define the sun dual $X^{\odot}$, where the adjoint of this semigroup is strongly continuous. Most of our work below will take place on its dual, the space $X^{\odot \star}$.

Let $T_{0}:=\left\{T_{0}(t)\right\}_{t \geq 0}$ be a $\mathcal{C}_{0}$-semigroup of bounded linear operators defined on a real or complex Banach space $X$ that has $A_{0}$ as (infinitesimal) generator with domain $\mathcal{D}\left(A_{0}\right)$. Then the dual semigroup $T_{0}^{\star}:=\left\{T_{0}^{\star}(t)\right\}_{t \geq 0}$, where $T_{0}^{\star}(t): X^{\star} \rightarrow X^{\star}$ is the adjoint of $T_{0}(t)$, is a semigroup on the topological dual space $X^{\star}$ of $X$. We denote the duality paring between $X$ and $X^{\star}$ as

$$
\left\langle x^{\star}, x\right\rangle:=x^{\star}(x), \quad \forall x^{\star} \in X^{\star}, x \in X
$$

If $X$ is not reflexive, then $T_{0}^{\star}$ is in general only weak ${ }^{\star}$ continuous on $X^{\star}$. This is also visible on the generator level, as the adjoint $A_{0}^{\star}$ of $A_{0}$ is only the weak ${ }^{\star}$ generator of $T_{0}^{\star}$ and has in general a non-dense domain. The maximal subspace of strong continuity

$$
X^{\odot}:=\left\{x^{\star} \in X^{\star}: t \mapsto T_{0}^{\star}(t) x^{\star} \text { is norm continuous on }[0, \infty)\right\}
$$

is a norm closed $T_{0}^{\star}(t)$-invariant weak ${ }^{\star}$ dense subspace of $X^{\star}$ and we have the characterization

$$
\begin{equation*}
X^{\odot}=\overline{\mathcal{D}\left(A_{0}^{\star}\right)}, \tag{3}
\end{equation*}
$$

where the bar denotes the norm closure in $X^{\star}$. The restriction of $T_{0}^{\star}$ to $X^{\odot}$ is a $\mathcal{C}_{0}$-semigroup on $X^{\odot}$ and its generator $A_{0}^{\odot}$ is the part of $A_{0}^{\odot}$ in $X^{\odot}$

$$
\mathcal{D}\left(A_{0}^{\odot}\right)=\left\{x^{\odot} \in \mathcal{D}\left(A_{0}^{\star}\right): A_{0}^{\star} x^{\odot} \in X^{\odot}\right\}, \quad A_{0}^{\odot} x^{\odot}=A_{0}^{\star} x^{\odot}
$$

We have at this moment a $\mathcal{C}_{0}$-semigroup $T_{0}^{\odot}$ with generator $A_{0}^{\odot}$ on the Banach space $X^{\odot}$, which are precisely the ingredients we started with. Repeating the construction once more, we obtain on the dual space $X^{\odot \star}$ the adjoint semigroup $T_{0}^{\odot \star}$ with weak ${ }^{\star}$ generator $A_{0}^{\odot \star}$. The restriction of $T_{0}^{\odot \star}$ to the maximal subspace of strong continuity $X^{\odot \odot}$ gives a $\mathcal{C}_{0}$-semigroup $T_{0}^{\odot \odot}$ with generator $A_{0}^{\odot \odot}$ that is the part of $A_{0}^{\odot \star}$ in $X^{\odot \odot}$.

The canonical embedding $j: X \rightarrow X^{\odot \star}$ defined by

$$
\begin{equation*}
\left\langle j x, x^{\odot}\right\rangle:=\left\langle x^{\odot}, x\right\rangle, \quad \forall x \in X, x^{\odot} \in X^{\odot} \tag{4}
\end{equation*}
$$

maps $X$ into $X^{\odot \odot}$. If $j$ maps $X$ onto $X^{\odot \odot}$ then $X$ is called $\odot$-reflexive with respect to $T_{0}$. $\odot$-reflexivity with respect to $T_{0}$ will be assumed throughout.

### 2.2 Time-dependent bounded linear perturbations

Let us now turn our attention to perturbations. We will show how a time-dependent perturbation is handled in the setting of dual perturbation theory.

A time-dependent bounded linear perturbation can be represented as a Lipschitz continuous map $B: J \rightarrow \mathcal{L}\left(X, X^{\odot \star}\right)$, where $J \subseteq \mathbb{R}$ is an interval and $\mathcal{L}\left(X, X^{\odot \star}\right)$ stands for the Banach space of all bounded linear operators from $X$ to $X^{\odot \star}$, equipped with the operator norm. We are interested in solutions of the initial value problem

$$
\begin{cases}d^{\star}(j \circ u)(t)=A_{0}^{\odot \star} j u(t)+B(t) u(t), & t \geq s  \tag{5}\\ u(s)=\varphi, & \varphi \in X\end{cases}
$$

where $d^{\star}$ stands for the weak ${ }^{\star}$ differential operator and $s \in J$ denotes a starting time. For the sake of completeness, let us recall the definition of the (partial) weak ${ }^{\star}$ differential operator.

Definition 1. Let $E$ be a Banach space, $J \subseteq \mathbb{R}$ an interval and $\Omega \subseteq J \times J$. We say that a function $f: J \rightarrow E^{\star}$ is weak${ }^{\star}$ differentiable with weak ${ }^{\star}$ derivative $d^{\star} f: J \rightarrow E^{\star}$ if

$$
\frac{d}{d t}\langle f(t), x\rangle=\left\langle d^{\star} f(t), x\right\rangle, \quad \forall x \in E, t \in J
$$

If in addition $d^{\star} f$ is weak ${ }^{\star}$ continuous, then $f$ is called weak ${ }^{\star}$ continuously differentiable.
Furthermore, we say that a function $g: \Omega \rightarrow E^{\star}$ has partial weak derivatives $\partial_{t}^{\star} g: \Omega \rightarrow E^{\star}$ and $\partial_{s}^{\star} g: \Omega \rightarrow E^{\star}$ if

$$
\begin{aligned}
\frac{\partial}{\partial t}\langle g(t, s), x\rangle & =\left\langle\partial_{t}^{\star} g(t, s), x\right\rangle,
\end{aligned} \quad \forall x \in E,(t, s) \in \Omega, ~ \begin{array}{ll}
\frac{\partial}{\partial s}\langle g(t, s), x\rangle & =\left\langle\partial_{s}^{\star} g(t, s), x\right\rangle,
\end{array} \quad \forall x \in E,(t, s) \in \Omega
$$

If in addition $\partial_{t}^{\star} g$ and $\partial_{s}^{\star} g$ are weak $k^{\star}$ continuous, then $g$ is called weak ${ }^{\star}$ continuously differentiable.
According to the literature [13], it is however more convenient to study the formally integrated problem as the abstract integral equation

$$
\begin{equation*}
u(t)=T_{0}(t-s) \varphi+j^{-1} \int_{s}^{t} T_{0}^{\odot \star}(t-\tau) B(\tau) u(\tau) d \tau, \quad \varphi \in X \tag{T-LAIE}
\end{equation*}
$$

where the integral has to be interpreted as a weak* Riemann integral [13, Chapter III] and takes values in $j(X)$ under the running assumption of $\odot$-reflexivity. If we define the set $\Omega_{J}:=\{(t, s) \in J \times J \mid t \geq$ $s\}$, it is known that (T-LAIE) uniquely defines a strongly continuous forward evolutionary system $U:=\{U(t, s)\}_{(t, s) \in \Omega_{J}}$ in the sense that $u(t)=U(t, s)$ for all $t \in[s, \infty) \cap J$, see [4, Definition 2.1] for more information. If one defines for any $s \in J$ the (generalized) generator $A^{\odot \star}(s): \mathcal{D}\left(A^{\odot \star}\right) \rightarrow X^{\odot \star}$ as

$$
A^{\odot \star}(s) j x:=\mathrm{w}^{\star}-\lim _{t \downarrow s} \frac{1}{t-s}(j U(t, s) x-j x)
$$

for any $j x$ in the (generalized) domain

$$
\mathcal{D}\left(A^{\odot \star}(s)\right):=\left\{j x \in X^{\odot \star} \left\lvert\, \mathrm{w}^{\star}-\lim _{t \downarrow s} \frac{1}{t-s}(j U(t, s) x-j x)\right. \text { exists in } X^{\odot \star}\right\}
$$

it is known that the perturbation $B$ enters additively in the action of the generator as

$$
\begin{equation*}
\mathcal{D}\left(A^{\odot \star}(s)\right)=\mathcal{D}\left(A_{0}^{\odot \star}\right), \quad A^{\odot \star}(s)=A_{0}^{\odot \star}+B(s) j^{-1}, \quad \forall s \in J \tag{6}
\end{equation*}
$$

We recover the generator $A(s)$ by considering the part of $A^{\odot \star}(s)$ in $X^{\odot \odot}$ as

$$
\begin{align*}
\mathcal{D}(A(s)) & =\left\{x \in X: j x \in \mathcal{D}\left(A_{0}^{\odot \star}\right) \text { and } A_{0}^{\odot \star} j x+B(s) x \in X^{\odot \odot}\right\} \\
A(s) x & =j^{-1}\left(A_{0}^{\odot \star} j x+B(s) x\right) \tag{7}
\end{align*}
$$

Let us now go back to (5). If the initial condition $\varphi \in j^{-1} \mathcal{D}\left(A_{0}^{\odot \star}\right)$ then, due to the Lipschitz continuity of $B$, it is known that $u=U(\cdot, s) \varphi:[s, \infty) \cap J \rightarrow X$ is continuously weak ${ }^{\star}$ differentiable, takes values in $j^{-1} \mathcal{D}\left(A_{0}^{\odot \star}\right)$ and is the unique solution of (5).

As we have defined $U(t, s)$ for all $(t, s) \in \Omega_{J}$, we are interested in the associated (sun) dual(s). It is clear that one can define $U^{\star}(s, t):=U(t, s)^{\star} \in \mathcal{L}\left(X^{\star}\right):=\mathcal{L}\left(X^{\star}, X^{\star}\right)$ and that $U^{\star}:=\left\{U^{\star}(s, t)\right\}_{(s, t) \in \Omega_{J}^{\star}}$ forms a backward evolutionary system on $X^{\star}$, with $\Omega_{J}^{\star}:=\left\{(s, t) \in J^{2}: t \geq s\right\}$. Furthermore, the Lipschitz continuity on $B$ ensures that the restriction $U^{\odot}(s, t):=\left.U^{\star}(s, t)\right|_{X \odot}$ is $X^{\odot}$-invariant and, by construction, $U^{\odot}:=\left\{U^{\odot}(s, t)\right\}_{(s, t) \in \Omega_{J}^{\star}}$ is a strongly continuous backward evolutionary system, see [4, Theorem 5.4]. This allows us to define $U^{\odot \star}(t, s):=\left(U^{\odot}(s, t)\right)^{\star}$ and it is clear that $U^{\odot \star}:=$ $\left\{U^{\odot \star}(t, s)\right\}_{(t, s) \in \Omega_{J}}$ is a forward evolutionary system on $X^{\odot \star}$ that extends $U$, which was previously defined on $X$.

In the upcoming sections, we will have to deal with a particular weak ${ }^{\star}$ integral involving $U^{\odot \star}$ that will be studied in the following lemma. This integral is crucial in the variation-of-constants formulation for the abstract delay equation.

Lemma 2. Let $g: J \rightarrow X^{\odot \star}$ be continuous and denote the set $\left\{(t, r, s) \in J^{3}: s \leq r \leq t\right\}$ by $\Theta_{J}$. Then the map $v(\cdot, \cdot, \cdot, g): \Theta_{J} \rightarrow X^{\odot \star}$ defined as the weak integral

$$
\begin{equation*}
v(t, r, s, g):=\int_{s}^{r} U^{\odot \star}(t, \tau) g(\tau) d \tau, \quad \forall(t, r, s) \in \Theta_{J} \tag{8}
\end{equation*}
$$

is continuous and takes values in $j(X)$. Furthermore, if $J$ is unbounded from below and $v(\cdot, \cdot, \cdot, g)$ is also bounded in norm on $\Theta_{J}$, then the limiting function $v(\cdot, \cdot,-\infty, g)$ converges in norm, is continuous, and its range is contained in $j(X)$.

Proof. Let $\left(t_{1}, r_{1}, s_{1}\right),\left(t_{2}, r_{2}, s_{2}\right) \in \Theta_{J}$ and performing the change of variables $\sigma=t-\tau$ yields

$$
v(t, r, s, g)=\int_{t-r}^{t-s} U^{\odot \star}(t, t-\sigma) g(t-\sigma) d \sigma
$$

Let $I_{i}=\left[t_{i}-r_{i}, t_{i}-s_{i}\right]$ for $i=1,2$. We can split the following difference into four integrals.

$$
\begin{aligned}
v\left(t_{1}, r_{1}, s_{1}, g\right)-v\left(t_{2}, r_{2}, s_{2}, g\right) & =\int_{I_{2} / I_{1}} U^{\odot \star}\left(t_{2}, t_{2}-\sigma\right) g\left(t_{2}-\sigma\right) d \sigma \\
& -\int_{I_{1} / I_{2}} U^{\odot \star}\left(t_{2}, t_{2}-\sigma\right) g\left(t_{2}-\sigma\right) d \sigma \\
& +\int_{I_{1} \cap I_{2}}\left(U^{\odot \star}\left(t_{2}, t_{2}-\sigma\right)-U^{\odot \star}\left(t_{1}, t_{1}-\sigma\right)\right) g\left(t_{2}-\sigma\right) d \sigma \\
& +\int_{I_{1} \cap I_{2}} U^{\odot \star}\left(t_{1}, t_{1}-\sigma\right)\left(g\left(t_{2}-\sigma\right)-g\left(t_{1}-\sigma\right)\right) d \sigma
\end{aligned}
$$

and using the triangle inequality, we get the following estimate

$$
\begin{aligned}
& \left\|v\left(t_{1}, r_{1}, s_{1}, g\right)-v\left(t_{2}, r_{2}, s_{2}, g\right)\right\| \\
& \leq\left(\left|I_{1} / I_{2}\right|+\left|I_{2} / I_{1}\right|\right) \sup _{\sigma \in I_{1} / I_{2} \cup I_{2} / I_{1}}\left\|U^{\odot \star}\left(t_{2}, t_{2}-\sigma\right) g\left(t_{2}-\sigma\right)\right\| \\
& +\left|I_{1} \cap I_{2}\right| \sup _{\sigma \in I_{1} \cap I_{2}}\left\|U^{\odot \star}\left(t_{2}, t_{2}-\sigma\right)-U^{\odot \star}\left(t_{1}, t_{1}-\sigma\right)\right\|\left\|g\left(t_{2}-\sigma\right)\right\| \\
& +\left|I_{1} \cap I_{2}\right| \sup _{t, \sigma \in I_{1} \cap I_{2}}\left\|U^{\odot \star}(t, t-\sigma)\right\|\left\|g\left(t_{2}-\sigma\right)-g\left(t_{1}-\sigma\right)\right\|
\end{aligned}
$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$. If we let $\left(t_{1}, r_{1}, s_{1}\right) \rightarrow\left(t_{2}, r_{2}, s_{2}\right)$ in norm, then the first term vanishes by definition. The second term vanishes as $U(t, s)$ is uniformly continuous along paths that keep $t-s$ constant [4, Lemma 5.2] and the last term vanishes due to the continuity of $g$. Hence, $v(\cdot, \cdot, \cdot, g)$ is continuous. Note that the second term does not appear for semigroups, as they are invariant under time translations.

Next, we will prove that the range of $v(\cdot, \cdot, \cdot, g)$ is contained in $j(X)$. Let $(t, r, s) \in \Theta_{J}$ and recall from (6) that $\mathcal{D}\left(A^{\odot \star}(t)\right)=\mathcal{D}\left(A_{0}^{\odot \star}\right)$. Taking the closure with respect to the norm defined on $X^{\odot \star}$ we get that

$$
\begin{aligned}
\left\{x^{\odot \star} \in X^{\odot \star} \mid \lim _{h \downarrow 0}\left\|U^{\odot \star}(t+h, t) x^{\odot \star}-x^{\odot \star}\right\|=0\right\} & =\overline{\mathcal{D}\left(A^{\odot \star}(t)\right)} \\
& =\overline{\mathcal{D}\left(A_{0}^{\odot \star}\right)}=X^{\odot \odot}=j(X),
\end{aligned}
$$

where these last two equations follow from the sun-variant of (3) and $\odot$-reflexivity of $X$ with respect to $T_{0}$. We want to show that $v(t, r, s, g)$ is an element of this first set. Using the continuity of $v(\cdot, \cdot, \cdot, g)$ we find that

$$
\lim _{h \downarrow 0}\left\|U^{\odot \star}(t+h, t) v(t, r, s, g)-v(t, r, s, g)\right\|=\lim _{h \downarrow 0}\|v(t+h, r, s)-v(t, r, s)\|=0
$$

and so we conclude that $v(t, r, s) \in j(X)$.
Finally, let $J$ be unbounded from below and suppose that $v(\cdot, \cdot, \cdot, g)$ in bounded in norm on $\Theta_{J}$. Define the map $w(\cdot, \cdot, g): \Omega_{J} \rightarrow X^{\odot \star}$ as

$$
w(t, r, g):=\lim _{n \rightarrow \infty} \int_{r-n}^{r} U^{\odot \star}(t, \tau) g(\tau) d \tau, \quad \forall(t, r) \in \Omega_{J}
$$

which is well-defined due to [21, Lemma 9] and the boundedness of $v(\cdot, \cdot, \cdot, g)$ in norm. To see this, notice that for any fixed $t \in J$, the integrand of $w(\cdot, \cdot, g)$ is weak ${ }^{\star}$ continuous, which implies weak ${ }^{\star}$ Lebesgue measurability, since for any $\tau \in J$ and $h \in \mathbb{R}$ such that $t \geq \max \{\tau, \tau+h\}$ and $\tau+h \in J$ we obtain that for all $x^{\odot} \in X^{\odot}$

$$
\begin{aligned}
& \left|\left\langle U^{\odot \star}(t, \tau+h) g(\tau+h), x^{\odot}\right\rangle-\left\langle U^{\odot \star}(t, \tau) g(\tau), x^{\odot}\right\rangle\right| \\
& \leq\left|\left\langle g(\tau+h), U^{\odot}(\tau+h, t) x^{\odot}\right\rangle-\left\langle g(\tau+h), U^{\odot}(\tau, t) x^{\odot}\right\rangle\right| \\
& +\left|\left\langle g(\tau+h), U^{\odot}(\tau, t) x^{\odot}\right\rangle-\left\langle g(\tau), U^{\odot}(\tau, t) x^{\odot}\right\rangle\right| \\
& \leq\|g(\tau+h)\|\left\|U^{\odot}(\tau+h, t) x^{\odot}-U^{\odot}(\tau, t) x^{\odot}\right\|+\|g(\tau+h)-g(\tau)\|\left\|U^{\odot}(\tau, t)\right\|\left\|x^{\odot}\right\| \\
& \rightarrow 0, \quad \text { as } h \rightarrow 0,
\end{aligned}
$$

since $g$ is norm continuous and $U \odot$ is a strongly continuous backward evolutionary system. Furthermore, boundedness of $v$ implies uniform continuity, hence $w$ also continuous. Since $[r-n, r]$ is compact for any fixed $n \in \mathbb{N}$, each integral inside the limit of $w(t, r, g)$ lies in $j(X)$ by the reasoning above. As $j(X)=X^{\odot \odot}$ is closed $w(t, r, g)=v(t, r,-\infty) \in j(X)$.

A similar statement holds for backward evolutionary systems and the proof is completely analogous.

### 2.3 Time-dependent nonlinear perturbations

The strongly continuous forward evolutionary system $U$ arises as a time-dependent bounded linear perturbation of the original $\mathcal{C}_{0}$-semigroup $T_{0}$, see (5). The next logical step is to introduce a timedependent nonlinear perturbation on $U$ itself. We can formulate solutions to this problem via a nonlinear abstract integral equation.

A time-dependent nonlinear perturbation on an interval $J \subseteq \mathbb{R}$ can be represented as a $C^{k}$-smooth operator $R: J \times X \rightarrow X^{\odot \star}$ for some $k \geq 1$ that satisfies

$$
\begin{equation*}
R(t, 0)=0, \quad D_{2} R(t, 0)=0, \quad \forall t \in J \tag{9}
\end{equation*}
$$

where the $D_{2} R(t, 0)$ denotes the partial Fréchet derivative of $R$ with respect to the second component evaluated at the point $(t, 0)$. Consider now the nonlinear abstract integral equation

$$
\begin{equation*}
u(t)=U(t, s) \varphi+j^{-1} \int_{s}^{t} U^{\odot \star}(t, \tau) R(\tau, u(\tau)) d \tau, \quad \varphi \in X \tag{T-AIE}
\end{equation*}
$$

with $t \geq s$ and $u(s)=\varphi$, where $s$ plays the role of a starting time. It follows from Lemma 2 that the weak ${ }^{\star}$ integral in (T-AIE) takes values in $j(X)$ and hence (T-AIE) is well-defined. Since the nonlinearity is $C^{k}$-smooth, we know from the mean value inequality in Banach spaces [8, Corollary 3.2] that $R$ is locally Lipschitz in the second component. Hence, for a given $\varphi \in X$ one can at most guarantee the existence of a unique solution $u_{\varphi}:\left[s, t_{\varphi}\right) \cap J \rightarrow X$ of (T-AIE) for some $s<t_{\varphi} \leq \infty$. In this setting, one expects the existence of a time-dependent semiflow, that is the nonlinear analogue of a forward evolutionary system and the time-dependent analogue of a semiflow introduced in [13, Definition VII.2.1]. Time-dependent semiflows also referred in the literature as a processes, see [2, 3].
Definition 3. Let $J \subseteq \mathbb{R}$ be an interval. A time-dependent semiflow on a Banach space $X$ is a map $S: \mathcal{D}(S) \rightarrow X$, where $\mathcal{D}(S) \subseteq \Omega_{\mathbb{R}} \times X$ that satisfies the following properties.

1. For any $s \in J$ and $x \in X$, there exists a $t_{x} \in[s, \infty]$ such that $\mathcal{D}(S)=\left\{(t, s, x) \in \Omega_{\mathbb{R}} \times X\right.$ : $\left.t \in\left[s, t_{x}\right) \cap J\right\}$.
2. For any $s \in J$ and $x \in X$ we have $S(s, s, x)=x$.
3. For any $t, v, s \in J$ with $t \geq v \geq s$ and $x \in X$ it holds

$$
S(t, s, x)=S(t, v, S(v, s, x))
$$

An example of a time-dependent semiflow on $X$ is the map $S: \mathcal{D}(S) \rightarrow X$ defined by

$$
\begin{equation*}
\mathcal{D}(S)=\left\{(t, s, \varphi) \in \Omega_{\mathbb{R}} \times X: t \in\left[s, t_{\varphi}\right) \cap J\right\}, \quad S(t, s, \varphi):=u_{\varphi}(t) \tag{10}
\end{equation*}
$$

## 3 Existence of the center manifold

In this section, we prove the existence of a periodic smooth finite-dimensional center manifold near the origin of the abstract integral equation (T-AIE) and apply the obtained results to classical DDEs.

To specify the setting, let $X$ be a real Banach space that is $\odot$-reflexive with respect to a given $\mathcal{C}_{0^{-}}$ semigroup $T_{0}$ defined on $X$. Let $B: J \rightarrow \mathcal{L}\left(X, X^{\odot \star}\right)$ be a time-dependent bounded linear perturbation defined on an interval $J \subseteq \mathbb{R}$ and define the strongly continuous forward evolutionary system $U$ as the unique solution of (T-LAIE) together with the (sun) dual(s) $U^{\star}, U^{\odot}$ and $U^{\odot \star}$. Assume that $R: J \times X \rightarrow X^{\odot \star}$ is a time-dependent nonlinear perturbation that is $C^{k}$-smooth for some $k \geq 1$. Furthermore, let $S: \mathcal{D}(S) \rightarrow X$ denote the time-dependent semiflow defined in (10) that generates a local unique solution of (T-AIE).

It turns out that these assumptions are not sufficient to prove the existence of a periodic smooth finite-dimensional center manifold for (T-AIE). Therefore, we invoke in Section 3.1 a hypothesis about the spectral structure of $X$ and $U$. We can lift the exponential structure of the spectral problem to $X^{\odot \star}$, using some technical lemmas presented in Appendix A. 1 and Appendix A.2. We show boundedness of solutions of the abstract integral equation in Section 3.2 and Section 3.3. This allows us to prove the existence of a Lipschitz center manifold in Section 3.4 using a fixed point argument. In Section 3.5 we show smoothness and periodicity using the theory of scales of Banach spaces, where the details can be found in Appendix B. Finally, in Section 3.6 we explain how the perturbation framework and center manifold theorem (Theorem 14) fits naturally into the setting of classical DDEs, see Corollary 17 for the main result.

### 3.1 Spectral decompositions of $X$ and $X^{\odot \star}$

The construction of a local center manifold has been established for equilibria under the assumption of the existence of a topological direct sum decomposition of $X^{\odot \star}$, see [13, Section IX.2]. The motivation behind this follows from the fact that the nonlinearity maps into $X^{\odot \star}$. However, depending on the evolution equation of interest, one should always first compute $X^{\odot \star}$ and its associated $\odot \star$-tools to check the underlying assumptions. It is however more convenient to state a hypothesis in $X$ and lift this towards $X^{\odot \star}$. It also turns out from Section 4 that we really need a decomposition in $X$ and $X^{\odot \star}$ that allows us to move back and forth between the two. The following hypothesis on the time-dependent spectral decompositions is inspired by [21, 2].
Hypothesis 1. The space $X$ and the forward evolutionary system $U$ have the following properties:

1. $X$ admits a direct sum decomposition

$$
\begin{equation*}
X=X_{-}(s) \oplus X_{0}(s) \oplus X_{+}(s), \quad \forall s \in \mathbb{R} \tag{11}
\end{equation*}
$$

where each summand is closed.
2. There exist three continuous time-dependent (spectral) projectors $P_{i}: \mathbb{R} \rightarrow \mathcal{L}(X)$ with $\operatorname{ran}\left(P_{i}(s)\right)=X_{i}(s)$ for any $s \in \mathbb{R}$ and $i \in\{-, 0,+\}$.
3. There exists a constant $N \geq 0$ such that $\sup _{s \in \mathbb{R}}\left(\left\|P_{-}(s)\right\|+\left\|P_{0}(s)\right\|+\left\|P_{+}(s)\right\|\right)=N<\infty$.
4. The projections are mutually orthogonal, meaning that $P_{i}(s) P_{j}(s)=0$ for all $i \neq j$ and $s \in \mathbb{R}$ with $i, j \in\{-, 0,+\}$.
5. The projections commute with the forward evolutionary system: $U(t, s) P_{i}(s)=P_{i}(t) U(t, s)$ for all $i \in\{-, 0,+\}$ and $t \geq s$.
6. Define the restrictions $U_{i}(t, s): X_{i}(s) \rightarrow X_{i}(t)$ for $i \in\{-, 0,+\}$ and $t \geq s$. The operators $U_{0}(t, s)$ and $U_{+}(t, s)$ are invertible and also backward evolutionary systems. Specifically, for any $t, \tau, s \in \mathbb{R}$ it holds

$$
\begin{equation*}
U_{0}(t, s)=U_{0}(t, \tau) U_{0}(\tau, s), \quad U_{+}(t, s)=U_{+}(t, \tau) U_{+}(\tau, s) \tag{12}
\end{equation*}
$$

7. The decomposition (11) is an exponential trichotomy on $\mathbb{R}$ meaning that there exist $a<0<b$ such that for every $\varepsilon>0$ there exists a $K_{\varepsilon}>0$ such that

$$
\begin{aligned}
\left\|U_{-}(t, s)\right\| & \leq K_{\varepsilon} e^{a(t-s)}, \quad t \geq s \\
\left\|U_{0}(t, s)\right\| & \leq K_{\varepsilon} e^{\varepsilon|t-s|}, \quad t, s \in \mathbb{R} \\
\left\|U_{+}(t, s)\right\| & \leq K_{\varepsilon} e^{b(t-s)}, \quad t \leq s
\end{aligned}
$$

We call $X_{-}(s), X_{0}(s)$ and $X_{+}(s)$ the stable subspace, center subspace and unstable subspace (at time s) respectively.

As the stable-, center- and unstable subspace are only defined at a specific time $s \in \mathbb{R}$, it is convenient to introduce the sets

$$
X_{i}:=\left\{(t, \varphi) \in \mathbb{R} \times X: \varphi \in X_{i}(t)\right\}
$$

for $i \in\{-, 0,+\}$ and call them the stable fiber bundle, center fiber bundle and unstable fiber bundle respectively. It is explained in Appendix A. 1 how Hypothesis 1 can be lifted to $X^{\odot \star}$, see Proposition 25 for the main result. We also impose the following.
Hypothesis 2. The subspaces $X_{0}^{\odot \star}(s)$ and $X_{+}^{\odot \star}(s)$ are contained in $j\left(X_{0}(s)\right)$ and $j\left(X_{+}(s)\right)$ respectively, for all $s \in \mathbb{R}$.

As part of the construction of a center manifold, we will be interested in solutions that exist for all time. It is therefore helpful to write (T-LAIE) in translation invariant form

$$
\begin{equation*}
u(t)=U(t, s) u(s)+j^{-1} \int_{s}^{t} U^{\odot \star}(t, \tau) R(\tau, u(\tau)) d \tau, \quad-\infty<s \leq t<\infty . \tag{13}
\end{equation*}
$$

One of the problems that occur in developing a center manifold theory for infinite-dimensional systems is that the linearized equation of (13) can have unbounded solutions in $X_{0}$. This leads to working in a function space that allows limited exponential growth both at plus and minus infinity. To do this, let $E$ be a Banach space, $\eta, s \in \mathbb{R}$ and define

$$
\mathrm{BC}_{s}^{\eta}(\mathbb{R}, E):=\left\{f \in C(\mathbb{R}, E): \sup _{t \in \mathbb{R}} e^{-\eta|t-s|}\|f(t)\|<\infty\right\}
$$

with the weighted supremum norm

$$
\|f\|_{\eta, s}:=\sup _{t \in \mathbb{R}} e^{-\eta|t-s|}\|f(t)\|,
$$

such that $\mathrm{BC}_{s}^{\eta}(\mathbb{R}, E)$ becomes a Banach space. Before we start working with the inhomogeneous equation (13), let us first derive some properties of the homogeneous equation

$$
\begin{equation*}
u(t)=U(t, s) u(s), \quad(t, s) \in \Omega_{J} \tag{14}
\end{equation*}
$$

on some interval $J \subseteq \mathbb{R}$. We say that $u: J \rightarrow X$ is a solution of (14) on $J$ if $u$ is a continuous function such that (14) holds. We have the following result that connects the center eigenspace $X_{0}(s)$ with $\mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ and the proof is inspired by [21, Lemma 29] and [2, Lemma 5.2.1].

Proposition 4. Let $\eta \in(0, \min \{-a, b\})$ and $s \in \mathbb{R}$. Then

$$
X_{0}(s)=\left\{\varphi \in X: \text { there exists a solution of (14) on } \mathbb{R} \text { through } \varphi \text { belonging to } \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)\right\}
$$

Proof. Let $\varphi \in X_{0}(s)$, then $u_{\varphi}: \mathbb{R} \rightarrow X$ defined by $u_{\varphi}(t):=U(t, s) \varphi=U_{0}(t, s) \varphi$ is a solution of (14) on $\mathbb{R}$ through $\varphi$. Let us now show that $u_{\varphi} \in \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$. Let $\varepsilon \in(0, \eta]$ be given. It follows from the exponential trichotomy of Hypothesis 1 that

$$
e^{-\eta|t-s|}\left\|u_{\varphi}(t)\right\|=e^{-\eta|t-s|}\left\|U_{0}(t, s) \varphi\right\| \leq K_{\varepsilon} e^{(\varepsilon-\eta)|t-s|}\|\varphi\| \leq K_{\varepsilon}\|\varphi\|, \quad \forall t, s \in \mathbb{R}
$$

since $\varepsilon-\eta<0$. Taking the supremum over $t \in \mathbb{R}$ yields $u_{\varphi} \in \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$.
Conversely, suppose that $\varphi \in X$ admits a solution $u_{\varphi} \in \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ of (14) on $\mathbb{R}$ that goes through $\varphi$ at time $s$ i.e. $u_{\varphi}(s)=\varphi$. We want to show that $P_{ \pm}(s) \varphi=0$ because then $\varphi=\left(P_{-}(s)+P_{0}(s)+\right.$ $\left.P_{+}(s)\right) \varphi=P_{0}(s) \varphi$ so $\varphi \in X_{0}(s)$. To do this, let us first show that $P_{+}(s) \varphi=0$. Take $t \geq s$ and $\varepsilon \in(0, \eta]$, then

$$
\left\|P_{+}(s) \varphi\right\|=\left\|U_{+}(s, t) P_{+}(t) u_{\varphi}(t)\right\| \leq K_{\varepsilon} e^{b(s-t)} N\left\|u_{\varphi}(t)\right\|, \quad \forall t \geq s
$$

It follows for $t \geq \max \{s, 0\}$ that

$$
e^{-\eta t}\left\|u_{\varphi}(t)\right\| \geq \frac{e^{-b s}}{K_{\varepsilon} N} e^{(b-\eta) t}\left\|P_{+}(s) \varphi\right\| \rightarrow \infty, \quad \text { as } t \rightarrow \infty
$$

unless $P_{+}(s) \varphi=0$. To prove $P_{-}(s) \varphi=0$, take $t \leq s$ and $\varepsilon \in(0, \eta]$, then

$$
\left\|P_{-}(s) \varphi\right\|=\left\|U_{-}(s, t) P_{-}(t) u_{\varphi}(t)\right\| \leq K_{\varepsilon} e^{a(s-t)} N\left\|u_{\varphi}(t)\right\| .
$$

It follows for $t \leq \min \{s, 0\}$ that

$$
e^{-\eta t}\left\|u_{\varphi}(t)\right\| \geq \frac{e^{-a s}}{K_{\varepsilon} N} e^{(a+\eta) t}\left\|P_{-}(s) \varphi\right\| \rightarrow \infty, \quad \text { as } t \rightarrow-\infty
$$

unless $P_{-}(s) \varphi=0$. Hence $P_{ \pm}(s)=0$ and so $\varphi \in X_{0}(s)$.

### 3.2 Bounded solutions of the linear inhomogeneous equation

Let $f: \mathbb{R} \rightarrow X^{\odot \star}$ be a continuous function. A solution of the linear inhomogeneous equation

$$
\begin{equation*}
u(t)=U(t, s) u(s)+j^{-1} \int_{s}^{t} U^{\odot \star}(t, \tau) f(\tau) d \tau, \quad(t, s) \in \Omega_{J} \tag{15}
\end{equation*}
$$

on an interval $J \subseteq \mathbb{R}$ is a continuous function $u: J \rightarrow X$ such that (15) holds. To prove existence of a center manifold, we need a pseudo-inverse of bounded solutions of (15). To do this, define (formally) for any $\eta \in(0, \min \{-a, b\})$ the operator $\mathcal{K}_{s}^{\eta}: \mathrm{BC}_{s}^{\eta}\left(\mathbb{R}, X^{\odot \star}\right) \rightarrow \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ as

$$
\begin{aligned}
\left(\mathcal{K}_{s}^{\eta} f\right)(t) & :=j^{-1} \int_{s}^{t} U^{\odot \star}(t, \tau) P_{0}^{\odot \star}(\tau) f(\tau) d \tau+j^{-1} \int_{\infty}^{t} U^{\odot \star}(t, \tau) P_{+}^{\odot \star}(\tau) f(\tau) d \tau \\
& +j^{-1} \int_{-\infty}^{t} U^{\odot \star}(t, \tau) P_{-}^{\odot \star}(\tau) f(\tau) d \tau, \quad \forall f \in \mathrm{BC}_{s}^{\eta}\left(\mathbb{R}, X^{\odot \star}\right)
\end{aligned}
$$

and we have to check that this is indeed a well-defined operator. This will be proven in the following proposition and also the fact that $\mathcal{K}_{s}^{\eta}$ is precisely the pseudo-inverse we are looking for. The proof is inspired by [21, Proposition 30] and [2, Lemma 5.2.3].

Proposition 5. Let $\eta \in(0, \min \{-a, b\})$ and $s \in \mathbb{R}$. The following properties hold.

1. $\mathcal{K}_{s}^{\eta}$ is a well-defined bounded linear operator. Moreover, the operator norm $\left\|\mathcal{K}_{s}^{\eta}\right\|$ is bounded above independent of $s$.
2. $\mathcal{K}_{s}^{\eta} f$ is the unique solution of (15) in $\mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ with vanishing $X_{0}(s)$-component at time $s$.
3. The map from $\mathrm{BC}_{s}^{0}\left(\mathbb{R}, X^{\odot *}\right)$ to $\mathrm{BC}_{s}^{0}(\mathbb{R}, X)$ given by $f \mapsto\left(I-P_{0}(\cdot)\right)\left(\mathcal{K}_{s}^{0} f\right)(\cdot)$ is well-defined, linear and bounded.

Proof. We start by proving the first assertion. Let $\varepsilon \in(0, \eta)$ be given and notice that for a given $f \in \mathrm{BC}_{s}^{\eta}\left(\mathbb{R}, X^{\odot \star}\right)$, the three integrals in the definition of $\mathcal{K}_{s}^{\eta}$ define functions $I_{0}(\cdot, s): \mathbb{R} \rightarrow X^{\odot \star}$ and $I_{i}: \mathbb{R} \rightarrow X^{\odot \star}$ for $i \in\{+,-\}$. We have to show that $I_{0}(\cdot, s)$ and $I_{i}$ are well-defined continuous functions that take values in $j(X)$ and satisfy certain estimates.
$I_{0}(\cdot, s)$ : The straightforward estimate

$$
\begin{equation*}
\left\|I_{0}(t, s)\right\| \leq K_{\varepsilon} N\|f\|_{\eta, s} \frac{e^{\eta|t-s|}}{\eta-\varepsilon}, \quad \forall t \in \mathbb{R} \tag{16}
\end{equation*}
$$

proves that $I_{0}(\cdot, s)$ is a well-defined weak ${ }^{\star}$ integral. Let $\tau \in[s, t]$ be given. By Hypothesis 2 we know that $P_{0}^{\odot \star}(\tau) f(\tau) \in j\left(X_{0}(\tau)\right)$ and so

$$
\begin{equation*}
U^{\odot \star}(t, \tau) P_{0}^{\odot \star}(\tau) f(\tau)=U^{\odot \star}(t, \tau) j j^{-1} P_{0}^{\odot \star}(\tau) f(\tau)=j U_{0}(t, \tau) j^{-1} P_{0}^{\odot \star}(\tau) f(\tau) \tag{17}
\end{equation*}
$$

Hence,

$$
I_{0}(t, s)=j \int_{s}^{t} U_{0}(t, \tau) j^{-1} P_{0}^{\odot \star}(\tau) f(\tau) d \tau \in j(X), \quad \forall t \in \mathbb{R}
$$

The map $I_{0}(\cdot, s)$ is continuous due to Lemma 2 because $[s, t]$ is compact and the maps $P_{0}^{\odot \star}$ and $f$ are is continuous.
$I_{+}$: Notice that

$$
\begin{equation*}
\left\|I_{+}(t)\right\| \leq K_{\varepsilon} N\|f\|_{\eta, s} e^{b t} \int_{t}^{\infty} e^{-b \tau+\eta|\tau-s|} d \tau \tag{18}
\end{equation*}
$$

and to prove norm boundedness of $I_{+}$, we have to evaluate the integral in the last estimate above. A calculation shows that

$$
\int_{t}^{\infty} e^{-b \tau+\eta|\tau-s|} d \tau= \begin{cases}\frac{e^{-b t}}{b-\eta} e^{\eta(t-s)}, & t \geq s  \tag{19}\\ \frac{e^{-b t}}{b+\eta} e^{\eta(s-t)}-\frac{e^{-b s}}{b+\eta}+\frac{e^{-b s}}{b-\eta}, & t \leq s\end{cases}
$$

We want to estimate the $t \leq s$ case. Notice that for real numbers $\alpha \geq \beta$ we have

$$
(\alpha-\beta)\left(\frac{1}{b+\eta}-\frac{1}{b-\eta}\right)=\frac{-2 \eta(\alpha-\beta)}{(b+\eta)(b-\eta)} \leq 0
$$

since $\eta<b$ by assumption. Hence,

$$
\frac{\alpha}{b+\eta}+\frac{\beta}{b-\eta} \leq \frac{\alpha}{b-\eta}+\frac{\beta}{b+\eta} .
$$

We want to replace $\alpha$ by $e^{-b t+\eta s-\eta t}$ and $\beta$ by $e^{-b s}$ and therefore we have to show that $-b t+\eta s-\eta t+b s \geq$ 0 which is true because $-b t+\eta s-\eta t+b s=(s-t)(b+\eta) \geq 0$ since $s-t \geq 0$. Filling this into (19) yields

$$
\int_{t}^{\infty} e^{-b \tau+\eta|\tau-s|} d \tau \leq \frac{e^{-b t}}{b-\eta} e^{\eta|t-s|}, \quad \forall t, s \in \mathbb{R}
$$

Filling this back into (18) yields for all $t \in \mathbb{R}$ that

$$
\begin{equation*}
\left\|I_{+}(t)\right\| \leq K_{\varepsilon} N\|f\|_{\eta, s} \frac{e^{\eta|t-s|}}{b-\eta}<\infty \tag{20}
\end{equation*}
$$

and so we conclude that $I_{+}$is well-defined. Let $\tau \in[t, \infty)$ be given. By Hypothesis 2 we know that $P_{+}^{\odot \star}(\tau) f(\tau) \in j\left(X_{+}(\tau)\right)$ and so

$$
I_{+}(t)=j \int_{t}^{\infty} U_{+}(t, \tau) j^{-1} P_{+}^{\odot \star}(\tau) f(\tau) d \tau \in j(X), \quad \forall t \in \mathbb{R}
$$

As $U^{\odot \star}(t, \tau)$ restricted to $j\left(X_{0}^{+}(\tau)\right)$ is invertible, we can adjust the proof from Lemma 2 to prove continuity of the limiting function $v(\cdot, \infty, \cdot, g)$ for a continuous function $g:[t, \infty) \rightarrow X^{\odot \star}$ under the assumption that $I_{+}$is bounded in norm. The fact that $I_{+}$is bounded in norm follows from (20) and the continuity of $g$ holds because $P_{+}^{\odot \star}$ and $f$ are continuous.
$I_{-}$: Notice that

$$
\left\|I_{-}(t)\right\| \leq K_{\varepsilon} N\|f\|_{\eta, s} e^{a t} \int_{-\infty}^{t} e^{-a \tau+\eta|\tau-s|} d \tau
$$

where this last integral is closely related to (19). A similar calculation shows that

$$
\begin{equation*}
\left\|I_{-}(t)\right\| \leq K_{\varepsilon} N\|f\|_{\eta, s} \frac{e^{\eta|t-s|}}{-a-\eta}, \quad \forall t \in \mathbb{R} \tag{21}
\end{equation*}
$$

which proves that $I_{-}$is well-defined. With the notation from Lemma 2 we have that $I_{-}(t)=$ $v(t, t,-\infty, g)$ with the continuous map $g$ defined as $g(\tau)=P_{-}^{\odot \star}(\tau) f(\tau)$ for all $\tau \in(-\infty, t]$, since $P_{-}^{\odot \star}$ and $f$ are assumed to be continuous. We conclude from this lemma that $I_{-}(t)$ takes values in $j(X)$ for all $t \in \mathbb{R}$ and that $I_{-}$is continuous.

Due to linearity, we have that $\mathcal{K}_{s}^{\eta} f \in C(\mathbb{R}, X)$ and combining the estimates (16), (20) and (21) yield

$$
\left\|\mathcal{K}_{s}^{\eta}\right\|_{\eta, s} \leq\left\|j^{-1}\right\| K_{\varepsilon} N\left(\frac{1}{\eta-\varepsilon}+\frac{1}{b-\eta}+\frac{1}{-a-\eta}\right)<\infty
$$

which implies $\mathcal{K}_{s}^{\eta}$ is a bounded linear operator from $\mathrm{BC}_{s}^{\eta}\left(\mathbb{R}, X^{\odot \star}\right)$ to $\mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$.
Let us now prove the second assertion by showing first that $\mathcal{K}_{s}^{\eta}$ is indeed a solution of (15). Let $f \in \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ and set $u=\mathcal{K}_{s}^{\eta} f$. Then, a straightforward computation shows that

$$
U(t, s) u(s)+j^{-1} \int_{s}^{t} U^{\odot \star}(t, \tau) f(\tau) d \tau=u(t)
$$

and so $u$ is indeed a solution of (15). Let us now prove that $u$ has vanishing $X_{0}(s)$-component at time $s$ i.e. $P_{0}(s) u(s)=0$. The mutual orthogonality of the projections implies

$$
\begin{aligned}
P_{0}(s) u(s) & =P_{0}(s)\left(j^{-1} \int_{\infty}^{s} U^{\odot \star}(s, \tau) P_{+}^{\odot \star}(\tau) f(\tau) d \tau+j^{-1} \int_{-\infty}^{s} U^{\odot \star}(s, \tau) P_{-}^{\odot \star}(\tau) f(\tau) d \tau\right) \\
& =j^{-1} \int_{\infty}^{s} U^{\odot \star}(s, \tau) P_{0}^{\odot \star}(\tau) P_{+}^{\odot \star}(\tau) f(\tau) d \tau+j^{-1} \int_{-\infty}^{s} U^{\odot \star}(s, \tau) P_{0}^{\odot \star}(\tau) P_{-}^{\odot \star}(\tau) f(\tau) d \tau \\
& =0
\end{aligned}
$$

It only remains to show that $u$ is the unique solution of (15) in $\mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$. Let $v \in \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ be another solution of (15) with vanishing $X_{0}(s)$-component at time $s$. Then the function $w:=u-v$ is an element of $\mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ and satisfies $w(t)=U(t, s) w(s)$ for $(s, t) \in \Omega_{\mathbb{R}}$. Proposition 4 shows us that $w(s) \in X_{0}(s)$ and notice that $P_{0}(s) w(s)=0$ since $u$ and $v$ have both vanishing $X_{0}(s)$-component at time $s$. From Hypothesis 1 we know that $w(t)=U_{0}(t, s) w(s)$ is in $X_{0}(t)$ for all $t \in \mathbb{R}$. Hence,

$$
P_{0}(t) w(t)=P_{0}(t) U_{0}(t, s) w(s)=U_{0}(t, s) P_{0}(s) w(s)=0, \quad \forall t \in \mathbb{R}
$$

and so $w=0$ i.e. $u=v$.
Let us now prove the third assertion. Take $f \in \mathrm{BC}_{s}^{0}\left(\mathbb{R}, X^{\odot \star}\right)$, then

$$
\left\|\left(\mathcal{K}_{s}^{0} f\right)(t)\right\| \leq\left\|j^{-1}\right\| K_{\varepsilon} N\|f\|_{0, s}\left(\frac{1}{-a}+\frac{1}{b}\right), \quad \forall t \in \mathbb{R}
$$

and because $\mathcal{K}_{s}^{0} f$ has vanishing $X_{0}(s)$-component at time $s$, we get

$$
\|\left(I-P_{0}(t)\left(\mathcal{K}_{s}^{0} f\right)(t)\|\leq\| j^{-1}\left\|K_{\varepsilon} N\right\| f \|_{0, s}\left(\frac{1}{-a}+\frac{1}{b}\right)\right.
$$

and so $\|\left(I-P_{0}(\cdot)\left(\mathcal{K}_{s}^{0} f\right)(\cdot)\left\|_{0, s} \leq\right\| j^{-1}\left\|K_{\varepsilon} N\right\| f \|_{0, s}\left(\frac{1}{-a}+\frac{1}{b}\right)\right.$ which shows that $\left(I-P_{0}(\cdot)\left(\mathcal{K}_{s}^{0} f\right)(\cdot)\right.$ is in $\mathrm{BC}_{s}^{0}(\mathbb{R}, X)$. Because the projections are linear and $\mathcal{K}_{s}^{0}$ is linear, we have that $f \mapsto\left(I-P_{0}(\cdot)\left(\mathcal{K}_{s}^{0} f\right)(\cdot)\right.$ is linear. Clearly the operator norm of $f \mapsto\left(I-P_{0}(\cdot)\right)\left(\mathcal{K}_{s}^{0} f\right)(\cdot)$ is bounded above by $\left\|j^{-1}\right\| K_{\varepsilon} N\left(\frac{1}{-a}+\frac{1}{b}\right)<$ $\infty$ and so this map is bounded, independent of $s$.

### 3.3 Modification of the nonlinearity

To prove the existence of a center manifold, a key step will be to use Banach fixed point theorem on some specific fixed point operator. This operator we will be of course linked to the inhomogeneous equation (15). However, we can not expect that any nonlinear operator $R(t, \cdot): X \rightarrow X^{\odot \star}$ for fixed $t \in \mathbb{R}$ will impose a Lipschitz condition on the fixed point operator that will be constructed. As we are only interested in the local behaviour of solutions near zero, we can modify the nonlinearity $R(t, \cdot)$ outside a ball of radius $\delta>0$ such that eventually the fixed point operator will become a contraction. To modify this nonlinearity, introduce the $C^{\infty}$-smooth cut-off function $\xi:[0, \infty) \rightarrow \mathbb{R}$ as

$$
\xi(s) \in \begin{cases}\{1\}, & 0 \leq s \leq 1 \\ {[0,1],} & 0 \leq s \leq 2 \\ 0, & s \geq 2\end{cases}
$$

and define then for any $\delta>0$ and $s \in \mathbb{R}$ the $\delta$-modification of $R$ as the operator $R_{\delta, s}: \mathbb{R} \times X \rightarrow X^{\odot \star}$ with action

$$
R_{\delta, s}(t, u):=R(t, u) \xi\left(\frac{\left\|P_{0}(s) u\right\|}{N \delta}\right) \xi\left(\frac{\left\|\left(P_{-}(s)+P_{+}(s)\right) u\right\|}{N \delta}\right), \quad \forall(t, u) \in \mathbb{R} \times X
$$

This $\delta$-modification of $R$ will ensure that the nonlinearity is globally Lipschitz. The proof is very similar to that of [21, Proposition 32] and therefore omitted.
Proposition 6. For $s \in \mathbb{R}$ and sufficiently small $\delta>0$, the operator $R_{\delta, s}(t, \cdot)$ is globally Lipschitz continuous for any $t \in \mathbb{R}$ with Lipschitz constant $L_{R_{\delta}} \rightarrow 0$ as $\delta \downarrow 0$ independent of $s$.

Let us introduce now for a given $\delta$-modification of $R$ the substitution operator $\tilde{R}_{\delta, s}: \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X) \rightarrow$ $\mathrm{BC}_{s}^{\eta}\left(\mathbb{R}, X^{\odot \star}\right)$ as

$$
\tilde{R}_{\delta, s}(u):=R_{\delta, s}(\cdot, u(\cdot)), \quad \forall u \in \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)
$$

and we show that this operator inherits the same properties as $R_{\delta, s}$. The proof is analogous to that of [21, Corollary 33] and therefore omitted.
Corollary 7. For $s \in \mathbb{R}$ and sufficiently small $\delta>0$, the map $\tilde{R}_{\delta, s}$ is well-defined, globally Lipschitz continuous with Lipschitz constant $L_{R_{\delta}} \rightarrow 0$ as $\delta \downarrow 0$ independent of $\eta$ and s.

### 3.4 Existence of a Lipschitz center manifold

Our next goal is to define a parameterized fixed point operator such that its fixed points correspond to exponentially bounded solutions on $\mathbb{R}$ of the modified equation

$$
\begin{equation*}
u(t)=U(t, s) u(s)+j^{-1} \int_{s}^{t} U^{\odot \star}(t, \tau) R_{\delta, s}(\tau, u(\tau)) d \tau, \quad-\infty<s \leq t<\infty \tag{22}
\end{equation*}
$$

for some small $\delta>0$. For a given $\eta \in(0, \min \{-a, b\})$ and $s \in \mathbb{R}$, we define the fixed point operator $\mathcal{G}_{s}: \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X) \times X_{0}(s) \rightarrow \mathrm{BC}_{s}^{\eta}\left(\mathbb{R}, X^{\odot \star}\right)$ as

$$
\begin{equation*}
\mathcal{G}_{s}(u, \varphi):=U(\cdot, s) \varphi+\mathcal{K}_{s}^{\eta}\left(\tilde{R}_{\delta, s}(u)\right), \quad \forall(u, \varphi) \in \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X) \times X_{0}(s) \tag{23}
\end{equation*}
$$

where its second argument in $X_{0}(s)$ is treated as a parameter.
We first show that $\mathcal{G}_{s}$ has a unique fixed point and is globally Lipschitz.
Theorem 8. Let $\eta \in(0, \min \{-a, b\})$ and $s \in \mathbb{R}$ be given. If $\delta>0$ is sufficiently small, then the following two statements hold.

1. For every $\varphi \in X_{0}(s)$ the equation $u=\mathcal{G}_{s}(u, \varphi)$ has a unique solution $u=u_{s}^{\star}(\varphi)$.
2. The map $u_{s}^{\star}: X_{0}(s) \rightarrow \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ is globally Lipschitz and satisfies $u_{s}^{\star}(0)=0$.

Proof. Let $\varepsilon \in(0, \eta)$ be given. Take $u, v \in \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ and $\varphi, \psi \in X_{0}(s)$ because then

$$
\begin{aligned}
\left\|\mathcal{G}_{s}(u, \varphi)-\mathcal{G}_{s}(v, \psi)\right\|_{\eta, s} & \leq \sup _{t \in \mathbb{R}} e^{-\eta|t-s|}\left\|U_{0}(t, s)(\varphi-\psi)\right\|+\left\|\mathcal{K}_{s}^{\eta}\right\| L_{R_{\delta}}\|u-v\|_{\eta, s} \\
& \leq K_{\varepsilon}\|\varphi-\psi\|+\left\|\mathcal{K}_{s}^{\eta}\right\| L_{R_{\delta}}\|u-v\|_{\eta, s}
\end{aligned}
$$

where we used the fact that $\varepsilon<\eta$ and the exponential trichotomy on the center eigenspace since $\varphi-\psi \in X_{0}(s)$. By Corollary 7 there exists a $\delta_{1}>0$ such that for all $0<\delta \leq \delta_{1}$ we have that $L_{R_{\delta}}\left\|\mathcal{K}_{s}^{\eta}\right\| \leq \frac{1}{2}$.

1. Set $\psi=\varphi$ in the previous estimate, because then for $0 \leq \delta \leq \delta_{1}$ we have that

$$
\left\|\mathcal{G}_{s}(u, \varphi)-\mathcal{G}_{s}(v, \varphi)\right\|_{\eta, s} \leq \frac{1}{2}\|u-v\|_{\eta, s}
$$

which means that $\mathcal{G}_{s}(\cdot, \varphi)$ is a contraction on the Banach space $\mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ equipped with the $\|\cdot\|_{\eta, s^{-}}$ norm. It follows from the contraction mapping principle that $\mathcal{G}_{s}(\cdot, \varphi)$ has a unique fixed point $u_{s}^{\star}(\varphi)$.
2. Let $u_{s}^{\star}(\varphi)$ and $u_{s}^{\star}(\psi)$ be unique fixed points of the operators $\mathcal{G}_{s}(u, \varphi)$ and $\mathcal{G}_{s}(u, \psi)$ respectively. Then,

$$
\left\|u_{s}^{\star}(\varphi)-u_{s}^{\star}(\psi)\right\|_{\eta, s}=\left\|\mathcal{G}_{s}\left(u_{s}^{\star}(\varphi), \varphi\right)-\mathcal{G}_{s}\left(u_{s}^{\star}(\psi), \psi\right)\right\|_{\eta, s} \leq K_{\varepsilon}\|\varphi-\psi\|+\frac{1}{2}\left\|u_{s}^{\star}(\varphi)-u_{s}^{\star}(\psi)\right\|_{\eta, s} .
$$

This implies that $\left\|u_{s}^{\star}(\varphi)-u_{s}^{\star}(\psi)\right\|_{\eta, s} \leq 2 K_{\varepsilon}\|\varphi-\psi\|$ and so $u_{s}^{\star}$ is globally Lipschitz. Since $u_{s}^{\star}(0)=$ $\mathcal{G}_{s}\left(u_{s}^{\star}(0), 0\right)=0$ the second assertion follows.

The $\operatorname{map} \mathcal{C}: X_{0} \rightarrow X$ defined by

$$
\begin{equation*}
\mathcal{C}(t, \varphi):=u_{t}^{\star}(\varphi)(t), \quad \forall(t, \varphi) \in X_{0}, \tag{24}
\end{equation*}
$$

ensures the existence of a center manifold in the following way.
Definition 9. The global center manifold for (22) is defined as

$$
\mathcal{W}^{c}:=\left\{(t, \mathcal{C}(t, \varphi)) \in \mathbb{R} \times X: \varphi \in X_{0}(t)\right\}
$$

whose $s$-fibers are defined as $\mathcal{W}^{c}(s):=\left\{\mathcal{C}(s, \varphi) \in X: \varphi \in X_{0}(s)\right\}$.

Proposition 10. If $\eta \in(0, \min \{-a, b\})$ and $s \in \mathbb{R}$, then

$$
\mathcal{W}^{c}(s)=\left\{\varphi \in X: \text { there exists a solution of }(22) \text { on } \mathbb{R} \text { through } \varphi \text { belonging to } \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)\right\}
$$

Proof. Let $\varphi \in \mathcal{W}^{c}(s)$, then $\varphi=\mathcal{C}(s, \psi)=u_{s}^{\star}(\psi)(s)$ for some $\psi \in X_{0}(s)$. We show that $u=u_{s}^{\star}(\psi)$ is a solution of (22) on $\mathbb{R}$ through $\varphi$ which belongs to $\mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$. Part 2 of Proposition 5 shows us that $\mathcal{K}_{s}^{\eta} \tilde{R}_{\delta, s}(u)$ is the unique solution of (15) in $\mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ with $f=\tilde{R}_{\delta, s}(u)$. Because $u=u_{s}^{\star}(\psi)$ is a fixed point of $\mathcal{G}_{s}(\cdot, \psi)$ we obtain

$$
\begin{aligned}
u(t) & =U(t, s) \psi+\left(\mathcal{K}_{s}^{\eta} \tilde{R}_{\delta, s}(u)\right)(t) \\
& =U(t, s) \psi+U(t, s)\left(\mathcal{K}_{s}^{\eta} \tilde{R}_{\delta, s}(u)\right)(s)+j^{-1} \int_{s}^{t} U^{\odot \star}(t, \tau) R_{\delta, s}(\tau, u(\tau)) d \tau \\
& =U(t, s) \psi+U(t, s)(u(s)-\psi)+j^{-1} \int_{s}^{t} U^{\odot \star}(t, \tau) R_{\delta, s}(\tau, u(\tau)) d \tau \\
& =U(t, s) u(s)+j^{-1} \int_{s}^{t} U^{\odot \star}(t, \tau) R_{\delta, s}(\tau, u(\tau)) d \tau
\end{aligned}
$$

for all $(t, s) \in \Omega_{\mathbb{R}}$. This shows that $u$ is a solution of $(22)$ on $\mathbb{R}$ through $\varphi=u_{s}^{\star}(\psi)(s)=u(s)$ which belongs to $\mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$.

To show the converse, let $\varphi \in X$ be given such that there exists a solution $u$ in $\mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ of (22) that satisfies $u(s)=\varphi$. For $(t, s) \in \Omega_{\mathbb{R}}$ it is possible to rewrite (22) as

$$
\begin{aligned}
u(t) & =U(t, s) P_{0}(s) u(s)+U(t, s)\left(I-P_{0}(s)\right) u(s)+j^{-1} \int_{s}^{t} U^{\odot \star}(t, \tau) R_{\delta, s}(\tau, u(\tau)) d \tau \\
& =U(t, s) P_{0}(s) u(s)+\left(\mathcal{K}_{s}^{\eta} \tilde{R}_{\delta, s}(u)\right)(t)
\end{aligned}
$$

where part 2 of Proposition 5 was used in the last equality. Hence, if we define $\psi:=P_{0}(s) u(s)$, then

$$
u(t)=U(t, s) \psi+\left(\mathcal{K}_{s}^{\eta} \tilde{R}_{\delta, s}(u)\right)(t), \quad \forall(t, s) \in \Omega_{\mathbb{R}}
$$

which implies $u=\mathcal{G}_{s}(u, \psi)$. As we know from Theorem 8 that this fixed point problem has a unique solution $u=u_{s}^{\star}(\psi)$, we have that $\varphi=u(s)=u_{s}^{\star}(\psi)=\mathcal{C}(s, \psi) \in \mathcal{W}^{c}(s)$, which completes the proof.

Recall from part 2 of Theorem 8 that for a fixed $t \in \mathbb{R}$ the map $u_{t}^{\star}: X_{0}(t) \rightarrow \mathrm{BC}_{t}^{\eta}(\mathbb{R}, X)$ is globally Lipschitz. Hence, from the definition of the map $\mathcal{C}$ given in (24), we see that the map $\mathcal{C}(t, \cdot): X_{0}(t) \rightarrow X$ is globally Lipschitz, where the Lipschitz constant depends on $t$ and so this shows that the map $\mathcal{C}$ is only fiberwise Lipschitz. However, it is proven in Corollary 28 that the Lipschitz constant can be chosen independently of the fiber, and so we can say that $\mathcal{W}^{c}$ is the global Lipschitz center manifold.

Let $B_{\delta}(X)$ denote the open ball centered around the origin in $X$ with radius $\delta>0$. From the cut-off function $\xi$ it is clear that the restrictions of $R(t, \cdot)$ and $R_{\delta, s}(t, \cdot)$ to this ball are equal for any $t \in \mathbb{R}$. Hence, if we restrict the unknown function $u$ to take only values in $B_{\delta}(X)$, then (13) and (22) coincide as well.

Definition 11. The local center manifold $\mathcal{W}_{\text {loc }}^{c}$ for (13) is defined as

$$
\mathcal{W}_{\mathrm{loc}}^{c}:=\left\{(t, \mathcal{C}(t, \varphi)) \in \mathbb{R} \times X: \varphi \in X_{0}(t) \text { and } \mathcal{C}(t, \varphi) \in B_{\delta}(X)\right\}
$$

whose $s$-fibers are defined as $\mathcal{W}_{\mathrm{loc}}^{c}(s):=\left\{\mathcal{C}(s, \varphi) \in X: \varphi \in X_{0}(s)\right.$ and $\left.\mathcal{C}(s, \varphi) \in B_{\delta}(X)\right\}$.
By construction, the center manifolds $\mathcal{W}^{c}$ and $\mathcal{W}_{\text {loc }}^{c}$ non-canonically depend on the choice of $\delta$ and the cut-off function $\xi$. This is the famous non-uniqueness property of the center manifold.

### 3.5 Properties of the center manifold

We will show some important properties that the local center manifold $\mathcal{W}_{\text {loc }}^{c}$ enjoys. We start off with the following result, that is inspired by [2, Theorem 5.4.2] and [21, Corollary 38].

Theorem 12. The local center manifold $\mathcal{W}_{\text {loc }}^{c}$ has the following properties.

1. $\mathcal{W}_{\text {loc }}^{c}$ is locally positively invariant: if $(s, \varphi) \in \mathcal{W}_{\text {loc }}^{c}$ and $s<t_{\varphi} \leq \infty$ are such that $S(t, s, \varphi) \in$ $B_{\delta}(X)$ for all $t \in\left[s, t_{\varphi}\right)$, then $(t, S(t, s, \varphi)) \in \mathcal{W}_{\mathrm{loc}}^{c}$.
2. $\mathcal{W}_{\text {loc }}^{c}$ contains every solution of (13) that exists on $\mathbb{R}$ and remains sufficiently small for all positive and negative time i.e. if $u: \mathbb{R} \rightarrow B_{\delta}(X)$ is a solution of (13), then $(t, u(t)) \in \mathcal{W}_{\text {loc }}^{c}$ for all $t \in \mathbb{R}$.
3. If $(s, \varphi) \in \mathcal{W}_{\text {loc }}^{c}$, then $S(t, s, \varphi)=u_{t}^{\star}\left(P_{0}(t) S(t, s, \varphi)\right)(t)=\mathcal{C}\left(t, P_{0}(t) S(t, s, \varphi)\right)$ for all $t \in\left[s, t_{\varphi}\right)$.
4. $\mathbb{R} \times\{0\} \in \mathcal{W}_{\text {loc }}^{c}$ and $\mathcal{C}(t, 0)=0$ for all $t \in \mathbb{R}$.

Proof. 1. Proposition 10 implies that there exists a solution $u \in \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ of (22) through $\varphi$ that can be chosen to be $u(s)=\varphi$. So, $S(\cdot, s, \varphi)$ and $u$ are both solutions of (22) on $\left[s, t_{\varphi}\right)$ and $S(s, s, \varphi)=$ $\varphi=u(s)$. This means $S(\cdot, s, \varphi)$ and $u$ coincide on $\left[s, t_{\varphi}\right)$ by uniqueness of solutions. This means $S(t, s, \varphi) \in \mathcal{W}^{c}(t)$ for all $t \in\left[s, t_{\varphi}\right)$ and so $(t, S(t, s, \varphi)) \in \mathcal{W}^{c}$. Since $\mathcal{W}_{\text {loc }}^{c}=\mathcal{W}^{c} \cap B_{\delta}(X)$ the result follows.
2. If $u$ is such a solution, then $u \in \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$. The assumption that $u$ takes values in $B_{\delta}(X)$ and Proposition 10 together imply the result.
3. Since $\varphi \in X_{0}(s)$ we have that $S(s, s, \varphi)=\varphi=u_{s}^{\star}\left(P_{0}(s) \varphi\right)(s)=\mathcal{C}\left(s, P_{0}(s) \varphi\right)$ and so the asserted equation holds at time $t=s$. Since $\mathcal{W}_{\text {loc }}^{c}$ is locally positively invariant, we have that $(t, S(t, s, \varphi))=$ $\left(t, u_{t}^{\star}(\psi(t))(t)\right) \in \mathcal{W}_{\text {loc }}^{c}$ for some $\psi(t) \in X_{0}(t)$ and we also have $\left(t, u_{t}^{\star}\left(P_{0}(t) S(t, s, \varphi)\right)(t) \in \mathcal{W}_{\text {loc }}^{c}\right.$. Because both solutions started at $\varphi$, we must have by uniqueness of solutions that $(t, S(t, s, \varphi))=$ $\left(t, u_{t}^{\star}\left(P_{0}(t) S(t, s, \varphi)\right)(t)\right)=\left(t, \mathcal{C}\left(t, P_{0}(t) S(t, s, \varphi)\right)\right)$.
4. Let $t \in \mathbb{R}$ be given. Notice that $\mathcal{C}(t, 0)=u_{t}^{\star}(0)(t)=0$, where the last equality follows from part 2 of Theorem 8. Clearly, $(t, 0)=(t, \mathcal{C}(t, 0)) \in \mathcal{W}_{\text {loc }}^{c}$.

The next step is to show that the map $\mathcal{C}$ inherits the same order of smoothness as the time-dependent nonlinear perturbation $R$, namely the preselected integer $k \geq 1$. Proving additional smoothness of center manifolds requires work. A well-known technique to achieve smoothness is via the theory of scales of Banach spaces that is presented in Appendix B. We refer to Appendix B for the statements of the results and additional proofs. The main result is the following, and the proofs can be found in Corollary 35 and Theorem 36 .
Theorem 13. The center manifold $\mathcal{W}^{c}$ is $C^{k}$-smooth and its tangent bundle is $X_{0}$ i.e. $D_{2} \mathcal{C}(t, 0) \varphi=\varphi$ for all $(t, \varphi) \in X_{0}$. Furthermore, if the time-dependent nonlinear perturbation $R: \mathbb{R} \times X \rightarrow X^{\odot \star}$ is $T$-periodic in the first variable, then there exists a $\delta>0$ such that $\mathcal{C}(t+T, \varphi)=\mathcal{C}(t, \varphi)$ for all $t \in \mathbb{R}$ whenever $\|\varphi\|<\delta$.

To summarize, we have proven the following center manifold theorem in a $T$-periodic setting.
Theorem 14 (Local center manifold). Let $T_{0}$ be a $\mathcal{C}_{0}$-semigroup on a $\odot$-reflexive real Banach space $X$ and let $U$ be the strongly forward evolutionary system defined by (T-LAIE) that satisfies Hypothesis 1 and Hypothesis 2, where B is a T-periodic time-dependent bounded linear perturbation. Suppose that the real center eigenspace $X_{0}(t)$, defined for all $t \in \mathbb{R}$, has dimension $1 \leq n_{0}+1<\infty$. Furthermore, suppose that the time-dependent nonlinear perturbation $R$ is $T$-periodic in the first component, $C^{k}$ smooth and satisfies (9).

Then there exists a $C^{k}$-smooth map $\mathcal{C}: \mathbb{R} \times V \rightarrow X$, where $\mathbb{R} \times V$ is a neighborhood of $\mathbb{R} \times\{0\}$ in the center fiber bundle $X_{0}$ such that the manifold $\mathcal{W}_{\text {loc }}^{c}:=\{(t, \mathcal{C}(t, \varphi)) \in \mathbb{R} \times X: t \in \mathbb{R}$ and $\mathcal{C}(t, \varphi) \in V\}$ is $T$-periodic, $C^{k}$-smooth, $\left(n_{0}+1\right)$-dimensional and locally positively invariant for the time-dependent semiflow $S$ generated by (13).

### 3.6 The special case of classical DDEs

Let us now specify the setting of classical DDEs, such that we can apply Theorem 14. Choose the Banach space $X:=C\left([-h, 0], \mathbb{R}^{n}\right)$ as the state space for some finite delay $h>0$ equipped with the supremum norm $\|\cdot\|_{\infty}$. Consider a $C^{k}$-smooth operator $F: X \rightarrow \mathbb{R}^{n}$ together with the initial value problem

$$
\begin{cases}\dot{x}(t)=F\left(x_{t}\right), & t \geq 0  \tag{DDE}\\ x_{0}=\varphi, & \varphi \in X\end{cases}
$$

where the history of $x$ at time $t \geq 0$, denoted by $x_{t} \in X$ is defined as

$$
\begin{equation*}
x_{t}(\theta):=x(t+\theta), \quad \forall \theta \in[-h, 0] \tag{25}
\end{equation*}
$$

By a solution of (DDE) we mean a continuous function $x:\left[-h, t_{\varphi}\right) \rightarrow \mathbb{R}^{n}$ for some final time $0<t_{\varphi} \leq$ $\infty$ that is continuously differentiable on $\left[0, t_{\varphi}\right.$ ) and satisfies (DDE). When $t_{\varphi}=\infty$, we call $x$ a global solution. We say that a function $\gamma: \mathbb{R} \rightarrow \mathbb{R}^{n}$ is a periodic solution of (DDE) if there exists a minimal $T>0$, called the period of $\gamma$ such that $\gamma_{T}=\gamma_{0}$. We call $\Gamma:=\left\{\gamma_{t} \in X: t \in \mathbb{R}\right\}$ a periodic orbit or (limit) cycle in $X$. It follows from [15, Corollary 10.3.1] that $\gamma \in C^{k+1}\left(\mathbb{R}, \mathbb{R}^{n}\right)$.

We want to study (DDE) near the periodic solution $\gamma$, and it is therefore more convenient to translate $\gamma$ towards the origin. More specifically, if $x$ is a solution of (DDE), then for $y$ defined as $x=\gamma+y$, we have that $y$ satisfies the nonlinear time-dependent DDE

$$
\begin{equation*}
\dot{y}(t)=L(t) y_{t}+G\left(t, y_{t}\right) \tag{T-DDE}
\end{equation*}
$$

where $L(t):=D F\left(\gamma_{t}\right)$ denotes the Fréchet derivative of $F$ evaluated at the point $\gamma_{t} \in X$ and $G(t, \cdot):=$ $F\left(\gamma_{t}+\cdot\right)-F\left(\gamma_{t}\right)-L(t)$ consists of solely nonlinear terms and is of the class $C^{k-1}$.

Before we can understand the relation between (T-DDE) and (T-AIE), we first have to apply the sun-star calculus machinery onto the setting of classical DDEs. The starting point is the trivial $D D E$

$$
\left\{\begin{array}{lr}
\dot{x}(t)=0, & t \geq 0  \tag{26}\\
x_{0}=\varphi, & \varphi \in X,
\end{array}\right.
$$

which has the unique global solution

$$
x(t)= \begin{cases}\varphi(t), & -h \leq t \leq 0  \tag{27}\\ \varphi(0), & t \geq 0\end{cases}
$$

Using this solution, we define the $\mathcal{C}_{0}$-semigroup $T_{0}$ on $X$, also called the shift semigroup, as

$$
\left(T_{0}(t) \varphi\right)(\theta):=\left\{\begin{array}{ll}
\varphi(t+\theta), & -h \leq t+\theta \leq 0,  \tag{28}\\
\varphi(0), & t+\theta \geq 0,
\end{array} \quad \forall \varphi \in X, t \geq 0, \theta \in[-h, 0]\right.
$$

Notice that $T_{0}$ generates the solution of (27) in the sense that $T_{0}(t) \varphi=x_{t}$ for all $t \geq 0$. For this specific combination of $X$ and $T_{0}$, the abstract duality structure from Section 2.1 can be constructed explicitly, see [13, Section II.5]. We only summarize here the basic results. A representation theorem by F. Riesz [25] enables us to identify $X^{\star}=C\left([-h, 0], \mathbb{R}^{n}\right)^{\star}$ with the Banach space NBV $\left([0, h], \mathbb{R}^{n \star}\right)$ consisting of functions $\zeta:[0, h] \rightarrow \mathbb{R}^{n \star}$ that are normalized by $\zeta(0)=0$, are continuous from the right on $(0, h)$ and have bounded variation. From (3) it turns out that

$$
X^{\odot} \cong \mathbb{R}^{n \star} \times L^{1}\left([0, h], \mathbb{R}^{n \star}\right)
$$

where $\cong$ stands for an isometric isomorphism and $\mathbb{R}^{n \star}$ denotes the linear space of row vectors over $\mathbb{R}$. Computing the dual of $X^{\odot}$ and afterwards the restriction to the maximal space of strong continuity yields

$$
X^{\odot \star} \cong \mathbb{R}^{n} \times L^{\infty}\left([-h, 0], \mathbb{R}^{n}\right), \quad X^{\odot \odot} \cong \mathbb{R}^{n} \times C\left([-h, 0], \mathbb{R}^{n}\right)
$$

The canonical embedding $j$ defined in (4) has action $j \varphi=(\varphi(0), \varphi)$ for $\varphi \in X$, mapping $X$ onto $X^{\odot \odot, ~}$ meaning that $X$ is $\odot$-reflexive with respect to the shift semigroup $T_{0}$.

Let us now specify the time-dependent bounded linear perturbation $B$ from Section 2.2. For $i=$ $1, \ldots, n$ we denote $r_{i}^{\odot \star}:=\left(e_{i}, 0\right)$, where $e_{i}$ is the $i$ th standard basic vector of $\mathbb{R}^{n}$. It is conventional and convenient to introduce the shorthand notation

$$
w r^{\odot \star}:=\sum_{i=1}^{n} w_{i} r_{i}^{\odot \star}, \quad \forall w=\left(w_{1}, \ldots, w_{n}\right) \in \mathbb{R}^{n}
$$

and note that $w r^{\odot \star}=(w, 0) \in X^{\odot \star}$. We specify the time-dependent bounded linear perturbation as

$$
\begin{equation*}
B(t) \varphi:=[L(t) \varphi] r^{\odot \star}, \quad \forall t \in \mathbb{R}, \varphi \in X \tag{29}
\end{equation*}
$$

and since $F \in C^{k}\left(X, \mathbb{R}^{n}\right), t \mapsto \gamma_{t} \in X$ is $T$-periodic and of the class $C^{k}$, we have that $B \in$ $C^{k-1}\left(\mathbb{R}, \mathcal{L}\left(X, X^{\odot \star}\right)\right)$ it $T$-periodic and Lipschitz continuous. It is shown in [13, Theorem 3.1] that there is a one-to-one correspondence between solutions of the time-dependent linear problem

$$
\begin{cases}\dot{y}(t)=L(t) y_{t}, & t \geq s  \tag{T-LDDE}\\ y_{s}=\varphi, & \varphi \in X\end{cases}
$$

which is (T-DDE) with $G=0$, and the linear abstract integral equation (T-LAIE). Hence, $y_{t}=U(t, s) \varphi$ and so $y(t)=(U(t, s) \varphi)(0)$ for all $t \geq s$. Recall from Section 2.2 that we only had that $t \mapsto U(t, s) \varphi$ is weak ${ }^{\star}$ continuously differentiable when $\varphi \in j^{-1} \mathcal{D}\left(A_{0}^{\odot \star}\right)$. The additional smoothness on $B$ ensures that $t \mapsto A^{\odot \star}(t)$ and $t \mapsto A(t)$ are of the class $C^{k-1}$, which follows directly from (6) and (7) as $A_{0}^{\odot *}$ is constant in time. The following proposition shows that the smoothness propagates through $U$ when the set of initial conditions is taken appropriate.

Proposition 15. The following two assertions hold.

1. If $\varphi \in \mathcal{D}(A(s))$, then the map $\mathbb{R} \ni s \mapsto U(t, s) \varphi \in X$ is of the class $C^{k}$.
2. If $\varphi \in X$ such that $U(t, s) \varphi \in \mathcal{D}(A(t))$ for all $t \geq s$, then the map $\mathbb{R} \ni t \mapsto U(t, s) \varphi \in X$ is of the class $C^{k}$.

Proof. We start by showing the first assertion. Let $\varphi \in \mathcal{D}(A(s))$, then due to [4, Lemma 2.4 and Theorem 2.5] we know

$$
\begin{equation*}
\frac{\partial}{\partial s} U(t, s) \varphi=-U(t, s) A(s) \varphi \tag{30}
\end{equation*}
$$

where the partial derivative should be interpreted in the norm topology of $X$. Because $U$ is strongly continuous, the map $s \mapsto U(t, s) \psi$ is continuous for any $\psi \in X$. Furthermore, since the map $s \mapsto A(s)$ is of the class $C^{k-1}$ we have that $s \mapsto-U(t, s) A(s) \varphi$ is continuous, hence $s \mapsto U(t, s) \varphi$ is of the class $C^{1}$ due to (30).

We show the claim by (strong) induction. For $k=1$ the assertion holds. Suppose that $s \mapsto U(t, s) \psi$ is of the class $C^{l}$ for all $\psi \in X$ and $l \in\{1, \ldots, k-1\}$, then

$$
\begin{equation*}
\frac{\partial^{l+1}}{\partial s^{l+1}} U(t, s) \varphi=-\frac{\partial^{l}}{\partial s^{l}}[U(t, s) A(s) \varphi]=-\sum_{m=0}^{l} \frac{l!}{m!(l-m)!} \frac{\partial^{l-m}}{\partial s^{l-m}} U(t, s) \frac{\partial^{m}}{\partial s^{m}} A(s) \varphi \tag{31}
\end{equation*}
$$

due to the general Leibniz rule. Because $s \mapsto U(t, s) \psi$ is of the class $C^{l}$ for all $\psi \in X$, it is also of the class $C^{l-m}$ for all $m \in\{0, \ldots, l\}$ and since $s \mapsto A(s)$ is of the class $C^{k-1}$ it is certainly of the class $C^{m}$ for $m \in\{1, \ldots, l\}$. Hence, the right-hand side of (31) is continuous and so $s \mapsto U(t, s) \varphi$ is of the class $C^{l+1}$. As $l \in\{1, \ldots, k-1\}$ was chosen arbitrary it follows that $s \mapsto U(t, s) \varphi$ is of the class $C^{k}$.

We now show the second assertion. From [4, Lemma 2.4] we know that

$$
\begin{equation*}
\frac{1}{h}(U(s+h, s) \varphi-\varphi) \rightarrow A(s) \varphi, \quad \text { as } h \downarrow 0 \tag{32}
\end{equation*}
$$

in norm for any $s \in \mathbb{R}$ and $\varphi \in \mathcal{D}(A(s))$. Let $\varphi \in X$ such that $y=U(t, s) \varphi \in \mathcal{D}(A(t))$, then

$$
\begin{aligned}
\frac{1}{h}(U(t+h, s) \varphi-U(t, s) \varphi)-A(t) U(t, s) \varphi & =\frac{1}{h}(U(t+h, t) U(t, s) \varphi-U(t, s) \varphi)-A(t) U(t, s) \varphi \\
& =\frac{1}{h}(U(t+h, t) y-y)-A(t) y \rightarrow 0, \quad \text { as } h \downarrow 0
\end{aligned}
$$

in norm due to (32) since $s \in \mathbb{R}$ was chosen here arbitrary. Hence,

$$
\frac{\partial}{\partial t} U(t, s) \varphi=A(t) U(t, s) \varphi
$$

The same arguments can be used now as in the previous part to conclude that $t \mapsto U(t, s) \varphi$ is of the class $C^{k}$. The induction argument is similar and notice that the domain of the (higher order) derivatives of $t \mapsto A(t)$ is $X$ since the $A_{0}^{\odot \star}$ part vanishes after differentiation.

Let us now specify the time-dependent nonlinear perturbation $R$ from Section 2.3 as

$$
\begin{equation*}
R(t, \varphi):=G(t, \varphi) r^{\odot \star}, \quad \forall t \in \mathbb{R}, \varphi \in X \tag{33}
\end{equation*}
$$

which is $T$ periodic in the first component and of the class $C^{k-1}$. As in the linear case, we have to show that there exists a one-to-one correspondence between solutions of (T-DDE) and (T-AIE). A proof for this could not be found in the literature, but is given in Theorem 41 with additional preparatory material presented in Appendix C. Hence, the time-dependent semiflow $S$ presented in (10) generates solutions of (T-DDE) in the sense that $y_{t}=S(t, s, \varphi)$ and so $y(t)=S(t, s, \varphi)(0)$ for all $t \in\left[s, t_{\varphi}\right)$.

We are in the position to verify Hypothesis 1 and Hypothesis 2. First we have to decompose $X$ in a topological direct sum (11). To do this, define for any $s \in \mathbb{R}$ the monodromy operator $U(s+T, s) \in \mathcal{L}(X)$ (at time $s$ ), and note that iterates of this map are compact, see [13, Corollary XII.3.4 and Corollary XIII.2.2]. Hence, the spectrum $\sigma(U(s+T, s))$ is a countable set consisting of 0 and isolated eigenvalues (called Floquet multipliers) that can possibly accumulate to 0 . The following remark regarding real state spaces and spectral theory is important.

Remark 16. For using spectral theory on the real Banach space $X=C([-h, 0], \mathbb{R})$, we have to complexify $X$ and all discussed operators on $X$. This is not entirely trivial and is discussed in [13, Section III. 7 and Section IV.2]. To clarify, by the spectrum of the real operator $U(s+T, s)$, we mean the spectrum of its complexification $U_{\mathbb{C}}(s+T, s)$ on the complexified Banach space $X_{\mathbb{C}}$. For the ease of notation, we omit the additional symbols.

The number $\sigma \in \mathbb{C}$ for which $\lambda=e^{\sigma T}$ is called the Floquet exponent and are only determined up to additive multiples of $i \frac{2 \pi}{T}$. The dimension of the eigenspace $\operatorname{ker}(U(s+T, s)-\lambda)$ associated to $\lambda$ is also called the geometric multiplicity of $\lambda$ and is finite. Furthermore, it is shown in [13, Theorem3.3] that the Floquet multipliers are independent of the starting time $s$ and thus well-defined. Moreover, it is known that 1 is always a Floquet multiplier (called the trivial Floquet multiplier) with associated as eigenfunction $\dot{\gamma}_{s}$, see [13, Proposition XIV.2.6]. By compactness, there exist two closed $U(s+T, s)$ invariant subspaces of $X$ denoted by $E_{\lambda}(s)$ and $R_{\lambda}(s)$ such that

$$
\begin{equation*}
X=E_{\lambda}(s) \oplus R_{\lambda}(s) \tag{34}
\end{equation*}
$$

The subspace $E_{\lambda}(s)$ is called the (generalized) eigenspace (at time $s$ ) associated to the Floquet multiplier $\lambda$. This (generalized) eigenspace is defined as the smallest closed linear subspace that contains
all $\operatorname{ker}\left((\lambda I-U(s+T, s))^{j}\right)$ for all integers $j \geq 1$. Due to compactness, it turns out that there exists a smallest integer $k_{\lambda}$ that $\cup_{j \in \mathbb{N}} \operatorname{ker}\left((\lambda I-U(s+T, s))^{j}\right)=\operatorname{ker}\left((\lambda I-U(s+T, s))^{k_{\lambda}}\right)$ and hence the dimension of the generalized eigenspace $E_{\lambda}(s)$ is finite and called the algebraic multiplicity. We call $R_{\lambda}(s)$ the complementary (generalized) eigenspace (at time $s$ ) associated to the Floquet multiplier $\lambda$ and notice that this subspace has finite codimension.

Due to compactness, the sets of Floquet multipliers outside the unit disk $\Lambda_{+}:=\{\lambda \in \sigma(U(s+T, s))$ : $|\lambda|>1\}$ and on the unit circle $\Lambda_{0}:=\{\lambda \in \sigma(U(s+T, s)):|\lambda|=1\}$ are both finite. With each of these sets, we define the unstable eigenspace (at time $s$ ) and center eigenspace (at time $s$ ) as

$$
X_{+}(s):=\bigoplus_{\lambda \in \Lambda_{+}} E_{\lambda}(s), \quad X_{0}(s):=\bigoplus_{\lambda \in \Lambda_{0}} E_{\lambda}(s)
$$

respectively and notice that both eigenspaces are finite-dimensional. The stable eigenspace (at time s) can be defined as

$$
\begin{equation*}
X_{-}(s):=\bigcap_{\lambda \in \Lambda_{0} \cup \Lambda_{+}} R_{\lambda}(s), \tag{35}
\end{equation*}
$$

and has finite codimension. From this construction, the unstable-, center- and stable eigenspace are all closed $T$-periodic $U(s+T, s)$-invariant subspaces of $X$. This decomposition is sufficient to prove that Hypothesis 1 and Hypothesis 2 hold in the setting of classical DDEs, presented in this subsection. The verification of both hypotheses is carried out in Appendix A. 2 and hence we obtain the following.

Corollary 17 (Local center manifold for DDEs). Consider (DDE) with a $C^{k}$-smooth right-hand side $F: X \rightarrow \mathbb{R}^{n}$ for a fixed $k \geq 1$ and a given T-periodic solution $\gamma$. Define the finite rank Lipschitz continuous T-periodic time-dependent bounded linear perturbation B as in (29) together with the timedependent nonlinear perturbation $R$ as in (33), that is $T$-periodic in the first component. Let $U$ denote the strongly continuous forward evolutionary system that generates solutions of (T-LDDE) with $L(t)=$ $D F\left(\gamma_{t}\right)$. Suppose that there are $1 \leq n_{0}+1<\infty$ Floquet multipliers on the unit circle, counted with algebraic multiplicity, with corresponding $\left(n_{0}+1\right)$-dimensional real center eigenspace $X_{0}(t)$ defined for all $t \in \mathbb{R}$.

Then there exists a $C^{k-1}$-smooth map $\mathcal{C}: \mathbb{R} \times V \rightarrow X$, where $\mathbb{R} \times V$ is a neighborhood of $\mathbb{R} \times\{0\}$ in the center fiber bundle $X_{0}$ such that the manifold $\mathcal{W}_{\mathrm{loc}}^{c}:=\{(t, \mathcal{C}(t, \varphi)) \in \mathbb{R} \times X: t \in \mathbb{R}$ and $\mathcal{C}(t, \varphi) \in V\}$ is T-periodic, $C^{k-1}$-smooth, $\left(n_{0}+1\right)$-dimensional and locally positively invariant for the time-dependent semiflow $S$ generated by (T-DDE).

## 4 Characterization of the center manifold and normal forms

In this section, we characterize the dynamics of (DDE) near the nonhyperbolic cycle $\Gamma:=\left\{\gamma_{t} \in X: t \in\right.$ $\mathbb{R}\}$, meaning that there are, except of the trivial Floquet multiplier, other Floquet multipliers present on the unit circle in the complex plane, or the trivial Floquet multiplier has an algebraic multiplicity higher than one. Recall from Section 1 that there are three generic codimension one bifurcation of limit cycles: the fold bifurcation, where the trivial Floquet multiplier has an algebraic multiplicity 2 and geometric multiplicity 1, the period-doubling bifurcations where there is a Floquet multiplier at -1 and the Neimark-Sacker bifurcation where there is a complex conjugate pair of Floquet multipliers with modulus 1 .

To study these bifurcations, we first construct the time-periodic smooth Jordan chains related to the monodromy operator of the cycle $\Gamma$ in Section 4.1 and separate the trivial Floquet multiplier from the rest of the dynamics in Section 4.2. However, we provide a framework that is also suited to study bifurcations of limit cycles of higher codimension. It is nevertheless helpful to keep these three codimension one bifurcations in mind. Then finally in Section 4.3 we prove the existence of a special coordinate system on the center manifold and provide in addition the periodic critical normal forms.

These results are an extension of the work by Iooss [18, 19] from finite-dimensional ODEs to infinitedimensional DDEs. In Section 4.4 we list the critical periodic normal forms for the fold, period-doubling and Neimark-Sacker bifurcations of limit cycles in DDEs using the results from Section 4.3.

For simplicity of notation, we work with a $C^{k+1}$-smooth right-hand side $F$ of (DDE) such that for the time-dependent system (T-DDE) in the setting of Corollary 17, there exists a $T$-periodic $C^{k}$ smooth $\left(n_{0}+1\right)$-dimensional center manifold $\mathcal{W}_{\text {loc }}^{c}$. Recall that (T-DDE) was just a time-dependent translation of (DDE) via the given periodic solution. Hence, if $x$ is a solution of (DDE) then $y=x+\gamma$ is a solution of (T-DDE) and so

$$
\begin{equation*}
\mathcal{W}_{\mathrm{loc}}^{c}(\Gamma):=\left\{\gamma_{t}+\mathcal{C}(t, \varphi) \in X: \varphi \in X_{0}(t) \text { and } \mathcal{C}(t, \varphi) \in B_{\delta}(X)\right\} \tag{36}
\end{equation*}
$$

is a $T$-periodic $C^{k}$-smooth $\left(n_{0}+1\right)$-dimensional manifold in $X$ defined in the vicinity of $\Gamma$ for a sufficiently small $\delta>0$. To see this, recall that $t \mapsto \gamma_{t}$ is $T$-periodic and $C^{k}$-smooth together with the fact that $\mathcal{C}$ is $T$-periodic in the first component and $C^{k}$-smooth (Corollary 17). Recall from Theorem 12 that $\mathcal{C}(t, 0)=0$ and so $\Gamma \subset \mathcal{W}_{\text {loc }}^{c}(\Gamma)$. We call $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$ a local center manifold around $\Gamma$ and notice that this manifold inherits all the properties of Theorem 12.

### 4.1 Time-periodic smooth Jordan chains

Let us focus on a specific Floquet multiplier $\lambda \in \sigma(U(s+T, s))$ for a fixed $s \in \mathbb{R}$. We know from (34) that $E_{\lambda}(s)$ is the finite-dimensional (generalized) eigenspace (at time $s$ ). By the construction given in [13, Section IV.4], it is possible to find a basis of $E_{\lambda}(s)$ that is in Jordan normal form. That is, there exists an ordered basis $\left\{\zeta_{s}^{1}, \ldots, \zeta_{s}^{m_{\lambda}}\right\}$ of $E_{\lambda}(s)$ called a Jordan chain such that

$$
(U(s+T, s)-\lambda I) \zeta_{s}^{i}= \begin{cases}0, & i=1  \tag{37}\\ \zeta_{s}^{i-1}, & i=2, \ldots, m_{\lambda}\end{cases}
$$

and $\zeta_{s}^{i}$ should be interpreted via the history property (25). As the map $U_{\lambda}(t, s):=$ $\left.U(t, s)\right|_{E_{\lambda}(s)}: E_{\lambda}(s) \rightarrow E_{\lambda}(t)$ is a topological isomorphism [13, Theorem XIII.3.3], we know that $\left\{U_{\lambda}(t, s) \zeta_{s}^{1}, \ldots, U_{\lambda}(t, s) \zeta_{s}^{m_{\lambda}}\right\}$ is a basis of $E_{\lambda}(t)$. Let $i=1, \ldots, m_{\lambda}$, it is clear from

$$
\zeta_{s+T}^{i}-\zeta_{s}^{i}=U(s+T, s) \zeta_{s}^{i}-\zeta_{s}^{i}=(\lambda-1) \zeta_{s}^{i}
$$

that $\zeta^{i}$ is not $T$-periodic unless $\lambda=1$. However, in the upcoming characterization of the center manifold, we explicitly need a $T$-periodic $C^{k}$-smooth (generalized) eigenbasis, and therefore we prove the following theorem. This result is a generalization from finite-dimensional ODEs [19, Proposition III.1] towards infinite-dimensional DDEs.

Theorem 18. Let $\lambda$ be a Floquet multiplier with $\sigma$ its associated Floquet exponent. Then there exist $T$-periodic $C^{k}$-smooth maps $\zeta^{i}: \mathbb{R} \rightarrow X$ for $i=1, \ldots, m_{\lambda}$ satisfying

$$
\left(-\frac{d}{d t}+A^{\odot \star}(t)-\sigma\right) j\left(\zeta^{i}(t)\right)= \begin{cases}0, & i=1  \tag{38}\\ j\left(\zeta^{i-1}(t)\right), & i=2, \ldots, m_{\lambda}\end{cases}
$$

or equivalently

$$
\left(-\frac{d}{d t}+A(t)-\sigma\right) \zeta^{i}(t)= \begin{cases}0, & i=1  \tag{39}\\ \zeta^{i-1}(t), & i=2, \ldots, m_{\lambda}\end{cases}
$$

whenever $\zeta^{i}(t) \in \mathcal{D}(A(t))$ for all $t \in \mathbb{R}$. Furthermore, for any $t \in \mathbb{R}$ the functions $\zeta^{1}(t), \ldots, \zeta^{m_{\lambda}}(t)$ form a basis of $E_{\lambda}(t)$.

Proof. Let $s \in \mathbb{R}$ be an initial starting time and consider the basis $\left\{\zeta_{s}^{1}, \ldots, \zeta_{s}^{m_{\lambda}}\right\}$ of $E_{\lambda}(s)$ in Jordan normal form. We show the claim by induction on $i \in\left\{1, \ldots, m_{\lambda}\right\}$. For the base case, consider the initial value problem

$$
\left\{\begin{array}{l}
\left(-d^{\star}+A^{\odot \star}(t)-\sigma\right) j\left(\zeta^{1}(t)\right)=0, \quad t \geq s  \tag{40}\\
\zeta^{1}(s)=\zeta_{s}^{1}
\end{array}\right.
$$

where $\zeta_{s}^{1}$ is the first basis vector of $E_{\lambda}(s)$ and $d^{\star}$ denotes the weak ${ }^{\star}$ differential operator. Notice that the differential equation in (40) may be rewritten as

$$
d^{\star}\left(j \circ \zeta^{1}\right)(t)=-\sigma j\left(\zeta^{1}(t)\right)+A^{\odot \star}(t) j\left(\zeta^{1}(t)\right),
$$

and so

$$
\begin{aligned}
d^{\star}\left(j \circ e^{\sigma(\cdot-s)} \zeta^{1}\right)(t) & =\partial_{t}^{\star} j\left(e^{\sigma(t-s)} \zeta^{1}(t)\right)=\sigma e^{\sigma(t-s)} j\left(\zeta^{1}(t)\right)+e^{\sigma(t-s)} d^{\star} j\left(\zeta^{1}(t)\right) \\
& =e^{\sigma(t-s)}\left(d^{\star}+\sigma\right) j\left(\zeta^{1}(t)\right)=A^{\odot \star}(t) j\left(e^{\sigma(t-s)} \zeta^{1}(t)\right)
\end{aligned}
$$

where we have used (40) in the last equality. The equation above is of the form (5) and hence by its unique solution is given by

$$
\begin{equation*}
e^{\sigma(t-s)} \zeta^{1}(t)=U(t, s) \zeta_{s}^{1} \tag{41}
\end{equation*}
$$

whenever $\zeta_{s}^{1} \in j^{-1} \mathcal{D}\left(A_{0}^{\odot \star}\right)$. To show this claim, choose $m \in \mathbb{N}$ large enough to guarantee $m T \geq h$ because then by [13, Corollary XIII.2.2 and Corollary XII.3.4] we know

$$
\lambda^{m} \zeta_{s}^{1}=U(s+T, s)^{m} \zeta_{s}^{1}=U(s+m T, s) \zeta_{s}^{1} \in j^{-1} \mathcal{D}\left(A_{0}^{\odot \star}\right)
$$

Since $j^{-1} \mathcal{D}\left(A_{0}^{\odot \star}\right)$ is a linear subspace of $X$ and $\lambda \neq 0$ we get $\zeta_{s}^{1} \in j^{-1} \mathcal{D}\left(A_{0}^{\odot \star}\right)$. Let us now prove the $T$-periodicity of $\zeta^{1}$. Choosing $t=s+T$ in (41) yields

$$
e^{\sigma T} \zeta^{1}(s+T)=U(s+T, s) \zeta_{s}^{1}=\lambda \zeta_{s}^{1}=\lambda \zeta^{1}(s)
$$

Because $\lambda=e^{\sigma T}$ is non-zero we get $\zeta^{1}(s+T)=\zeta^{1}(s)$ and so $\zeta^{1}$ is $T$-periodic.
To show the smoothness property, we want to apply Proposition 15 and therefore have to verify that $U(t, s) \zeta_{s}^{1} \in \mathcal{D}(A(t))$. Recall that $\zeta_{s}^{1} \in j^{-1} \mathcal{D}\left(A_{0}^{\odot \star}\right)$ and since $t \mapsto U(t, s) \zeta_{s}^{1}$ is the unique solution of (5) we know $U(t, s) \zeta_{s}^{1} \in j^{-1} \mathcal{D}\left(A_{0}^{\odot \star}\right)$.

It remains to show that $A^{\odot \star}(t) j U(t, s) \zeta_{s}^{1} \in X^{\odot \odot}=j(X)$, due to $\odot$-reflexivity of $X$ with respect to the shift semigroup. From (5) it is clear that

$$
d^{\star}\left(j \circ U(\cdot, s) \zeta_{s}^{1}\right)(t)=A^{\odot \star}(t) j U(t, s) \zeta_{s}^{1}
$$

Recall now from Section 3.6 that $t \mapsto U(t, s) \zeta_{s}^{1}$ is the unique solution of (T-LDDE) and hence it is at least continuously differentiable. Using Definition 1, a small computation gives

$$
d^{\star}\left(j \circ U(\cdot, s) \zeta_{s}^{1}\right)(t)=j\left(\partial_{t} U(t, s) \zeta_{s}^{1}\right) \in j(X)
$$

and so $t \mapsto U(t, s) \zeta_{s}^{1}$ is of the class $C^{k}$. Since $t \mapsto e^{-\sigma(t-s)}$ is analytic, $t \mapsto \zeta^{1}(t)=e^{-\sigma(t-s)} U(t, s) \zeta_{s}^{1}$ is of the class $C^{k}$. Hence, $\zeta^{1}$ satisfies (40) in the sense that the weak differential operator $d^{\star}$ can be replaced by $\frac{d}{d t}$ which proves (38) for the base case. Because the map $j$ is a linear isomorphism on $X^{\odot \odot}=j(X)$, we get

$$
\left(-\frac{d}{d t}+A^{\odot \star}(t)-\sigma\right) j\left(\zeta^{1}(t)\right)=j\left[\left(-\frac{d}{d t}+A(t)-\sigma\right) \zeta^{1}(t)\right]
$$

under the extra assumption that $\zeta^{1}(t) \in \mathcal{D}(A(t))$. Since the left hand-side is equal to $0=j(0)$ we must have that $\left(-\frac{d}{d t}+A(t)-\sigma\right) \zeta^{1}(t)=0$ and so (39) is proven for the base case.

Now to complete the induction, assume that the $T$-periodic $C^{k}$-smooth maps $\zeta^{1}, \ldots, \zeta^{i-1}$ are constructed for some $i \geq 2$ and consider the initial value problem

$$
\left\{\begin{array}{l}
\left(-d^{\star}+A^{\odot \star}(t)-\sigma\right) j\left(\zeta^{i}(t)\right)=j\left(\zeta^{i-1}(t)\right), \quad t \geq s  \tag{42}\\
\zeta^{i}(s)=\sum_{k=1}^{i} \alpha_{i k} \zeta_{s}^{k}
\end{array}\right.
$$

where $\zeta_{s}^{1}, \ldots, \zeta_{s}^{i}$ are the first $i$ basis vectors of $E_{\lambda}(s)$. Notice that the differential equation from (42) can be rewritten as

$$
d^{\star}\left(j \circ \zeta^{i}\right)(t)=-\sigma j\left(\zeta^{i}(t)\right)+A^{\odot \star}(t) j\left(\zeta^{i}(t)\right)+j\left(\zeta^{i-1}(t)\right)
$$

The goal is to find scalars $\alpha_{i k}$ such that $\zeta^{i}$ becomes $T$-periodic. A similar computation as done for the $i=1$ case tells us by using (42) that

$$
d^{\star}\left(j \circ e^{\sigma(\cdot-s)} \zeta^{i}\right)(t)=\left(d^{\star}+\sigma\right) j\left(e^{\sigma(t-s)} \zeta^{i}(t)\right)=A^{\odot \star}(t) j\left(e^{\sigma(t-s)} \zeta^{i}(t)\right)-j\left(e^{\sigma(t-s)} \zeta^{i-1}(t)\right),
$$

which is a differential equation of the form (87) with inhomogeneous term $f=j \circ e^{\sigma(\cdot-s)} \zeta^{i-1}$. Since $\zeta_{s}^{1}, \ldots, \zeta_{s}^{i} \in \mathcal{D}(A(s))$ we have that $\zeta^{i}(s)=\sum_{k=1}^{i} \alpha_{i k} \zeta_{s}^{k} \in \mathcal{D}(A(s)) \subseteq j^{-1} \mathcal{D}\left(A_{0}^{\odot \star}\right)$ due to linearity. Because $\zeta^{i-1}$ is of the class $C^{k}$ for a fixed $k \geq 1$, it is certainly continuous and so it follows from Proposition 37 that

$$
\begin{equation*}
e^{\sigma(t-s)} \zeta^{i}(t)=U(t, s) \zeta_{s}^{i}-j^{-1} \int_{s}^{t} U^{\odot \star}(t, \tau) j\left(e^{\sigma(\tau-s)} \zeta^{i-1}(\tau)\right) d \tau \tag{43}
\end{equation*}
$$

which is well-defined, i.e. the weak ${ }^{\star}$ integral takes values in $j(X)$ due to Lemma 2 as clearly the map $\tau \mapsto j\left(e^{\sigma(\tau-s)} \zeta^{i-1}(\tau)\right)$ from $\mathbb{R}$ to $j(X) \subseteq X^{\odot \star}$ is continuous. Applying (43) again, we get

$$
\begin{aligned}
& e^{\sigma(t-s)} \zeta^{i}(t) \\
& =U(t, s) \zeta_{s}^{i}-j^{-1} \int_{s}^{t} U^{\odot \star}(t, \tau) j\left(U(\tau, s) \zeta_{s}^{i-1}-j^{-1} \int_{s}^{\tau} U^{\odot \star}(\tau, \theta) j\left(e^{\sigma(\theta-s)} \zeta^{i-2}(\theta)\right) d \theta\right) d \tau \\
& =U(t, s) \zeta_{s}^{i}-j^{-1} \int_{s}^{t} j\left(U(t, s) \zeta_{s}^{i-1}\right) d \tau+j^{-1} \int_{s}^{t} \int_{s}^{\tau} U^{\odot \star}(t, \theta) j\left(e^{\sigma(\theta-s)} \zeta^{i-2}(\theta)\right) d \theta d \tau
\end{aligned}
$$

Notice that the integrand in the first integral of the last equality is independent of $\tau$, which makes this integral easy to evaluate. Applying Fubini's theorem yields

$$
\begin{aligned}
e^{\sigma(t-s)} \zeta^{i}(t) & =U(t, s) \zeta_{s}^{i}-(t-s) U(t, s) \zeta_{s}^{i-1}+j^{-1} \int_{s}^{t} \int_{\theta}^{t} U^{\odot \star}(t, \theta) j\left(e^{\sigma(\theta-s)} \zeta^{i-2}(\theta)\right) d \tau d \theta \\
& =U(t, s) \zeta_{s}^{i}-(t-s) U(t, s) \zeta_{s}^{i-1}+j^{-1} \int_{s}^{t}(t-\theta) U^{\odot \star}(t, \theta) j\left(e^{\sigma(\theta-s)} \zeta^{i-2}(\theta)\right) d \theta
\end{aligned}
$$

because the integrand in the first equation was independent of $\tau$. By recurrence, we obtain

$$
\begin{equation*}
e^{\sigma(t-s)} \zeta^{i}(t)=U(t, s) \sum_{k=0}^{i-1}(-1)^{k} \frac{(t-s)^{k}}{k!} \zeta_{s}^{i-k} \tag{44}
\end{equation*}
$$

Putting $t=s+T$ in (44), we see that $\zeta^{i}(s)=\zeta^{i}(s+T)$ if and only if

$$
\begin{equation*}
(U(s+T, s)-\lambda) \zeta^{i}(s)=U(s+T, s) \sum_{k=1}^{i-1}(-1)^{k+1} \frac{T^{k}}{k!} \zeta_{s}^{i-k} \tag{45}
\end{equation*}
$$

Recall from (42) that $\zeta^{i}(s)=\sum_{k=1}^{i} \alpha_{i k} \zeta_{s}^{k}$ and retrieving (37) yields

$$
\begin{aligned}
\sum_{k=2}^{i} \alpha_{i k} \zeta_{s}^{k-1} & =U(s+T, s) \sum_{l=1}^{i-1}(-1)^{l+1} \frac{T^{l}}{l!} \zeta_{s}^{i-l} \\
& =\sum_{p=1}^{i-1}(-1)^{p+1} \frac{T^{i-p}}{(i-p)!} \begin{cases}\lambda \zeta_{s}^{p}, & p=1 \\
\lambda \zeta_{s}^{p}+\zeta_{s}^{p-1}, & p=2, \ldots, i-1\end{cases}
\end{aligned}
$$

Because the right-hand side is a known element in the subspace spanned by $\zeta_{s}^{1}, \ldots, \zeta_{s}^{i-1}$ the $\alpha_{i k}$ 's are uniquely determined for $k=2, \ldots, i$ and so we have proven that $\zeta^{i}(s)=\zeta^{i}(s+T)$ i.e. $\zeta^{i}$ is $T$-periodic.

Let us now show the smoothness property. Similar to the case that $i=1$, we will first show that $U(t, s) \zeta_{s}^{1}, \ldots, U(t, s) \zeta_{s}^{i-1} \in \mathcal{D}(A(t))$. By assumption of the induction hypothesis we already know that $\zeta_{s}^{1}, \ldots, \zeta_{s}^{i-1} \in \mathcal{D}(A(t)) \subseteq j^{-1} \mathcal{D}\left(A_{0}^{\odot \star}\right)$ and so $U(t, s) \zeta_{s}^{1}, \ldots, U(t, s) \zeta_{s}^{i-1} \in j^{-1} \mathcal{D}\left(A_{0}^{\odot \star}\right)$ since they are solutions of (5). It also follows from (5) that

$$
d^{\star}\left(j \circ U(\cdot, s) \zeta_{s}^{l}\right)(t)=A^{\odot \star}(t) j U(t, s) \zeta_{s}^{l}
$$

for $l \in\{1, \ldots, i-1\}$ and the same argument from the base case implies, together with the induction step, that $U(t, s) \zeta_{s}^{1}, \ldots, U(t, s) \zeta_{s}^{i-1} \in \mathcal{D}(A(t))$. It follows from Proposition 15 now that $t \mapsto U(t, s) \zeta_{s}^{l}$ is of the class $C^{k}$ for all $l \in\{1, \ldots, i-1\}$. As $t \mapsto e^{-\sigma(t-s)}$ is analytic we have that the map

$$
\begin{equation*}
t \mapsto \zeta^{i}(t)=e^{-\sigma(t-s)} U(t, s) \sum_{k=0}^{i-1}(-1)^{k} \frac{(t-s)^{k}}{k!} \zeta_{s}^{i-k} \tag{46}
\end{equation*}
$$

is $C^{k}$-smooth due to linearity. As we have proven that $\zeta^{i}$ is $C^{k}$-smooth we can replace the weak ${ }^{\star}$ differential operator $d^{\star}$ by $\frac{d}{d t}$ in (42) which proves (38). The same reasoning from the base case applies now for $i \geq 2$ to get the result for (39).

Notice also that $\alpha_{i 1}$ is already determined in the base case. Expanding out all the terms and comparing them yields for example

$$
\alpha_{i i}= \begin{cases}\lambda T \alpha_{i-1, i-1}, & i>2 \\ \lambda T, & i=2\end{cases}
$$

Hence $\alpha_{i i}=(\lambda T)^{i-1} \neq 0$ and so $\zeta_{s}^{1}, \ldots, \zeta_{s}^{m_{\lambda}}$ are all linearly independent. Furthermore, $\zeta^{1}(t), \ldots, \zeta^{m_{\lambda}}(t)$ are all linearly independent because they are all solutions to abstract ODE

$$
\left(-\frac{d}{d t}+A^{\odot \star}(t)-\sigma\right)^{m_{\lambda}} j(\zeta(t))=0
$$

which completes the proof.
Let us now take some time to discuss a connection between the $T$-periodicity and the history property (25). Let $\left\{\zeta_{s}^{1}, \ldots, \zeta_{s}^{m_{\lambda}}\right\}$ be a basis of $E_{\lambda}(s)$. Hence, $\left\{\zeta_{t}^{1}, \ldots, \zeta_{t}^{m_{\lambda}}\right\}$ is (in general) a non- $T$ periodic basis of $E_{\lambda}(t)$ that has the history property (25), where $\zeta_{t}^{i}=U(t, s) \zeta_{s}^{i}$. On the other hand, Theorem 18 shows us that $\left\{\zeta^{1}(t), \ldots, \zeta^{m_{\lambda}}(t)\right\}$ is a $T$-periodic basis of $E_{\lambda}(t)$, but how is this basis related to the history property (25)?

Notice that the function $\zeta^{1}(t) \in X$ would have the history property if and only if it satisfies the transport equation

$$
\begin{equation*}
\frac{\partial}{\partial t} \zeta^{1}(t)(\theta)=\frac{\partial}{\partial \theta} \zeta^{1}(t)(\theta) \tag{47}
\end{equation*}
$$

but a small calculation directly shows that

$$
\begin{aligned}
\frac{\partial}{\partial t} \zeta^{1}(t)(\theta) & =e^{-\sigma(t-s)}\left(-\sigma+\dot{\zeta}_{t}^{1}(\theta)\right) \\
\frac{\partial}{\partial \theta} \zeta^{1}(t)(\theta) & =e^{-\sigma(t-s)} \dot{\zeta}_{t}^{1}(\theta)
\end{aligned}
$$

and so $\zeta^{1}$ satisfies the transport equation (47) if and only if $\sigma=0$ i.e. $\lambda=1$. A similar analysis for the $T$-periodic generalized eigenfunctions (46) shows that these never have the history property. Hence, the only solution of (39) which satisfies the history property is the derivative of the periodic orbit $\dot{\gamma}_{t}$ itself. It is however the $T$-periodic basis $\left\{\zeta^{1}(t), \ldots, \zeta^{m_{\lambda}}(t)\right\}$ of $E_{\lambda}(t)$ that is needed for the characterization of the center manifold.

In the upcoming construction of the characterization of the center manifold, we also need the Floquet operator (at time $t$ ) associated to $\lambda$, defined as the coordinate map $Q_{\lambda}(t): \mathbb{C}^{m_{\lambda}} \rightarrow E_{\lambda}(t)$ by

$$
\begin{equation*}
Q_{\lambda}(t) \xi:=\sum_{i=1}^{m_{\lambda}} \xi_{i} \zeta^{i}(t), \quad \forall \xi=\left(\xi_{1}, \ldots, \xi_{m_{\lambda}}\right) \in \mathbb{C}^{m_{\lambda}} \tag{48}
\end{equation*}
$$

It is clear from Theorem 18 that the map $t \mapsto Q_{\lambda}(t)$ is $T$-periodic, $C^{k}$-smooth and takes values in $\mathcal{L}\left(\mathbb{C}^{m_{\lambda}}, E_{\lambda}(t)\right)$. Furthermore, a small calculation shows that

$$
\left(-\frac{d}{d t}+A^{\odot \star}(t)\right) j\left(Q_{\lambda}(t) \xi\right)=j\left(Q_{\lambda}(t) M_{\lambda} \xi\right)
$$

where $M_{\lambda}$ is the $m_{\lambda} \times m_{\lambda}$ Jordan matrix defined by

$$
M_{\lambda}:=\left(\begin{array}{cccc}
\sigma & 1 & \cdots & 0  \tag{49}\\
0 & \sigma & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
0 & \ldots & 0 & \sigma
\end{array}\right)
$$

This result is an extension of [19, Proposition III.3] from finite-dimensional ODEs to infinitedimensional DDEs. Because we are dealing with the real state space $X=C\left([-h, 0], \mathbb{R}^{n}\right)$, the linear operator $M_{\lambda}$, written in matrix form in (49), should represent a real operator. Depending on the location of $\lambda$ in the complex plane, we have three options [19]:

- If $\lambda$ is real and positive, we choose $\sigma$ real and $\zeta^{1}(t), \ldots, \zeta^{m_{\lambda}}(t)$ real.
- If $\lambda$ is not real, then its complex conjugate $\bar{\lambda} \neq \lambda$ is also a Floquet multiplier. Hence, we choose $\sigma$ and $\zeta^{1}(t), \ldots, \zeta^{m_{\lambda}}(t)$ complex, introduce $\bar{\sigma}$ and $\overline{\zeta^{1}}(t), \ldots, \overline{\zeta^{m_{\lambda}}}(t)$ for the complex conjugate.
- If $\lambda$ is real and negative, both methods describe above do not succeed. Indeed, the Floquet exponents $\sigma$ are of the form $\frac{\pi}{T}+\frac{2 l i \pi}{T}$ with $l \in \mathbb{Z}$. The standard way to deal with this situation is to double the period, since if $\lambda \in \sigma(U(s+T, s))$ is a Floquet multiplier then $\lambda^{2} \in \sigma(U(s+2 T, s))$. To see this, let $\zeta$ be a (generalized) eigenfunction of $U(s+T, s)$ associated to the eigenvalue $\lambda$, then

$$
U(s+2 T, s) \zeta=U(s+2 T, s+T) U(s+T, s) \zeta=U(s+T, s) \lambda \zeta=\lambda^{2} \zeta
$$

In this last case, we have to adjust some results due to the change of periodicity.

Proposition 19. Let $\lambda$ be a real and negative Floquet multiplier with $\sigma$ its associated Floquet exponent. Then there exist $2 T$-periodic $C^{k}$-smooth maps $\zeta^{i}: \mathbb{R} \rightarrow X$ for $i=1, \ldots, m_{\lambda}$ satisfying

$$
\begin{aligned}
\zeta^{i}(t+T) & =-\zeta^{i}(t), \\
\left(-\frac{d}{d t}+A^{\odot \star}(t)-\sigma\right) j\left(\zeta^{i}(t)\right) & = \begin{cases}0, & i=1 \\
j\left(\zeta^{i-1}(t)\right), & i=2, \ldots, m_{\lambda}\end{cases}
\end{aligned}
$$

or the equivalent differential equation

$$
\left(-\frac{d}{d t}+A(t)-\sigma\right) \zeta^{i}(t)= \begin{cases}0, & i=1, \\ \zeta^{i-1}(t), & i=2, \ldots, m_{\lambda}\end{cases}
$$

whenever $\zeta^{i}(t) \in \mathcal{D}(A(t))$ for all $t \in \mathbb{R}$. The number $\sigma \in \mathbb{R}$ is defined as $e^{\sigma T}=|\lambda|$. If $E_{\lambda}(t)$ is the subspace spanned by $\zeta^{1}(t), \ldots, \zeta^{m_{\lambda}}(t)$, then there exists a real T-periodic projector $P_{\lambda}: \mathbb{R} \rightarrow \mathcal{L}(X)$ onto $E_{\lambda}(t)$. Moreover, the Floquet operator at $\lambda$ satisfies $Q_{\lambda}(t+T)=-Q_{\lambda}(t)$ and the differential equation (48), where $M_{\lambda}$ is now a linear operator on $\mathbb{R}^{n}$.

Proof. To prove the first assertions, we copy the proof of Proposition 18 but in the $2 T$-periodic setting. The proof goes identical up to (41). If we set $t=s+T$ in (41) we get

$$
|\lambda| \zeta^{1}(s+T)=e^{\sigma T} \zeta^{1}(s+T)=U(s+T, s) \zeta^{1}(s)=\lambda \zeta^{1}(s)
$$

and so $\zeta^{1}(s)=\operatorname{sign}(\lambda) \zeta^{1}(s)=-\zeta^{1}(s)$. This automatically shows the $2 T$-periodicity of $\zeta^{1}$.
Consider now (42) and suppose that its right-hand side of this equation satisfies $\zeta^{i-1}(s+T)=$ $-\zeta^{i-1}(s)$. Our goal now is to find the $\alpha_{i k}$ such that $\zeta^{i}(s+T)=-\zeta^{i}(s)$. The proof from this point on goes exactly the same up to (44). Instead of requiring the $T$-periodicity of $\zeta^{i}$ we require now that $\zeta^{i}(s+T)=-\zeta^{i}(s)$. we see that $\zeta^{i}(s+T)=-\zeta^{i}(s)$ if and only if

$$
(U(s+T, s)-\lambda) \zeta^{i}(s)=U(s+T, s) \sum_{k=1}^{i-1}(-1)^{k+1} \frac{T^{k}}{k!} \zeta_{s}^{i-k}
$$

which is precisely (45). Hence, the same procedure in Proposition 18 can be followed to find the associated $\alpha_{i k}$ 's uniquely and get $\zeta^{i}(s+T)=-\zeta^{i}(s)$.

The real spectral projection $P_{\lambda}(t) \in \mathcal{L}(X)$ onto $E_{\lambda}(t)$ for $t \in \mathbb{R}$ is constructed in the same way as the Dunford integral in (84). For the Floquet operator at $\lambda$, it follows from linearity and $\zeta^{i}(t+T)=-\zeta^{i}(t)$ for all $i=1, \ldots, m_{\lambda}$ that $Q_{\lambda}(t+T)=-Q_{\lambda}(t)$ for all $t \in \mathbb{R}$.

### 4.2 Separating the dynamics of the periodic orbit

The coordinate system and normal forms we will present consist of two parts and is inspired by [18, 19]. The first part expresses the dynamics along $\Gamma$ by a time-dependent phase and the other part expresses the dynamics transverse to $\Gamma$ in terms of this phase. The normal forms depend on the location and multiplicities of the Floquet multipliers on the unit circle. Let us first separate the dynamics of the periodic orbit via coordinates along this phase and transverse to this phase.

Recall that $X_{0}(t)$ is a $\left(n_{0}+1\right)$-dimensional subspace of $X$ for all $t \in \mathbb{R}$. For each $\lambda \in \Lambda_{0}$, we know that the (generalized) eigenspace $E_{\lambda}(t)$ has a basis that satisfies the conditions from Theorem 18 or Proposition 19, depending on the location of $\lambda$ on the unit circle. Recall that the trivial Floquet multiplier is always present on the unit circle and $\dot{\gamma}_{s}$ is the associated eigenfunction of $U(s+T, s)$. We choose $\zeta^{0}(t)$ to be $\dot{\gamma}_{t}$ and denote by $\tilde{X}_{0}(t)$ the space spanned by $\zeta^{1}(t), \ldots, \zeta^{n_{0}}(t)$ that forms a $T$-periodic $C^{k}$-smooth basis as presented in Theorem 18. Define for any $t \in \mathbb{R}$ the operator $\tilde{Q}_{0}(t): \mathbb{R}^{n_{0}} \rightarrow \tilde{X}_{0}(t)$ as

$$
\begin{equation*}
\tilde{Q}_{0}(t) \xi:=\sum_{i=1}^{n_{0}} \xi_{i} \zeta^{i}(t), \quad \forall \xi=\left(\xi_{1}, \ldots, \xi_{n_{0}}\right) \in \mathbb{R}^{n_{0}} . \tag{50}
\end{equation*}
$$

With this notation, it is clear that the Floquet operator $Q_{0}(t): \mathbb{R} \times \mathbb{R}^{n_{0}} \rightarrow X_{0}(t)$ (at time $t$ ) associated to $\Lambda_{0}$ is given by

$$
Q_{0}(t)\left(\xi_{0}, \xi\right)=\xi_{0} \dot{\gamma}_{t}+\tilde{Q}_{0}(t) \xi, \quad \forall \xi_{0} \in \mathbb{R}, \xi=\left(\xi_{1}, \ldots, \xi_{n_{0}}\right) \in \mathbb{R}^{n_{0}}
$$

The $\left(n_{0}+1\right) \times\left(n_{0}+1\right)$ matrix $M_{0}$ from takes the form

$$
M_{0}=\left(\begin{array}{c|cc}
0 & \star & \cdots \tag{51}
\end{array} 0\right.
$$

where $\star \in\{0,1\}$ depends on the algebraic multiplicity of the trivial Floquet multiplier.

### 4.3 Characterization and normal form theorems

To characterise $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$, we prove the existence of a normalizing coordinate system on $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$ and provide the critical normal forms. Depending on the algebraic multiplicity of the trivial Floquet multiplier and the location of the other Floquet multipliers on the unit circle, the normal forms will have a different shape and therefore three different normal form theorems will be presented.

The main idea to prove the existence of suitable coordinates on $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$ is to use the invariance property of $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$ around the periodic orbit $\Gamma$ to the fullest. We try to parametrize the history $x_{t}$ in the vicinity of the periodic orbit $\Gamma$ as

$$
\begin{equation*}
x_{t}=\gamma_{\tau}+\tilde{Q}_{0}(\tau) \xi+H(\tau, \xi) \tag{52}
\end{equation*}
$$

where $\tau$ is a function of $t$, expresses the dynamics along $\Gamma$ by a time-dependent phase and $\xi$ is a function of $\tau$ that expresses the dynamics transverse to $\Gamma$ in terms of this phase. Such a coordinate system is visualized for a two-dimensional local center manifold in Figure 1.


Figure 1: Illustration of a two-dimensional center manifolds $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$ together with the coordinate $\operatorname{system}(\tau, \xi)$. The left figure represents the case when $-1 \notin \Lambda_{0}$ and then $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$ is locally diffeomorphic to a cylinder in a neighborhood of $\Gamma$, see Theorem 21. The right figure represents the case when $-1 \in \Lambda_{0}$ and then $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$ is locally diffeomorphic to a Möbius band in a neighborhood of $\Gamma$, see Theorem 22.

The only unknown in (52) is the nonlinear operator $H: \mathbb{R} \times \mathbb{R}^{n_{0}} \rightarrow X$ and to obtain the Taylor expansion of this operator, we use again the invariance property of $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$. To be more precise, if we take an initial condition $\varphi \in \mathcal{W}_{\text {loc }}^{c}(\Gamma)$, then we must have that $x_{t} \in \mathcal{W}_{\text {loc }}^{c}(\Gamma)$ for all $t$ in the time domain of definition, say $I \subseteq \mathbb{R}$. By [6, Theorem 3.6] we know that the history $x_{t}$ satisfies the abstract ODE

$$
\begin{equation*}
\frac{d}{d t} j\left(x_{t}\right)=A_{0}^{\odot \star} j\left(x_{t}\right)+G\left(x_{t}\right), \quad t \in I, \tag{53}
\end{equation*}
$$

where $G(\varphi)=F(\varphi) r^{\odot \star}$ for $\varphi \in X$ and $F \in C^{k+1}\left(X, \mathbb{R}^{n}\right)$ for some $k \geq 1$ is the right-hand side of (DDE). The idea is then to show the existence of each $q$ th order term in the Taylor expansion of $H$ for $q=2, \ldots, k$ by using (53) and the invariance property of $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$.

First we consider the case where the trivial Floquet multiplier has algebraic multiplicity 1 and there is no Floquet multiplier located at -1 . This is for example the case in the Neimark-Sacker bifurcation.

Theorem 20 (Normal Form I). Assume that the algebraic multiplicity of the trivial Floquet multiplier is one and that -1 is not a Floquet multiplier. Then there exist $C^{k}$-smooth maps $H: \mathbb{R} \times \mathbb{R}^{n_{0}} \rightarrow X, p$ : $\mathbb{R} \times \mathbb{R}^{n_{0}} \rightarrow \mathbb{R}$ and $P: \mathbb{R} \times \mathbb{R}^{n_{0}} \rightarrow \mathbb{R}^{n_{0}}$ such that the history $x_{t} \in \mathcal{W}_{\text {loc }}^{c}(\Gamma)$ may be represented as

$$
x_{t}=\gamma_{\tau}+\tilde{Q}_{0}(\tau) \xi+H(\tau, \xi), \quad t \in I
$$

where the time dependence of the coordinates $(\tau, \xi)$ describing the dynamics of ( $\mathrm{DDE)}$ on $\mathcal{W}_{\mathrm{loc}}^{c}(\Gamma)$ is defined by the normal form

$$
\left\{\begin{array}{l}
\frac{d \tau}{d t}=1+p(\tau, \xi)+\mathcal{O}\left(|\xi|^{k+1}\right) \\
\frac{d \xi}{d \tau}=\tilde{M}_{0} \xi+P(\tau, \xi)+\mathcal{O}\left(|\xi|^{k+1}\right)
\end{array}\right.
$$

Here the functions $H, p$, and $P$ are $T$-periodic in $\tau$ and at least quadratic in $\xi$. The $\mathcal{O}$-terms are also $T$-periodic in $\tau$. Moreover, $p$ and $P$ are polynomials in $\xi$ of degree less than or equal to $k$ such that

$$
\frac{d}{d \tau} p\left(\tau, e^{-\tau \tilde{M}_{0}^{\star}} \xi\right)=0 \quad \text { and } \quad \frac{d}{d \tau}\left(e^{\tau \tilde{M}_{0}^{\star}} P\left(\tau, e^{-\tau \tilde{M}_{0}^{\star}} \xi\right)\right)=0
$$

for all $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^{n_{0}}$.
The proof of this theorem is quite long and technical. Essentially, it is a carefull generalization of [19, Theorem III.7]. Therefore we first sketch the idea of the proof and break it up into several steps. The final goal is to characterize the map $H$ by its Taylor expansion. In Step 1 of the proof, we assume this Taylor expansion and start in Step 2 with collecting terms in powers of $\xi^{q}$ for $q=0, \ldots, k$ from both sides of the resulted equation, obtained from the invariance property of the center manifold. We get for $q=2, \ldots, k$ an equation for the coefficient $H_{q}$ and we must show that this can be uniquely solved. This will be done via decomposing $H_{q}$ into the decomposition provided in (11) together with the separation Section 4.2, see Step 3. Hence, we get for each $q=2, \ldots, k$ the terms $H_{q}^{00}, \tilde{H}_{q}^{0}$, $H_{q}^{-}$and $H_{q}^{+}$and then we prove the existence of each of these terms in Step $\mathbf{4}\left(H_{q}^{+}\right)$, Step $\mathbf{5}\left(H_{q}^{-}\right)$and Step 6 $\left(H_{q}^{00}\right.$ and $\left.\tilde{H}_{q}^{0}\right)$. The provided normal forms are partially derived in part 6 of the proof in combination with [19, Theorem III.7].

Proof of Theorem 20. We follow the outlined route of the proof as indicated above.
Step 1: Taylor expansion. Let us write (DDE) in the form of (53) and notice that

$$
\begin{equation*}
G\left(\gamma_{\tau}+\varphi\right)=G\left(\gamma_{\tau}\right)+B(\tau) \varphi+\sum_{q=2}^{k} G_{q}\left(\tau, \varphi^{(q)}\right)+\mathcal{O}\left(\|\varphi\|_{\infty}^{k+1}\right), \quad \forall \varphi \in X \tag{54}
\end{equation*}
$$

where $B(\tau) \varphi=\left[D F\left(\gamma_{\tau}\right) \varphi\right] r^{\odot \star}$ is the time-dependent bounded linear perturbation and the nonlinear term is given by $G_{q}\left(\tau, \varphi^{(q)}\right)=\frac{1}{q!} D^{q} F\left(\gamma_{\tau}\right)\left(\varphi^{(q)}\right) r^{\odot \star}$, where $D^{q} F\left(\gamma_{\tau}\right): X^{q} \rightarrow \mathbb{R}^{n}$ is the $q$ th order Fréchet derivative evaluated at $\gamma_{\tau}$ for $q=2, \ldots, k$ and $\varphi^{(q)}:=(\varphi, \ldots, \varphi) \in X^{q}$. We also expand the maps $H, p$ and $P$ as

$$
H(\tau, \xi)=\sum_{q=2}^{k} H_{q}\left(\tau, \xi^{(q)}\right)+\mathcal{O}\left(|\xi|^{k+1}\right), \quad p(\tau, \xi)=\sum_{q=2}^{k} p_{q}\left(\tau, \xi^{(q)}\right), \quad P(\tau, \xi)=\sum_{q=2}^{k} P_{q}\left(\tau, \xi^{(q)}\right)
$$

with the coefficients $H_{q}\left(\tau, \xi^{(q)}\right) \in X, p_{q}\left(\tau, \xi^{(q)}\right) \in \mathbb{R}$ and $P_{q}\left(\tau, \xi^{(q)}\right) \in \mathbb{R}^{n_{0}}$, where $\xi^{(q)}:=(\xi, \ldots, \xi) \in$ $\left[\mathbb{R}^{n_{0}}\right]^{q}$. As already announced, we will use the invariance property of $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$ to show existence of the coefficients $H_{q}\left(\tau, \xi^{(q)}\right)$ for all $q=2, \ldots, k$. Hence, we compare the expansions of

$$
\frac{d}{d t} j\left(\gamma_{\tau}+\tilde{Q}_{0}(\tau) \xi+H(\tau, \xi)\right)=j\left(\dot{\gamma}_{\tau}+\frac{\partial \tilde{Q}_{0}(\tau)}{\partial \tau} \xi+\frac{\partial H(\tau, \xi)}{\partial \tau}+\left(\tilde{Q}_{0}(\tau)+D_{\xi} H(\tau, \xi)\right) \frac{d \xi}{d \tau}\right) \frac{d \tau}{d t}
$$

and

$$
A_{0}^{\odot \star} j\left(x_{t}\right)+G\left(x_{t}\right)=A_{0}^{\odot \star} j\left(\gamma_{\tau}+\tilde{Q}_{0}(\tau) \xi+H(\tau, \xi)\right)+G\left(\gamma_{\tau}+\tilde{Q}_{0}(\tau) \xi+H(\tau, \xi)\right)
$$

by subsituting

$$
\frac{d \tau}{d t}=1+p(\tau, \xi)+\mathcal{O}\left(|\xi|^{k+1}\right)
$$

and

$$
\frac{d \xi}{d \tau}=\tilde{M}_{0} \xi+P(\tau, \xi)+\mathcal{O}\left(|\xi|^{k+1}\right)
$$

Using the expansions of $H, p$ and $P$ together with (54) where now $\varphi$ must be substituted by $\tilde{Q}_{0}(\tau) \xi+$ $H(\tau, \xi)$, we get

$$
\begin{aligned}
& j\left[\dot{\gamma}_{\tau}+\frac{\partial \tilde{Q}_{0}(\tau)}{\partial \tau} \xi+\sum_{q=2}^{k} \frac{\partial H_{q}\left(\tau, \xi^{(q)}\right)}{\partial \tau}+\left(\tilde{Q}_{0}(\tau)+\sum_{q=2}^{k} D_{\xi} H_{q}\left(\tau, \xi^{(q)}\right)\right)\left(\tilde{M}_{0}+\sum_{q=2}^{k} P_{q}\left(\tau, \xi^{(q)}\right)\right)\right] \\
& \left(1+\sum_{q=2}^{k} p_{q}\left(\tau, \xi^{(q)}\right)\right)+\mathcal{O}\left(|\xi|^{k+1}\right) \\
& =A_{0}^{\odot \star} j\left(\gamma_{\tau}\right)+G\left(\gamma_{\tau}\right)+A^{\odot \star}(\tau) j\left(\tilde{Q}_{0}(\tau) \xi+\sum_{q=2}^{k} H_{q}\left(\tau, \xi^{(q)}\right)\right) \\
& +\sum_{q=2}^{k} G_{q}\left(\tau,\left[\tilde{Q}_{0}(\tau) \xi+\sum_{p=2}^{k} H_{q}\left(\tau, \xi^{(p)}\right)\right]^{(q)}\right)+\mathcal{O}\left(|\xi|^{k+1}\right) .
\end{aligned}
$$

Step 2: Collecting terms. Let us now compare the terms in powers of $\xi$ on both side of this equation. Collecting the $\xi^{0}$-terms give us

$$
\frac{d}{d \tau} j\left(\gamma_{\tau}\right)=A_{0}^{\odot \star} j\left(\gamma_{\tau}\right)+G\left(\gamma_{\tau}\right)
$$

which means that $\gamma$ is a solution (53). This was already known since $\gamma$ is a periodic solution of (DDE). The $\xi^{1}$-terms give us

$$
\begin{equation*}
\left(-\frac{\partial}{\partial \tau}+A^{\odot \star}(\tau)\right) j\left(\tilde{Q}_{0}(\tau) \xi\right)=j\left(\tilde{Q}_{0}(\tau) \tilde{M}_{0} \xi\right) \tag{55}
\end{equation*}
$$

which is exactly the result established in (48), but now for all Floquet multipliers on the unit circle and this characterizes the linear part. After collecting the $\xi^{(2)}$-terms, we get the expression

$$
\begin{aligned}
& \left(-\frac{\partial}{\partial \tau}+A^{\odot \star}(\tau)\right) j\left(H_{2}\left(\tau, \xi^{(2)}\right)\right) \\
& =j\left(D_{\xi} H_{2}\left(\tau, \xi^{(2)}\right) \tilde{M}_{0} \xi+p_{2}\left(\tau, \xi^{(2)}\right) \dot{\gamma}_{\tau}+\tilde{Q}_{0}(\tau) P_{2}\left(\tau, \xi^{(2)}\right)\right)-R_{2}\left(\tau, \xi^{(2)}\right)
\end{aligned}
$$

where $R_{2}\left(\tau, \xi^{(2)}\right)=G_{2}\left(\tau,\left(\tilde{Q}_{0}(\tau) \xi\right)^{(2)}\right)$. Finally, after collecting the $\xi^{(q)}$-terms for $q=3, \ldots, k$ one obtains

$$
\begin{align*}
& \left(-\frac{\partial}{\partial \tau}+A^{\odot \star}(\tau)\right) j\left(H_{q}\left(\tau, \xi^{(q)}\right)\right) \\
& =j\left(D_{\xi} H_{q}\left(\tau, \xi^{(q)}\right) \tilde{M}_{0} \xi+p_{q}\left(\tau, \xi^{(q)}\right) \dot{\gamma}_{\tau}+\tilde{Q}_{0}(\tau) P_{q}\left(\tau, \xi^{(q)}\right)\right)-R_{q}\left(\tau, \xi^{(q)}\right) \tag{56}
\end{align*}
$$

where $R_{q}\left(\tau, \xi^{(q)}\right)$ depends on $G_{q^{\prime}}(\tau, \cdot)$ for $2 \leq q^{\prime} \leq q$ and $j H_{q^{\prime}}(\tau, \cdot), j\left(p_{q^{\prime}}(\tau, \cdot) \dot{\gamma}_{\tau}\right)$ and $j\left(\tilde{Q}_{0}(\tau) P_{q^{\prime}}(\tau, \cdot)\right)$ for $q^{\prime}=2, \ldots, q-1$.

Step 3: Projecting on subspaces. We want to project (56) onto the spaces $\mathbb{R} j \dot{\gamma}_{\tau}, j \tilde{X}_{0}(\tau)$ and $X_{ \pm}^{\odot \star}(\tau)$, where $X_{+}^{\odot \star}(\tau)=j\left(X_{+}(\tau)\right)$ to show the existence of $H_{q}$ separately on each individual space. Because $X=\mathbb{R} \dot{\gamma}_{\tau} \oplus \tilde{X}_{0}(\tau) \oplus X_{-}(\tau) \oplus X_{+}(\tau)$ for any $\tau \in \mathbb{R}$, we can decompose for any $q=2, \ldots, k$ the function $H_{q}$ as

$$
H_{q}\left(\tau, \xi^{(q)}\right)=H_{q}^{00}\left(\tau, \xi^{(q)}\right) \dot{\gamma}_{\tau}+\tilde{Q}_{0}(\tau) \tilde{H}_{q}^{0}\left(\tau, \xi^{(q)}\right)+H_{q}^{-}\left(\tau, \xi^{(q)}\right)+H_{q}^{+}\left(\tau, \xi^{(q)}\right)
$$

where $H_{q}^{ \pm}\left(\tau, \xi^{(q)}\right)=P_{ \pm}(\tau) H_{q}\left(\tau, \xi^{(q)}\right) \in X_{ \pm}(\tau)$ together with $H_{q}^{00}\left(\tau, \xi^{(q)}\right) \in \mathbb{R}$ and $\tilde{H}_{q}^{0}\left(\tau, \xi^{(q)}\right) \in \mathbb{R}^{n_{0}}$ for all $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^{n_{0}}$. Notice due to Proposition 27 and Proposition 25 that we also have the decomposition $X^{\odot \star}=\mathbb{R} j \dot{\gamma}_{\tau} \oplus j \tilde{X}_{0}(\tau) \oplus X_{-}^{\odot \star}(\tau) \oplus j\left(X_{+}(\tau)\right)$ for any $\tau \in \mathbb{R}$ such that

$$
R_{q}\left(\tau, \xi^{(q)}\right)=R_{q}^{00}\left(\tau, \xi^{(q)}\right) j \dot{\gamma}_{\tau}+j\left(\tilde{Q}_{0}(\tau) \tilde{R}_{q}^{0}\left(\tau, \xi^{(q)}\right)\right)+R_{q}^{-}\left(\tau, \xi^{(q)}\right)+R_{q}^{+}\left(\tau, \xi^{(q)}\right)
$$

where $R_{q}^{ \pm}\left(\tau, \xi^{(q)}\right)=P_{ \pm}^{\odot \star}(\tau) R_{q}\left(\tau, \xi^{(q)}\right) \in X_{ \pm}^{\odot \star}(\tau)$ together with $R_{q}^{00}\left(\tau, \xi^{(q)}\right) \in \mathbb{R}$ and $\tilde{R}_{q}^{0}\left(\tau, \xi^{(q)}\right) \in \mathbb{R}^{n_{0}}$ for all $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^{n_{0}}$. Substituting these decompositions into (56) yields for the left-hand side of this equation

$$
\begin{aligned}
\left(-\frac{\partial}{\partial \tau}+A^{\odot \star}(\tau)\right) j\left(H_{q}\left(\tau, \xi^{(q)}\right)\right) & =-j\left(\frac{\partial H_{q}^{00}\left(\tau, \xi^{(q)}\right)}{\partial \tau} \dot{\gamma}_{\tau}+H_{q}^{00}\left(\tau, \xi^{(q)}\right) \ddot{\gamma}_{\tau}\right) \\
& +A^{\odot \star}(\tau) j\left(H_{q}^{00}\left(\tau, \xi^{(q)}\right) \dot{\gamma}_{\tau}\right) \\
& -j\left(\frac{\partial \tilde{Q}_{0}(\tau)}{\partial \tau} \tilde{H}_{q}^{0}\left(\tau, \xi^{(q)}\right)+\tilde{Q}_{0}(\tau) \frac{\partial \tilde{H}_{q}^{0}\left(\tau, \xi^{(q)}\right)}{\partial \tau}\right) \\
& +A^{\odot \star}(\tau) j\left(\tilde{Q}_{0}(\tau) \tilde{H}_{q}^{0}\left(\tau, \xi^{(q)}\right)\right) \\
& +\left(-\frac{\partial}{\partial \tau}+A^{\odot \star}(\tau)\right) j\left(H_{q}^{-}\left(\tau, \xi^{(q)}\right)+H_{q}^{+}\left(\tau, \xi^{(q)}\right)\right)
\end{aligned}
$$

where we twice used the product rule for differentiation. Since $\dot{\gamma}_{\tau}$ is a $T$-periodic eigenfunction of the monodromy operator $U(\tau+T, \tau)$, we get from Theorem 18 that $\left(-\frac{d}{d \tau}+A^{\odot \star}(\tau)\right) j \dot{\gamma}_{\tau}=0$. Using (55), we arrive at

$$
\begin{align*}
\left(-\frac{\partial}{\partial \tau}+A^{\odot \star}(\tau)\right) j\left(H_{q}\left(\tau, \xi^{(q)}\right)\right) & =j\left(-\frac{\partial H_{q}^{00}\left(\tau, \xi^{(q)}\right)}{\partial \tau} \dot{\gamma}_{\tau}\right)  \tag{57}\\
& +j\left(\tilde{Q}_{0}(\tau)\left(\frac{\partial \tilde{H}_{q}^{0}\left(\tau, \xi^{(q)}\right)}{\partial \tau}\right)+\tilde{M}_{0} \tilde{H}_{q}^{0}\left(\tau, \xi^{(q)}\right)\right) \\
& +\left(-\frac{\partial}{\partial \tau}+A^{\odot \star}(\tau)\right) j\left(H_{q}^{-}\left(\tau, \xi^{(q)}\right)+H_{q}^{+}\left(\tau, \xi^{(q)}\right)\right)
\end{align*}
$$

and this must be equal to the right-hand side of (56). Let us first show existence of $H_{q}^{ \pm}$via projecting on the spaces $X_{ \pm}^{\odot \star}(\tau)$. On these subspaces, we get the equation

$$
\begin{equation*}
\left(-\frac{\partial}{\partial \tau}+A^{\odot \star}(\tau)\right) j\left(H_{q}^{ \pm}\left(\tau, \xi^{(q)}\right)\right)=j\left(D_{\xi} H_{q}^{ \pm}\left(\tau, \xi^{(q)}\right) \tilde{M}_{0} \xi\right)-R_{q}^{ \pm}\left(\tau, \xi^{(q)}\right) \tag{58}
\end{equation*}
$$

Substituting $\tau \rightarrow \theta$ and $\xi \rightarrow e^{(\theta-\tau) \tilde{M}_{0}} \xi=\tilde{\xi}$ leads us to

$$
-\frac{\partial}{\partial \theta} j\left(H_{q}^{ \pm}\left(\theta, \tilde{\xi}^{(q)}\right)\right)+A^{\odot \star}(\theta) j\left(H_{q}^{ \pm}\left(\theta, \tilde{\xi}^{(q)}\right)\right)-j\left(D_{\tilde{\xi}} H_{q}^{ \pm}\left(\theta, \tilde{\xi}^{(q)}\right) \tilde{M}_{0} \xi\right)=-R_{q}^{ \pm}\left(\theta, \tilde{\xi}^{(q)}\right)
$$

When the operator $-U^{\odot \star}(\tau, \theta)$ acts on both side of the equation, we obtain

$$
\begin{align*}
& -U^{\odot \star}(\tau, \theta)\left[-\frac{\partial}{\partial \theta} j\left(H_{q}^{ \pm}\left(\theta, \tilde{\xi}^{(q)}\right)\right)+A^{\odot \star}(\theta) j\left(H_{q}^{ \pm}\left(\theta, \tilde{\xi}^{(q)}\right)\right)-j\left(D_{\tilde{\xi}} H_{q}^{ \pm}\left(\theta, \tilde{\xi}^{(q)}\right) \tilde{M}_{0} \xi\right)\right] \\
& =U^{\odot \star}(\tau, \theta) R_{q}^{ \pm}\left(\theta, \tilde{\xi}^{(q)}\right) \tag{59}
\end{align*}
$$

Let us focus on the left-hand-side of this equation. It follows from [4, Theorem 5.5] that

$$
-U^{\odot \star}(\tau, \theta) A^{\odot \star}(\theta) j\left(H_{q}^{ \pm}\left(\theta, \tilde{\xi}^{(q)}\right)\right)=-\left[\partial_{\theta}^{\star} U^{\odot \star}(\tau, \theta)\right] j\left(H_{q}^{ \pm}\left(\theta, \tilde{\xi}^{(q)}\right)\right)
$$

Filling this result back into (59) and using the partial weak derivative operator yields

$$
U^{\odot \star}(\tau, \theta)\left[\partial_{\theta}^{\star} j\left(H_{q}^{ \pm}\left(\theta, \tilde{\xi}^{(q)}\right)\right)\right]+\left[\partial_{\theta}^{\star} U^{\odot \star}(\tau, \theta)\right] j\left(H_{q}^{ \pm}\left(\theta, \tilde{\xi}^{(q)}\right)\right)=U^{\odot \star}(\tau, \theta) R_{q}^{ \pm}\left(\theta, \tilde{\xi}^{(q)}\right)
$$

where we have used the product rule for differentiation, but essentially in the dual pairings due to the partial weak ${ }^{\star}$ derivative. Using the product rule again and recalling that $\tilde{\xi}=e^{(\theta-\tau) \tilde{M}_{0}} \xi$, we get the identity

$$
\partial_{\theta}^{\star}\left[U^{\odot \star}(\tau, \theta) j\left(H_{q}^{ \pm}\left(\theta,\left(e^{(\theta-\tau) \tilde{M}_{0}}\right)^{(q)}\right)\right)\right]=U^{\odot \star}(\tau, \theta) R_{q}^{ \pm}\left(\theta,\left(e^{(\theta-\tau) \tilde{M}_{0}}\right)^{(q)}\right)
$$

Using the definition of the weak ${ }^{\star}$ derivative, we get for all $x^{\odot} \in X^{\odot}$ that

$$
\begin{equation*}
\frac{\partial}{\partial \theta}\left\langle j\left(U(\tau, \theta) H_{q}^{ \pm}\left(\theta,\left(e^{(\theta-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right)\right), x^{\odot}\right\rangle=\left\langle U^{\odot \star}(\tau, \theta) R_{q}^{ \pm}\left(\theta,\left(e^{(\theta-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right), x^{\odot}\right\rangle \tag{60}
\end{equation*}
$$

Step 4: Existence of $H_{q}^{+}$. Let us first find an expression for $H_{q}^{+}\left(\tau, \xi^{(q)}\right)$. As $X_{+}(s)$ is finitedimensional, $U(\tau, s)$ extends to all $\tau, s \in \mathbb{R}$ on the subspace $X_{+}(s)$. So let $s \geq \tau$ and integrate (60) over the interval $[\tau, s]$ to obtain

$$
\begin{align*}
\left\langle j\left(H_{q}^{+}\left(\tau, \xi^{(q)}\right)\right), x^{\odot}\right\rangle & =\left\langle j\left(U(\tau, s) H_{q}^{+}\left(s,\left(e^{(s-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right)\right), x^{\odot}\right\rangle \\
& -\int_{\tau}^{s}\left\langle U^{\odot \star}(\tau, \theta) R_{q}^{+}\left(\theta,\left(e^{(\theta-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right), x^{\odot}\right\rangle d \theta \tag{61}
\end{align*}
$$

Let us focus on the first term of the right-hand side. Notice that

$$
H_{q}^{+}\left(s,\left(e^{(s-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right)=\sum_{|\alpha|=q} \frac{1}{\alpha!} P_{+}(s) H_{s}^{\alpha}\left(\left(e^{(s-\tau) \tilde{M}_{0}} \xi\right)^{\alpha}\right)
$$

where $H_{s}^{\alpha}\left(\left(e^{(s-\tau) \tilde{M}_{0}} \xi\right)^{\alpha}\right) \in X$. The notation from Hypothesis 1 implies

$$
U(\tau, s) H_{q}^{+}\left(s,\left(e^{(s-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right)=\sum_{|\alpha|=q} \frac{1}{\alpha!} U_{+}(\tau, s) P_{+}(s) H_{s}^{\alpha}\left(\left(e^{(s-\tau) \tilde{M}_{0}} \xi\right)^{\alpha}\right)
$$

and using the exponential trichotomy property of the forward evolutionary system (criterium 7 of Hypothesis 1), there is a $b>0$ such that for a given $\varepsilon>0$ there exists a $K_{\varepsilon}>0$ with the property

$$
\left\|U(\tau, s) H_{q}^{+}\left(s,\left(e^{(s-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right)\right\|_{\infty} \leq K_{\varepsilon} e^{b(\tau-s)} \sum_{|\alpha|=q} \frac{1}{\alpha!}\left\|H_{s}^{\alpha}\left(\left(e^{(s-\tau) \tilde{M}_{0}} \xi\right)^{\alpha}\right)\right\|_{\infty}
$$

where the number $N$ from criterium 3 of Hypothesis 1 is absorbed in the $K_{\varepsilon}$ constant. Since the diagonal elements of the matrix $\tilde{M}_{0}$ have real part zero, $e^{(s-\tau)} \tilde{M}_{0} \xi$ is a polynomial in $\xi$ and so $\left\|H_{s}^{\alpha}\left(\left(e^{(s-\tau) \tilde{M}_{0}} \xi\right)^{\alpha}\right)\right\|_{\infty}$ can grow at most polynomially for $s \rightarrow \pm \infty$. With this in mind, we get

$$
\begin{aligned}
\left|\left\langle j\left(U(\tau, s) H_{q}^{+}\left(s,\left(e^{(s-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right)\right), x^{\odot}\right\rangle\right| & \leq K_{\varepsilon} e^{b(\tau-s)}\left\|x^{\odot}\right\| \sum_{|\alpha|=q} \frac{1}{\alpha!}\left\|H_{s}^{\alpha}\left(\left(e^{(s-\tau) \tilde{M}_{0}} \xi\right)^{\alpha}\right)\right\|_{\infty} \\
& \leq M_{\varepsilon} e^{b(\tau-s)} \max _{|\alpha|=q}\left\|H_{s}^{\alpha}\left(\left(e^{(s-\tau) \tilde{M}_{0}} \xi\right)^{\alpha}\right)\right\|_{\infty} \\
& \rightarrow 0, \quad \text { as } s \rightarrow \infty
\end{aligned}
$$

Using this convergence, taking the limit in (61) yields

$$
\begin{equation*}
\left\langle j\left(H_{q}^{+}\left(\tau, \xi^{(q)}\right)\right), x^{\odot}\right\rangle=\left\langle\int_{\tau}^{\infty}-U^{\odot \star}(\tau, \theta) R_{q}^{+}\left(\theta,\left(e^{(\theta-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right) d \theta, x^{\odot}\right\rangle \tag{62}
\end{equation*}
$$

if we can show that for any $x^{\odot} \in X^{\odot}$ and fixed $\tau \in \mathbb{R}, \xi \in \mathbb{R}^{n_{0}}$ the map $g_{q, \tau, \xi}^{+}:[\tau, \infty) \rightarrow \mathbb{R}$ defined by $g_{q, \tau, \xi}^{+}(\theta)=\left\langle-U^{\odot \star}(\tau, \theta) R_{q}^{+}\left(\theta,\left(e^{(\theta-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right), x^{\odot}\right\rangle$ is in $L^{1}([\tau, \infty), \mathbb{R})$. Let $\tau \in \mathbb{R}, \xi \in \mathbb{R}^{n_{0}}$ and $x^{\odot} \in X^{\odot}$ be given. From criterium 3 and 7 of Proposition 25 we get

$$
\int_{\tau}^{\infty}\left|g_{q, \tau, \xi}^{+}(\theta)\right| d \theta \leq K_{\varepsilon} N\left\|x^{\odot}\right\| e^{b \tau} \int_{\tau}^{\infty} e^{-b \theta}\left\|R_{q}\left(\theta,\left(e^{(\theta-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right)\right\| d \theta
$$

Recall that $e^{(\theta-\tau) \tilde{M}_{0}} \xi$ is a polynomial in $\xi$ and that $R_{q}\left(\tau, \xi^{(q)}\right)$ depends on $G_{q^{\prime}}(\tau, \cdot)$ for $2 \leq q^{\prime} \leq q$ and $j H_{q^{\prime}}(\tau, \cdot), j\left(p_{q^{\prime}}(\tau, \cdot) \dot{\gamma}_{\tau}\right)$ and $j\left(\tilde{Q}_{0}(\tau) P_{q^{\prime}}(\tau, \cdot)\right)$ for $q^{\prime}=2, \ldots, q-1$. Since $G_{q^{\prime}}$ is periodic in the first variable and evaluated at a polynomial, $G_{q^{\prime}}$ grows at most polynomially for $2 \leq q^{\prime} \leq q$. As we can assume that $H_{q^{\prime}}$ is $T$-periodic in the first variable for $q^{\prime}=2, \ldots, q-1$ (we will show this later for $q^{\prime}=q$ ) and evaluated at a polynomial it follows that $j H_{q^{\prime}}(\tau, \cdot)$ grows at most polynomially for $q^{\prime}=2, \ldots, q-1$. As $p_{q^{\prime}}$ and $P_{q^{\prime}}$ are $T$-periodic in the first variable for $q^{\prime}=2, \ldots, q-1$ (we will show this later for $q^{\prime}=q$ ), it follows that $j\left(p_{q^{\prime}}(\tau, \cdot) \dot{\gamma}_{\tau}\right)$ and $j\left(\tilde{Q}_{0}(\tau) P_{q^{\prime}}(\tau, \cdot)\right)$ grows at most polynomially. To conclude, there exists a polynomial $r_{q, \tau, \xi}^{+}: \mathbb{R} \rightarrow \mathbb{R}$ such that $\left\|R_{q}^{+}\left(\theta,\left(e^{(\theta-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right)\right\| \leq r_{q, \tau, \xi}^{+}(\theta)$ for all $\theta \geq \tau$. Hence,

$$
\begin{equation*}
\int_{\tau}^{\infty}\left|g_{q, \tau, \xi}^{+}(\theta)\right| d \theta \leq K_{\varepsilon} N\left\|x^{\odot}\right\| e^{b \tau} \int_{\tau}^{\infty} e^{-b \theta} r_{q, \tau, \xi}^{+}(\theta) d \theta<\infty \tag{63}
\end{equation*}
$$

because the map $[\tau, \infty) \ni \theta \mapsto e^{-b \theta} g_{q, \tau, \xi}^{+}(\theta) \in \mathbb{R}$ decays fast enough to zero as $\theta \rightarrow \infty$. We have proven that the weak ${ }^{\star}$ integral in (62) exists. Because $R_{q}^{+}\left(\theta,\left(e^{(\theta-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right) \in j\left(X_{+}(\tau)\right)$ and (62) holds for any $x^{\odot} \in X^{\odot}$, we obtain

$$
j\left(H_{q}^{+}\left(\tau, \xi^{(q)}\right)\right)=j \int_{\tau}^{\infty}-U(\tau, \theta) j^{-1} R_{q}^{+}\left(\theta,\left(e^{(\theta-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right) d \theta
$$

By $\odot$-reflexivity we have that $j$ is an isomorphism on $j(X)=X^{\odot \odot}$ and hence

$$
\begin{equation*}
H_{q}^{+}\left(\tau, \xi^{(q)}\right)=-\int_{\tau}^{\infty} U(\tau, \theta) j^{-1} R_{q}^{+}\left(\theta,\left(e^{(\theta-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right) d \theta \tag{64}
\end{equation*}
$$

can be evaluated as a standard Riemann integral. It can easily be checked that $H_{q}^{+}$is $T$-periodic in the first variable because $P_{+}^{\odot \star}$ is $T$-periodic and $R_{q}$ is $T$-periodic in the first variable. Let us now prove the continuity of the map $H_{q}^{+}$. As $U^{\odot \star}(t, \tau)$ restricted to $j\left(X_{+}(\tau)\right)$ is invertible, we can adjust the proof from Lemma 2 to prove continuity of the limiting function $v(\cdot, \infty, \cdot, g)$ for a continuous function $g:[\tau, \infty) \rightarrow X^{\odot \star}$ under the assumption that $H_{q}^{+}$is bounded in norm. Since proved in (63) that $H_{q}^{+}$is bounded in norm and noticing that $P_{+}^{\odot \star}$ and $R_{q}$ are continuous for all $q \in\{1, \ldots, k\}$, the result follows.

Step 5: Existence of $H_{q}^{-}$. Now, we can look for an explicit expression of $H_{q}^{-}\left(\tau, \xi^{(q)}\right)$. Integrating (60) over $[s, \tau]$ for a fixed $s \in \mathbb{R}$, yields for any $x^{\odot} \in X^{\odot}$, due to the definition of the weak integral

$$
\begin{align*}
\left\langle j\left(H_{q}^{-}\left(\tau, \xi^{(q)}\right)\right), x^{\odot}\right\rangle & =\left\langle j\left(U(\tau, s) H_{q}^{-}\left(s,\left(e^{(s-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right)\right), x^{\odot}\right\rangle \\
& +\int_{\tau}^{s}\left\langle U^{\odot \star}(\tau, \theta) R_{q}^{-}\left(\theta,\left(e^{(\theta-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right), x^{\odot}\right\rangle d \theta \tag{65}
\end{align*}
$$

Similar to the $H_{q}^{+}$-case, we want to show that the first term goes to zero, but now as $s \rightarrow-\infty$. Recall that $\left\|H_{s}^{\alpha}\left(\left(e^{(s-\tau) \tilde{M}_{0}} \xi\right)^{\alpha}\right)\right\|_{\infty}$ can grow at most polynomially for $s \rightarrow \pm \infty$ and so due to the exponential
trichotomy of the forward evolutionary system (criterium 7 of Hypothesis 1 ), there exists an $a<0$ such that for a given $\varepsilon>0$ there is a $M_{\varepsilon}>0$ with the property

$$
\begin{aligned}
\left\langle j\left(U(\tau, s) H_{q}^{-}\left(s,\left(e^{(s-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right)\right), x^{\odot}\right\rangle & \leq M_{\varepsilon} e^{a(\tau-s)} \max _{|\alpha|=q}\left\|H_{s}^{\alpha}\left(s,\left(e^{(s-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right)\right\|_{\infty} \\
& \rightarrow 0, \quad \text { as } s \rightarrow-\infty
\end{aligned}
$$

where the other constants are already absorbed in $M_{\varepsilon}$. We conclude that

$$
\begin{equation*}
\left\langle j\left(H_{q}^{-}\left(\tau, \xi^{(q)}\right)\right), x^{\odot}\right\rangle=\left\langle\int_{-\infty}^{\tau} U^{\odot \star}(\tau, \theta) R_{q}^{-}\left(\theta,\left(e^{(\theta-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right) d \theta, x^{\odot}\right\rangle \tag{66}
\end{equation*}
$$

if we are able to show that for any $x^{\odot} \in X^{\odot}$ and fixed $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^{n_{0}}$ that the map $g_{q, \tau, \xi}^{-}$: $(-\infty, \tau] \rightarrow \mathbb{R}$ defined by $g_{q, \tau, \xi}^{-}(\theta)=\left\langle U^{\odot \star}(\tau, \theta) R_{q}^{-}\left(\theta,\left(e^{(\theta-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right), x^{\odot}\right\rangle$ is in $L^{1}((-\infty, \tau], \mathbb{R})$. The exponential trichotomy implies that for a given $\varepsilon>0$ one can find a $K_{\varepsilon}>0$ such that

$$
\int_{\tau}^{\infty}\left|g_{q, \tau, \xi}^{-}(\theta)\right| d \theta \leq K_{\varepsilon} N\left\|x^{\odot}\right\| e^{a \tau} \int_{-\infty}^{\tau} e^{-a \theta}\left\|R_{q}\left(\theta,\left(e^{(\theta-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right)\right\| d \theta
$$

From the same reasoning as in the $H_{q}^{+}$-case, there exists a polynomial $r_{q, \tau, \xi}^{-}: \mathbb{R} \rightarrow \mathbb{R}$ that satisfies the estimate $\left\|R_{q}\left(\theta,\left(e^{(\theta-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right)\right\| \leq r_{q, \tau, \xi}^{-}(\theta)$ for all $\theta \leq \tau$. Hence,

$$
\begin{equation*}
\int_{\tau}^{\infty}\left|g_{q, \tau, \xi}^{-}(\theta)\right| d \theta \leq K_{\varepsilon} N\left\|x^{\odot}\right\| e^{a \tau} \int_{-\infty}^{\tau} e^{-a \theta} r_{q, \tau, \xi}^{-}(\theta) d \theta<\infty \tag{67}
\end{equation*}
$$

because the map $\theta \mapsto e^{-a \theta} r_{q, \tau, \xi}^{-}(\theta)$ decays fast enough to zero as $\theta \rightarrow-\infty$. Hence, $g_{q, \tau, \xi}^{-} \in$ $L^{1}((-\infty, \tau], \mathbb{R})$ and so the weak ${ }^{\star}$ integral exists. Since (66) holds for all $x^{\odot} \in X^{\odot}$ we get

$$
\begin{equation*}
H_{q}^{-}\left(\tau, \xi^{(q)}\right)=j^{-1} \int_{-\infty}^{\tau} U^{\odot \star}(\tau, \theta) R_{q}^{-}\left(\theta,\left(e^{(\theta-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right) d \theta \tag{68}
\end{equation*}
$$

if we can prove that the weak ${ }^{\star}$ integral takes values in $j(X)$. Notice that we proved in (67) that $H_{q}^{-}$ is bounded in norm. With the notation from Lemma 2 we have that $j\left(H_{q}^{-}\left(\tau, \xi^{(q)}\right)\right)=v(\tau, \tau,-\infty, g)$ with the continuous map $g$ defined by $g(\theta)=P_{-}^{\odot \star}(\theta) R_{q}\left(\theta,\left(e^{(\theta-\tau) \tilde{M}_{0}} \xi\right)^{(q)}\right)$ for all $\theta \in(-\infty, \tau]$ since $P_{-}^{\odot \star}$ and $R_{q}$ are continuous for all $q \in\{1, \ldots, k\}$. It follows from Lemma 2 that $H_{q}$ takes values in $j(X)$ and is continuous It is not difficult to show that $H_{q}^{-}$is $T$-periodic in the first variable because $P_{-}^{\odot \star}$ is $T$-periodic and $R_{q}$ is $T$-periodic in the first variable.

Step 6: Existence of $H_{q}^{00}$ and $H_{q}^{0}$. To obtain $H_{q}^{00}\left(\tau, \xi^{(q)}\right)$ and $H_{q}^{0}\left(\tau, \xi^{(q)}\right)$, we project (56) onto $\mathbb{R} j \dot{\gamma}_{\tau}$ and $j \tilde{X}_{0}(\tau)$. Since $j$ is an isomorphism on $j(X)=X^{\odot \odot}$ we get from combining (56) and (57) that the coefficients must satisfy

$$
\begin{array}{r}
-\frac{\partial H_{q}^{00}\left(\tau, \xi^{(q)}\right)}{\partial \tau}-D_{\xi} H_{q}^{00}\left(\tau, \xi^{(q)}\right) \tilde{M}_{0} \xi=p_{q}\left(\tau, \xi^{(q)}\right)-R_{q}^{00}\left(\tau, \xi^{(q)}\right)  \tag{69}\\
-\frac{\partial \tilde{H}_{q}^{0}\left(\tau, \xi^{(q)}\right)}{\partial \tau}+\tilde{M}_{0} \tilde{H}_{q}^{0}\left(\tau, \xi^{(q)}\right)-D_{\xi} \tilde{H}_{q}^{0}\left(\tau, \xi^{(q)}\right) \tilde{M}_{0} \xi=P_{q}\left(\tau, \xi^{(q)}\right)-R_{q}^{0}\left(\tau, \xi^{(q)}\right)
\end{array}
$$

These are precisely the equations obtained in [19, Theorem III.7] and hence from the results of [19, Theorem III.7], the provided normal forms follow. In adition, it is proven in [19, Theorem III.7] that $H_{q}^{00}, \tilde{H}_{q}^{0}, p_{q}, P_{q}$ are continuous and so we conclude that $H, p$ and $P$ are $C^{k}$-smooth maps.

Recall that the map $\tau \mapsto \gamma_{\tau}$ is $T$-periodic and $C^{k}$-smooth. Furthermore, from (48) in combination with (50) we have that $\tau \mapsto \tilde{Q}_{0}(\tau)$ is $T$-periodic and $C^{k}$-smooth. It also follows from previous theorem
that $(\tau, \xi) \mapsto H(\tau, \xi)$ is $T$-periodic in the first component and $C^{k}$-smooth. Hence, $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$ can be also written as

$$
\begin{equation*}
\mathcal{W}_{\text {loc }}^{c}(\Gamma)=\left\{\gamma_{\tau}+\tilde{Q}_{0}(\tau) \xi+H(\tau, \xi): \tau \in \mathbb{R} \text { and } \xi \in \mathbb{R}^{n_{0}}\right\} \subset X, \tag{70}
\end{equation*}
$$

and has exactly the same properties as the description of $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$ given in (36). Hence, $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$ is the center manifold for (DDE) around the periodic orbit $\Gamma$ whenever $U(s+T, s)$ fulfils the requirements of Theorem 20. This center manifold is $T$-periodic in the sense that for any $\xi \in \mathbb{R}^{n_{0}}$ the map $\mathbb{R} \ni \tau \mapsto$ $\gamma_{\tau}+\tilde{Q}_{0}(\tau) \xi+H(\tau, \xi) \in X$ is $T$-periodic.

Next we consider the case where the trivial Floquet multiplier has algebraic multiplicity larger than 1 and there is no Floquet multiplier located at -1 . This is for example the case in the fold bifurcation.

Theorem 21 (Normal Form II). Assume that the algebraic multiplicity of the trivial Floquet multiplier is more than one and that -1 is not a Floquet multiplier. Then there exist $C^{k}$-smooth maps $H$ : $\mathbb{R} \times \mathbb{R}^{n_{0}} \rightarrow X, p: \mathbb{R} \times \mathbb{R}^{n_{0}} \rightarrow \mathbb{R}$ and $P: \mathbb{R} \times \mathbb{R}^{n_{0}} \rightarrow \mathbb{R}^{n_{0}}$ such that the history $x_{t} \in \mathcal{W}_{\text {loc }}^{c}(\Gamma)$ may be represented as

$$
x_{t}=\gamma_{\tau}+\tilde{Q}_{0}(\tau) \xi+H(\tau, \xi), \quad t \in I
$$

where the time dependence of the coordinates $(\tau, \xi)$ describing the dynamics of (DDE) on $\mathcal{W}_{\mathrm{loc}}^{c}(\Gamma)$ is defined by the normal form

$$
\left\{\begin{array}{l}
\frac{d \tau}{d t}=1+\xi_{1} p(\tau, \xi)+\mathcal{O}\left(|\xi|^{k+1}\right) \\
\frac{d \xi}{d \tau}=\tilde{M}_{0} \xi+P(\tau, \xi)+\mathcal{O}\left(|\xi|^{k+1}\right)
\end{array}\right.
$$

Here the functions $H, p$ and $P$ are $T$-periodic in $\tau$ and at least quadratic in $\xi$. The $\mathcal{O}$-terms are also $T$-periodic in $\tau$. Moreover, $p$ and $P$ are polynomials in $\xi$ of degree less than or equal to $k$ such that

$$
\frac{d}{d \tau} p\left(\tau, e^{-\tau \tilde{M}_{0}^{*}} \xi\right)=0 \text { and } \frac{d}{d \tau}\left(e^{\tau \tilde{M}_{0}^{*}} P\left(\tau, e^{-\tau \tilde{M}_{0}^{*}} \xi\right)\right)=0,
$$

for all $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^{n_{0}}$.
Notice the appearance of the $\xi_{1}$-term in the normal form description. This is due to the fact that the $\star$ in (51) is now replaced by 1 instead of 0 compared to Theorem 20 . The proof of this theorem is very similar to that of Theorem 20.

Proof of Theorem 21. We proceed in the same way as the proof of Theorem 20 and start by comparing the expansions of

$$
\frac{d}{d t} j\left(\gamma_{\tau}+\tilde{Q}_{0}(\tau) \xi+H(\tau, \xi)\right)=j\left(\dot{\gamma}_{\tau}+\frac{\partial \tilde{Q}_{0}(\tau)}{\partial \tau} \xi+\frac{\partial H(\tau, \xi)}{\partial \tau}+\left(\tilde{Q}_{0}(\tau)+D_{\xi} H(\tau, \xi)\right) \frac{d \xi}{d \tau}\right) \frac{d \tau}{d t}
$$

and

$$
A_{0}^{\odot \star} j\left(\gamma_{\tau}+\tilde{Q}_{0}(\tau) \xi+H(\tau, \xi)\right)+G\left(\gamma_{\tau}+\tilde{Q}_{0}(\tau) \xi+H(\tau, \xi)\right)
$$

by subsituting

$$
\frac{d \tau}{d t}=1+\xi_{1}+p(\tau, \xi)+\mathcal{O}\left(|\xi|^{k+1}\right)
$$

and

$$
\frac{d \xi}{d \tau}=\tilde{M}_{0} \xi+P(\tau, \xi)+\mathcal{O}\left(|\xi|^{k+1}\right)
$$

We copy the same notation from the proof of Theorem 20 and use the expansions of $H, p$ and $P$ together with (54) where now $\varphi$ must be substituted by $\tilde{Q}_{0}(\tau) \xi+H(\tau, \xi)$. Eventually,

$$
\begin{aligned}
& j\left[\dot{\gamma}_{\tau}+\frac{\partial \tilde{Q}_{0}(\tau)}{\partial \tau} \xi+\sum_{q=2}^{k} \frac{\left.\partial H_{q}\left(\tau, \xi^{(q)}\right)\right)}{\partial \tau}+\left(\tilde{Q}_{0}(\tau)+\sum_{q=2}^{k} D_{\xi} H_{q}\left(\tau, \xi^{(q)}\right)\right)\left(\tilde{M}_{0}+\sum_{q=2}^{k} P_{q}\left(\tau, \xi^{(q)}\right)\right)\right] \\
& \left(1+\xi_{1}+\sum_{q=2}^{k} p_{q}\left(\tau, \xi^{(q)}\right)\right)+\mathcal{O}\left(|\xi|^{k+1}\right) \\
& =A_{0}^{\odot \star} j\left(\gamma_{\tau}\right)+G\left(\gamma_{\tau}\right)+A^{\odot \star}(\tau) j\left(\tilde{Q}_{0}(\tau) \xi+\sum_{q=2}^{k} H_{q}\left(\tau, \xi^{(q)}\right)\right) \\
& +\sum_{q=2}^{k} G_{q}\left(\tau,\left[\tilde{Q}_{0}(\tau) \xi+\sum_{p=2}^{k} H_{q}\left(\tau, \xi^{(p)}\right)\right]^{(q)}\right)+\mathcal{O}\left(|\xi|^{k+1}\right)
\end{aligned}
$$

Let us now compare the terms in powers of $\xi$ on both side of the equation. The $\xi^{0}$-terms give us

$$
\frac{d}{d \tau} j\left(\gamma_{\tau}\right)=A_{0}^{\odot \star} j\left(\gamma_{\tau}\right)+G\left(\gamma_{\tau}\right)
$$

which means that $\gamma$ is a solution (53). The $\xi^{1}$-terms tell us

$$
\begin{equation*}
\left(-\frac{\partial}{\partial \tau}+A^{\odot \star}(\tau)\right) j\left(\tilde{Q}_{0}(\tau) \xi\right)=j\left(\left(\tilde{Q}_{0}(\tau) \tilde{M}_{0}+\gamma_{\tau} \Pi_{1}\right) \xi\right) \tag{71}
\end{equation*}
$$

which is exactly the result established in (48), but now for all Floquet multipliers on the unit circle and characterizes the linear part. Here $\Pi_{1}: \mathbb{R}^{n_{0}} \rightarrow \mathbb{R}$ is the projection on the first component, defined as $\Pi_{1} \xi:=\xi_{1}$, where $\xi=\left(\xi_{1}, \ldots, \xi_{n_{0}}\right)$. After collecting the $\xi^{(2)}$-terms, we get

$$
\begin{aligned}
& \left(-\frac{\partial}{\partial \tau}+A^{\odot \star}(\tau)\right) j\left(H_{2}\left(\tau, \xi^{(2)}\right)\right) \\
& =j\left(D_{\xi} H_{2}\left(\tau, \xi^{(2)}\right) \tilde{M}_{0} \xi+p_{2}\left(\tau, \xi^{(2)}\right) \dot{\gamma}_{\tau}+\tilde{Q}_{0}(\tau) P_{2}\left(\tau, \xi^{(2)}\right)\right)-R_{2}\left(\tau, \xi^{(2)}\right)
\end{aligned}
$$

where $R_{2}\left(\tau, \xi^{(2)}\right)=G_{2}\left(\tau,\left(\tilde{Q}_{0}(\tau) \xi\right)^{2}\right)-\xi_{1}\left(\frac{d \tilde{Q}_{0}(\tau)}{d \tau} \xi+\tilde{Q}_{0}(\tau) \tilde{M}_{0} \xi\right)$. Finally, after collecting the $\xi^{(q)}$-terms for $q=3, \ldots, k$, we get

$$
\begin{align*}
& \left(-\frac{\partial}{\partial \tau}+A^{\odot \star}(\tau)\right) j\left(H_{q}\left(\tau, \xi^{(q)}\right)\right)  \tag{72}\\
& =j\left(D_{\xi} H_{q}\left(\tau, \xi^{(q)}\right) \tilde{M}_{0} \xi+p_{q}\left(\tau, \xi^{(q)}\right) \dot{\gamma}_{\tau}+\tilde{Q}_{0}(\tau) P_{q}\left(\tau, \xi^{(q)}\right)\right)-R_{q}\left(\tau, \xi^{(q)}\right) \tag{73}
\end{align*}
$$

where $R_{q}\left(\tau, \xi^{(q)}\right)$ depends on $F_{q^{\prime}}$ for $2 \leq q^{\prime} \leq q$ and $H_{q^{\prime}}, p_{q^{\prime}}$ and $P_{q^{\prime}}$ for $q^{\prime}=2, \ldots, q-1$.
We want to project (72) onto the spaces $\mathbb{R} \dot{\gamma}_{\tau}, \tilde{Q}_{0}(\tau)$ and $X_{ \pm}(\tau)$ to show the existence of $H_{q}$ separately on each individual space. We decompose for any $q=2, \ldots, k$ the functions $H_{q}$ and $R_{q}$ as

$$
\begin{aligned}
H_{q}\left(\tau, \xi^{(q)}\right) & =H_{q}^{00}\left(\tau, \xi^{(q)}\right) \dot{\gamma}_{\tau}+\tilde{Q}_{0}(\tau) \tilde{H}_{q}^{0}\left(\tau, \xi^{(q)}\right)+H_{q}^{-}\left(\tau, \xi^{(q)}\right)+H_{q}^{+}\left(\tau, \xi^{(q)}\right) \\
R_{q}\left(\tau, \xi^{(q)}\right) & =R_{q}^{00}\left(\tau, \xi^{(q)}\right) \dot{\gamma}_{\tau}+\tilde{Q}_{0}(\tau) \tilde{R}_{q}^{0}\left(\tau, \xi^{(q)}\right)+R_{q}^{-}\left(\tau, \xi^{(q)}\right)+R_{q}^{+}\left(\tau, \xi^{(q)}\right)
\end{aligned}
$$

where $H_{q}^{ \pm}\left(\tau, \xi^{(q)}\right)=P_{ \pm}(\tau) H_{q}\left(\tau, \xi^{(q)}\right) \in X_{ \pm}(\tau)$ and $R_{q}^{ \pm}\left(\tau, \xi^{(q)}\right)=P_{ \pm}^{\odot \star}(\tau) R_{q}\left(\tau, \xi^{(q)}\right) \in X_{ \pm}^{\odot \star}(\tau)$ with coordinates $H_{q}^{00}\left(\tau, \xi^{(q)}\right), R_{q}^{00}\left(\tau, \xi^{(q)}\right) \in \mathbb{R}$ and $\tilde{H}_{q}^{0}\left(\tau, \xi^{(q)}\right), \tilde{R}_{q}^{0}\left(\tau, \xi^{(q)}\right) \in \mathbb{R}^{n_{0}}$ for all $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^{n_{0}}$.

Carrying out the calculations in the same way as the proof of Theorem 21 and recalling that $\left(-\frac{d}{d \tau}+A^{\odot \star}(\tau)\right) j \dot{\gamma}_{\tau}=0$ together with (71), we obtain

$$
\begin{aligned}
\left(-\frac{\partial}{\partial \tau}+A^{\odot \star}(\tau)\right) j\left(H_{q}\left(\tau, \xi^{(q)}\right)\right) & =j\left(-\frac{\partial H_{q}^{00}\left(\tau, \xi^{(q)}\right)}{\partial \tau} \dot{\gamma}_{\tau}+\Pi_{1} \tilde{H}_{q}^{0}\left(\tau, \xi^{(q)}\right) \dot{\gamma}_{\tau}\right) \\
& +j\left(\tilde{Q}_{0}(\tau)\left(\frac{\partial \tilde{H}_{q}^{0}\left(\tau, \xi^{(q)}\right)}{\partial \tau}\right)+\tilde{M}_{0} \tilde{H}_{q}^{0}\left(\tau, \xi^{(q)}\right)\right) \\
& +\left(-\frac{\partial}{\partial \tau}+A^{\odot \star}(\tau)\right) j\left(H_{q}^{-}\left(\tau, \xi^{(q)}\right)+H_{q}^{+}\left(\tau, \xi^{(q)}\right)\right)
\end{aligned}
$$

and this must be equal to the right-hand side of (72). To obtain the coefficients, we project onto the spaces $\mathbb{R} j \dot{\gamma}_{\tau}, j \tilde{X}_{0}(\tau), j\left(X_{+}(\tau)\right)$ and $X_{-}^{\odot \star}(\tau)$. This yields the equations

$$
\begin{aligned}
& -\frac{\partial H_{q}^{00}\left(\tau, \xi^{(q)}\right)}{\partial \tau} D_{\xi} H_{q}^{00}\left(\tau, \xi^{(q)}\right) \tilde{M}_{0} \xi=p_{q}\left(\tau, \xi^{(q)}\right)-\Pi_{1} \tilde{H}_{q}^{0}\left(\tau, \xi^{(q)}\right)-R_{q}^{00}\left(\tau, \xi^{(q)}\right) \\
& -\frac{\partial \tilde{H}_{q}^{0}\left(\tau, \xi^{(q)}\right)}{\partial \tau}+\tilde{M}_{0} \tilde{H}_{q}^{0}\left(\tau, \xi^{(q)}\right)-D_{\xi} \tilde{H}_{q}^{0}\left(\tau, \xi^{(q)}\right) \tilde{M}_{0} \xi=P_{q}\left(\tau, \xi^{(q)}\right)-\tilde{R}_{q}^{0}\left(\tau, \xi^{(q)}\right) \\
& \left(-\frac{\partial}{\partial \tau}+A^{\odot \star}(\tau)\right) j\left(H_{q}^{ \pm}\left(\tau, \xi^{(q)}\right)\right)=j\left(D_{\xi} H_{q}^{ \pm}\left(\tau, \xi^{(q)}\right) \tilde{M}_{0} \xi\right)-R_{q}^{ \pm}\left(\tau, \xi^{(q)}\right) .
\end{aligned}
$$

We see that the equations for the $X_{ \pm}^{\odot \star}(\tau)$-component are the same as in the proof of Theorem 20. Hence, we obtain $H_{q}^{ \pm}$as (64) and (68) respectively. To solve the remaining part of this hierarchy of equations, notice these equations are solvable in exactly the same way as the proof of Theorem 20 and the proposed normal forms follow. One should make the observation that $\tilde{H}_{q}^{0}$ has to be computed before $H_{q}^{00}$.

Under these assumptions on the Floquet multipliers, we have also proven that $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$ can also be parametrized as (70).

The last normal form theorem is more involved because we have to deal with the Floquet multiplier -1 that induces $2 T$-periodic maps due to Proposition 19. Introduce the decomposition

$$
\tilde{X}_{0}(t)=\tilde{X}_{0}^{\prime}(t) \oplus \tilde{X}_{0}^{\prime \prime}(t)
$$

where $\tilde{X}_{0}^{\prime}(t)$ is spanned by $T$-periodic maps $\zeta^{1}(t), \ldots, \zeta^{n_{0}^{\prime}}(t)$ and where $\tilde{X}_{0}^{\prime \prime}(t)$ is spanned by $2 T$-periodic maps $\zeta^{n_{0}^{\prime}+1}(t), \ldots, \zeta^{n_{0}^{\prime}+n_{0}^{\prime \prime}}(t)$ and $n_{0}^{\prime}+n_{0}^{\prime \prime}=n_{0}$, corresponding to all (generalized) eigenfunctions of the monodromy operator belonging to the Floquet multiplier -1 . Define the symmetry $\tilde{S}_{0}: \mathbb{R}^{m} \rightarrow \mathbb{R}^{m}$ as

$$
\xi=\left(\xi^{\prime}, \xi^{\prime \prime}\right) \mapsto \tilde{S}_{0} \xi=\left(\xi^{\prime},-\xi^{\prime \prime}\right)
$$

then we have the following theorem, which is for example the case in the period-doubling bifurcation.
Theorem 22 (Normal form III). Assume that -1 is a Floquet multiplier. Then the results of Theorem 20 or Theorem 21, depending on the location and algebraic multiplicity of other the Floquet multipliers on the unit circle hold with the following modification: the maps $H, p$ and $P$ are 2T-periodic in the first component and additionally satisfy

$$
H(\tau+T, \xi)=H\left(\tau, \tilde{S}_{0} \xi\right)
$$

and

$$
p(\tau+T, \xi)=p\left(\tau, \tilde{S}_{0} \xi\right), \quad P(\tau+T, \xi)=P\left(\tau, \tilde{S}_{0} \xi\right)
$$

for all $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^{m}$.

Proof. The proof of this theorem is analogous to that of Theorem 20 or Theorem 21 but in a $2 T$ periodic setting. Hence, we obtain the results from Theorem 20 or Theorem 21, depending on the location and algebraic multiplicity of the Floquet multipliers on the unit circle where now the maps $H, p$ and $P$ being $2 T$-periodic in $\tau$. It remains to show the additional symmetries on the maps $H, p$ and $P$. Because the structure of the parts in the normal form are similar to that of the ODE case, treated in Theorem III. 13 of [19] this part will be omitted since the proof is identical by making the substitution of $\tau \mapsto \gamma(\tau)$ towards $\tau \mapsto \gamma_{\tau}$.

Via this theorem we obtain a $2 T$-periodic $\left(n_{0}+1\right)$-dimensional $C^{k}$-submanifold $\mathcal{W}_{\text {loc }}^{c}(\Gamma) \subset X$ that is a center manifold for (DDE) around the periodic orbit $\Gamma$.

### 4.4 Codimension 1 critical normal forms

Here we list the critical periodic normal forms for all three codim 1 bifurcations of limit cycles in DDEs. Of course, these normal forms are exactly the same as in ODEs and are given only for convenience of the reader.

### 4.4.1 Fold bifurcation

Suppose that (DDE) has a periodic solution $\gamma$ with $\Lambda_{0}=\{1\}$, where the trivial Floquet multiplier on the unit circle has algebraic multiplicity two and geometric multiplicity one.

The restriction of (DDE) to the corresponding 2-dimensional center manifold $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$ can be parametrized locally near $\Gamma$ by $(\tau, \xi)$, where $\tau \in[0, T]$ and $\xi$ is a real coordinate on $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$ that is transverse to $\Gamma$, see the left picture of Figure 1 for a visualization. With this information one can deduce via Theorem 21 that the critical periodic normal form at the fold bifurcation is

$$
\left\{\begin{array}{l}
\frac{d \tau}{d t}=1-\xi+a \xi^{2}+\mathcal{O}\left(\xi^{3}\right) \\
\frac{d \xi}{d \tau}=b \xi^{2}+\mathcal{O}\left(\xi^{3}\right)
\end{array}\right.
$$

or equivalently

$$
\left\{\begin{array}{l}
\frac{d \tau}{d t}=1-\xi+a \xi^{2}+\mathcal{O}\left(\xi^{3}\right)  \tag{74}\\
\frac{d \xi}{d t}=b \xi^{2}+\mathcal{O}\left(\xi^{3}\right)
\end{array}\right.
$$

where $a, b \in \mathbb{R}$ and the $\mathcal{O}\left(\xi^{3}\right)$-terms are time-independent and $T$-periodic in $\tau$. An explicit calculation of this normal form can be found in Section III. 2 of [19].

### 4.4.2 Period-doubling bifurcation

Suppose that (DDE) has a periodic solution $\gamma$ with $\Lambda_{0}=\{-1,1\}$, where all the Floquet multipliers on the unit circle have algebraic multiplicity one.

The restriction of (DDE) to the corresponding 2-dimensional center manifold $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$ can be parametrized locally near $\Gamma$ by $(\tau, \xi)$, where $\tau \in[0,2 T]$ and $\xi$ is a real coordinate on $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$ that is transverse to $\Gamma$, see the right picture of Figure 1 for a visualization. With this information one can deduce via Theorem 22 that the critical periodic normal form at the period-doubling bifurcation is

$$
\left\{\begin{array}{l}
\frac{d \tau}{d t}=1+a \xi^{2}+\mathcal{O}\left(\xi^{4}\right) \\
\frac{d \xi}{d \tau}=c \xi^{3}+\mathcal{O}\left(\xi^{4}\right)
\end{array}\right.
$$

or equivalently

$$
\left\{\begin{array}{l}
\frac{d \tau}{d t}=1+a \xi^{2}+\mathcal{O}\left(\xi^{4}\right)  \tag{75}\\
\frac{d \xi}{d t}=c \xi^{3}+\mathcal{O}\left(\xi^{4}\right)
\end{array}\right.
$$

where $a, c \in \mathbb{R}$ and the $\mathcal{O}\left(\xi^{4}\right)$-terms are time-independent and $2 T$-periodic in $\tau$. An explicit calculation of this normal form can be found in Section III. 2 of [19].

### 4.4.3 Neimark-Sacker bifurcation

Suppose that (DDE) has a periodic solution $\gamma$ with $\Lambda_{0}=\left\{1, e^{ \pm i \omega}\right\}$, where all the Floquet multiplier on the unit circle have algebraic multiplicity one and

$$
e^{i q \omega} \neq 1, \quad q=1,2,3,4 \quad \text { (no strong resonances). }
$$

The restriction of (DDE) to the corresponding 3-dimensional center manifold $\mathcal{W}_{\text {loc }}^{c}(\Gamma)$ can be parametrized locally near $\Gamma$ by $(\tau, \xi)$, where $\tau \in[0, T]$ and $\xi$ is a complex coordinate complementary to $\tau$. With this information one can deduce via Theorem 20 that the critical periodic normal form at the Neimark-Sacker bifurcation is

$$
\left\{\begin{array}{l}
\frac{d \tau}{d t}=1+a|\xi|^{2}+\mathcal{O}\left(|\xi|^{4}\right) \\
\frac{d \xi}{d \tau}=\frac{i \omega}{T} \xi+\tilde{d} \xi|\xi|^{2}+\mathcal{O}\left(|\xi|^{4}\right)
\end{array}\right.
$$

where $a \in \mathbb{R}, \tilde{d} \in \mathbb{C}$ and the $\mathcal{O}\left(|\xi|^{4}\right)$-terms are time-independent and $T$-periodic in $\tau$. An explicit calculation of this normal form can be found in Section III. 2 of [19]. Equivalently, this normal form can be written as

$$
\left\{\begin{array}{l}
\frac{d \tau}{d t}=1+a|\xi|^{2}+\mathcal{O}\left(|\xi|^{4}\right)  \tag{76}\\
\frac{d \xi}{d t}=\frac{i \omega}{T} \xi+d \xi|\xi|^{2}+\mathcal{O}\left(|\xi|^{4}\right)
\end{array}\right.
$$

where

$$
d=\tilde{d}-\frac{i a \omega}{T} \in \mathbb{C}
$$

## 5 Conclusions

We have proven the existence of a smooth finite-dimensional periodic center manifold near a nonhyperbolic cycle and the existence of a normalizing coordinate system on this center manifold for general classical delay differential equations. This coordinate system allowed us to describe the dynamics on the center manifold by means of the standard normal forms. The next logical step is to derive explicit formulas for the critical normal form coefficients for all codimension one bifurcation of limit cycles (i.e., $b$ for fold, $c$ for period-doubling and $d$ for Neimark-Sacker, see Section 4.4). A paper providing such explicit formulas, along the lines of the periodic normalization method [23, 30], is in preparation. We already note that from an abstract point of view, the formulas in [23, 30] for finite-dimensional ODEs look strikingly similar to the ones we obtained for classical DDEs. However, as one would expect, the (generalized) eigenfunctions and adjoint eigenfunctions are more involved to compute numerically for DDEs than for finite-dimensional ODEs due to the infinite-dimensionality of the problem.

Recall from Section 2.1 that we assumed $\odot$-reflexivitiy throughout this article, because we were only interested in the classical DDEs. The interesting question arises how the assumptions and results have to be adapted such that a non- $\odot$-reflexive variant of Theorem 14 still holds. We are already
inspired by the work of [21] where the existence of a smooth finite-dimensional center manifold near a nonhyperbolic equilibrium has been proven in the non- $\odot$-reflexive setting by means of admissible ranges and perturbations. The existence of a center manifold near a nonhyperbolic cycle would be interesting for studying abstract DDEs, see [27, 20, 21] for examples of such DDEs describing neural fields.

We formulated Section 4 in a classical delay setting. However, due to the strength of dual perturbation theory, it looks like the results also hold for a much broader class of delay equations such as renewal equations [10, 13], systems with infinite delay [11], and probably even mixed systems [17]. Furthermore, if it is possible to put Section 4 in a general setting, how must the results and proofs be adapted for a non- $\odot$-reflexive setting?

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## A Spectral decomposition

This appendix consists of two parts. In the first part, we will lift the spectral decomposition (Hypothesis 1) from $X$ to $X^{\odot \star}$ and in the second part we show that classical DDEs fulfill the requirements of Hypothesis 1 and Hypothesis 2.

## A. 1 Lifting the spectral decomposition from $X$ to $X^{\odot *}$

We consider the setting from the preface of Section 3 and prove that the spectral decomposition on $X$ from Hypothesis 1 induces a spectral decomposition on $X^{\star}, X^{\odot}$ and most importantly on $X^{\odot \star}$.

Proposition 23. Under the assumption of Hypothesis 1, the space $X^{\star}$ and the backward evolutionary system $U^{\star}$ have the following properties:

1. $X^{\star}$ admits a direct sum decomposition

$$
\begin{equation*}
X^{\star}=X_{-}^{\star}(s) \oplus X_{0}^{\star}(s) \oplus X_{+}^{\star}(s), \quad \forall s \in \mathbb{R} \tag{77}
\end{equation*}
$$

where each summand is closed.
2. There exist three continuous time-dependent projectors $P_{i}^{\star}: \mathbb{R} \rightarrow \mathcal{L}\left(X^{\star}\right)$ with $\operatorname{ran}\left(P_{i}^{\star}(s)\right)=$ $X_{i}^{\star}(s)$ for any $s \in \mathbb{R}$ and $i \in\{-, 0,+\}$.
3. There exists a constant $N \geq 0$ such that $\sup _{s \in \mathbb{R}}\left(\left\|P_{-}^{\star}(s)\right\|+\left\|P_{0}^{\star}(s)\right\|+\left\|P_{+}^{\star}(s)\right\|\right)=N<\infty$.
4. The projections are mutually orthogonal meaning that $P_{i}^{\star}(s) P_{j}^{\star}(s)=0$ for all $i \neq j$ and $s \in \mathbb{R}$ with $i, j \in\{-, 0,+\}$.
5. The projections commute with the backward evolutionary system: $U^{\star}(s, t) P_{i}^{\star}(t)=P_{i}^{\star}(s) U^{\star}(s, t)$ for all $i \in\{-, 0,+\}$ and $s \leq t$.
6. Define the restrictions $U_{i}^{\star}(s, t): X_{i}^{\star}(t) \rightarrow X_{i}^{\star}(s)$ for $i \in\{-, 0,+\}$ and $t \geq s$. The operators $U_{0}^{\star}(s, t)$ and $U_{+}^{\star}(s, t)$ are invertible and also forward evolutionary systems. Specifically, for any $t, \tau, s \in \mathbb{R}$ it holds

$$
\begin{equation*}
U_{0}^{\star}(s, t)=U_{0}^{\star}(s, \tau) U_{0}^{\star}(\tau, s), \quad U_{+}^{\star}(s, t)=U_{+}^{\star}(s, \tau) U_{+}^{\star}(\tau, t) . \tag{78}
\end{equation*}
$$

7. The decomposition (77) is an exponential trichotomy on $\mathbb{R}$ with the same constants as in Hypothesis 1.

Proof. We prove this proposition by separately showing that each statement holds. Throughout the proof, we assume that $s \in \mathbb{R}$ is given.

1. It follows from part 1 and 2 of Hypothesis 1 that by taking duals

$$
X^{\star}=\left[\operatorname{ran}\left(P_{-}(s)\right)\right]^{\star} \oplus\left[\operatorname{ran}\left(P_{0}(s)\right)\right]^{\star} \oplus\left[\operatorname{ran}\left(P_{+}(s)\right)\right]^{\star} .
$$

If $i \in\{-, 0,+\}$, then it follows from [21, Lemma A.1] that the map $\iota_{i}(s): \operatorname{ran}\left(P_{i}(s)^{\star}\right) \rightarrow\left[\operatorname{ran}\left(P_{i}(s)\right)\right]^{\star}$ defined as $\iota_{i}(s) y^{\star}=\left.y^{\star}\right|_{\operatorname{ran}\left(P_{i}(s)\right)}$ is an isometric isomorphism and $P_{i}(s)^{\star} \in \mathcal{L}\left(X^{\star}\right)$. From this isometric isomorphism, the space $\left[X_{i}(s)\right]^{\star}=\left[\operatorname{ran}\left(P_{i}(s)\right)\right]^{\star}$ can be identified with $\operatorname{ran}\left(P_{i}^{\star}(s)\right)=: X_{i}^{\star}(s)$ where we defined $P_{i}^{\star}(s):=P_{i}(s)^{\star}$ for any $s \in \mathbb{R}$. Because $P_{i}^{\star}(s)$ has closed range, $X_{i}^{\star}(s)$ is closed.
2. It only remains to show that $P_{i}$ is continuous for each $i \in\{-, 0,+\}$. Consider $h \in \mathbb{R}$, then

$$
\left\|P_{i}^{\star}(s+h)-P_{i}^{\star}(s)\right\|=\left\|\left[P_{i}(s+h)-P_{i}(s)\right]^{\star}\right\|=\left\|P_{i}(s+h)-P_{i}(s)\right\| \rightarrow 0, \text { as } h \rightarrow 0
$$

because $P_{i}$ is continuous by part 2 of Hypothesis 1 .
3. Since $\left\|P_{i}^{\star}(s)\right\|=\left\|P_{i}(s)^{\star}\right\|=\left\|P_{i}(s)\right\|$ we have that part 3 holds with the same constant $N$ as in part 3 Hypothesis 1.
4. Let $i \neq j$, then $P_{i}^{\star}(s) P_{j}^{\star}(s)=P_{i}(s)^{\star} P_{j}(s)^{\star}=\left(P_{j}(s) P_{i}(s)\right)^{\star}=0$ because $P_{j}(s) P_{i}(s)=0$ due to part 4 of Hypothesis 1.
5. Notice that for any $s \leq t$ we have that

$$
U^{\star}(s, t) P_{i}^{\star}(t)=\left(P_{i}(t) U(t, s)\right)^{\star}=\left(U(t, s) P_{i}(s)\right)^{\star}=P_{i}^{\star}(s) U(t, s)^{\star}=P_{i}^{\star}(s) U^{\star}(s, t)
$$

where we used part 5 of Hypothesis 1 in the third equality.
6. The restrictions are well-defined. Because $U_{0}(t, s)$ and $U_{+}(t, s)$ are invertible we also have that $U_{0}^{\star}(s, t)=U_{0}(t, s)^{\star}$ and $U_{+}^{\star}(s, t)^{\star}=U_{+}(t, s)^{\star}$ are invertible and so forward evolutionary systems. Let us now prove (78). Let $t, \tau, s \in \mathbb{R}$ be given, then

$$
U_{0}^{\star}(s, t)=U_{0}(t, s)^{\star}=\left(U_{0}(t, \tau) U_{0}(\tau, s)\right)^{\star}=U_{0}(\tau, s)^{\star} U_{0}(t, \tau)^{\star}=U_{0}^{\star}(s, \tau) U_{0}^{\star}(\tau, s)
$$

where we used (12) in the second equality. The proof for $U_{+}^{\star}$ is analogous.
7. Let $i=-$ and suppose that $t \geq s$. Let $x^{\star} \in X_{i}^{\star}(s)=\operatorname{ran}\left(P_{i}^{\star}(s)\right)$ be given. Since $\iota_{i}(t)$ is an isometry for any $t \in \mathbb{R}$,

$$
\left\|U^{\star}(s, t) x^{\star}\right\|=\left\|\iota_{i}(t)\left[U^{\star}(s, t) x^{\star}\right]\right\|=\sup _{\substack{x \in X_{i}(s) \\\|x\| \leq 1}}\left|\left\langle U_{i}(t, s) x, x^{\star}\right\rangle\right| \leq\left\|U_{i}(t, s)\right\|\left\|x^{\star}\right\|
$$

Taking the supremum over all $x^{\star}$ that satisfies $\left\|x^{\star}\right\| \leq 1$ we obtain $\left\|U_{i}^{\star}(s, t)\right\| \leq\left\|U_{i}(t, s)\right\|$ and this last part can be bounded by one of the three estimates in part 7 of Hypothesis 1. The cases for $i \in\{0,+\}$ are analogous. This completes the proof.

Proposition 24. Under the assumption of Hypothesis 1 , the space $X^{\odot}$ and the backward evolutionary system $U \odot$ have the following properties:

1. $X^{\odot}$ admits a direct sum decomposition

$$
\begin{equation*}
X^{\odot}=X_{-}^{\odot}(s) \oplus X_{0}^{\odot}(s) \oplus X_{+}^{\odot}(s), \quad \forall s \in \mathbb{R}, \tag{79}
\end{equation*}
$$

where each summand is closed.
2. There exist three continuous time-dependent projectors $P_{i}^{\odot}: \mathbb{R} \rightarrow \mathcal{L}\left(X^{\odot}\right)$ with $\operatorname{ran}\left(P_{i}^{\odot}(s)\right)=$ $X_{i}^{\odot}(s)$ for any $s \in \mathbb{R}$ and $i \in\{-, 0,+\}$.
3. There exists a constant $N \geq 0$ such that $\sup _{s \in \mathbb{R}}\left(\left\|P_{-}^{\odot}(s)\right\|+\left\|P_{0}^{\odot}(s)\right\|+\left\|P_{+}^{\odot}(s)\right\|\right)=N<\infty$.
4. The projections are mutually orthogonal meaning that $P_{i}^{\odot}(s) P_{j}^{\odot}(s)=0$ for all $i \neq j$ and $s \in \mathbb{R}$ with $i, j \in\{-, 0,+\}$.
5. The projections commute with the backward evolutionary system: $U^{\odot}(s, t) P_{i}^{\odot}(t)=$ $P_{i}^{\odot}(t) U^{\odot}(s, t)$ for all $i \in\{-, 0,+\}$ and $s \leq t$.
6. Define the restrictions $U_{i}^{\odot}(s, t): X_{i}^{\odot}(t) \rightarrow X_{i}^{\odot}(s)$ for $i \in\{-, 0,+\}$ and $t \geq s$. The operators $U_{0}^{\odot}(s, t)$ and $U_{+}^{\odot}(s, t)$ are invertible and also forward evolutionary systems. Specifically, for any $t, \tau, s \in \mathbb{R}$ it holds

$$
\begin{equation*}
U_{0}^{\odot}(s, t)=U_{0}^{\odot}(s, \tau) U_{0}^{\odot}(\tau, s), \quad U_{+}^{\odot}(s, t)=U_{+}^{\odot}(s, \tau) U_{+}^{\odot}(\tau, t) \tag{80}
\end{equation*}
$$

7. The decomposition (79) is an exponential trichotomy on $\mathbb{R}$ with the same constants as in Hypothesis 1.

Proof. Let $s \in \mathbb{R}$ and $i \in\{-, 0,+\}$ be given. Notice directly that the Lipschitz continuity of $B$ implies that $U^{\odot}(s, t)$ is well-defined and $X^{\odot}$-invariant. We define for any $s$ the map $P_{i}^{\odot}(s):=\left.P_{i}^{\star}(s)\right|_{X \odot}$ and notice that part 6 of Proposition 23 implies that $P_{i}^{\star}(s)$ maps $X^{\odot}$ into itself. We denote the range of $P_{i}^{\odot}(s)$ by $X_{i}^{\odot}(s)$ and it is clear that

$$
\begin{equation*}
X_{i}^{\odot}(s)=X_{i}^{\star}(s) \cap X^{\odot} \tag{81}
\end{equation*}
$$

Let us now prove the seven assertions.

1. Notice that $X_{i}^{\odot}(s)$ is closed because $X_{i}^{\star}(s)$ is closed (part 1 of Proposition 23) and $X^{\odot}$ is closed. The result follows from (81).
2. As $X^{\odot}$ is a subspace of $X^{\star}$, we have for any $h \in \mathbb{R}$ that

$$
\left\|P_{i}^{\odot}(s+h)-P_{i}^{\odot}(s)\right\|=\left\|\left[P_{i}(s+h)-P_{i}(s)\right]^{\odot}\right\| \leq\left\|\left[P_{i}(s+h)-P_{i}(s)\right]^{\star}\right\| \rightarrow 0, \text { as } h \rightarrow 0
$$

due to part 2 of Proposition 23. Hence, $P_{i}^{\odot}$ is continuous.
3. This follows from part 3 of Proposition 23 because $\left\|P_{i}^{\odot}(s)\right\| \leq\left\|P_{i}^{\star}(s)\right\|$ due to the restriction.
4. This follows from part 4 of Proposition 23 due to the restriction.
5. This claim follows from part 4 of Proposition 23 and recalling the fact that $U^{\odot}(s, t)$ is $X^{\odot_{-}}$ invariant.
6. For the well-definedness of the restriction, we have to check that $U_{i}^{\odot}(s, t)$ takes values in $X_{i}^{\odot}(s)$. Since $U_{i}^{\odot}(s, t)=\left.U_{i}^{\star}(s, t)\right|_{X \odot}$ we get from part 6 of Proposition 23 that $U_{i}^{\odot}(s, t)$ maps into $X_{i}^{\star}(s)$. Because $U^{\odot}(s, t)$ is $X^{\odot}$-invariant we also have that the restriction $U_{i}^{\odot}(s, t)$ is $X^{\odot}$-invariant and so $U_{i}^{\odot}(s, t)$ takes values in $X^{\odot}$. To conclude, $U_{i}^{\odot}(s, t)$ takes values in $X_{i}^{\star}(s) \cap X^{\odot}=X_{i}^{\odot}(s)$ by (81). The remaining claims follow immediately because of the restriction.
7. Because of the restriction we have that $\left\|U_{i}^{\odot}(s, t)\right\|=\left\|U_{i}(t, s)^{\odot}\right\| \leq\left\|U_{i}(t, s)^{\star}\right\|=\left\|U_{i}^{\star}(s, t)\right\|$ and the right-hand side can now be estimated by the upper bounds given in part 7 of Proposition 23.

Proposition 25. Under the assumption of Hypothesis 1, the space $X^{\odot \star}$ and the forward evolutionary system $U^{\odot \star}$ have the following properties:

1. $X^{\odot \star}$ admits a direct sum decomposition

$$
\begin{equation*}
X=X_{-}^{\odot \star}(s) \oplus X_{0}^{\odot \star}(s) \oplus X_{+}^{\odot \star}(s), \quad \forall s \in \mathbb{R} \tag{82}
\end{equation*}
$$

where each summand is closed.
2. There exist three continuous time-dependent projectors $P_{i}^{\odot \star}: \mathbb{R} \rightarrow \mathcal{L}\left(X^{\odot \star}\right)$ with $\operatorname{ran}\left(P_{i}^{\odot \star}(s)\right)=$ $X_{i}^{\odot \star}(s)$ for any $s \in \mathbb{R}$ and $i \in\{-, 0,+\}$.
3. There exists a constant $N \geq 0$ such that $\sup _{s \in \mathbb{R}}\left(\left\|P_{-}^{\odot \star}(s)\right\|+\left\|P_{0}^{\odot \star}(s)\right\|+\left\|P_{+}^{\odot \star}(s)\right\|\right)=N<\infty$.
4. The projections are mutually orthogonal meaning that $P_{i}^{\odot \star}(s) P_{j}^{\odot \star}(s)=0$ for all $i \neq j$ and $s \in \mathbb{R}$ with $i, j \in\{-, 0,+\}$.
5. The projections commute with the forward evolutionary system: $U^{\odot \star}(t, s) P_{i}^{\odot \star}(s)=$ $P_{i}^{\odot \star}(t) U^{\odot \star}(t, s)$ for all $i \in\{-, 0,+\}$ and $t \geq s$.
6. Define the restrictions $U_{i}^{\odot \star}(t, s): X_{i}^{\odot \star}(s) \rightarrow X_{i}^{\odot \star}(t)$ for $i \in\{-, 0,+\}$ and $t \geq s$. The operators $U_{0}^{\odot \star}(t, s)$ and $U_{+}^{\odot \star}(t, s)$ are invertible and also backward evolutionary systems. Specifically, for any $t, \tau, s \in \mathbb{R}$ it holds

$$
\begin{equation*}
U_{0}^{\odot \star}(t, s)=U_{0}^{\odot \star}(t, \tau) U_{0}^{\odot \star}(\tau, s), \quad U_{+}^{\odot \star}(t, s)=U_{+}^{\odot \star}(t, \tau) U_{+}^{\odot \star}(\tau, s) \tag{83}
\end{equation*}
$$

7. The decomposition (82) is an exponential trichotomy on $\mathbb{R}$ with the same constants as in Hypothesis 1.

Proof. Recall that $X^{\odot}$ is a Banach space and $U^{\odot}$ a backward evolutionary system on $X^{\odot}$. Therefore, we can apply Proposition 23 with $X$ replaced by $X^{\odot}$ and $U$ replaced by $U^{\star}$ by going over from a forward towards a backward evolutionary system. Hence, we obtain the desired result.

## A. 2 Verification of Hypothesis 1 and Hypothesis 2 for classical DDEs

In order to verify both hypotheses, we have to construct three time-dependent projectors $P_{i}$ with $i \in\{-, 0,+\}$. Before we do this, let us first define the time-dependent spectral projection (at time $s)$ as $P_{\lambda}(s) \in \mathcal{L}(X)$ with range $E_{\lambda}(s)$ and kernel $R_{\lambda}(s)$ that can be represented via the holomorphic functional calculus as the Dunford integral

$$
\begin{equation*}
P_{\lambda}(s):=\frac{1}{2 \pi i} \oint_{\partial C_{\lambda}}(z I-U(s+T, s))^{-1} d z \tag{84}
\end{equation*}
$$

where $C_{\lambda} \subset \mathbb{C}$ is a sufficiently small open disk centered at $\lambda$ with $\partial C_{\lambda}$ its boundary such that $\lambda$ is the only Floquet multiplier inside $C_{\lambda}$. Recall from the compactness property of $U(s+T, s)$ that the Floquet multipliers are isolated and hence making such a contour $\partial C_{\lambda}$ in the complex plane is possible.

Proposition 26. The map $P_{\lambda}: \mathbb{R} \rightarrow \mathcal{L}(X)$ is continuous and T-periodic.
Proof. Let an initial starting time $s \in \mathbb{R}$ be given with arbitrary $h \in \mathbb{R}$. Let $C_{\lambda}$ be an open disk in $\mathbb{C}$ centered at $\lambda$ such that $\partial C_{\lambda}$ is a circle with sufficiently small radius $r>0$ such that $\lambda$ is the only Floquet multiplier in $C_{\lambda}$. Hence,

$$
\left\|P_{\lambda}(s+h)-P_{\lambda}(s)\right\|=\frac{1}{2 \pi}\left\|\oint_{\partial C_{\lambda}}(z I-U(s+T+h, s+h))^{-1}-(z I-U(s+T, s))^{-1} d z\right\|
$$

because the Floquet multipliers are independent of the starting time. Notice that the integrand is just a difference of resolvents and due to the second resolvent identity [16, Theorem 4.8.2] we notice that the integrand equals

$$
R(z, h)[U(s+T+h, s+h)-U(s+T, s)] R(z, 0), \quad \forall z \in \partial C_{\lambda}
$$

where for any $h \in \mathbb{R}$ the resolvent map $R(\cdot, h): \partial C_{\lambda} \rightarrow \mathcal{L}(X)$ is defined as $R(z, h)=(z I-U(s+$ $T+h, s+h))^{-1}$. Notice that $R(\cdot, h)$ indeed takes values in $\mathcal{L}(X)$ due to the bounded inverse theorem. Filling this back into the expression above yields

$$
\left\|P_{\lambda}(s+h)-P_{\lambda}(s)\right\| \leq \frac{1}{2 \pi}\|U(s+T+h, s+h)-U(s+T, s)\| \oint_{\partial C_{\lambda}}\|R(z, h)\|\|R(z, 0)\| d z
$$

We claim that for any fixed $h \in \mathbb{R}$ the map $\partial C_{\lambda} \ni z \mapsto\|R(z, h)\| \in \mathbb{R}$ is continuous. Indeed, fix a $h \in \mathbb{R}$ and choose $u \in C_{\lambda}$ such that $|z-u| \rightarrow 0$, where $|\cdot|$ represents the arc length on the circle $C_{\lambda}$. The reverse triangle inequality and the first resolvent identity [16, Theorem 4.8.1] implies

$$
|\|R(u, h)\|-\|R(z, h)\|| \leq|z-u|\|R(u, h)\|\|R(z, h)\| \rightarrow 0, \quad \text { as }|z-u| \rightarrow 0
$$

Since $C_{\lambda}$ is compact, we have that the image $\left\{\|R(z, h)\|: z \in C_{\lambda}\right\}$ is a compact subset of $\mathbb{R}$ and hence this set is bounded, say it is contained in the interval $\left[-M_{h}, M_{h}\right]$ for some constant $M_{h}>0$, for a fixed $h \in \mathbb{R}$. We obtain

$$
\left\|P_{\lambda}(s+h)-P_{\lambda}(s)\right\| \leq r M_{0} M_{h}\|U(s+T+h, s+h)-U(s+T, s)\| \rightarrow 0, \quad \text { as } h \rightarrow 0
$$

by [4, Lemma 5.2] since $(s+T, s) \in \Omega_{\mathbb{R}}$. The $T$-periodicity holds due to [13, Corollary XIII.2.2] and the fact that the Floquet multipliers are independent of the starting time [13, Theorem XIII.3.3].

We also need the associated spectral projections on the unstable, center and stable eigenspace. For the unstable and center eigenspace, denote the spectral projection on the unstable eigenspace (at time $s$ ) and the spectral projection on the center eigenspace (at time $s$ ) as the operators $P_{+}(s) \in \mathcal{L}(X)$ with range $X_{+}(s)$ and $P_{0}(s) \in \mathcal{L}(X)$ with range $X_{0}(s)$ defined as

$$
P_{+}(s):=\sum_{\lambda \in \Lambda_{+}} P_{\lambda}(s), \quad P_{0}(s):=\sum_{\lambda \in \Lambda_{0}} P_{\lambda}(s)
$$

Define the spectral projection on the stable eigenspace (at time $s$ ) as $P_{-}(s):=I-P_{0}(s)-P_{+}(s) \in \mathcal{L}(X)$ and it holds that $P_{-}(s)$ is indeed the projection on the stable eigenspace $X_{-}(s)$, see [2, Lemma 7.2.2]. The proof of the following result is almost the same as [2, Theorem 7.2.1], but we give it for the sake of completeness.

Proposition 27. The setting of (DDE) satisfies Hypothesis 1.
Proof. We verify the seven criteria step by step.

1. The decomposition (11) can be also used in the case where $E_{\lambda}(s)$ is replaced with the finitedimensional vector space $X_{+}(s) \oplus X_{0}(s)$. Then, $X=X_{+}(s) \oplus X_{0}(s) \oplus R(s)$ for some vector space $R(s)$. We have to show $R(s)=X_{-}(s)$. By the decomposition (11) we know that $P_{+0}(s):=P_{+}(s)+P_{0}(s)$ is a projection with range $X_{+0}(s)=X_{-}(s) \oplus X_{0}(s)$ and $R(s)=\operatorname{ker} P_{+0}(s)$ and notice that $R(s)=$ $\cap_{\lambda \in \Lambda_{0+}} \operatorname{ker}\left(P_{\lambda}(s)\right)=X_{-}(s)$. The spaces $X_{+}(s)$ and $X_{0}(s)$ are automatically closed since they are finite-dimensional. To show that $X_{-}(s)$ is closed, notice that for each $\lambda \in \Lambda_{0+}$ the space $R_{\lambda}(s)$ is closed and because the finite intersection of closed sets is closed, the result follows from (35).
2. For $P_{+}$and $P_{0}$ the claim about the range follows immediately from their definition and the claim about $P_{-}$follows from the fact that $P_{-}(s)$ is the projection on $X_{-}(s)$. To show the continuity statement, recall from Proposition 26 that for any Floquet multiplier $\lambda$, the map $P_{\lambda}$ is continuous. As $P_{+}$and $P_{0}$ are finite sums of such continuous projectors, it follows that both projectors are continuous. Since $P_{-}=I-P_{0}-P_{+}$it follows that $P_{-}$is also continuous.
3. Since $P_{-}+P_{0}+P_{+}=I$, we have $\left\|P_{-}(t)\right\| \leq\left\|P_{0}(t)\right\|+\left\|P_{+}(t)\right\|$ for all $t \in \mathbb{R}$, and so it remains to prove that $t \mapsto\left\|P_{0}(t)\right\|$ and $t \mapsto\left\|P_{+}(t)\right\|$ are uniformly bounded on $[0, T]$ by $T$-periodicity. We will only show the claim for $P_{0}$ since the proof is similar for $P_{+}$.

Suppose for a moment that part 5 and 7 are satisfied. They will be proven later, independently of this property. Assume that $t \mapsto\left\|P_{0}(t)\right\|$ is not uniformly bounded on $[0, T]$, then there exist sequences $\left(x_{n}\right)_{n \in \mathbb{N}} \subset X$ and $\left(t_{n}\right)_{n \in \mathbb{N}} \subset[0, T]$ such that $\left\|x_{n}\right\|_{\infty}=1$ and $\left\|P_{0}\left(t_{n}\right) x_{n}\right\|_{\infty}=n$. Then for a given $\varepsilon>0$, there is a constant $K_{\varepsilon}>0$ such that

$$
n=\left\|P_{0}\left(t_{n}\right) x_{n}\right\|_{\infty} \leq\left\|U_{0}\left(t_{n}, T\right)\right\|\left\|P_{0}(T)\right\|\left\|U_{0}\left(T, t_{n}\right)\right\| \leq K_{\varepsilon}^{2} e^{2 \varepsilon T}\left\|P_{0}(T)\right\|
$$

which is a contradiction, since $n \in \mathbb{N}$ can be taken arbitrary large.
4. Let $i, j \in\{-, 0,+\}$ with $i \neq j$ and let $\varphi \in X$. By the decomposition proved in criterium one we have that $\varphi=\varphi_{i}(s)+\varphi_{j}(s)+\varphi_{k}(s)$, where $k \in\{-, 0,+\}$ such that $k \neq i$ and $k \neq j$. Then from the interplay between the ranges and kernels of the projections it follows that

$$
P_{i}(s) P_{j}(s) \varphi=P_{i}(s) P_{j}(s)\left[\varphi_{i}(s)+\varphi_{j}(s)+\varphi_{k}(s)\right]=P_{i}(s) \varphi_{j}(s)=0
$$

which proves this part.
5. It is proven in [13, Theorem XIII.3.3] that

$$
\begin{equation*}
P(t) U(t+T, s+j T)=U(t+T, s+j T) P(s) \tag{85}
\end{equation*}
$$

for $j \in \mathbb{N}$ chosen in such a way that $s+(j-1) T \leq t<s+j T$ and for $P \in\left\{P_{-}, P_{0}, P_{+}\right\}$. Hence,

$$
P(t) U(t, s)=P(t) U(t, s+j T) U(s+j T, s)=U(t, s+j T) U(s+T, s)^{j} P^{j}(s)=U(t, s) P(s)
$$

where we have used that $P(s)$ is a projection that commutes with $U(s+T, s)$. This last claim follows from setting $s=t$ and $j=1$ in (85) together with [13, Corollary XIII.2.2].
6. Notice that $U_{+}(t, s)$ and $U_{0}(t, s)$ are defined for all $t, s \in \mathbb{R}$ because they are restricted to a finitedimensional space. Since $U_{+}(t, s) U_{+}(s, t)=I=U_{+}(s, t) U_{+}(t, s)$ we have that $U_{+}(t, s)$ is invertible with inverse $U_{+}(t, s)^{-1}=U_{+}(s, t)$. Similarly $U_{0}(t, s)^{-1}=U_{0}(s, t)$. To show the remaining part, that is (12), we have six different cases depending on the location of $t, \tau, s \in \mathbb{R}$. This is a straightforward computation and will be omitted.
7. We will start with the center part. The stable and unstable part will then follow from a similar reasoning. Let $\varepsilon>0$ and $s \in \mathbb{R}$ be given. As the map $t \mapsto U_{0}(t, s) \varphi$ is continuous for any $\varphi \in X$ and $t \geq s$, we know

$$
\sup _{s \leq t \leq s+T}\left\|U_{0}(t, s) \varphi\right\|_{\infty}<\infty, \quad \forall \varphi \in X
$$

By the principle of uniform boundedness, we get

$$
\sup _{s \leq t \leq s+T}\left\|U_{0}(t, s)\right\| \leq K
$$

for some $K>0$. Because the spectrum of $U_{0}(s+T, s)$ lies on the unit circle, we have by the spectral radius formula also known as the Gelfand-Beurling formula that

$$
1=\max _{\lambda \in \sigma\left(U_{0}(s+T, s)\right)}|\lambda|=\lim _{j \rightarrow \infty}\left\|U_{0}(s+T, s)^{j}\right\|^{\frac{1}{j}}
$$

and so there exists an integer $k_{\varepsilon}>0$ such that $\left\|U_{0}(s+T, s)^{k_{\varepsilon}}\right\|<1+\varepsilon T$ and denote

$$
K_{\varepsilon}:=K \max _{j=0, \ldots, k_{\varepsilon}-1}\left\|U_{0}(s+T, s)^{j}\right\|
$$

Now, let $m_{t}$ be the largest integer such that $s+m_{t} k_{\varepsilon} T \leq t$ and $m_{t}^{\star} \in\left\{0, \ldots, k_{\varepsilon}-1\right\}$ the largest integer such that $s+m_{t} k_{\varepsilon} T+m_{t}^{\star} \leq t$. Then,

$$
\begin{aligned}
U_{0}(t, s) & =U_{0}\left(t, s+m_{t} k_{\varepsilon} T+m_{t}^{\star} T\right) U_{0}\left(s+m_{t} k_{\varepsilon} T+m_{t}^{\star} T, s+m_{t} k_{\varepsilon} T\right) U_{0}\left(s+m_{t} k_{\varepsilon} T, s\right) \\
& =U_{0}\left(t-m_{t} k_{\varepsilon} T-m_{t}^{\star} T, s\right) U_{0}\left(s+m_{t}^{\star} T, s\right) U_{0}\left(s+m_{t} k_{\varepsilon} T, s\right) \\
& =U_{0}\left(t-m_{t} k_{\varepsilon} T-m_{t}^{\star} T, s\right) U_{0}(s+T, s)^{m_{t}^{\star}} U_{0}(s+T, s)^{m_{t} k_{\varepsilon}}
\end{aligned}
$$

We can make the estimate

$$
\left\|U_{0}(t, s)\right\| \leq K_{\varepsilon}\left\|U_{0}(s+T, s)^{k_{\varepsilon}}\right\|^{m_{t}} \leq K_{\varepsilon}(1+\varepsilon T)^{\frac{t-s}{T}}=K_{\varepsilon}\left[(1+\varepsilon T)^{\frac{1}{\varepsilon T}}\right]^{\varepsilon(t-s)} \leq K_{\varepsilon} e^{\varepsilon(t-s)}
$$

since the function $(0, \infty) \ni x \mapsto\left(1+\frac{1}{x}\right)^{x} \in \mathbb{R}$ is monotonically increasing. The proof is analogous when $t \leq s$ and so we obtain $\left\|U_{0}(t, s)\right\| \leq K_{\varepsilon} e^{\varepsilon|t-s|}$. The proofs for the stable and unstable part are analogous.

Denote for any Floquet multiplier $\lambda$ and any $s \in \mathbb{R}$ the time-dependent extended spectral projection $P_{\lambda}^{\odot \star}(s) \in \mathcal{L}\left(X^{\odot \star}\right)$ with range $j E_{\lambda}(s)$ and kernel $R_{\lambda}^{\odot \star}(s)$, where $R_{\lambda}^{\odot \star}(s)$ is the called the extended complementary (generalized) eigenspace (at time s) coming from the decomposition $X^{\odot \star}=j E_{\lambda}(s) \oplus$ $R_{\lambda}^{\odot \star}(s)$. Define the extended unstable eigenspace (at time $s$ ) and extended center eigenspace (at time s) as

$$
X_{+}^{\odot \star}(s):=j\left(X_{+}(s)\right)=\bigoplus_{\lambda \in \Lambda_{+}} j E_{\lambda}(s), \quad X_{0}^{\odot \star}(s):=j\left(X_{0}(s)\right)=\bigoplus_{\lambda \in \Lambda_{0}} j E_{\lambda}(s)
$$

and notice via extended complementary (generalized) eigenspaces that the extended stable eigenspace (at time $s$ ) can be defined as

$$
X_{-}^{\odot \star}(s):=\bigcap_{\lambda \in \Lambda_{0} \cup \Lambda_{+}} R_{\lambda}^{\odot \star}(s)
$$

The construction of $X_{+}^{\odot \star}(s)$ and $X_{0}^{\odot \star}(s)$ directly shows that Hypothesis 2 is satisfied.

## B Smoothness and periodicity of the center manifold

This section of the appendix consists of three parts. Firstly, we show that the map $\mathcal{C}$ is not only fiberwise Lipschitz, but Lipschitz continuous in the second component where the Lipschitz constant is independent of the fiber. The proof of this claim is inspired by [2, Corollary 5.4.1.1]. Secondly, we prove via the theory of contractions on scales of Banach spaces, see [13, Section IX.6, Appendix IV] and [29] that the $\operatorname{map} \mathcal{C}$ is $C^{k}$-smooth. To do this, we combine the ideas from [13, Section IX.7], [2, Section 8] and [17]. Lastly, under the assumption of $T$-periodicity of the time-dependent nonlinear perturbation $R$ in the first component, we show that there exists a neighborhood of 0 in $X$ such that the center manifold is $T$-periodic in this neighborhood. The proof of this result is inspired by [2, Lemma 8.3.1 and Theorem 8.3.1].

Corollary 28. There exists a constant $L>0$ such that $\|\mathcal{C}(t, \varphi)-\mathcal{C}(t, \psi)\| \leq L\|\varphi-\psi\|$ for all $t \in \mathbb{R}$ and $\varphi, \psi \in X_{0}(t)$.

Proof. Let $t \in \mathbb{R}$ and $\varphi, \psi \in X_{0}(t)$ be given. Notice that

$$
\mathcal{C}(t, \varphi)=u_{t}^{\star}(\varphi)(t)=\left[\mathcal{G}_{t}\left(u_{t}^{\star}(\varphi)(t), \varphi\right)\right](t)=\varphi+\mathcal{K}_{t}^{\eta}\left[\tilde{R}_{\delta, t}\left(u_{t}^{\star}(\varphi)(t)\right)\right](t)
$$

By Proposition 5, we know there exists a constant $C_{\eta}>0$, independent of $t$ such that $\left\|\mathcal{K}_{t}^{\eta}\right\| \leq C_{\eta}$. Hence, from Corollary 7 and Theorem 8 we get

$$
\begin{aligned}
\|\mathcal{C}(t, \varphi)-\mathcal{C}(t, \psi)\| & \leq\|\varphi-\psi\|+\left\|\mathcal{K}_{t}^{\eta}\left[\tilde{R}_{\delta, t}\left(u_{t}^{\star}(\varphi)(t)\right)-\tilde{R}_{\delta, t}\left(u_{t}^{\star}(\psi)(t)\right)\right](t)\right\| \\
& \leq\|\varphi-\psi\|+\left\|\mathcal{K}_{t}^{\eta}\right\| \sup _{s \in \mathbb{R}}\left\|\left[\tilde{R}_{\delta, t}\left(u_{t}^{\star}(\varphi)(t)\right)-\tilde{R}_{\delta, t}\left(u_{t}^{\star}(\psi)(t)\right)\right](s)\right\| e^{-\eta|t-s|} \\
& \leq\|\varphi-\psi\|+C_{\eta}\left\|\tilde{R}_{\delta, t}\left(u_{t}^{\star}(\varphi)(t)\right)-\tilde{R}_{\delta, t}\left(u_{t}^{\star}(\psi)(t)\right)\right\|_{\eta, t} \\
& =\left(1+2 C_{\eta} L_{R_{\delta}} K_{\varepsilon}\right)\|\varphi-\psi\| .
\end{aligned}
$$

Hence $L=1+2 C_{\eta} L_{R_{\delta}} K_{\varepsilon}>0$ is the Lipschitz constant we were looking for.
The following lemma will be important to prove smoothness of $\mathcal{C}$ and $\mathcal{W}^{c}$.
Lemma 29 ([13, Lemma XII.6.6 and XII.6.7]). Let $Y_{0}, Y, Y_{1}$ and $\Lambda$ be Banach spaces with continuous embeddings $J_{0}: Y_{0} \hookrightarrow Y$ and $J: Y \hookrightarrow Y_{1}$. Consider the fixed point problem $y=f(y, \lambda)$ for $f: Y \times \Lambda \rightarrow$ $Y$. Suppose that the following conditions hold.

1. The function $g: Y_{0} \times \Lambda \rightarrow Y_{1}$ defined as $g\left(y_{0}, \lambda\right):=J f\left(J_{0} y_{0}, \lambda\right)$ is of the class $C^{1}$ and there exist mappings

$$
\begin{aligned}
& f^{(1)}: J_{0} Y_{0} \times \Lambda \rightarrow \mathcal{L}(Y) \\
& f_{1}^{(1)}: J_{0} Y_{0} \times \Lambda \rightarrow \mathcal{L}\left(Y_{1}\right)
\end{aligned}
$$

such that

$$
D_{1} g\left(y_{0}, \lambda\right) \xi=J f^{(1)}\left(J_{0} y_{0}, \lambda\right) J_{0}, \quad \forall\left(y_{0}, \lambda, \xi\right) \in Y_{0} \times \Lambda \times Y_{0}
$$

and

$$
J f^{(1)}\left(J_{0} y_{0}, \lambda\right) y=f_{1}^{(1)}\left(J_{0} y_{0}, \lambda\right) J y, \quad \forall\left(y_{0}, \lambda, y\right) \in Y_{0} \times \Lambda \times Y
$$

2. There exists a $\kappa \in[0,1)$ such that for all $\lambda \in \Lambda$ the map $f(\cdot, \lambda): Y \rightarrow Y$ is Lipschitz continuous with Lipschitz constant $\kappa$, independent of $\lambda$. Furthermore, for any $\lambda \in \Lambda$ the maps $f^{(1)}(\cdot, \lambda)$ and $f_{1}^{(1)}(\cdot, \lambda)$ are uniformly bounded by $\kappa$.
3. Under the previous condition, the unique fixed point $\Psi: \Lambda \rightarrow Y$ satisfies $\Psi(\lambda)=f(\Psi(\lambda), \lambda)$ and can be written as $\Psi=J_{0} \circ \Psi$ for some continuous $\Psi: \Lambda \rightarrow Y_{0}$.
4. The function $f_{0}: Y_{0} \times \Lambda \rightarrow Y$ defined by $f_{0}\left(y_{0}, \lambda\right)=f\left(J_{0} y_{0}, \lambda\right)$ has continuous partial derivative

$$
D_{2} f: Y_{0} \times \Lambda \rightarrow \mathcal{L}(\Lambda, Y)
$$

5. The mapping $Y_{0} \times \Lambda \ni(y, \lambda) \mapsto J \circ f^{(1)}\left(J_{0} y, \lambda\right) \in \mathcal{L}\left(Y, Y_{1}\right)$ is continuous.

Then the map $J \circ \Psi$ is of the class $C^{1}$ and $D(J \circ \Psi)(\lambda)=J \circ \mathcal{A}(\lambda)$ for all $\lambda \in \Lambda$, where $A=\mathcal{A}(\lambda) \in$ $\mathcal{L}(\Lambda, Y)$ is the unique solution of the fixed point equation

$$
A=f^{(1)}(\Psi(\lambda), \lambda) A+D_{2} f_{0}(\Psi(\lambda), \lambda)
$$

formulated in $\mathcal{L}(\Lambda, Y)$.
An important observation between the dependence of $u_{s}^{\star}$ on $\delta$ is presented in the following lemma. To make the notation a bit simpler, we define the map $\hat{P}_{0}: \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X) \rightarrow \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ pointwise as $\left(\hat{P}_{0} \varphi\right)(t):=\left(P_{0}(t) \varphi\right)(t) \in X_{0}(t)$ for all $t \in \mathbb{R}$ and have the following lemma.
Lemma 30. If $\delta>0$ is sufficiently small, then $\left\|\left(I-\hat{P}_{0}\right) u_{s}^{\star}(\varphi)\right\|_{0, s}<N \delta$.
Proof. Since $u_{s}^{\star}(\varphi)=\mathcal{G}_{s}\left(u_{s}^{\star}(\varphi), \varphi\right)=U(\cdot, s) \varphi+\mathcal{K}_{s}^{\eta}\left(\tilde{R}_{\delta, s}\left(u_{s}^{\star}(\varphi)\right)\right)$ we have that

$$
\left(I-\hat{P}_{0}\right) u_{s}^{\star}(\varphi)=\left(I-\hat{P}_{0}\right)\left[\mathcal{K}_{s}^{\eta}\left(\tilde{R}_{\delta, s}\left(u_{s}^{\star}(\varphi)\right)\right)\right],
$$

because for any $t \in \mathbb{R}$ we have that

$$
\left[\left(I-\hat{P}_{0}\right) U(\cdot, s) \varphi\right](t)=U(t, s) \varphi-P_{0}(t) U(t, s) \varphi=0
$$

since $U(t, s) \varphi=U_{0}(t, s) \varphi \in X_{0}(t)$ due to part 6 of Hypothesis 1 and the fact hat $\varphi \in X_{0}(s)$. It follows from the operator norm bounds in Proposition 5, and the bound for $\tilde{R}_{\delta, s}$ in Corollary 7 that

$$
\left\|\left(I-\hat{P}_{0}\right) u_{s}^{\star}(\varphi)\right\|_{0, s}=\left\|\left(I-\hat{P}_{0}\right)\left[\mathcal{K}_{s}^{\eta}\left(\tilde{R}_{\delta, s}\left(u_{s}^{\star}(\varphi)\right)\right)\right]\right\|_{0, s} \leq 4 \delta\left\|j^{-1}\right\| K_{\varepsilon} N L_{R_{\delta}}\left(\frac{1}{-a}+\frac{1}{b}\right)
$$

which is less than or equal to $N \delta$ if we choose

$$
L_{R_{\delta}} \leq \frac{1}{4\left\|j^{-1}\right\| K_{\varepsilon}}\left(\frac{1}{-a}+\frac{1}{b}\right)^{-1}
$$

which is possible since $L_{R_{\delta}} \rightarrow 0$ as $\delta \downarrow 0$.

Let us introduce some notation. For a Banach space $X$, define the sets $\mathrm{BC}_{s}^{\infty}(\mathbb{R}, X):=$ $\cup_{\eta>0} \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ and $\mathrm{BC}_{s}^{\infty}\left(\mathbb{R}, X^{\odot \star}\right):=\cup_{\eta>0} \mathrm{BC}_{s}^{\eta}\left(\mathbb{R}, X^{\odot \star}\right)$ together with the space

$$
V_{s}^{\eta}(\mathbb{R}, X):=\left\{u \in \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X):\left\|\left(I-\hat{P}_{0}\right) u\right\|_{0, s}<\infty\right\}
$$

with the norm

$$
\|u\|_{V_{s}^{\eta}}:=\left\|\hat{P}_{0} u\right\|_{\eta, s}+\left\|\left(I-\hat{P}_{0}\right) u\right\|_{0, s}
$$

such that $V_{s}^{\eta}(\mathbb{R}, X)$ becomes a Banach space and is continuously embedded in $\mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$. Define in addition for a sufficiently small $\delta>0$ the open set

$$
V_{\delta, s}^{\eta}(\mathbb{R}, X):=\left\{u \in V_{s}^{\eta}(\mathbb{R}, X):\left\|\left(I-\hat{P}_{0}\right) u\right\|_{0, s}<N \delta\right\}
$$

and notice that this set is non-empty due to Lemma 30. Define similarly as before the set $V_{\delta, s}^{\infty}(\mathbb{R}, X):=$ $\cup_{\eta>0} V_{\delta, s}^{\eta}(\mathbb{R}, X)$. For Banach spaces $E, E_{1}, E_{2}, \ldots, E_{p}$ with $p \geq 1$ we denote by $\mathcal{L}^{p}\left(E_{1} \times \cdots \times E_{p}, E\right)$ the Banach space of $E$-valued continuous $p$-linear maps defined on the $E_{1} \times \cdots \times E_{p}$. When there are $p$ identical copies in this Cartesian product, we simply write $E^{p}:=E \times \cdots \times E$, where this notation will also be used with $E$ is just simply a set.

If we chose $\delta$ as in Lemma 30, then the map $u \mapsto \tilde{R}_{\delta, s}(u)$ is of the class $C^{k}$, when $u \in V_{\delta, s}^{\infty}(\mathbb{R}, X)$. Consider any pair of integers $p, q \geq 0$ with $p+q \leq k$ and notice that the norm $\left\|D_{1}^{p} D_{2}^{q} R_{\delta, s}(t, \varphi)\right\|$ is uniformly bounded on $\mathbb{R} \times V_{\delta, s}^{\infty}(\mathbb{R}, X)$. Hence, for any $u \in V_{\delta, s}^{\infty}(\mathbb{R}, X)$ we can define the map $R_{\delta, s}^{(p, q)}(u): \mathrm{BC}_{s}^{\infty}(\mathbb{R}, X)^{p} \rightarrow \mathrm{BC}_{s}^{\infty}\left(\mathbb{R}, X^{\odot \star}\right)$ as

$$
R_{\delta, s}^{(p, q)}(u)\left(v_{1}, \ldots, v_{q}\right)(t):=D_{1}^{p} D_{2}^{q} R_{\delta, s}(t, u(t))\left(v_{1}(t), \ldots, v_{q}(t)\right), \quad \forall v_{1}, \ldots, v_{q} \in \mathrm{BC}_{s}^{\infty}(\mathbb{R}, X)
$$

The following two lemmas will be crucial for the proof of Theorem 33.
Lemma 31 ([13, Lemma XII.7.3] and [17, Proposition 8.1]). Consider integers $p \geq 0$ and $q \geq 0$ with $p+q \leq k$ together with integers $\mu_{1}, \ldots, \mu_{q}>0$ such that $\mu=\mu_{1}+\cdots+\mu_{q}$ and consider $\eta>q \mu>0$. Then,

$$
\tilde{R}_{\delta, s}^{(p, q)}(u) \in \mathcal{L}^{q}\left(\mathrm{BC}_{s}^{\mu_{1}}(\mathbb{R}, X) \times \cdots \times \mathrm{BC}_{s}^{\mu_{q}}(\mathbb{R}, X), \mathrm{BC}^{\eta}\left(\mathbb{R}, X^{\odot \star}\right)\right), \quad \forall u \in V_{\delta, s}^{\infty}(\mathbb{R}, X)
$$

Furthermore, consider any $0 \leq l \leq k-(p+q)$ and $\sigma>0$. If $\eta>q \mu+l \sigma$, then the map $R_{\delta, s}^{(p, q)}$ : $V_{\delta, s}^{\sigma}(\mathbb{R}, X) \rightarrow \mathcal{L}^{q}\left(\mathrm{BC}_{s}^{\mu_{1}}(\mathbb{R}, X) \times \cdots \times \mathrm{BC}_{s}^{\mu_{p}}(\mathbb{R}, X), \mathrm{BC}^{\eta}\left(\mathbb{R}, X^{\odot \star}\right)\right)$ is $C^{l}$-smooth, with $D^{l} R_{\delta, s}^{(p, q)}=R_{\delta, s}^{(p, q+l)}$.

Lemma 32 ([13, Lemma XII.7.6] and [17, Proposition 8.2]). Consider integers $p \geq 0$ and $q \geq 0$ with $p+q<k$ together with integers $\mu_{1}, \ldots, \mu_{q}>0$ such that $\mu=\mu_{1}+\cdots+\mu_{q}$. Let $\eta>q \mu+\sigma$ for some $\sigma>0$ and consider a $C^{1}$-smooth map $\Phi_{s}: X_{0}(s) \rightarrow V_{\delta, s}^{\sigma}(\mathbb{R}, X)$. Then the map $\tilde{R}_{\delta, s}^{(p)} \circ \Phi_{s}: X_{0}(s) \rightarrow$ $\mathcal{L}^{q}\left(\mathrm{BC}_{s}^{\mu_{1}}(\mathbb{R}, X) \times \cdots \times \mathrm{BC}_{s}^{\mu_{q}}(\mathbb{R}, X), \mathrm{BC}^{\eta}\left(\mathbb{R}, X^{\odot \star}\right)\right)$ is $C^{1}$-smooth with

$$
D\left(\tilde{R}_{\delta, s}^{(p, q)} \circ \Phi_{s}\right)(\varphi)\left(v_{1}, \ldots, v_{q}\right)(t)=\tilde{R}_{\delta, s}^{(p, q+1)}\left(\Phi_{s}(\varphi)\right)\left(\Phi_{s}^{\prime}(\varphi)(t), v_{1}(t), \ldots, v_{q}(t)\right)
$$

So far we have only proven that the center manifold is Lipschitz continuous. Recall from Theorem 8 that we solved the fixed point problem $u=\mathcal{G}_{s}(u, \varphi)$ for a given $\varphi \in X_{0}(s)$ in the space $\mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ for a given $\eta \in(0, \min \{-a, b\})$. It turns out that the space $\mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ is not really suited to study additional smoothness of the center manifold. The idea to obtain this is by working with another exponent, say $\tilde{\eta}$, which is chosen high enough to guarantee smoothness, but not too high to loose the contraction property. Hence, a trade-off has to be made. To do this, choose an interval $\left[\eta_{\text {min }}, \eta_{\text {max }}\right] \subset$ $(0, \min \{-a, b\})$ such that $k \eta_{\min }<\eta_{\max }$ and choose $\delta>0$ small enough to guarantee that

$$
L_{R_{\delta}}\left\|\mathcal{K}_{s}^{\eta}\right\|_{\eta, s}<\frac{1}{4}, \quad \forall \eta \in\left[\eta_{\min }, \eta_{\max }\right], s \in \mathbb{R}
$$

which is possible since $L_{R_{\delta}} \rightarrow 0$ as $\delta \downarrow 0$ proven in Proposition 6. Following the proof again of Theorem 8 we obtain for any $\eta \in\left[\eta_{\min }, \eta_{\max }\right]$ a unique fixed point $u_{\eta, s}^{\star}: X_{0}(s) \rightarrow \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ of the equation $u=\mathcal{G}_{s}(u, \varphi)$. Denote for real numbers $\eta_{1} \leq \eta_{2}$ the continuous embedding operator as $\mathcal{J}_{s}^{\eta_{2}, \eta_{1}}: \mathrm{BC}_{s}^{\eta_{1}}(\mathbb{R}, X) \hookrightarrow \mathrm{BC}_{s}^{\eta_{2}}(\mathbb{R}, X)$, then for $\eta_{1}, \eta_{2} \in\left[\eta_{\text {min }}, \eta_{\text {max }}\right]$ we have that $u_{\eta_{2}, s}^{\star}=\mathcal{J}_{s}^{\eta_{2}, \eta_{1}} \circ u_{\eta_{1}, s}^{\star}$. These embedding operators will play the role of $J_{0}$ and $J$ defined in Lemma 29. The following proof is a combination of [13, Theorem IX.7.7], [2, Theorem 7.1.1] and [17, Theorem 7.1] to our setting.

Theorem 33. For each $l \in\{1, \ldots, k\}$ and $\eta \in\left(l \eta_{\min }, \eta_{\max }\right]$, the mapping $\mathcal{J}_{s}^{\eta, \eta_{\min }} \circ u_{\eta_{\min }, s}^{\star}: X_{0}(s) \rightarrow$ $\mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ is of the class $C^{l}$ provided that $\delta>0$ is sufficiently small.

Proof. We prove this by induction on $l$. Let $l=k=1$ and $\eta \in\left(\eta_{\min }, \eta_{\max }\right]$. We show that Lemma 29 applies with the Banach spaces

$$
Y_{0}=V_{\delta}^{\eta_{\min }, s}(\mathbb{R}, X), \quad Y=\mathrm{BC}_{s}^{\eta_{\min }}(\mathbb{R}, X), \quad Y_{1}=\mathrm{BC}_{s}^{\eta}(\mathbb{R}, X), \quad \Lambda=X_{0}(s)
$$

and operators

$$
\begin{aligned}
f(u, \varphi) & =U(\cdot, s) \varphi+\mathcal{K}_{s}^{\eta_{\min }}\left(\tilde{R}_{\delta, s}(u)\right), \quad \forall(u, \varphi) \in \mathrm{BC}_{s}^{\eta_{\min }}(\mathbb{R}, X) \times X_{0}(s), \\
f^{(1)}(u, \varphi) & =\mathcal{K}_{s}^{\eta} \circ \tilde{R}_{\delta, s}^{(0,1)}(u) \in \mathcal{L}\left(\mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)\right), \quad \forall(u, \varphi) \in V_{\delta, s}^{\eta}(\mathbb{R}, X) \times X_{0}(s), \\
f_{1}^{(1)}(u, \varphi) & =\mathcal{K}_{s}^{\eta_{\min }} \circ \tilde{R}_{\delta, s}^{(0,1)}(u) \in \mathcal{L}\left(\mathrm{BC}_{s}^{\eta_{\min }}(\mathbb{R}, X)\right), \quad \forall(u, \varphi) \in V_{\delta, s}^{\eta_{\min }}(\mathbb{R}, X) \times X_{0}(s),
\end{aligned}
$$

with embeddings $J=\mathcal{J}_{s}^{\eta, \eta_{\min }}$ and $J_{0}: V_{\delta}^{\eta_{\min }, s}(\mathbb{R}, X) \hookrightarrow \mathrm{BC}_{s}^{\eta_{\min }}(\mathbb{R}, X)$. To verify condition 1 of Lemma 29, we must show that the map

$$
V_{\delta}^{\eta_{\min }, s}(\mathbb{R}, X) \times X_{0}(s) \ni(u, \varphi) \mapsto g(u, \varphi)=\mathcal{J}_{s}^{\eta, \eta_{\min }}\left[U(\cdot, s) \varphi+\mathcal{K}_{s}^{\eta_{\min }}\left(\tilde{R}_{\delta, s}\left(J_{0} u\right)\right)\right] \in \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)
$$

is $C^{1}$-smooth. Notice that the embedding operator $J$ is $C^{1}$-smooth, as well as $\varphi \mapsto U(\cdot, s) \varphi$. Furthermore, from Lemma 31 the map $J_{0} u \mapsto \tilde{R}_{\delta, s}\left(J_{0} u\right)$ is $C^{1}$-smooth and hence $g$ is $C^{1}$-smooth by the continuity of the linear embedding $J_{0}$. Verification of the equalities $D_{1} g\left(y_{0}, \lambda\right) \xi=J f^{(1)}\left(J_{0} y_{0}, \lambda\right) J_{0}$ and $J f^{(1)}\left(J_{0} y_{0}, \lambda\right) y=f_{1}^{(1)}\left(J_{0} y_{0}, \lambda\right) J y$ is straightforward.

Let us now verify condition 2. The Lipschitz claim follows immediately from the small Lipschitz constant for $U(\cdot, s) \varphi+\mathcal{K}_{s}^{\eta_{\min }}\left(\tilde{R}_{\delta, s}(u)\right)$ by choosing $\delta$ sufficiently small. Furthermore, the uniform boundedness claims hold because the embedding operators are bounded.

For condition 3, the unique fixed point is $u_{\eta_{\min }, s}^{\star}=J_{0} \circ \Phi$, where $\Phi: X_{0}(s) \rightarrow V_{\delta, s}^{\eta_{\min }}(\mathbb{R}, X)$ is defined by $\Phi(\varphi):=u_{\eta_{\text {min }}, s}^{\star}(\varphi)$ for all $\varphi \in X_{0}(s)$. The $\operatorname{map} \Phi$ is well-defined due to Lemma 30 and is continuous due to Theorem 8.

To verify condition 4 , we must check that the map

$$
V_{\delta}^{\eta_{\min }, s}(\mathbb{R}, X) \times X_{0}(s) \ni(u, \varphi) \mapsto f\left(J_{0} u, \varphi\right)=U(\cdot, s) \varphi+\mathcal{K}_{s}^{\eta_{\min }}\left(\tilde{R}_{\delta, s}\left(J_{0} u\right)\right) \in \mathrm{BC}_{s}^{\eta_{\min }}(\mathbb{R}, X)
$$

has continuous partial derivative in the second variable. This is clear since the map $\varphi \mapsto f\left(J_{0} u, \varphi\right)$ is linear.

To verify condition 5 , we have to check that the map

$$
(u, \varphi) \mapsto J \circ f^{(1)}\left(J_{0} u, \varphi\right)=\mathcal{J}_{s}^{\eta, \eta_{\min }} \circ \mathcal{K}_{s}^{\eta} \circ \tilde{R}_{\delta, s}^{(1)}(u)
$$

from $V_{\delta}^{\eta_{\min }, s}(\mathbb{R}, X) \times X_{0}(s)$ to $\mathcal{L}\left(X_{0}(s), \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)\right)$ is continuous. This again follows from the fact that the embedding operators are continuous and the smoothness of $\tilde{R}_{\delta, s}$ from Lemma 31.

Since all conditions of Lemma 29 are satisfied, we conclude that $\mathcal{J}_{s}^{\eta, \eta_{\text {min }}} \circ u_{\eta_{\text {min }}, s}^{\star}$ is $C^{1}$-smooth and the Fréchet derivative $D\left(\mathcal{J}_{s}^{\eta, \eta_{\min }} \circ u_{\eta_{\min }, s}^{\star}\right) \in \mathcal{L}\left(X_{0}(s), \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)\right)$ is the unique solution $w^{(1)}$ of the equation

$$
\begin{equation*}
w^{(1)}=\mathcal{K}_{s}^{\eta_{\min }} \circ \tilde{R}_{\delta, s}^{(1)}\left(u_{s, \eta_{\min }}^{\star}(\varphi)\right) w^{(1)}+U(\cdot, s)=: F_{\eta_{\min }}^{(1)}\left(w^{(1)}, \varphi\right) \tag{86}
\end{equation*}
$$

where $F_{\eta_{\text {min }}}^{(1)}: \mathcal{L}\left(X_{0}(s), \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)\right) \times X_{0}(s) \rightarrow \mathcal{L}\left(X_{0}(s), \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)\right)$. Notice that $F_{\eta_{\min }}^{(1)}(\cdot, \varphi)$ is a uniform contraction for each $\eta \in\left[\eta_{\min }, \eta_{\max }\right]$ and hence its unique fixed point $u_{\eta_{\min }, s, s(1)}^{\star(\varphi)} \in$ $\left.\mathcal{L}\left(X_{0}(s), \mathrm{BC}_{s}^{\eta_{\min }}(\mathbb{R}, X)\right) \subseteq \mathcal{L}\left(X_{0}(s), \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)\right)\right)$ for $\eta \geq \eta_{\text {min }}$. Also, the mapping $u_{\eta_{\min }, s}^{\star,(1)}: X_{0}(s) \rightarrow$ $\left.\mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)\right)$ is continuous if $\eta \in\left(\eta_{\min }, \eta_{\max }\right]$.

Now, consider any integer $1 \leq l<k$ and suppose that for all $1 \leq q \leq l$ and all $\eta \in\left(q \eta_{\min }, \eta_{\max }\right)$ the mapping $\mathcal{J}_{s}^{\eta, \eta_{\text {min }}} \circ u_{\eta_{\text {min }}, s}^{\star}$ is $C^{q}$-smooth with $D^{q}\left(\mathcal{J}_{s}^{\eta, \eta_{\text {min }}} \circ u_{\eta_{\text {min }}, s}^{\star}\right)=\mathcal{J}_{s}^{\eta, \eta_{\text {min }}} \circ u_{\eta_{\text {min }}, s}^{\star,(q)}$ and $u_{\eta_{\text {min }}, s}^{\star,(q)}(\varphi) \in$ $\mathcal{L}^{q}\left(X_{0}(s)^{q}, \mathrm{BC}_{s}^{q \eta_{\min }}(\mathbb{R}, X)\right)$ such that the mapping $\mathcal{J}_{s}^{\eta, \eta_{\min }} \circ u_{\eta_{\min }, s}^{\star,(q)}: X_{0}(s) \rightarrow \mathcal{L}^{q}\left(X_{0}(s)^{q}, \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)\right)$ is continuous for $\eta \in\left(q \eta_{\min }, \eta_{\max }\right]$. Suppose also for any $\varphi \in X_{0}(s)$ that $u_{\eta_{\min }, s}^{\star(l)}(\varphi)$ is the unique solution $w^{(l)}$ of an equation of the form

$$
w^{(l)}=\mathcal{K}_{s}^{\eta_{\min } p} \circ \tilde{R}_{\delta, s}^{(l)}\left(u_{s, \eta_{\min }^{\star}}^{\star}(\varphi)\right) w^{(l)}+H_{\eta_{\text {min }}}^{(l)}(\varphi)=: F_{\eta_{\text {min }}}^{(l)}\left(w^{(l)}, \varphi\right),
$$

with $H_{\eta_{\min }}^{(1)}(\varphi)=0$ and for $\nu \in\left[\eta_{\min }, \eta_{\max }\right]$ and $l \geq 2$ the $\operatorname{map} H_{\nu}^{(l)}(\varphi)$ is a finite sum of terms of the form

$$
\mathcal{K}_{s}^{l \nu} \circ \tilde{R}_{\delta, s}^{(0, q)}\left(u_{\nu, s}^{\star}(\varphi)\right)\left(u_{\nu, s}^{\star,\left(r_{1}\right)}(\varphi), \ldots, u_{\nu, s}^{\star,\left(r_{q}\right)}(\varphi)\right),
$$

with $2 \leq q \leq l$ and $1 \leq r_{i}<l$ for $i=1, \ldots, q$ such that $r_{1}+\cdots+r_{q}=l$. Under these assumptions we have that the mapping $F_{\eta}^{(l)}: \mathcal{L}^{l}\left(X_{0}(s)^{l}, \mathrm{BC}_{s}^{l \eta}(\mathbb{R}, X)\right) \times X_{0}(s) \rightarrow \mathcal{L}^{l}\left(X_{0}(s)^{l}, \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)\right)$ is a uniform contraction for all $\eta \in\left[\eta_{\text {min }}, \frac{1}{l} \eta_{\text {max }}\right]$ due to Lemma 31.

Fix some $\eta \in\left((l+1) \eta_{\min }, \eta_{\max }\right]$ and choose $\eta_{\min }<\sigma<(l+1) \sigma<\mu<\eta$. We show that Lemma 29 applies with the Banach spaces

$$
\begin{aligned}
& Y_{0}=\mathcal{L}^{l}\left(X_{0}(s)^{l}, \mathrm{BC}_{s}^{l \sigma}(\mathbb{R}, X)\right), \quad Y=\mathcal{L}^{l}\left(X_{0}(s)^{l}, \mathrm{BC}_{s}^{\mu}(\mathbb{R}, X)\right), \\
& Y_{1}=\mathcal{L}^{l}\left(X_{0}(s)^{l}, \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)\right), \quad \Lambda=X_{0}(s)
\end{aligned}
$$

and operators

$$
\begin{aligned}
f(u, \varphi) & =\mathcal{K}_{s}^{\mu} \circ \tilde{R}_{\delta, s}^{(0,1)}\left(u_{\eta_{\text {min }}, s}^{\star}(\varphi)\right) u+H_{\mu / l}^{(l)}(\varphi), \quad \forall(u, \varphi) \in \mathcal{L}^{l}\left(X_{0}(s)^{l}, \mathrm{BC}_{s}^{\mu}(\mathbb{R}, X)\right) \times X_{0}(s), \\
f^{(1)}(u, \varphi) & =\mathcal{K}_{s}^{\mu} \circ \tilde{R}_{\delta, s}^{(0,1)}\left(u_{\eta_{\min }, s}^{\star}(\varphi)\right) \in \mathcal{L}\left(\mathcal{L}^{l}\left(X_{0}(s)^{l}, \mathrm{BC}_{s}^{\mu}(\mathbb{R}, X)\right)\right), \\
f_{1}^{(1)}(u, \varphi) & =\mathcal{K}_{s}^{\eta} \circ \tilde{R}_{\delta, s}^{(0,1)}\left(u_{\eta_{\min }, s}^{\star}(\varphi)\right) \in \mathcal{L}\left(\mathcal{L}^{l}\left(X_{0}(s)^{l}, \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)\right)\right),
\end{aligned}
$$

We start with verifying condition 1 . We have to check that the map

$$
(u, \varphi) \mapsto \mathcal{J}_{s}^{\eta, \mu}\left[\mathcal{K}_{s}^{\mu} \circ \tilde{R}_{\delta, s}^{(0,1)}\left(u_{\eta_{\min }, s}^{\star}(\varphi)\right) u+H_{\mu / l}^{(l)}(\varphi)\right]
$$

from $\mathcal{L}^{l}\left(X_{0}(s)^{l}, \mathrm{BC}_{s}^{p \sigma}(\mathbb{R}, X)\right) \times X_{0}(s)$ to $\mathcal{L}^{l}\left(X_{0}(s)^{l}, \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)\right)$ is $C^{1}$-smooth, where now $\mathcal{J}_{s}^{\eta, \mu}$ : $\mathcal{L}^{l}\left(X_{0}(s)^{l}, \mathrm{BC}_{s}^{\mu}(\mathbb{R}, X)\right) \hookrightarrow \mathcal{L}^{l}\left(X_{0}(s)^{l}, \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)\right)$ is a continuous embedding. The mapping defined above $C^{1}$-smooth in the first variable since it is linear. For the second variable, notice that the map $\varphi \mapsto \mathcal{J}_{s}^{\eta, \mu} \circ \mathcal{K}_{s}^{\mu} \circ \tilde{R}_{\delta, s}^{(1)}\left(u_{\eta_{\text {min }}, s}^{\star}(\varphi)\right)$ is $C^{1}$ due to Lemma 32 with $\mu>(l+1) \sigma$ and the $C^{1}$ smoothness of $\varphi \mapsto \mathcal{J}_{s}^{\sigma \eta_{\min }} \circ u_{\eta_{\text {min }}, s}^{\star}(\varphi)$ with $\sigma>\eta_{\min }$. For the $C^{1}$ smoothness of $\varphi \mapsto \mathcal{J}_{s}^{\eta, \mu} \circ H_{\mu / l}^{(l)}(\varphi)$, we get differentiability from Lemma 32 and hence we have that the derivative of $\varphi \mapsto H_{\mu / l}^{(l)}(\varphi)$ is a sum of terms of the form

$$
\begin{aligned}
& \mathcal{K}_{s}^{\mu} \circ \tilde{R}_{\delta, s}^{(0, q+1)}\left(u_{\eta_{\min }, s}^{\star}(\varphi)\right)\left(u_{\eta_{\min }, s}^{\star,\left(r_{1}\right)}(\varphi), \ldots, u_{\eta_{\min }, s}^{\star\left(r_{q}\right)}(\varphi)\right) \\
& +\sum_{j=1}^{q} \mathcal{K}_{s}^{\mu} \circ \tilde{R}_{\delta, s}^{(q)}\left(u_{\eta_{\min }, s}^{\star}(\varphi)\right)\left(u_{\eta_{\min }, s}^{\star\left(r_{1}\right)}(\varphi), \ldots, u_{\eta_{\min }, s}^{\star,\left(r_{j}+1\right)}(\varphi), \ldots, u_{\eta_{\min },\left(, r_{q}\right), s}^{\star\left(r_{2}\right)}(\varphi)\right)
\end{aligned}
$$

and each $u_{\eta_{\text {min }}}^{\star\left(r_{j}\right)}$, is a map from $X_{0}(s)$ into $\mathrm{BC}_{s}^{j \sigma}(\mathbb{R}, X)$. Applying Lemma 31 with $\mu>(l+1) \sigma$ ensures continuity of $D H_{\mu / l}^{(l)}(\varphi)$ and also then continuity of $\mathcal{J}_{s}^{\eta, \mu} D H_{\mu / l}^{(l)}(\varphi)$. The remaining calculations from condition 1 are easily checked. Condition 4 can be proven similarly.

The Lipschitz condition and boundedness for condition 2 follows by the choice of $\delta>0$ defined at the beginning and the uniform contractivity of $H_{\mu / l}^{(l)}$ described above. Let us now prove condition 3 . Let us write

$$
\mathcal{K}_{s}^{\eta} \circ \tilde{R}_{\delta, s}^{(0,1)}\left(u_{\eta_{\min }, s}^{\star}(\varphi)\right)=\mathcal{J}_{s}^{\eta, \mu} \circ \mathcal{K}_{s}^{\mu} \circ R_{\delta, s}^{(0,1)}\left(u_{\eta_{\min }, s}^{\star}\right)(\varphi)
$$

and by applying Lemma 31 together with the $C^{1}$-smoothness of $u_{\eta_{\text {min }}, s}^{\star}$ to obtain continuity of $\varphi \mapsto$ $\tilde{R}_{\delta, s}^{(0,1)}\left(u_{\eta_{\text {min }}, s}^{\star}(\varphi)\right)$. This also proves condition 5. All the conditions from Lemma 29 are satisfied, and so we conclude that $u_{\eta_{\min }, s}^{(l)}: X_{0}(s) \rightarrow \mathcal{L}^{l}\left(X_{0}(s)^{l}, \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)\right)$ is of the class $C^{1}$ with derivative $u_{\eta_{\min }, s}^{(l+1)}=D u_{\eta_{\min }, s}^{(l)} \in \mathcal{L}^{l+1}\left(X_{0}(s)^{l+1}, \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)\right)$ given by the unique solution $w^{(l+1)}$ of the equation

$$
w^{(l+1)}=\mathcal{K}_{s}^{\mu} \circ \tilde{R}_{\delta, s}^{(1)}\left(u_{\eta_{\min }, s}^{\star}(\varphi)\right) w^{(l+1)}+H_{\mu /(l+1)}^{(l+1)}(\varphi),
$$

where $H_{\mu /(l+1)}^{(l+1)}(\varphi)=\mathcal{K}_{s}^{\mu} \circ \tilde{R}_{\delta, s}^{(0,2)}\left(u_{\eta_{\min }, s}^{\star}(\varphi)\right)\left(u_{\eta_{\min }, s}^{\star,(l)}(\varphi), u_{\eta_{\min }, s}^{\star,(1)}(\varphi)\right)+D H_{\mu / l}^{(l)}(\varphi)$. Similar arguments of the proof of the $l=k=1$ case show that the unique fixed point $u_{\eta_{\min }, s}^{\star,(l+1)} \in$ $\mathcal{L}^{l+1}\left(X_{0}(s)^{l+1}, \mathrm{BC}_{s}^{\eta_{\text {min }}(l+1)}(\mathbb{R}, X)\right)$. Hence, the map $\mathcal{J}_{s}^{\eta, \eta_{\text {min }}} \circ u_{\eta_{\text {min }}, s}^{\star}: X_{0}(s) \rightarrow \mathrm{BC}_{s}^{\eta}(\mathbb{R}, X)$ is of the class $C^{l+1}$ if $\eta \in\left((l+1) \eta_{\min }, \eta_{\max }\right]$ which completes the proof.

We also show that each partial derivative of the center manifold in the second component is uniformly Lipschitz continuous. The proof is inspired by [2, Corollary 8.2.1.2].

Corollary 34. For each $l \in\{0, \ldots, k\}$, there exists a constant $L(l)>0$ such that $\| D_{2}^{l} \mathcal{C}(t, \varphi)-$ $D_{2}^{l} \mathcal{C}(t, \psi)\|\leq L(l)\| \varphi-\psi \|$ for all $t \in \mathbb{R}$ and $\varphi, \psi \in X_{0}(t)$.

Proof. For $l=0$, the result is already proven in Corollary 28. Now let $l \in\{1, \ldots, k\}$. Then, from the proof of Theorem 33 we see that $u_{\eta_{\min }, s}^{\star(l)}$ is the unique solution of a fixed point problem, where the right hand-side is a contraction with a Lipschitz constant $L(l)$ independent of $s$. Using the same strategy as the proof of Corollary 28, we obtain the desired result.

Corollary 35. The center manifold $\mathcal{W}^{c}$ is $C^{k}$-smooth and its tangent bundle is $X_{0}$ i.e. $D_{2} \mathcal{C}(t, 0) \varphi=\varphi$ for all $(t, \varphi) \in X_{0}$.

Proof. Let $\eta \in\left[\eta_{\min }, \eta_{\max }\right] \subset(0, \min \{-a, b\})$ such that $k \eta_{\min }<\eta_{\max }$. Define for any $t \in \mathbb{R}$ the evolution map $\mathrm{ev}_{t}: \mathrm{BC}_{t}^{\eta}(\mathbb{R}, X) \rightarrow X$ as $\mathrm{ev}_{t}(f):=f(t)$. Then, for all $(t, \varphi) \in X_{0}$ we get

$$
\mathcal{C}(t, \varphi)=\operatorname{ev}_{t}\left(u_{\eta_{\min }, t}^{\star}(\varphi)\right)=\operatorname{ev}_{t}\left(\mathcal{J}_{t}^{\eta, \eta_{\min }} u_{\eta_{\min }, t}^{\star}(\varphi)\right)
$$

It is clear that $\mathrm{ev}_{t} \in \mathcal{L}\left(\mathrm{BC}_{t}^{\eta}(\mathbb{R}, X), X\right)$ and hence it follows from Theorem 33 that $\mathcal{C}$ is of the class $C^{k}$. This shows that the center manifold $\mathcal{W}^{c}$ is $C^{k}$-smooth. Moreover,

$$
D_{2} \mathcal{C}(t, 0) \varphi=\operatorname{ev}_{t}\left(D\left(\mathcal{J}_{t}^{\eta, \eta_{\min }} \circ u_{\eta_{\min }, t}^{\star}\right)(0) \varphi\right)=\operatorname{ev}_{t}\left(u_{\eta_{\min }, t}^{\star,(1)}(0) \varphi\right)
$$

As $D \tilde{R}_{\delta, t}(0)=0$ and $u_{\eta_{\min }, t}^{\star}(0)=0$ for all $t \in \mathbb{R}$, we get from (86) that $u_{\eta_{\min }, t}^{\star,(1)}(0)=U(\cdot, t)$ and so $D_{2} \mathcal{C}(t, 0) \varphi=\operatorname{ev}_{t}(U(\cdot, t) \varphi)=\varphi$, as claimed.

It follows from the previous corollary that the local center manifold $\mathcal{W}_{\text {loc }}^{c}$ is also $C^{k}$-smooth and has $X_{0}$ as a tangent bundle. Let us now take a look into periodicity.
Theorem 36. If the time-dependent nonlinear perturbation $R: \mathbb{R} \times X \rightarrow X^{\odot \star}$ is T-periodic in the first variable, then there exists a $\delta>0$ such that $\mathcal{C}(t+T, \varphi)=\mathcal{C}(t, \varphi)$ for all $t \in \mathbb{R}$ whenever $\|\varphi\|<\delta$.

Proof. The proof of this theorem is essentially the same as [2, Lemma 8.3.1], which was obtained for impulsive DDEs. To obtain the result for classical DDEs, one has to essentialy ignore the discontinuous impulses and make the logical subsitution $\mathcal{R C \mathcal { R }} \rightarrow X$ and put the formal adjoint setting towards the sun-star setting. The result follows.

## C Variation-of-constants formulas and one-to-one correspondences

This section of the appendix deals translating solutions of delay differential equations, abstract integral equations and abstract ordinary differential for particular problems. Firstly, we take a look on the interplay between inhomogeneous time-dependent linear abstract ODEs and inhomogeneous time-dependent linear AIEs. Secondly, we prove the one-to-one correspondence between solutions of (T-DDE) and (T-AIE) that is crucial to obtain Corollary 17.

Applying an inhomogeneous perturbation $f: J \rightarrow X^{\odot \star}$ on the generator $A^{\odot \star}(t)$ to the problem (5) yields

$$
\begin{cases}d^{\star}(j \circ u)(t)=A^{\odot \star}(t) j u(t)+f(t), & t \geq s  \tag{87}\\ u(s)=\varphi, & \varphi \in X\end{cases}
$$

which suggest the variation-of-constants formula

$$
\begin{equation*}
u(t)=U(t, s) \varphi+j^{-1} \int_{s}^{t} U^{\odot \star}(t, \tau) f(\tau) d \tau, \quad \varphi \in X \tag{88}
\end{equation*}
$$

This suggestion, with the additional assumptions on $f$, is verified in the following proposition. The proof is inspired by [21, Proposition 21], where this was proven for the autonomous setting, i.e. $B$ does not depend on time.

Proposition 37. Let $J \subseteq \mathbb{R}$ be an interval and assume that $f \in C\left(J, X^{\odot \star}\right)$. If $u$ is a solution of (87) on $J$ then $u$ is given by (88).

Proof. Let $(t, s) \in \Omega_{J}$ with $t>s$ be arbitrary. Define the function $w:[s, t] \rightarrow X^{\odot \star}$ by $w(\tau):=$ $U^{\odot \star}(t, \tau) j u(\tau)$ for all $\tau \in[s, t]$. We claim that $w$ is weak ${ }^{\star}$ differentiable with weak ${ }^{\star}$ derivative

$$
\begin{equation*}
d^{\star} w(\tau)=U^{\odot \star}(t, \tau) d^{\star}(j \circ u)(\tau)-U^{\odot \star}(t, \tau) A^{\odot \star}(\tau) j u(\tau), \quad \forall \tau \in[s, t] \tag{89}
\end{equation*}
$$

To show this claim, let $\tau \in[s, t]$ and $x^{\odot} \in X^{\odot}$ be given. For any $h \in \mathbb{R}$ such that $\tau+h \in[s, t]$ we have

$$
\begin{aligned}
\left\langle w(\tau+h)-w(\tau), x^{\odot}\right\rangle & =\left\langle U^{\odot \star}(t, \tau+h) j u(\tau+h)-U^{\odot \star}(t, \tau) j u(\tau), x^{\odot}\right\rangle \\
& =\left\langle U^{\odot \star}(t, \tau+h)[j u(\tau+h)-j u(\tau)], x^{\odot}\right\rangle \\
& +\left\langle\left[U^{\odot \star}(t, \tau+h)-U^{\odot \star}(t, \tau)\right] j u(\tau), x^{\odot}\right\rangle \\
& =\left\langle j u(\tau+h)-j u(\tau), U^{\odot}(\tau+h, t) x^{\odot}\right\rangle \\
& +\left\langle\left[U^{\odot \star}(t, \tau+h)-U^{\odot \star}(t, \tau)\right] j u(\tau), x^{\odot}\right\rangle
\end{aligned}
$$

Because $U^{\odot}$ is a strongly continuous backward evolutionary system we have that $U^{\odot}(\tau+h, t) x^{\odot} \rightarrow$ $U \odot(\tau, t) x^{\odot}$ in norm as $h \rightarrow 0$. Moreover, from the definition of the weak ${ }^{\star}$ derivative we obtain

$$
\frac{1}{h}(j u(\tau+h)-j u(\tau)) \rightarrow d^{\star}(j \circ u)(\tau) \quad \text { weakly }{ }^{\star} \text { as } h \rightarrow 0
$$

while the difference quotients remains bounded because $j \circ u$ is Lipschitz on $[s, t]$. Combining these two facts, we get

$$
\frac{1}{h}\left\langle j u(\tau+h)-j u(\tau), U^{\odot}(\tau+h, t) x^{\odot}\right\rangle \rightarrow\left\langle d^{\star}(j \circ u)(\tau), U^{\odot}(t, \tau) x^{\odot}\right\rangle \quad \text { as } h \rightarrow 0
$$

Furthermore, since $j u(\tau) \in \mathcal{D}\left(A^{\odot \star}(\tau)\right)=\mathcal{D}\left(A_{0}^{\odot \star}\right)$, it follows from [4, Theorem 5.5] that

$$
\frac{1}{h}\left\langle\left[U^{\odot \star}(t, \tau+h)-U^{\odot \star}(t, \tau)\right] j u(\tau), x^{\odot}\right\rangle \rightarrow\left\langle-U^{\odot \star}(t, \tau) A^{\odot \star}(\tau) j u(\tau), x^{\odot}\right\rangle \quad \text { as } h \rightarrow 0
$$

Consequently, it holds

$$
\frac{1}{h}\left\langle w(\tau+h)-w(\tau), x^{\odot}\right\rangle \rightarrow\left\langle U^{\odot \star}(t, \tau) d^{\star}(j \circ u)(\tau)-U^{\odot \star}(t, \tau) A^{\odot \star}(\tau) j u(\tau), x^{\odot}\right\rangle \quad \text { as } h \rightarrow 0
$$

which proves (89). Substituting the abstract integral equation from (88) into (89) yields

$$
d^{\star} w(\tau)=U^{\odot \star}(t, \tau) f(\tau), \quad \forall \tau \in[s, t]
$$

and it is clear from the estimates in Lemma 2 that $d^{\star} w$ is weak ${ }^{\star}$ continuous. Now, for any $x^{\odot} \in X^{\odot}$ we get

$$
\begin{aligned}
\left\langle j u(t)-U^{\odot \star}(t, s) j u(s), x^{\odot}\right\rangle & =\left\langle w(t), x^{\odot}\right\rangle-\left\langle w(s), x^{\odot}\right\rangle \\
& =\int_{s}^{t}\left\langle d^{\star} w(\tau), x^{\odot}\right\rangle d \tau \\
& =\left\langle\int_{s}^{t} U^{\odot \star}(t, \tau) f(\tau) d \tau, x^{\odot}\right\rangle
\end{aligned}
$$

As $x^{\odot} \in X^{\odot}$ and $(s, t) \in \Omega_{J}$ were arbitrary, we conclude that

$$
j u(t)-U^{\odot \star}(t, s) j u(s)=\int_{s}^{t} U^{\odot \star}(t, \tau) f(\tau) d \tau
$$

and so

$$
j[u(t)-U(t, s) u(s)]=\int_{s}^{t} U^{\odot \star}(t, \tau) f(\tau) d \tau
$$

By $\odot$-reflexivity of $X$ with respect to $\left\{T_{0}(t)\right\}_{t \geq 0}$, and recalling that $j$ is an isomorphism on its image $X^{\odot \odot}$ we get

$$
\begin{equation*}
u(t)=U(t, s) u(s)+j^{-1} \int_{s}^{t} U^{\odot \star}(t, \tau) f(\tau) d \tau \tag{90}
\end{equation*}
$$

which shows the claim since $\varphi=u(s)$. The continuity of $f$ ensures from Lemma 2 that the weak ${ }^{\star}$ integral takes values in $j(X)$ and so (90) is well-defined.

The proof of the important one-to-one correspondence between solutions of (T-DDE) and (T-AIE) will be presented in several steps. The key to solve this problem is the interplay between solutions of DDEs, AIEs and abstract ODEs. We start of by going back for a moment to (87) and noticing that one can also perturb the generator $A_{0}^{\odot \star}$ by $\varphi \mapsto B(t) \varphi+f$, for some fixed $t \in J$ that gives

$$
\begin{cases}d^{\star}(j \circ u)(t)=A_{0}^{\odot \star} j u(t)+B(t) u(t)+f(t), & t \geq s  \tag{91}\\ u(s)=\varphi, & \varphi \in X\end{cases}
$$

which suggest the variation-of-constants formula

$$
\begin{equation*}
u(t)=T_{0}(t-s) \varphi+j^{-1} \int_{s}^{t} T_{0}^{\odot \star}(t-\tau)[B(\tau) u(\tau)+f(\tau)] d \tau, \quad \varphi \in X \tag{92}
\end{equation*}
$$

which is well defined because the weak ${ }^{\star}$ integral in (92) takes values in $j(X)$ due to [13, Lemma XII.2.8] since $[s, t] \ni \tau \mapsto B(\tau) u(\tau)+f(\tau) \in X^{\odot \star}$ is continuous. It is clear by (6) that (87) and (91) are equivalent, but the same can not be directly said about (88) and (92). This was also the observation made in [21, Section 3] for an autonomous perturbation $B$, and therefore we will follow his approach here, but in a time-dependent setting. We start off with a proposition, that is inspired by [21, Corollary 19].

Proposition 38. Suppose that $\varphi \in j^{-1} \mathcal{D}\left(A_{0}^{\odot \star}\right)$ and $f: J \rightarrow X^{\odot \star}$ is locally Lipschitz. If $u$ is a locally Lipschitz solution of (92) on $J$ then $u$ is a solution of (91) on $J$.

Proof. If we apply $j$ to the abstract integral equation in (92), we get for any $t \in J$ that

$$
\begin{equation*}
j u(t)=T_{0}^{\odot \star}(t-s) j \varphi+\int_{s}^{t} T_{0}^{\odot \star}(t-\tau)[B(\tau) u(\tau)+f(\tau)] d \tau \tag{93}
\end{equation*}
$$

The first term on the right side takes values in $\mathcal{D}\left(A_{0}^{\odot \star}\right)$ and notice that this term is weak ${ }^{\star}$ continuously differentiable with (partial) weak ${ }^{\star}$ derivative

$$
\partial_{t}^{\star} T_{0}^{\odot \star}(t-s) j \varphi=A_{0}^{\odot \star} T_{0}^{\odot \star}(t-s) j \varphi,
$$

where $\partial_{t}^{\star}$ stands dor the weak ${ }^{\star}$ partial differential operator. Now, $f$ and $u$ are locally Lipschitz functions and $B$ is by definition of the time-dependent bounded linear perturbation. Hence, $g: J \rightarrow X^{\odot \star}$ defined by $g(\tau):=B(\tau) u(\tau)+f(\tau)$ for all $\tau \in J$ is locally Lipschitz and denote the second term of (93) by $v_{1}(t, s, g)$. From [6, Proposition 2.2] it is clear that $v_{1}(\cdot, s, g)$ is weak ${ }^{\star}$ continuously differentiable, takes values in $\mathcal{D}\left(A_{0}^{\odot \star}\right)$ and has the partial weak ${ }^{\star}$ derivative

$$
\begin{equation*}
\partial_{t}^{\star} v_{1}(t, s, g)=A_{0}^{\odot \star} v_{1}(t, s, g)+g(t), \quad \forall t \in[s, \infty) \cap J \tag{94}
\end{equation*}
$$

Hence, $u$ takes values in $j^{-1} \mathcal{D}\left(A_{0}^{\odot \star}\right)$ and combining all these results yield

$$
\begin{aligned}
d^{\star}(j \circ u)(t) & =A_{0}^{\odot \star} T_{0}^{\odot \star}(t-s) j \varphi+A_{0}^{\odot \star} \int_{s}^{t} T_{0}^{\odot \star}(t-\tau)[B(\tau) u(\tau)+f(\tau)] d \tau+B(t) u(t)+f(t) \\
& =A_{0}^{\odot \star} j u(t)+B(t) u(t)+f(t)
\end{aligned}
$$

This shows $j \circ u$ is weak ${ }^{\star}$ continuously differentiable and satisfies (91) on $J$ since $u(s)=\varphi$.
The following result is inspired by [21, Proposition 20]
Proposition 39. Let $J$ be compact. The following two statements hold.

1. For every $\varphi \in X$ there exists a unique solution $u_{\varphi, f}$ of (92) on $J$ and the map

$$
X \times C\left(J, X^{\odot \star}\right) \ni(\varphi, f) \mapsto u_{\varphi, f} \in C(J, X)
$$

is continuous.
2. If $\varphi \in j^{-1} \mathcal{D}\left(A_{0}^{\odot \star}\right)$ and $f: J \rightarrow X^{\odot \star}$ is locally Lipschitz, then there exist sequences of Lipschitz functions $u_{m}: J \rightarrow X$ and $f_{m}: J \rightarrow X^{\odot \star}$ such that

$$
\begin{equation*}
u_{m}(t)=T_{0}(t) \varphi+j^{-1} \int_{s}^{t} T_{0}^{\odot \star}(t-\tau)\left[B(\tau) u_{m}(\tau)+f_{m}(\tau)\right] d \tau, \quad \forall t \in J \tag{95}
\end{equation*}
$$

and $f_{m} \rightarrow f$ and $u_{m} \rightarrow u_{\varphi, f}$ as $m \rightarrow \infty$, uniformly on $J$.
Proof. We show the first claim by a fixed point argument. Choose $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\left\|T_{0}(t)\right\| \leq M e^{\omega t}$. On the space $C(J, X)$, we introduce the one-parameter family of equivalent norms

$$
\|u\|_{\eta}:=\sup _{t \in J} e^{-\eta t}\|u(t)\|, \quad \eta \in \mathbb{R}
$$

that makes $\left(C(J, X),\|\cdot\|_{\eta}\right)$ a Banach space for each $\eta \in \mathbb{R}$. For each fixed $(\varphi, f) \in X \times C\left(J, X^{\odot \star}\right)$ define the operator $K_{\varphi, f}: C(J, X) \rightarrow C(J, X)$ as

$$
\begin{equation*}
\left(K_{\varphi, f} u\right)(t):=T_{0}(t-s) \varphi+j^{-1} \int_{s}^{t} T_{0}^{\odot \star}(t-\tau)[B(\tau) u(\tau)+f(\tau)] d \tau, \quad \forall t \in J \tag{96}
\end{equation*}
$$

Define $N:=\sup _{(t, s) \in \Omega_{J}} W(t, s)$ and notice that $N$ is finite because $J$ is compact and $K$ is continuous, where $K: \Omega_{J} \rightarrow \mathbb{R}$ is defined as $W(t, s):=\sup _{s \leq \tau \leq t}\|B(\tau)\|$. Let $\eta>\omega$, then for all $u_{1}, u_{2} \in C(J, X)$ and $t \in J$ we get

$$
\begin{aligned}
e^{-\eta t}\left\|\left(K_{\varphi, f} u_{1}\right)(t)-\left(K_{\varphi, f} u_{2}\right)(t)\right\| & \leq\left\|j^{-1}\right\| M N \int_{s}^{t} e^{-(\eta-\omega)(t-\tau)} e^{-\eta \tau}\left\|u_{1}(\tau)-u_{2}(\tau)\right\| d \tau \\
& \leq\left\|j^{-1}\right\| M N\left\|u_{1}-u_{2}\right\|_{\eta} \int_{s}^{t} e^{-(\eta-\omega)(t-\tau)} d \tau \\
& =\frac{\left\|j^{-1}\right\| M N\left(1-e^{-(t-s)(\eta-\omega)}\right)}{\eta-\omega}\left\|u_{1}-u_{2}\right\|_{\eta} \\
& \leq \frac{\left\|j^{-1}\right\| M N}{\eta-\omega}\left\|u_{1}-u_{2}\right\|_{\eta}
\end{aligned}
$$

If we choose $\eta>\omega$ large enough such that $\frac{\left\|j^{-1}\right\| M N}{\eta-\omega} \leq \frac{1}{2}$, then $K_{\varphi, f}$ is a contraction with respect to the $\|\cdot\|_{\eta}$-norm. The uniqueness of $u$ now follows from the Banach fixed point theorem. For a fixed $u \in C(J, X)$, it follows that the map

$$
X \times C\left(J, X^{\odot \star}\right) \ni(\varphi, f) \mapsto K_{\varphi, f} u \in C(J, X)
$$

is continuous.
Let us now show the second assertion. Let $\varphi \in j^{-1} \mathcal{D}\left(A_{0}^{\odot \star}\right)$ and $f$ be locally Lipschitz, we will show that $K_{\varphi, f}$ maps $\operatorname{Lip}(J, X)$ into itself, where $\operatorname{Lip}(J, X)$ denotes the closed subspace of $C(J, X)$ consisting of $X$-valued Lipschitz continuous functions defined on $J$. From the theory of Favard classes of $\mathcal{C}_{0}$-semigroups and the important equalities [21, Equation 19], it follows immediately that $T_{0}(\cdot) \varphi$ is locally Lipschitz. Let $u \in \operatorname{Lip}(J, X)$ be given, since $B$ is Lipschitz continuous and $f$ is assumed to be locally Lipschitz we know that $t \mapsto B(t) u(t)+f(t)$ is locally Lipschitz on $J$ and takes values in $X^{\odot \star}$. Hence, with the notation from the proof of Proposition 38 we have $v_{1}(\cdot, s, B(\cdot) u+f)$ is weak $^{\star}$ continuously differentiable and so locally Lipschitz by [21, Remark 16]. It follows that $K_{\varphi, f} u=$ $T_{0}(\cdot) \varphi+j^{-1} v_{1}(\cdot, s, B(\cdot) u+f)$ is Lipschitz continuous.

Now, let $u_{0} \in \operatorname{Lip}(J, X)$ be given. The sequence $\left(u_{m}\right)_{m \in \mathbb{N}}$ defined by

$$
u_{m}:=K_{\varphi, f} u_{m-1}, \quad m \geq 1
$$

is in $\operatorname{Lip}(J, X)$ and converges to the unique fixed point $u_{\varphi, f} \in \operatorname{Lip}(J, X)$ due to the same computations as done in part 1 of this proposition and the closedness of $\operatorname{Lip}(J, X)$ with respect to the $\|\cdot\|_{\eta^{\prime}}$-norm for all $\eta \in \mathbb{R}$. We only have to show that there exists a sequence of $X^{\odot \star}$-valued Lipschitz continuous functions $\left(f_{m}\right)_{m \in \mathbb{N}}$ defined on $J$ that satisfies the integral formula. It follows from (96) that for any $t \in J$ and $m \geq 1$ we have

$$
\begin{aligned}
u_{m}(t) & =K_{\varphi, f} u_{m-1}(t) \\
& =T_{0}(t-s) \varphi+j^{-1} \int_{s}^{t} T_{0}^{\odot \star}(t-\tau)\left[B(\tau) u_{m}(\tau)+f(\tau)+B(\tau)\left[u_{m-1}(\tau)-u_{m}(\tau)\right]\right] d \tau
\end{aligned}
$$

If we define for any $m \geq 1$ the functions $f_{m}: J \rightarrow X^{\odot \star}$ as $f_{m}:=f+B(\cdot)\left(u_{m-1}-u_{m}\right)$, then each $f_{m}$ is Lipschitz continuous and $f_{m} \rightarrow f$ uniformly on $J$ because

$$
\begin{aligned}
\left\|f_{m}-f\right\| & \leq \sup _{(t, s) \in \Omega_{J}} W(t, s)\left\|u_{m-1}-u_{m}\right\| \\
& \leq N\left[\left\|u_{m-1}-u_{\varphi, f}\right\|+\left\|u_{\varphi, f}-u_{m}\right\|\right] \\
& \rightarrow 0, \quad \text { as } m \rightarrow \infty
\end{aligned}
$$

as both $u_{m-1}$ and $u_{m}$ converge to $u_{\varphi, f}$ uniformly on $J$ as $m \rightarrow \infty$.

The following result is inspired by [21, Theorem 22].
Proposition 40. The unique solution of (92) is given by (88).
Proof. We first prove the statement for a compact interval $I \subseteq J$ such that $s \in I$ and later extend it towards $J$ by a continuity and density argument.

Let $\varphi \in j^{-1} \mathcal{D}\left(A_{0}^{\odot \star}\right)$ and $f: J \rightarrow X^{\odot \star}$ be locally Lipschitz. From Proposition 39 we get a unique solution $u_{\varphi, f}: I \rightarrow X$ of (92) and sequences of Lipschitz functions $u_{m}: I \rightarrow X$ and $f_{m}: I \rightarrow X^{\odot \star}$ that satisfy (95). For each $m \in \mathbb{N}$, let $\hat{f}_{m}: I \rightarrow X^{\odot \star}$ be a Lipschitz extension of $f_{m}$ such that $\left.\hat{f}_{m}\right|_{I}=f_{m}$. Substituting $f$ with $\hat{f}_{m}$ and $u$ with $u_{m}$ in Proposition 38 shows us that each $u_{m}$ is a solution to the initial value problem

$$
\left\{\begin{array}{l}
d^{\star}\left(j \circ u_{m}\right)(t)=A_{0}^{\odot \star} j u_{m}(t)+B(t) u_{m}(t)+\hat{f}_{m}(t), \quad t \geq s \\
u_{m}(s)=\varphi
\end{array}\right.
$$

Recall from (6) that each $u_{m}$ also is a solution of

$$
\left\{\begin{array}{l}
d^{\star}\left(j \circ u_{m}\right)(t)=A^{\odot \star}(t) j u_{m}(t)+\hat{f}_{m}(t), \quad t \geq s \\
u_{m}(s)=\varphi
\end{array}\right.
$$

It follows from Proposition 37, with $u$ replaced by $u_{m}$ and $f$ replaced by $\hat{f}_{m}$, that

$$
\begin{equation*}
u_{m}(t)=U(t, s) \varphi+j^{-1} \int_{s}^{t} U^{\odot \star}(t, \tau) f_{m}(\tau) d \tau, \quad \forall m \in \mathbb{N}, t \geq s \tag{97}
\end{equation*}
$$

since $\hat{f}_{m}$ restricted to $I$ precisely is $f_{m}$. Let us take the limit as $m \rightarrow \infty$ in (97) to obtain

$$
\begin{equation*}
u_{\varphi, f}(t)=U(t, s) \varphi+j^{-1} \int_{s}^{t} U^{\odot \star}(t, \tau) f(\tau) d \tau \tag{98}
\end{equation*}
$$

for all $(\varphi, f) \in j^{-1} \mathcal{D}\left(A_{0}^{\odot \star}\right) \times \operatorname{Lip}\left(I, X^{\odot \star}\right)$. As $j^{-1} \mathcal{D}\left(A_{0}^{\odot \star}\right) \times \operatorname{Lip}\left(I, X^{\odot \star}\right)$ is dense in $X \times C\left(I, X^{\odot \star}\right)$, the continuity statement from Proposition 39 implies that (98) also holds for all $\varphi \in X$ and $f \in C\left(I, X^{\odot \star}\right)$. Hence, the unique solution of (92) on $I$ is given by (88) on $I$. To extend this result towards $J$ the same proof can be followed as in [21, Theorem 22].

Let us now prove the important one-to-one correspondence between solutions of (T-DDE) and (T-AIE). To prove this result, we assume weaker assumptions on the (nonlinear) time-dependent perturbations becuase this is not needed for the proof.

Theorem 41. Consider (T-LDDE) with $L \in C\left(\mathbb{R}, \mathcal{L}\left(X, \mathbb{R}^{n}\right)\right)$ and $G \in C\left(\mathbb{R} \times X, \mathbb{R}^{n}\right)$.

1. Suppose that $y:\left[s-h, t_{\varphi}\right) \rightarrow \mathbb{R}^{n}$ is a solution of (T-DDE), then the function $u_{\varphi}:\left[s, t_{\varphi}\right) \rightarrow X$ defined by

$$
u_{\varphi}(t):=y_{t}, \quad \forall t \in\left[s, t_{\varphi}\right)
$$

is a solution of (T-AIE).
2. Suppose that $u_{\varphi}:\left[s, t_{\varphi}\right) \rightarrow X$ is a solution of (T-AIE), then the function $y:\left[s-h, t_{\varphi}\right) \rightarrow \mathbb{R}^{n}$ defined by

$$
y(t):= \begin{cases}\varphi(t-s), & s-h \leq t \leq s \\ u_{\varphi}(t)(0), & s \leq t \leq t_{\varphi}\end{cases}
$$

is a solution of (T-DDE).

Proof. Before we start proving the first assertion, notice that the differential equation from (T-DDE) is equivalent to the integral equation

$$
\begin{equation*}
y(t)=\varphi(0)+\int_{s}^{t} L(\tau) y_{\tau}+G\left(\tau, y_{\tau}\right) d \tau, \quad t \geq s \tag{99}
\end{equation*}
$$

due to the fundamental theorem of calculus. Let us start with proving the first assertion.

1. Notice that the right-hand side of the abstract integral equation in (92) with a $C^{k}$-smooth function $f=R\left(\cdot, u_{\varphi}(\cdot)\right)$ is equivalent to

$$
T_{0}(t-s) \varphi+j^{-1} \int_{s}^{t} T_{0}^{\odot \star}(t-\tau)\left[L(\tau) u_{\varphi}(\tau)+G\left(\tau, u_{\varphi}(\tau)\right)\right] r^{\odot \star} d \tau, \quad \forall t \in\left[s, t_{\varphi}\right)
$$

It then follows from the action of the shift semigroup (28), the assumption $u_{\varphi}(t)=y_{t}$ and [13, Lemma XII.3.3] where in this lemma the map $g$ must be replaced by the continuous map $\left.L(\cdot) u_{\varphi}(\cdot)+G\left(\cdot, u_{\varphi}(\cdot)\right)\right)$, since $L \in C\left(\mathbb{R}, \mathcal{L}\left(X, \mathbb{R}^{n}\right)\right), u_{\varphi} \in C\left(\left[s, t_{\varphi}\right), X\right)$ and $G \in C\left(\mathbb{R} \times X, \mathbb{R}^{n}\right)$, that this right-hand side evaluated at $\theta \in[-h, 0]$ is equivalent to

$$
\begin{aligned}
& \left(T_{0}(t-s) \varphi\right)(\theta)+j^{-1}\left(\int_{s}^{t} T_{0}^{\odot \star}(t-\tau)\left[L(\tau) u_{\varphi}(\tau)+G\left(\tau, u_{\varphi}(\tau)\right)\right] r^{\odot \star} d \tau\right)(\theta) \\
& =\left(T_{0}(t-s) \varphi\right)(\theta)+\int_{s}^{\max \{s, t+\theta\}} L(\tau) u_{\varphi}(\tau)+G\left(\tau, u_{\varphi}(\tau)\right) d \tau \\
& =\left(T_{0}(t-s) \varphi\right)(\theta)+\int_{s}^{\max \{s, t+\theta\}} L(\tau) y_{\tau}+G\left(\tau, y_{\tau}\right) d \tau \\
& = \begin{cases}\varphi(t+\theta), & s-h \leq t+\theta \leq s, \\
\varphi(0)+\int_{s}^{t+\theta} L(\tau) y_{\tau}+G\left(\tau, y_{\tau}\right) d \tau, & s \leq t+\theta \leq t_{\varphi}\end{cases} \\
& =y(t+\theta)=u_{\varphi}(t)(\theta),
\end{aligned}
$$

where the fourth equality holds due to (99). Hence, $u_{\varphi}$ is a solution to (92) with $f=R\left(\cdot, u_{\varphi}(\cdot)\right)$. It follows from Proposition 40 that $u_{\varphi}$ then also is a solution of (88) with $f=R\left(\cdot, u_{\varphi}(\cdot)\right)$, which is equivalent to saying that $u_{\varphi}$ is a solution of (T-AIE).
2. Let us first prove that the function $y$ is continuous on $\left[s-h, t_{\varphi}\right)$. As $\varphi \in X$, it is clear that $y$ is continuous for $t \in[s-h, s]$. As point evaluation acts continuously on elements in $X \ni u_{\varphi}(t)$, it follows that $y$ is continuous on $\left[s, t_{\varphi}\right)$. Since $u_{\varphi}(s)(0)=\varphi(0)$ we have that $y \in C\left(\left[s-h, t_{\varphi}\right), \mathbb{R}^{n}\right)$.

Our next goal is to show that $y$ satisfies (T-DDE) or equivalently (99). Because $u_{\varphi}$ is a solution of (88) with $f=R\left(\cdot, u_{\varphi}(\cdot)\right)$, we know from Proposition 40 that $u_{\varphi}$ is then also a solution of (92) with $f=R\left(\cdot, u_{\varphi}(\cdot)\right)$. It follows from (28) and [13, Lemma XII.3.3] that

$$
\begin{align*}
y(t) & =u_{\varphi}(t)(0) \\
& =\left(T_{0}(t-s) \varphi\right)(0)+j^{-1}\left(\int_{s}^{t} T_{0}^{\odot \star}(t-\tau)\left[L(\tau) u_{\varphi}(\tau)+G\left(\tau, u_{\varphi}(\tau)\right)\right] r^{\odot \star} d \tau\right)(0)  \tag{0}\\
& =\varphi(0)+\int_{s}^{t} L(\tau) u_{\varphi}(\tau)+G\left(\tau, u_{\varphi}(\tau)\right) d \tau
\end{align*}
$$

It remains to show that $u_{\varphi}(\tau)=y_{\tau}$ for all $\tau \in\left[s, t_{\varphi}\right)$. Because, then we have shown that $y$ indeed satisfies (99). Let $\theta \in[-h, 0]$ be given. If $\tau+\theta \in[s-h, s]$ then we have that

$$
y_{\tau}(\theta)=y(\tau+\theta)=\varphi(\tau+\theta-s)=\left(T_{0}(\tau-s) \varphi\right)(\theta)=u_{\varphi}(\tau)(\theta)
$$

due to (28). When $\tau+\theta \in\left[s, t_{\varphi}\right)$, it again follows from (28) and [13, Lemma XII.3.3] that

$$
\begin{aligned}
y_{\tau}(\theta) & =y(\tau+\theta) \\
& =u_{\varphi}(\tau+\theta)(0) \\
& =\left(T_{0}(\tau+\theta-s) \varphi\right)(0) \\
& +j^{-1}\left(\int_{s}^{\tau+\theta} T_{0}^{\odot \star}(\tau+\theta-\sigma)\left[L(\sigma) u_{\varphi}(\sigma)+G\left(\sigma, u_{\varphi}(\sigma)\right)\right] r^{\odot \star} d \sigma\right)(0) \\
& =\varphi(0)+\int_{0}^{\tau+\theta} L(\sigma) u_{\varphi}(\sigma)+G\left(\sigma, u_{\varphi}(\sigma)\right) d \sigma \\
& =\left(T_{0}(\tau-s) \varphi\right)(\theta)+j^{-1}\left(\int_{s}^{\tau} T_{0}^{\odot \star}(\tau-\sigma)\left[L(\sigma) u_{\varphi}(\sigma)+G\left(\sigma, u_{\varphi}(\sigma)\right)\right] r^{\odot \star} d \sigma\right)(\theta) \\
& =u_{\varphi}(\tau)(\theta)
\end{aligned}
$$

and so $y_{\tau}=u_{\varphi}(\tau)$ for all $\tau \in\left[s, t_{\varphi}\right)$. To conclude,

$$
y(t)=\varphi(0)+\int_{s}^{t} L(\tau) y_{\tau}+G\left(\tau, y_{\tau}\right) d \tau
$$

and so $y$ satisfies the differential equation of (T-DDE). By the history property, and the fact that $\varphi \in X$, it follows by the method of steps applied to (99) that $y \in C^{1}\left(\left[s, t_{\varphi}\right), \mathbb{R}^{n}\right)$. This shows that $y$ indeed is a solution to (T-DDE).

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