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A FLUID RESERVOIR REGULATED BY A BIRTH-DEATH PROCESS

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ABSTRACT

This paper is concerned with a reservoir which receives and releases fluid flows at variable rates such that the net input rate of fluid is uniquely determined by the state of a birth-death process. We derive some structural properties which generalize and justify earlier findings for special cases of the model, and obtain explicit results for the equilibrium distribution of the content of the reservoir.

1. INTRODUCTION

We consider a reservoir with infinite capacity which receives and releases fluid flows at variable rates in such a way that the net input rate of fluid into the reservoir (which is negative when fluid is flowing out of the reservoir) is uniquely determined by the state of a birth-death process. That is, the net input rate is constant during an interval of time which corresponds to the sojourn time of the associated birth-death process in a state, with the evident restriction that the content of the reservoir cannot decrease whenever the reservoir is empty.

Several recent papers ([1], [4], [8], [9]) deal with models which fit into the general setting described above. By way of illustration we shall briefly describe two examples. First, Gaver and Lehoczky [4] study an integrated circuit- and packet-switched multiplexer through which voice and data traffic can be transmitted on the same link in a communications network. They propose a fluid flow approximation for data traffic which is stored in a buffer until capacity is available on the link to transmit the data. The input rate of data into the buffer is assumed constant, but the output rate varies with the number of voice conversations carried on the link. The number of conversations is modelled as a birth-death process where the birth rate is constant and the death rate depends linearly on the state of the process.

Secondly, Anick et al. [1] describe a fluid flow model for a buffer which receives messages from a finite number of identical information sources that asynchronously alternate between exponentially distributed periods in the "on" and "off" states. While on, a source transmits at a uniform rate. The buffer depletes through an output channel with a fixed rate of transmission. So here the output rate is constant while the input rate varies with the number of information sources in the "on" state. The latter is a birth-death process where birth rates as well as death rates depend linearly on the state of the process.

Both of these papers are concerned with determining the distribution of the buffer content in equilibrium. A crucial step in obtaining this distribution is the observation that the system's eigenvalues (to be defined later) are real and that there are as many strictly negative eigenvalues as there are states of the birth-death process for which the net input rate is positive. Anick et al. [1] proved this result by exploiting the specific structure of the birth and death rates in their model; in [4] the result is mentioned without proof. The main purpose of this paper is to show that this key-result is valid for any set of birth and death rates, see Section 3. This enables us in Section 4 to obtain a scheme for calculating the continuous-time equilibrium distribution of the content of the reservoir in our more general setting. Exploiting a conditional variant of PASTA (Poisson Arrivals See Time Averages), we relate the equilibrium distribution at epochs where the birth-death process changes states to the continuous-time distribution in Section 5. Section 2 is devoted to preliminary results, and Section 6 to numerical aspects. Finally, Section 7 contains some concluding remarks, in particular with regard to generalizations of our model.

2. PRELIMINARIES

Let us denote by $\{X(t),\ t\geq 0\}$ the birth-death process which regulates the content of the reservoir and let $S \equiv \{0,1,\ldots,N\}$ be its state space. The generator of the process will be denoted by Q, that is

$$Q \equiv \begin{pmatrix} -\lambda_0 & \lambda_0 & & & & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & & \bigcirc & \\ & \mu_2 & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & \bigcirc & & & \mu_{N-1} & -(\lambda_{N-1}+\mu_{N-1}) & \lambda_{N-1} \\ & & & & & \mu_N & -\mu_N \end{pmatrix}$$

where $\lambda_i > 0$ is the birth rate in state $i \in S\backslash\{N\}$ and $\mu_i > 0$ the death rate in state $i \in S\backslash\{0\}$. It will be convenient to define $\lambda_{-1} = \lambda_N = \mu_0 = \mu_{N+1} = 0$. We let

$$\pi_i \equiv \prod_{j=1}^{i} \frac{\lambda_{j-1}}{\mu_j}, \quad i \in S,$$

where the empty product is interpreted as unity. The vector $\mathbf{p} \equiv (p_0, p_1, \ldots, p_N)^T$ of stationary state probabilities of the birth-death process satisfies $Q^T\mathbf{p} = \mathbf{0}$ and $\mathbf{1}^T\mathbf{p} = 1$, whence

$$p_i = \pi_i \Big(\sum_{j=0}^{N} \pi_j \Big)^{-1}, \quad i \in S. \tag{1}$$

Here $\mathbf{0}$ and $\mathbf{1}$ denote the column vectors with all components equal to 0 and 1, respectively, and a superscript T denotes transpose.

Whenever $X(t) = i$, $i \in S$, the net input rate of fluid into the reservoir is r_i. We assume that a state $k \in S\backslash\{N\}$ exists such that $r_i < 0$ if $i \in S^- \equiv \{0,1,\ldots,k\}$ and $r_i > 0$ if $i \in S^+ \equiv \{k+1,k+2,\ldots,N\}$. The content of the reservoir at time t is denoted by $C(t)$.

In order that a limit distribution for $C(t)$ exists as $t \to \infty$, the stationary net input rate should be negative, that is, $\sum p_i r_i < 0$, or, equivalently,

$$\sum_{i=0}^{N} \pi_i r_i < 0. \tag{2}$$

We shall assume throughout that this stability condition is satisfied.

With $F_i(t,u)$, $i \in S$, $t \geq 0$, $u \geq 0$, denoting the probability that at time t the birth-death process is in state i and the content of the reservoir does not exceed u, it can easily be shown that for $i \in S$, $t \geq 0$, $u > 0$ and $h > 0$

$$F_i(t+h,u) = \big(1-h(\lambda_i+\mu_i)\big)F_i(t,u-r_i h) + \lambda_{i-1}hF_{i-1}(t,u) + \mu_{i+1}hF_{i+1}(t,u) + o(h),$$

where $F_i(t,u) \equiv 0$ if $i \notin S$. Passing to the limit $h \to 0$ we obtain the forward Kolmogorov equations

$$\frac{\partial F_i(t,u)}{\partial t} = -r_i\frac{\partial F_i(t,u)}{\partial u} - (\lambda_i+\mu_i)F_i(t,u) + \lambda_{i-1}F_{i-1}(t,u) + \mu_{i+1}F_{i+1}(t,u), \quad i \in S. \tag{3}$$

Since we are interested in time-independent equilibrium probabilities, we set $\partial F_i(t,u)/\partial t = 0$ in (3) and obtain

$$r_iF_i'(u) = \lambda_{i-1}F_{i-1}(u) - (\lambda_i+\mu_i)F_i(u) + \mu_{i+1}F_{i+1}(u), \quad i \in S, \tag{4}$$

where $F_i(u)$ denotes the equilibrium probability that the birth-death process is in state i and the content of the reservoir does not exceed u, again with the convention $F_i(u) \equiv 0$ if $i \notin S$. In matrix notation (4) may be written as

$$R\mathbf{F}'(u) = Q^T\mathbf{F}(u), \tag{5}$$

where $R \equiv \text{diag}(r_0,r_1,\ldots,r_N)$ and $\mathbf{F}(u) \equiv (F_0(u),F_1(u),\ldots,F_N(u))^T$.

With $\xi_0,\xi_1,\ldots,\xi_N$ denoting the eigenvalues of the matrix $R^{-1}Q^T$ and $\mathbf{y}^{(0)},\mathbf{y}^{(1)},\ldots,\mathbf{y}^{(N)}$ the corresponding right eigenvectors, the solution of (5) is readily seen to be given by

$$\mathbf{F}(u) = \sum_{j=0}^{N} c_j\exp(\xi_j u)\mathbf{y}^{(j)}, \tag{6}$$

provided the eigenvalues are distinct (which they are indeed as we will show). Here c_j, $j \in S$, are constants which are to be determined from boundary conditions.

In the next section we will address the problem of obtaining the eigenvalues and right eigenvectors of $R^{-1}Q^T$.

3. THE EIGENVALUES AND RIGHT EIGENVECTORS OF $R^{-1}Q^T$

Our first task in this section will be to establish the following result.

Theorem 1. The eigenvalues ξ_i, $i \in S$, of the matrix $R^{-1}Q^T$ are real and simple. Ordering them in decreasing order of magnitude, one also has $\xi_N < \xi_{N-1} < \ldots < \xi_{k+1} < \xi_k = 0 < \xi_{k-1} < \ldots < \xi_0$, provided the stability condition (2) is satisfied.

Before we can prove this theorem we must introduce some notation and derive some preparatory results. For any square matrix M we number rows and columns 0,1,2,... and we denote by $M_{i:j}$, $j \geq i$, the principal submatrix of M determined by the rows and columns numbered $i,i+1,\ldots,j$. Now let $A \equiv R^{-1}Q^T$ and define

$$D_{i:j}(x) \equiv \det(xI_{j-i+1} - A_{i:j}), \quad 0 \leq i \leq j \leq N, \tag{7}$$

where I_n is the nxn identity matrix. It will be convenient to let $D_{i:i-1}(x) \equiv 1$. Clearly, the eigenvalues of $R^{-1}Q^T$ are the zeros of $D_{0:N}(x)$.

We shall have use for the recurrence relation

$$D_{0:n}(x) = \left(x + \frac{\lambda_n}{r_n} + \frac{\mu_n}{r_n}\right)D_{0:n-1}(x) - \frac{\lambda_{n-1}\mu_n}{r_{n-1}r_n} D_{0:n-2}(x), \quad n \in S\backslash\{0\}$$

$$D_{0:0}(x) = x + \frac{\lambda_0}{r_0}, \quad D_{0:-1}(x) = 1, \tag{8}$$

which is obtained by expanding $\det(xI_{n+1} - A_{0:n})$ by its last row. From (8) one easily derives by induction that

$$D_{0:n}(0) = \prod_{i=0}^{n} \frac{\lambda_i}{r_i}, \quad n \in S. \tag{9}$$

Analogously, by expanding $\det(xI_{N-n+1} - A_{n:N})$ by its first row one gets a recurrence relation for $D_{n:N}(x)$, $n \in S$, from which one obtains

$$D_{n:N}(0) = \prod_{i=n}^{N} \frac{\mu_i}{r_i}, \quad n \in S. \tag{10}$$

Lemma 1. $D_{0:N}(x)$ has the following properties:

(i) $D_{0:N}(0) = 0$;

(ii) $D'_{0:N}(0) = \frac{1}{r_0}\left(\prod_{i=1}^{N}\frac{\mu_i}{r_i}\right)\sum_{n=0}^{N}\pi_n r_n$.

Proof. Assertion (i) follows immediately from (9) or (10) (recall that $\mu_0 = \lambda_N = 0$). Next observe that

$$D'_{0:N}(0) = \sum_{n=0}^{N}(\text{nth principal minor of } -A)$$

$$= \sum_{n=0}^{N} D_{0:n-1}(0)D_{n+1:N}(0),$$

which upon substitution of (9) and (10) yields the second property. □

Lemma 2. (i) The zeros of $D_{0:k}(x)$ and $D_{0:k-1}(x)$ are real, simple and strictly positive; the zeros of $D_{0:k-1}(x)$ strictly separate those of $D_{0:k}(x)$.

(ii) The zeros of $D_{k+1:N}(x)$ and $D_{k+2:N}(x)$ are real, simple and strictly negative; the zeros of $D_{k+2:N}(x)$ strictly separate those of $D_{k+1:N}(x)$.

Proof. Assertion (i), with the exception of the positivity of the zeros, follows from the fact that $A_{0:k}$ is a sign-symmetric tri-diagonal matrix ([6, p.166] or [3]). To prove positivity we let

$$\tilde{A}_{0:j} = E_j A_{0:j} E_j^{-1}, \quad j \in S^-,$$

where $E_j \equiv \text{diag}(e_0, e_1, \ldots, e_j)$, $e_0 = 1$ and $e_{i+1} = e_i\sqrt{\mu_{i+1}r_{i+1}}/\sqrt{\lambda_i r_i}$, $i = 0,1,\ldots,j-1$. It is easily seen that $\tilde{A}_{0:j}$ is symmetric, while

$$\det(\tilde{A}_{0:j}) = \det(A_{0:j}) = (-1)^{j+1}D_{0:j}(0) > 0, \quad j \in S^-,$$

by (9) (recall that $r_i < 0$ for $i \in S^-$). We conclude that $\tilde{A}_{0:k}$ is positive definite [6, p.70]. Since $A_{0:k}$ and $\tilde{A}_{0:k}$ are similar, it follows that the eigenvalues of $A_{0:k}$, and hence the zeros of $D_{0:k}(x)$, are positive.

Assertion (ii) is proven similarly with the help of (10). □

We shall denote the zeros of $D_{0:k}(x)$ by α_j, $j = 0,1,\ldots,k$, those of $D_{k+1:N}(x)$ by α_j, $j = k+1,k+2,\ldots,N$, those of $D_{0:k-1}(x)$ by β_j, $j = 0,1,\ldots,k-1$, and those of $D_{k+2:N}(x)$ by β_j, $j = k+2,k+3,\ldots,N$, each time in decreasing order of magnitude. Lemma 2 then tells us that

$$\begin{aligned}\alpha_N < \beta_N < \alpha_{N-1} < \ldots < \alpha_{k+2} < \beta_{k+2} < \alpha_{k+1} < 0 \\ 0 < \alpha_k < \beta_{k-1} < \alpha_{k-1} < \ldots < \alpha_1 < \beta_0 < \alpha_0 .\end{aligned} \tag{11}$$

We are now ready to prove Theorem 1. By Laplace's expansion theorem [6, p.14] we have

$$\begin{aligned}D_{0:N}(x) &= D_{0:k}(x)D_{k+1:N}(x) - \frac{\lambda_k\mu_{k+1}}{r_k r_{k+1}} D_{0:k-1}(x)D_{k+2:N}(x) \\ &= \prod_{i=0}^{N}(x-\alpha_i) - \frac{\lambda_k\mu_{k+1}}{r_k r_{k+1}} \prod_{\substack{i=0 \\ i\neq k,k+1}}^{N} (x-\beta_i) .\end{aligned} \tag{12}$$

Recalling that $r_k r_{k+1} < 0$, we see from (12) that

$$\operatorname{sgn}(D_{0:N}(\alpha_i)) = -\operatorname{sgn}(D_{0:N}(\beta_i)), \quad i \in S\backslash\{k,k+1\}, \tag{13}$$

that is, $D_{0:N}(x)$ has a negative zero in each of the N-k-1 intervals (α_i,β_i), $i = k+2,k+3,\ldots,N$, and a positive zero in each of the k intervals (β_i,α_i), $i = 0,1,\ldots,k-1$. By Lemma 1 (i), $D_{0:N}(0) = 0$ as well, so $D_{0:N}(x)$ has at least N real zeros, and therefore all of its N+1 zeros are real. It remains to determine the sign of the unlocated zero. Recalling the stability condition (2), we see from

Lemma 1 (ii) that $\mathrm{sgn}[D'_{0:N}(0)] = (-1)^k$. But (12) tells us that $\mathrm{sgn}[D_{0:N}(\alpha_{k+1})] = (-1)^k$ as well. It follows that the unlocated zero must be in the interval $(\alpha_{k+1},0)$ which concludes the proof of Theorem 1.

To obtain the right eigenvectors of the matrix $R^{-1}Q^T$ we argue as follows. Writing

$$
\begin{aligned}
Y_n(x) &= \Big(\prod_{i=0}^{n-1}\frac{r_i}{\mu_{i+1}}\Big)D_{0:n-1}(x), \quad n \in S \\
Y_{N+1}(x) &= \Big(\prod_{i=0}^{N-1}\frac{r_i}{\mu_{i+1}}\Big)D_{0:N}(x),
\end{aligned}
\tag{14}
$$

where, as usual, the empty product is interpreted as unity, we get from (8)

$$
\begin{aligned}
-\lambda_0 Y_0(x) + \mu_1 Y_1(x) &= x r_0 Y_0(x) \\
\lambda_{n-1}Y_{n-1}(x) - (\lambda_n+\mu_n)Y_n(x) + \mu_{n+1}Y_{n+1}(x) &= x r_n Y_n(x), \quad n \in S\backslash\{0,N\} \\
\lambda_{N-1}Y_{N-1}(x) - \mu_N Y_N(x) + r_N Y_{N+1}(x) &= x r_N Y_N(x).
\end{aligned}
\tag{15}
$$

Since $Y_{N+1}(\xi_j) = 0$, $j \in S$, it follows that the vectors $\mathbf{Y}(\xi_j) = [Y_0(\xi_j), Y_1(\xi_j), \ldots, Y_N(\xi_j)]^T$ satisfy the relation

$$R^{-1}Q^T\mathbf{Y}(\xi_j) = \xi_j\mathbf{Y}(\xi_j), \quad j \in S,$$

that is, $\mathbf{Y}(\xi_j)$ is a right eigenvector of $R^{-1}Q^T$ corresponding to the eigenvalue ξ_j. It will be convenient to normalize the right eigenvectors such that the first component equals $p_0 = (\sum\pi_i)^{-1}$. Thus we have the following result.

Theorem 2. Let $\mathbf{y}^{(j)} = [y_0^{(j)}, y_1^{(j)}, \ldots, y_N^{(j)}]^T$ denote the right eigenvector of $R^{-1}Q^T$ corresponding to the eigenvalue ξ_j, $j \in S$, normalized such that $y_0^{(j)} = p_0$, then

$$y_n^{(j)} = p_0 Y_n(\xi_j) = p_0 \Big(\prod_{i=0}^{n-1} \frac{r_i}{\mu_{i+1}}\Big) D_{0:n-1}(\xi_j), \tag{16}$$

where $D_{0:n}(x)$, $n \in S$, are the polynomials satisfying the recurrence relation (8).

In passing we note that (16) is very similar to the corresponding result for a symmetric tri-diagonal matrix [10, (7-10-1)]. We finally observe that $\xi_k = 0$ together with (9) imply

$$y_n^{(k)} = p_0 \Big(\prod_{i=0}^{n-1} \frac{r_i}{\mu_{i+1}}\Big)\Big(\prod_{i=0}^{n-1} \frac{\lambda_i}{r_i}\Big) = p_0 \pi_n = p_n, \quad n \in S, \tag{17}$$

so that $\mathbf{y}^{(k)} = \mathbf{p}$, as it should be, for $Q^T\mathbf{p} = \mathbf{0}$ and hence $R^{-1}Q^T\mathbf{p} = \mathbf{0}$. This explains why we have chosen the particular normalization in (16).

4. THE CONTENT OF THE RESERVOIR IN CONTINUOUS TIME

We recall that $F_n(u)$, the continuous-time equilibrium probability that the birth-death process is in state n and the content of the reservoir does not exceed u, satisfies

$$F_n(u) = \sum_{j=0}^{N} c_j y_n^{(j)} \exp(\xi_j u), \quad n \in S, \ u \geq 0, \tag{18}$$

while $\xi_N < \ldots < \xi_{k+1} < \xi_k = 0 < \xi_{k-1} < \ldots < \xi_0$. Evidently, $y_0^{(j)} = p_0 > 0$ and $0 \leq F_0(u) \leq 1$ for all u, so that necessarily

$$c_0 = c_1 = \ldots = c_{k-1} = 0. \tag{19}$$

Furthermore, when u goes to infinity, $F_n(u)$ must tend to p_n, the equilibrium probability that the birth-death process is in state n.

With (17), (18) and (19) it follows that $c_k = 1$, so that

$$F_n(u) = p_n + \sum_{j=k+1}^{N} c_j y_n^{(j)} \exp(\xi_j u), \quad n \in S, \ u \geq 0. \tag{20}$$

The remaining constants $c_{k+1},\ldots,c_N$ are determined by the fact that $F_n(0)$ must be equal to zero if n is one of the states where the net input of fluid into the reservoir is positive, that is, if $n \in S^+ = \{k+1,k+2,\ldots,N\}$. So these constants must be solved from

$$\sum_{j=k+1}^{N} c_j y_n^{(j)} = -p_n, \quad n \in S^+. \tag{21}$$

Now (15) and (16) imply that, for $j \in S$,

$$\begin{aligned} -\lambda_0 y_0^{(j)} + \mu_1 y_1^{(j)} &= \xi_j r_0 y_0^{(j)} \\ \lambda_{n-1} y_{n-1}^{(j)} - (\lambda_n+\mu_n) y_n^{(j)} + \mu_{n+1} y_{n+1}^{(j)} &= \xi_j r_n y_n^{(j)}, \quad n \in S\backslash\{0,N\} \\ \lambda_{N-1} y_{N-1}^{(j)} - \mu_N y_N^{(j)} &= \xi_j r_N y_N^{(j)}. \end{aligned} \tag{22}$$

Specifically, by taking $j = k$ in (22) and using (17) we regain

$$\lambda_{n-1} p_{n-1} - (\lambda_n+\mu_n) p_n + \mu_{n+1} p_{n+1} = 0, \quad n \in S, \tag{23}$$

where $p_i = 0$ if $i \notin S$. With the help of (22) and (23) we can transform the set of equations (21) as follows, where by equation no. i we mean the equation corresponding to $n = i$. First, for each $n \in S^+$, $n > k+1$ we replace equation no. n-1 in (21) by λ_{n-1} times equation no. n-1, minus $\lambda_n+\mu_n$ times equation no. n, plus (if $n < N$) μ_{n+1} times equation no. n+1. With (22) and (23) it then follows that (21) is equivalent to the system

$$\begin{aligned} \sum_{j=k+1}^{N} c_j y_n^{(j)} \xi_j &= 0, \quad n = k+2,k+3,\ldots,N \\ \sum_{j=k+1}^{N} c_j y_N^{(j)} &= -p_N. \end{aligned} \tag{24}$$

Secondly, for each $n \in S^+$, $n > k+2$ we replace equation no. n-1 in (24) by a linear combination of equations nos. n-1, n and n+1 similar to the one in the first step, etc. After N-k-1 such steps it follows that (21) is equivalent to the system

$$\sum_{j=k+1}^{N} c_j y_N^{(j)} \xi_j^i = -p_N \delta_{i0}, \quad i = 0,1,\ldots,N-k-1, \tag{25}$$

which admits the explicit solution (see [1])

$$c_j y_N^{(j)} = -p_N \prod_{\substack{i=k+1 \\ i \neq j}}^{N} \xi_i (\xi_i - \xi_j)^{-1}, \quad j \in S^+. \tag{26}$$

As an aside we remark that the argument used in [1] to obtain (25) can be used in our more general context as well.

5. THE CONTENT OF THE RESERVOIR IN DISCRETE TIME

It may be of interest to know the equilibrium distribution of the content of the reservoir at epochs where the underlying birth-death process makes transitions of a particular type; for instance at epochs where the net input rate of fluid changes from positive to negative, since these epochs mark the end of a build-up period. A direct derivation of results of this type is possible but rather elaborate. We shall see, however, that such equilibrium results in discrete time can readily be obtained from the results of the preceding section with the help of a conditional variant of PASTA (Poisson Arrivals See Time Averages), see [12]. First we need some notation.

Let T_n, $n = 1,2,\ldots,$ denote the epochs at which the birth-death process $\{X(t),\ t \geq 0\}$ makes transitions and let $Y_n \equiv X(T_n-0)$, $n = 1,2,\ldots$. With $\mathbf{q} \equiv (q_0,q_1,\ldots,q_N)^T$ denoting the vector of stationary state probabilities of the Markov chain

$\{Y_n, n = 1,2,\ldots\}$, it is well known that

$$q_i = \frac{(\lambda_i+\mu_i)p_i}{\sum_{j=0}^{N}(\lambda_j+\mu_j)p_j}, \quad i \in S. \tag{27}$$

Next let $D_n = C(T_n-0)$, $n = 1,2,\ldots,$ and $G_i(n,u) = \Pr\{D_n \le u, Y_n=i\}$. As in the continuous-time case we take for granted the intuitively obvious fact that under the stability condition (2) the limit $G_i(u) = \lim_{n\to\infty} G_i(n,u)$ exists and equals the equilibrium probability that, just before a transition epoch, the birth-death process is in state i and the content of the reservoir does not exceed u. Corollary 1 in [12] then tells us that

$$q_i^{-1}G_i(u) = p_i^{-1}F_i(u), \quad i \in S,\ u \ge 0, \tag{28}$$

which is the result we were referring to.

Returning to the example mentioned in the beginning of this section, we see that the distribution of the content of the reservoir at epochs where the birth-death process makes a transition from state k+1 to state k (which, of course, equals the distribution $q_{k+1}^{-1}G_{k+1}(u)$ of the content of the reservoir at epochs where the birth death process leaves state k) can be expressed as $p_{k+1}^{-1}F_{k+1}(u)$.

6. NUMERICAL ASPECTS

To compute the equilibrium probabilities $F_n(u)$ one can of course ignore the fact that the underlying Markov process is a birth-death process and use standard routines from software packages to determine the eigenvalues and eigenvectors of $R^{-1}Q^T$ and solve the system of equations (21). Results of this approach with values of N ranging up to 100 have been reported in the literature [9]. The question is whether something can be gained by exploiting the structural properties of the model at hand. As for the eigenvalues of $R^{-1}Q^T$, it is not difficult to see that for $x \leq 0$ the sequence $D(x) = (D_{0:k}(x), D_{0:k+1}(x), \ldots, D_{0:N}(x))$ resembles a Sturm sequence in that the number of zeros of $D_{0:N}(x)$ (i.e., the number of eigenvalues of $R^{-1}Q^T$) which is smaller than x equals N-k minus the number of sign changes in the sequence D(x). This observation enables us to exploit, in slightly adapted form, the very stable and efficient bisection algorithm [2] to obtain the eigenvalues ξ_j, $j \in S^+$. We wrote a program in BASIC and employed a simple personal computer to execute it with values of N ranging up to 100.

Once the system's eigenvalues have been calculated, one can further exploit the structure of the model at hand, and obtain the remaining quantities via a very simple scheme based on (22) and (26). Unfortunately, for larger values of N the recursion based on (22) becomes numerically unstable. (A similar phenomenon has been observed for symmetric tri-diagonal matrices [10, p. 131].) We found the procedure effective for $N \leq 10$ and sometimes, depending on the values of the other parameters of the model, for N up to 20. In this way we verified the numerical results of Regterschot [11, p.156].

7. CONCLUDING REMARKS

The main features of the models studied in [1], [4] and [9] are the reality of the system's eigenvalues and the fact that precisely N-k of them are negative. We have shown that these features are maintained in our more general setting. Obviously, one loses some explicitness by our generalization. Thus Anick et al. [1] derive a simple expression for the important eigenvalue ξ_{k+1} and the associated eigenvector, an accomplishment which does not seem possible for our model.

Further generalizations of the model in which the background Markov process remains irreducible but is no longer a birth-death process, will generally involve complex eigenvalues. It has been shown, however, that then the real parts of precisely N-k of the eigenvalues are negative (Regterschot [11], Sonneveld [13]), so that it is still possible to obtain the formal solution (20), although the explicit evaluation will be more cumbersome (cf. Kosten [5]). Regterschot [11, Ch.4] chooses an entirely different approach to analyse this generalization of our model by formulating it as a matrix factorization problem.

While completing the revised version of this manuscript we became aware of Mitra's paper [7]. He shows that the eigenvalues of the matrix $R^{-1}Q^{T}$ are all real when the background Markov process is reversible; explicit results like (22) and (26), however, seem out of reach in this more general case.

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