Static Balance of a Flexure-Based Four-Bar Mechanism: Less Torque with More Preload

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Abstract. Static balance eliminates the inherent actuation stiffness of flexure-based mechanisms. In this paper we investigate how, and under which conditions, preloading in the joints can approximately balance a flexure-based four-bar mechanism. We show that actuation reduction is possible when the preload leverages kinematic non-linearities of the system, resulting in a negative kinematic stiffness. The static balance conditions are formulated such that they can be solved using linear algebra and also permit kinematic insight. Application to a flexure-based Watt's linkage shows a reduction factor of 500 in actuator torque.

Keywords: Compliant mechanisms · Zero stiffness · Geometric non-linearities

1 Introduction

Flexure-based mechanisms—also known as compliant mechanisms—consist of thin strips of deforming material that allow motion in compliant directions while supporting forces in other stiff directions [6]. Since this allows for a backlash-free motion without friction, these type of hinges are used in precision applications where high repeatability is desired. However, the inherent stiffness of flexures requires a continuous actuation force to deflect the mechanism, necessitating larger motors with higher energy consumption and unwanted thermal loads.

To eliminate the actuation stiffness, a mechanism may be statically balanced. In that case it has no preference pose and is therefore in an indifferent equilibrium position [5]. Historically, static balance was used to counteract the pull of gravity, but it can also be used to counteract other conservative energy fields such as magnetic force and joint compliance. A mechanism is statically balanced when its potential energy is constant over deflection. Therefore, a potential energy storage, i.e. pretension, is needed with a synchronized energy release to balance the positive energy build-up in the flexures.

Static balance of compliant structures is often performed on joint level by adding different type of energy storages like, prestressed flexures [7], buckled beams [4] and linear springs [9]. Design methodologies on mechanism level are

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A. Müller and M. Brandstötter (Eds.): RAAD 2022, MMS 120, pp. 1–8, 2022.
https://doi.org/10.1007/978-3-031-04870-8_36

typically based on optimization or visual inspection [8]. In [3] analytic rigid body approximation methods were used to statically balance compliant mechanisms with an additional zero-free-length spring.

In [1,2] redundantly actuated mechanisms with flexure joints were presented that show the potential of static balancing. Initially, actuator redundancy of a 2-DoF compliant mechanism was used to increase its effective workspace and support stiffness. Surprisingly, optimization showed and measurements confirmed that the actuation stiffness of this mechanism could be reduced with a factor of 1.9 by applying torsional preload in the flexure joints [1]. From the cited literature it is not clear how it is possible that more parallel connected flexure joints could lead to a reduction of actuation stiffness, nor is it clear how such mechanism could be designed.

In this paper we show how torsional preload in the flexure joints of a four-bar mechanism can lead to approximate static balance. We present a new formalism that allows for a closed form description of the balancing solution. Through simulation we demonstrate the procedure on a flexure-based Watt's linkage. This method eliminates the actuation stiffness in one desired pose, which is considered to be sufficient for most applications as compliant mechanisms typically are used in a rather limited working range. We rely on the pseudo-rigid body model (PRBM) [6] to be able to leverage kinematic insights from screw theory. Although the findings here are confined to a four-bar linkage, the procedure may be readily extended to other (multi-DOF) flexure-based structures as long as the PRBM assumptions are valid.

2 Methods

2.1 Notation

In this paper we will make use of screw theory notation to describe the kinematics and statics of the mechanism. Here, the generalized velocity of body i is denoted by the twist t_i which is the concatenation of the angular velocity ω_i of the body and the linear velocity v_i of the origin of the reference frame. When the body is in a kinematic chain, its twist may be described by the joint velocities $\dot{q}$ and screw vectors s_j of the joints lower in the chain

$$t_i = \begin{bmatrix} \omega_i \\ v_i \end{bmatrix} = \begin{bmatrix} s_1 \cdots s_i \ 0 \end{bmatrix} \dot{q}, \qquad s_i = \begin{bmatrix} n_z \\ o_i \times n_z \end{bmatrix}. \tag{1}$$

Here, the screw vectors are the twist of the connected bodies if only that particular joint rotates with unit velocity. In this paper we confine ourselves to planar revolute joints where n_z is the out-of-plane unit vector and o_i is the joint location.

The acceleration twist of a body under constant joint velocities $\ddot{q} = 0$ may be expressed similarly

$$\dot{t}_i = \begin{bmatrix} \dot{s}_1 \cdots \dot{s}_i \ 0 \end{bmatrix} \dot{q}, \qquad \dot{s}_i = \sum_{j=1}^{i-1} \mathrm{ad}(s_j) s_i \dot{q}_j, \tag{2}$$

where s_j are the screw vectors of the joints lower in the chain. The $\mathrm{ad(s_j)}$ denotes the adjoint twist representation of s_j

$$\mathrm{ad}(s_j) = \begin{bmatrix} [n_z\times] & 0 \\ [(o_j \times n_z)\times] & [n_z\times] \end{bmatrix}, \tag{3}$$

and $[a\times]$ is the skew symmetric matrix.

2.2 Kinematics of the Four-Bar Mechanism

The four-bar linkage under investigation (Fig. 1) has four revolute joints and one degree of freedom denoted by x. To study the static balance conditions we need the relation between the joint angles on velocity level, i.e. Jacobian J, and on acceleration level, i.e. Hessian H. These may be obtained from the velocity loop closure condition matrix F. This states that the fourth body is fixed and its twist is zero

$$t_4 = FJ\dot{x} = 0, \qquad F = \begin{bmatrix} s_1 \cdots s_4 \end{bmatrix}, \qquad \dot{q} = J\dot{x}. \tag{4}$$

The Jacobian follows from the choice of independent coordinates. Here we choose the angle of the first joint as independent coordinate $x = q_1$. The twist of the coupler t_2 passes through pole g, while the difference in twist between the crank and the follower $t_1 - t_3$ passes through pole h.

The Hessian H may obtained by equating the acceleration of the loop closure to zero with the condition that $\ddot{x} = 0$

$$\frac{\partial}{\partial x}(FJ) = \underbrace{\sum_{i=1}^{4} \frac{\partial}{\partial q_i}(F)J\frac{\partial q_i}{\partial x}}_{a_v} + FH = 0, \qquad H = \frac{\partial}{\partial x}(J), \tag{5}$$

Although this equation looks daunting it permits a geometrical interpretation. The a_v is the acceleration violation of the loop closure constraint (in this case, the acceleration of body 4) if the joint velocities were kept constant, i.e. if $\ddot{q} = 0$. The Hessian part FH then compensates this acceleration in order to satisfy the loop closure under acceleration. The a_v term may be found by observing that

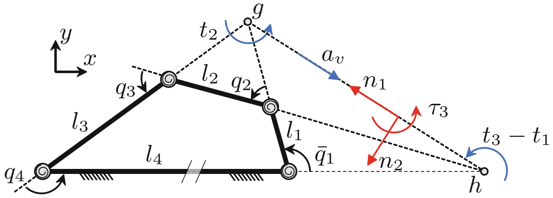

Fig. 1. The definition of a pseudo rigid model of a flexure-based four-bar linkage where all joints are modelled as pin joints with rotational springs.

only the location of joint 2 and 3 are changing, depending on the velocity of joints 1 and 4 respectively. By including the actual twists of the bodies i.e. $s_1\dot{q}_1 = t_1$, $s_2\dot{q}_2 = t_2 - t_1$, $s_3\dot{q}_3 = t_3 - t_2$, and $s_4\dot{q}_4 = t_3$ we find the violation term to be

$$a_v = \mathrm{ad}(s_1)s_2\dot{q}_1\dot{q}_2 + \mathrm{ad}(s_3)s_4\dot{q}_3\dot{q}_4 = \mathrm{ad}(t_2)(t_1 - t_3) = \alpha_v \begin{bmatrix} 0 \\ n_1 \end{bmatrix}. \tag{6}$$

This is a pure translational acceleration along vector n_1, which points from pole g to pole h (Fig. 1). When either g or h is at infinity this n_1 can still be found by inspecting the limit case. Yet, when both g and h are at infinity, i.e. the mechanism is a parallelogram, the Hessian H becomes zero and a_v is zero as the crank and the follower have the same velocity $t_1 - t_3 = 0$.

2.3 Static Balance Conditions

A mechanism is statically balanced when the potential energy V is constant for any motion. The potential energy of a mechanism with compliance in the joints is given by

$$V = \sum_{i=1}^{4} \frac{1}{2} k_i (e_i + q_i)^2 = \sum_{i=1}^{4} \frac{1}{2k_i} (p_i + k_i q_i)^2, \tag{7}$$

where q_i is the deflection of each joint, e_i its rest length/angle, and k_i its actuation stiffness. Later on we use the preload $p_i = k_i e_i$ in the joint as a design variable. The flexure joints are here assumed to be ideal joints with no pivot shift, constant actuation stiffness and no load dependencies. Notice also that we assume that gravity and other conservative forces do not play a role.

In this paper we are interested in a Taylor-like approximate balance where we set the first (resultant force/torque) and second derivatives (effective stiffness) of the potential energy with respect to the generalized coordinates x to zero. For first-order approximate static balance it is then required that the system is 1) in equilibrium and 2) that this equilibrium is locally indifferent i.e. that the stiffness is zero. Therefore, we impose two conditions on the potential energy. The "equilibrium condition" reads

$$\frac{\partial V}{\partial x} = \sum_{i=1}^{4} \frac{\partial V}{\partial q_i} \frac{\partial q_i}{\partial x} = \sum_{i=1}^{4} (p_i + k_i q_i) \frac{\partial q_i}{\partial x} = \sum_{i=1}^{4} p_i \frac{\partial q_i}{\partial x} = 0, \tag{8}$$

where we evaluated this condition around the origin ($x = 0$ and $q = 0$). In here the partial derivate $\frac{\partial q_i}{\partial x}$ are the elements of J, the joint Jacobian.

The "zero stiffness condition" is given by the second derivative of V

$$\frac{\partial^2 V}{\partial x^2} = \sum_{i=1}^{4} k_i \left(\frac{\partial q_i}{\partial x} \right)^2 + p_i \frac{\partial^2 q_i}{\partial x^2} = 0, \tag{9}$$

where $\frac{\partial^2 q_i}{\partial x^2}$ are the elements of H, the Hessian. Here it should be noticed that the mixed derivatives vanish since $\frac{\partial^2 V}{\partial q_i \partial q_j} = 0$ for $i \neq j$, while $\frac{\partial^2 V}{\partial q_i^2} = k_i$.

These balance conditions are linear in the parameters p_i and k_i, which may therefore be directly solved for using linear algebra, particularly pseudo-inverse and null-space operations. In matrix format these conditions read

$$J^\top p = 0, \qquad H^\top p + (J^{\circ 2})^\top k = 0, \tag{10}$$

where $p^\top = \begin{bmatrix} p_1 \cdots p_4 \end{bmatrix}^\top$ is the array of preloads in the joints, $k^\top = \begin{bmatrix} k_1 \cdots k_4 \end{bmatrix}^\top$ is an array with the joint stiffnesses, and $J^{\circ 2}$ denotes the element-wise square of the Jacobian.

It may be noticed that the positive elastic stiffness induced by k is to be balanced by the preload p. This negative stiffness leverages the non-linearities in the geometric transfer function as captured by the Hessian. This indicates that if this non-linearity is weak, a large preload is required for static balance. It should be emphasized that by selection of coordinates and reference frame, the origin $q = 0$ can always be selected as the evaluation point.

2.4 Static Balance Solution

Although the solution for p may be readily found through linear algebra, it may offer little insight in the meaning of the solution space of p. In this section we include the loop closure conditions F to retain some of this insight which might help to find the desired solution. The first-order loop closure condition denotes that a wrench w on the constrained body does not generate any work. Therefore without loss of generality we may therefore select the preload as

$$p = F^\top w, \tag{11}$$

where w is our new design parameter. It may be observed that w is the reaction wrench induced by p at the loop closure. So, this is the wrench required to keep the loop intact. Therefore, with Eq. 4 we eliminate the equilibrium condition of Eq. 10 and we obtain the following "zero-stiffness" condition

$$(FH)^\top w + (J^{\circ 2})^\top k = 0. \tag{12}$$

The FH term is the opposite of the acceleration violation term (Eq. 5 and Eq. 6) and the zero-stiffness condition finally reads

$$-a_v^\top w + (J^{\circ 2})^\top k = 0. \tag{13}$$

Clearly, if the closure reaction wrench w is selected to be orthogonal to a_v, the preload does not affect the stiffness of the system. However, if w is selected along a_v a geometric stiffness is induced and the effective stiffness of the four-bar is altered or even eliminated. We define the matrix W to span the reaction wrenches that are orthogonal to a_v. Finally, for first-order static balance we select w as

$$w = \frac{(J^{\circ 2})^\top k}{|a_v|^2} a_v + W\gamma, \qquad W = \begin{bmatrix} 0 & n_z \\ n_2 & 0 \end{bmatrix}, \qquad a_v^\top W = 0. \tag{14}$$

This solution approach shows that a parallelogram cannot be statically balanced through this approach as its a_v and Hessian are zero. Therefore, it does not possess sufficient non-linearity for the preload to act on. Other types of four-bar do permit such a static balance approach.

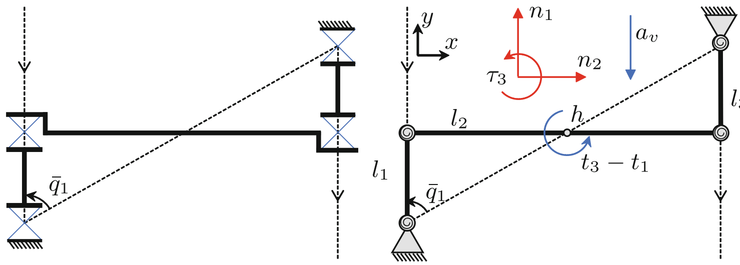

Fig. 2. The flexure-based Watt's linkage (left) consisting of cross-flexure pivots and its pseudo-rigid body model (right).

3 Case Study: Balancing the Watt's Linkage

To show the effectiveness of the static balance procedure we will balance a particular four-bar mechanism, the Watt's linkage (Fig. 2). This mechanism can be used as an approximate straight line guidance in precision applications. The joints consist of cross-flexures that have a maximum bending angle of 8°. The dimensions and design parameters can be found in Table 1. We first derive the PRBM balance conditions after which we simulate the PRBM as well as a fully non-linear flexible multibody system (FMBS) describing the mechanism for different preloads.

Due to the simple structure the Jacobian and the Hessian are

$$F = \begin{bmatrix} n_z & n_z & n_z & n_z \\ -l_1 & 0 & 0 & l_1 \\ -l_2/2 & l_2/2 & l_2/2 & l_2/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad J = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \qquad H = 2\begin{bmatrix} 0 \\ -l_1/l_2 \\ -l_1/l_2 \\ 0 \end{bmatrix}. \tag{15}$$

Table 1. Parameters of the statically balanced flexure-based Watt's linkage.

Name	Symbol	Value	Unit
Crank and follower length	$l_1 = l_3$	100	mm
Coupler length	l_2	200	mm
Base length	l_4	282	mm
Flexure thickness	t_f	0.40	mm
Flexure length	l_f	7.07	mm
Flexure width	w_f	6.66	mm
Young's modulus	E	200	MPa
Joint stiffness	k_i	1.00	Nm/rad
Midpoint displacement	x	14.0	mm
PRMB preload	$p_2 = -p_3$	−2.00	Nm
FMBS preload	$p_2 = -p_3$	−2.15	Nm

As the quadratic Jacobian is an all-ones matrix, the original unbalanced stiffness of the system is $k = \sum k_i$. The acceleration violation twist is $a_v^\top = -2l_1 \begin{bmatrix} 0^\top & n_y^\top \end{bmatrix}$ such that reaction wrench w and the corresponding torsional preload p become

$$w = -\frac{k}{2l_1}\begin{bmatrix} 0 \\ n_y \end{bmatrix} + \begin{bmatrix} 0 & n_z \\ n_x & 0 \end{bmatrix}\gamma, \qquad p = \frac{kl_2}{4l_1}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \gamma_1 + l_1\gamma_2 \\ \gamma_1 \\ \gamma_1 \\ \gamma_1 - l_1\gamma_2 \end{bmatrix}. \tag{16}$$

Figure 3 shows PRMB and FMBS simulation results of a flexure-based Watt's linkage under no load condition and under PRBM static balance preload. Preload in base joints is set to zero $p_1 = p_4 = 0$. The mechanism is actuated by prescribing an x–displacement of the coupler's midpoint. It may be clearly seen that the maximum force required for deflection is reduced significantly from a maximum of 5.7 N to a maximum of 9.9 mN for the PRBM. This implies a reduction of a factor more than 500. When the preload computed for the PRBM is applied in the FMBS-model the balance quality is significantly less, i.e. a maximum force of 0.4 N and reduction factor of only 14. A potential explanations for this difference stems from the fact that PRBM does not take pivot-shift into account, leading to a significant different kinematic behaviour at this stroke. When the preload is slightly increased, the FMBS preload in Table 1, static balance is restored and the maximum force is reduced to 7.5 mN, a reduction factor of 760.

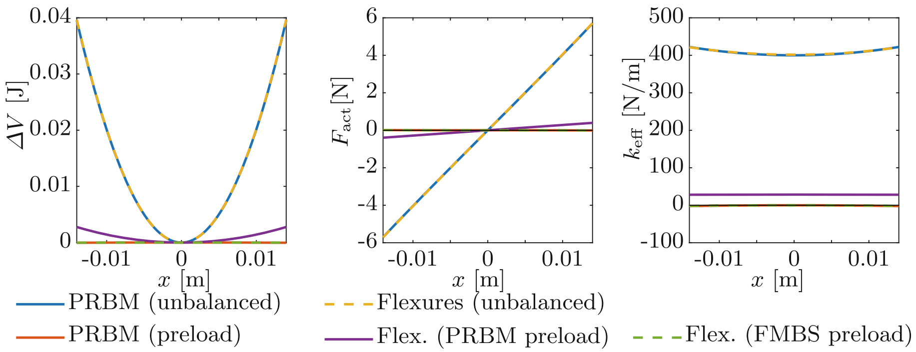

Fig. 3. The potential energy (left), actuator force (middle) and effective stiffness (right) of the flexure-based Watt's linkage with and without preload.

4 Conclusion

This paper untangles the paradoxical behaviour of static balance where higher preloads may lead to less driving torques. It is observed that if the preload is applied in a particular manner, the change in geometry results in a negative stiffness effect. To alter the effective stiffness, the preload-induced reaction forces at the loop closure should be directed along the acceleration violation twist vector, i.e. the acceleration of the loop closure constraint when joint-velocities are kept constant. Any preload that causes reaction force in another direction does not change the effective stiffness. The magnitude of these non-linear effects is inversely proportional to the required preload; highly linear systems require large preloads. In particular, the parallelogram cannot be statically balanced with just torsionally preloaded joints as it has a constant linear relation between the joint velocities. It is shown that the validity of the PRBM assumptions are critical in the effectiveness of the proposed approach. A more advanced FMBS analysis may be needed to account for e.g. pivot-shift. Numerical simulation of a compliant Watt's linkage show a potential actuation force reduction of more than 99.8%, indicating that static balance is obtained.

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