Queueing Systems
https://doi.org/10.1007/s11134-022-09804-4

Birth–death fluid queues and orthogonal polynomials

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Received: 6 February 2022 / Accepted: 28 February 2022

1 Introduction We consider a fluid queue as, e.g., in [1], but driven by a birth–death process $\{X(t), t \geq 0\}$ which has an infinite state space $\mathbb{N} = \{0, 1, \ldots\}$. The birth and death rates λ_i and μ_{i+1}, $i \in \mathbb{N}$, are such that $p_i = \lim_{t\to\infty} P[X(t) = i] = \pi_i / \left(\sum_{j\in\mathbb{N}} \pi_j\right)$ exists, where

$$\pi_i = \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}}, \; i \in \mathbb{N}, \qquad \text{with} \sum_{i\in\mathbb{N}} \pi_i < \infty \;\text{ and }\; \sum_{i\in\mathbb{N}} (\lambda_i \pi_i)^{-1} = \infty.$$

Let $C(t) \geq 0$ be the content of the fluid queue at time t. We denote the fluid rates by r_i, $i \in \mathbb{N}$, that is, we have $d/dt\; C(t) = r_i$ at times t when $X(t) = i$ (unless $C(t) = 0$ and $r_i < 0$). Assuming that $\sum_{i\in\mathbb{N}} r_i \pi_i < 0$, the joint process $\{(X(t), C(t)), t \geq 0\}$ is known to be positive recurrent so that its stationary distribution exists, given by the functions

$$F_i(y) \equiv \lim_{t\to\infty} P[X(t) = i,\; C(t) \leq y], \qquad y \geq 0,\; i \in \mathbb{N},$$

and our goal is to find expressions for these functions in terms of λ_i, μ_{i+1} and r_i, $i \in \mathbb{N}$.

If the state space is finite, say $\{0, \ldots, N-1\}$ for some $N \geq 2$, the $F_i(y)$ can be found by solving a system of linear differential equations, given by $F'(y) = F(y)QR^{-1}$, where $F(y)$ is a row vector with entries $F_i(y)$, Q is the (tri-diagonal) generator matrix of $\{X(t)\}$, and $R = \mathrm{diag}(r_0, \ldots, r_{N-1})$; we assume $r_i \neq 0$ for all i, so that R^{-1} exists. Thus, the solution $F(y)$ can be expressed in terms of eigenvalues and (left) eigenvectors of QR^{-1}, with coefficients determined by boundary conditions, and we might wonder what happens as N increases to infinity.

This work is dedicated to the memory of Erik A. van Doorn (August 12, 1949–November 1, 2019). For an obituary and some remembrances of him and his work, see [3, 5]. His encouraging supervision and close involvement resulted among other things in [7] and [6]. I will always be grateful I had the privilege of being one of his students.

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Published online: 28 April 2022

In [6, 7], it was shown how the birth–death structure of $\{X(t)\}$ in many cases allows structural expressions for $F_i(y)$ by using the theory of orthogonal polynomials (inspired by results in [4]). Interestingly, the form of these expressions varies considerably, depending on N_+, the number of so-called upstates (i.e., states i for which $r_i > 0$), and N_-, the number of downstates (states i for which $r_i < 0$). It is clear that at least one of N_+ and N_- equals infinity, and the structural results in [6, 7] are precisely about the cases in which one of them is finite (and the other is infinite).

When we have finitely many upstates ($N_+ < \infty$), the solution can be found by taking a limit $N \to \infty$ in a suitable sequence of finite-state models with state space $\{0, 1, \dots, N-1\}$. The expression for $F_i(y)$ then takes the form of a finite summation, as in

$$F_i(y) = p_i + \sum_{j=0}^{N_+-1} c_j v_i^{(j)} \exp\{\xi_j y\}. \tag{1}$$

Here, ξ_j are limits of eigenvalues of QR^{-1} in the finite-state models, $v_i^{(j)}$ are elements of the corresponding limiting left 'eigenvectors', and the c_j are coefficients that can be determined from boundary conditions. See [6,Section 2.4] for more details, where also an example is presented with $N_+ = 1$ and explicit expressions for ξ_0, $v_i^{(0)}$ and c_0.

On the other hand, when we have finitely many downstates ($N_- < \infty$) the analysis is more involved and we end up with an integral expression for $F_i(y)$ given by

$$F_i(y) = p_i + \frac{\pi_i}{r_i} \int_{-\infty}^{0-} e^{xy} P_i(x) R(x) \psi(\mathrm{d}x). \tag{2}$$

Here, the $P_i(x)$, $i \in \mathbb{N}$, are polynomials that are orthogonal with respect to ψ, which is actually a signed measure (due to the negative entries in R, unlike the situation in [4]). The function $R(x)$ is a linear combination of the N_- polynomials $P_i(x)$ corresponding to downstates. See [6,Section 2.5] for more details, where also an example is presented with $N_- = 1$ and explicit expressions for $P_i(x)$, ψ and $R(x)$.

2 Problem statement Although the presentation in [6, 7] was quite satisfying, some questions remained:

- For the case $N_- < \infty$, can examples (i.e., sequences λ_i, μ_{i+1} and r_i, $i \in \mathbb{N}$) be found, other than the one presented in [6, 7], for which the corresponding signed measure, and hence $F_i(y)$, can be found explicitly?
- Can a more generic structural result for $F_i(y)$ be given along the lines of [6, 7], not necessarily assuming that either $N_+ < \infty$ as in (1) or $N_- < \infty$ as in (2)?

3 Discussion

Examples for the case $N_- < \infty$

In the example in [6] where ψ is identified, $\{X(t)\}$ is the queue size of an $M/M/1$ queue, so $\lambda_i \equiv \lambda$ and $\mu_{i+1} \equiv \mu$, $i \in \mathbb{N}$, while $r_0 < 0$ and $r_i \equiv r > 0$, $i \neq 0$. For that case, the polynomials $P_i(x)$ can be transformed to a sequence of perturbed Chebyshev polynomials, for which the orthogonalizing (positive) measure is known using results in [2]. Inverse transformation then gives the form of the (signed) measure ψ, and hence the form of $F_i(y)$. Interestingly, in this particular example the support of ψ on $\mathbb{R}^-$ has a continuum separated from zero and possibly a single isolated point mass. For other parameters λ_i, μ_{i+1} and r_i, $i \in \mathbb{N}$, one might follow the same procedure and hope to find (transformed) polynomials with a known corresponding (positive) measure. If parameters can be found for which this works, the results will probably be interesting since many different shapes of ψ, and thus outcomes for $F_i(y)$, could emerge.

Generic result, including the case $N_+ = N_- = \infty$

Though the expressions in (1) and (2) look different, each of them can be seen as a limit for $N \to \infty$ in some sequence of fluid queues with finite state space $\{0, 1, \ldots, N-1\}$ while fixing, respectively, N_+ or N_- to some (finite) value. In each of these models, the solution is given in the same form as (1), since we always have a single eigenvalue 0 (corresponding to the term p_i), N_+ negative eigenvalues (corresponding to the summation), and $N_- - 1$ positive eigenvalues (which do not play a role in the solution). This explains the form of (1) when fixing N_+ and letting N (and hence N_-) grow large, but it also explains the form of (2) when fixing N_- and letting N (and hence N_+) grow large. In the latter case, we have a fixed number of positive eigenvalues (which do not play a role in the solution), and an increasing number of negative eigenvalues, which form the negative part of the support of the corresponding measure as $N \to \infty$.

In fact, (1) could be viewed as a special case of (2), noting that when $N_+ < \infty$ the negative part of the support of ψ only consists of N_+ (limits of) eigenvalues. For each of these values, say ξ_j, $j = 0, \ldots, N_+ - 1$, the role of the components $v_i^{(j)}$ is then played by the constants $\frac{\pi_i}{r_i} P_i(\xi_j)$, and the role of the coefficient c_j by $R(\xi_j)\psi(\{\xi_j\})$. Thus, one might expect that also for the case $N_+ = N_- = \infty$ an expression such as (2) would hold.

References

1. Anick, D., Mitra, D., Sondhi, M.: Stochastic theory of a data-handling system with multiple sources. Bell Syst. Tech. J. **61**(8), 1871–1894 (1982)
2. Chihara, T.: An Introduction to Orthogonal Polynomials. Gordon and Breach, New York (1978)
3. Chihara, T., Scheinhardt, W., Coolen, F., Littlejohn, L., Marcellán, F.: Some remembrances of Erik van Doorn. Newsletter of the SIAM Activity Group on OPSF **27**(1), 6–10 (2020). https://math.nist.gov/opsf/nl271.pdf
4. Karlin, S., McGregor, J.: The differential equations of birth-and-death processes, and the Stieltjes moment problem. Trans. Am. Math. Soc. **85**, 489–546 (1957)
5. Littlejohn, L.: Obituary for Erik van Doorn. Newsletter of the SIAM Activity Group on OPSF **26**(6), 6–7 (2019). https://math.nist.gov/opsf/nl266.pdf
6. Scheinhardt, W.: Markov-modulated and Feedback Fluid Queues. Ph.D. thesis, University of Twente, The Netherlands (1998)
7. van Doorn, E.A.., Scheinhardt, W.R.W.: A fluid queue driven by an infinite-state birth–death process. In: Ramaswami, V., Wirth, P. (eds.) Teletraffic Contributions for the Information Age, Proceedings of ITC 15, pp. 465–475. Elsevier, Amsterdam (1997)