Simple odd β-cycle inequalities for binary polynomial optimization

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Abstract

We consider the multilinear polytope which arises naturally in binary polynomial optimization. Del Pia and Di Gregorio introduced the class of odd β-cycle inequalities valid for this polytope, showed that these generally have Chvátal rank 2 with respect to the standard relaxation and that, together with flower inequalities, they yield a perfect formulation for cycle hypergraph instances. Moreover, they describe a separation algorithm in case the instance is a cycle hypergraph. We introduce a weaker version, called simple odd β-cycle inequalities, for which we establish a strongly polynomial-time separation algorithm for arbitrary instances. These inequalities still have Chvátal rank 2 in general and still suffice to describe the multilinear polytope for cycle hypergraphs.

1 Introduction

In *binary polynomial optimization* our task is to find a binary vector that maximizes a given multivariate polynomial function. In order to give a mathematical formulation, it is useful to use a hypergraph $G = (V, E)$, where the node set V represents the variables in the polynomial function, and the edge set E represents the monomials with nonzero coefficients. In a binary polynomial optimization problem, we are then given a hypergraph $G = (V, E)$, a profit vector $p \in \mathbb{R}^{V \cup E}$, and our goal is to solve the optimization problem

$$\max \left\{ \sum_{v \in V} p_v z_v + \sum_{e \in E} p_e \prod_{v \in e} z_v : z \in \{0,1\}^V \right\}. \tag{1}$$

Using Fortet's linearization [12, 14], we introduce binary auxiliary variables z_e, for $e \in E$, which are linked to the variables z_v, for $v \in V$, via the linear inequalities

$$z_v - z_e \geq 0 \qquad \forall e \in E,\ \forall v \in e \tag{2a}$$

$$(z_e - 1) + \sum_{v \in e} (1 - z_v) \geq 0 \qquad \forall e \in E. \tag{2b}$$

It is simple to see that

$$\left\{ z \in \{0,1\}^{V \cup E} : z_e = \prod_{v \in e} z_v \ \forall e \in E \right\} = \left\{ z \in \{0,1\}^{V \cup E} : (2) \right\}.$$

Hence, we can reformulate (1) as the integer linear optimization problem

$$\max \left\{ \sum_{v \in V} p_v z_v + \sum_{e \in E} p_e z_e : (2),\ z \in \{0,1\}^{V \cup E} \right\}. \tag{3}$$

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We define the *multilinear polytope* $\mathrm{ML}(G)$ [6], which is the convex hull of the feasible points of (3), and its *standard relaxation* $\mathrm{SR}(G)$:

$$\mathrm{ML}(G) := \mathrm{conv}\left\{z \in \{0,1\}^{V \cup E} : (2)\right\},$$
$$\mathrm{SR}(G) := \left\{z \in [0,1]^{V \cup E} : (2)\right\}.$$

Recently, several classes of inequalities valid for $\mathrm{ML}(G)$ have been introduced, including 2-link inequalities [4], flower inequalities [7], running intersection inequalities [8], and odd β-cycle inequalities [5]. On a theoretical level, these inequalities fully describe the multilinear polytope for several hypergraph instances: flower inequalities for γ-acyclic hypergraphs, running intersection inequalities for kite-free β-acyclic hypergraphs, and flower inequalities together with odd β-cycle inequalities for cycle hypergraphs. Furthermore, these cutting planes greatly reduce the integrality gap of (3) [8, 5] and their addition leads to a significant reduction of the runtime of the state-of-the-art solver BARON [9]. Unfortunately, we are not able to separate efficiently over most of these inequalities. In fact, while the simplest 2-link inequalities can be trivially separated in polynomial time, there is no known polynomial-time algorithm to separate the other classes of cutting planes, and it is known that separating flower inequalities is NP-hard [9].

Contribution. In this paper we introduce a novel class of cutting planes called *simple odd β-cycle inequalities*. As the name suggests, these inequalities form a subclass of the odd β-cycle inequalities introduced in [5]. The main result of this paper is that simple odd β-cycle inequalities can be separated in strongly polynomial time. While our inequalities form a subclass of the inequalities introduced in [5], they still inherit the two most interesting properties of the odd β-cycle inequalities. First, simple odd β-cycle inequalities can have Chvátal rank 2. To the best of our knowledge, our algorithm is the first known polynomial-time separation algorithm over an exponential class of inequalities with Chvátal rank 2. Second, simple odd β-cycle inequalities, together with standard linearization inequalities and flower inequalities with at most two neighbors, provide a perfect formulation of the multilinear polytope for cycle hypergraphs. Finally, we believe that our separation algorithm could lead to significant speedups in solving several applications that can be formulated as (1) with a hypergraph that contains β-cycles. These applications include the image restoration problem in computer vision [4, 5], and the low auto-correlation binary sequence problem in theoretical physics [2, 15, 5, 18, 17].

Outline. We first introduce certain simple inequalities in Section 2 that are then combined to form the simple odd β-cycle inequalities in Section 3. Section 4 is dedicated to the polynomial-time separation algorithm. In Section 5 we briefly address the question of redundancy since our inequalities are formally defined for a more general structure than a β-cycle. Finally, Section 6 relates the simple odd β-cycle inequalities to the general (non-simple) odd β-cycle inequalities in [5].

2 Building block inequalities

We consider certain affine linear functions $s : \mathbb{R}^{V \cup E} \to \mathbb{R}$ defined as follows.

$$z_v - z_e \quad \forall e \in E,\ \forall v \in e \qquad (s^{\mathrm{inc}}_{e,v})$$

$$2z_e - 1 + \sum_{u \in U}(1 - z_u) + \sum_{w \in W}(1 - z_w) + \sum_{v \in e \setminus (U \cup W)}(2 - 2z_v) \quad \forall e \in E,\ \forall U, W : \varnothing \neq U, W \subseteq e,\ U \cap W = \varnothing \qquad (s^{\mathrm{odd}}_{e,U,W})$$

$$2z_e - 1 + \sum_{u \in U}(1 - z_u) + (1 - z_f) + \sum_{v \in e \setminus (U \cup f)}(2 - 2z_v) \quad \forall e, f \in E : e \cap f \neq \varnothing,\ \forall U : \varnothing \neq U \subseteq e,\ U \cap f = \varnothing \qquad (s^{\mathrm{one}}_{e,U,f})$$

$$2z_e - 1 + (1 - z_f) + (1 - z_g) + \sum_{v \in e \setminus (f \cup g)}(2 - 2z_v) \quad \forall e, f, g \in E : e \cap f \neq \varnothing,\ e \cap g \neq \varnothing,\ e \cap f \cap g = \varnothing \qquad (s^{\mathrm{two}}_{e,f,g})$$

In this paper we often refer to $s^{\mathrm{inc}}_{e,v}$, $s^{\mathrm{odd}}_{e,U,W}$, $s^{\mathrm{one}}_{e,U,f}$, $s^{\mathrm{two}}_{e,f,g}$ as *building blocks*. Although in these definitions U and W can be arbitrary subsets of an edge e, in the following U and W will always correspond to the intersection of e with another edge. In the next lemma we will show that all building blocks are nonnegative on a relaxation of $\mathrm{ML}(G)$ obtained by adding some flower inequalities [7] to $\mathrm{SR}(G)$, which we will define now. For ease of notation, in this paper, we denote by $[m]$ the set $\{1,\dots,m\}$, for any nonnegative integer m.

Let $f \in E$ and let e_i, $i \in [m]$, be a collection of distinct edges in E, adjacent to f, such that $f \cap e_i \cap e_j = \emptyset$ for all $i, j \in [m]$ with $i \neq j$. Then the *flower inequality* [7, 5] centered at f with neighbors e_i, $i \in [m]$, is defined by

$$(z_f - 1) + \sum_{i \in [m]} (1 - z_{e_i}) + \sum_{v \in f \setminus \cup_{i \in [m]} e_i} (1 - z_v) \geq 0.$$

We denote by $\mathrm{FR}(G)$ the polytope obtained from $\mathrm{SR}(G)$ by adding all flower inequalities with at most two neighbors. Clearly $\mathrm{FR}(G)$ is a relaxation of $\mathrm{ML}(G)$. Furthermore, $\mathrm{FR}(G)$ is defined by a number of inequalities that is bounded by a polynomial in $|V|$ and $|E|$.

Lemma 1. *Let $G = (V, E)$ be a hypergraph and let s be one of $s^{\mathrm{inc}}_{e,v}$, $s^{\mathrm{odd}}_{e,U,W}$, $s^{\mathrm{one}}_{e,U,f}$, $s^{\mathrm{two}}_{e,f,g}$. Then $s(z) \geq 0$ is valid for $\mathrm{FR}(G)$. Furthermore, if $z \in \mathrm{ML}(G) \cap \mathbb{Z}^{V \cup E}$ and $s(z) = 0$, then the implication given in Table 1 holds.*

Table 1: Implications of tight building block inequalities for integer solutions z.

Condition	**Implication**
$s^{\mathrm{inc}}_{e,v}(z) = 0$	$z_v = z_e$
$s^{\mathrm{odd}}_{e,U,W}(z) = 0$	$\prod_{u \in U} z_u + \prod_{w \in W} z_w = 1$
$s^{\mathrm{one}}_{e,U,f}(z) = 0$	$z_f + \prod_{u \in U} z_u = 1$
$s^{\mathrm{two}}_{e,f,g}(z) = 0$	$z_f + z_g = 1$

Proof. First, $s^{\mathrm{inc}}_{e,v}(z) \geq 0$ is part of the standard relaxation and the implication is obvious.

Second, $s^{\mathrm{odd}}_{e,U,W}(z) \geq 0$ is the sum of the following inequalities from the standard relaxation: $z_e \geq 0$, $1 - z_v \geq 0$ for all $v \in e \setminus (U \cup W)$, and $(z_e - 1) + \sum_{v \in e} (1 - z_v) \geq 0$. If $z \in \mathrm{ML}(G) \cap \mathbb{Z}^{V \cup E}$ and $s^{\mathrm{odd}}_{e,U,W}(z) = 0$, then each of these inequalities must be tight, thus $z_e = 0$, $z_v = 1$ for each $v \in e \setminus (U \cup W)$. The last (tight) inequality yields $-1 + \sum_{v \in U \cup W} (1 - z_v) = 0$, i.e., precisely one variable z_v, for $v \in U \cup W$, is 0, while all others are 1, which yields the implication from Table 1.

Third, $s^{\mathrm{one}}_{e,U,f}(z) \geq 0$ is the sum of the following inequalities: $z_e \geq 0$, $1 - z_v \geq 0$ for all $v \in e \setminus (U \cup f)$ and $(z_e - 1) + (1 - z_f) + \sum_{v \in e \setminus f} (1 - z_v) \geq 0$. The latter is the flower inequality centered at e with neighbor f. If $z \in \mathrm{ML}(G) \cap \mathbb{Z}^{V \cup E}$ and $s^{\mathrm{one}}_{e,U,f}(z) = 0$, then each of these inequalities must be tight, thus $z_e = 0$, $z_v = 1$ for each $v \in e \setminus (U \cup f)$. The last (tight) inequality yields $-1 + (1 - z_f) + \sum_{u \in U} (1 - z_u) = 0$, i.e., either $z_f = 1$ and $z_u = 0$ for exactly one $u \in U$, or $z_f = 0$ and $z_u = 1$ holds for all $u \in U$. Both cases yield the implication from Table 1.

Fourth, we consider $s^{\mathrm{two}}_{e,f,g}(z) \geq 0$. Note that due to $e \cap f \neq \varnothing$, $e \cap g \neq \varnothing$ and $e \cap f \cap g = \varnothing$, the three edges e, f, g must all be different. Thus, $s^{\mathrm{two}}_{e,f,g}(z) \geq 0$ is the sum of $z_e \geq 0$, $1 - z_v \geq 0$ for all $v \in e \setminus (f \cup g)$ and of $(z_e - 1) + (1 - z_f) + (1 - z_g) + \sum_{v \in e \setminus (f \cup g)} (1 - z_v) \geq 0$. The latter is the flower inequality centered at e with neighbors f and g. If $z \in \mathrm{ML}(G) \cap \mathbb{Z}^{V \cup E}$ and $s^{\mathrm{two}}_{e,f,g}(z) = 0$ holds, then each of the involved inequalities must be tight, thus $z_e = 0$ and $z_v = 1$ for each $v \in e \setminus (f \cup g)$. The last (tight) inequality implies $-1 + (1 - z_f) + (1 - z_g) = 0$, i.e., $z_f + z_g = 1$. □

3 Simple odd β-cycle inequalities

We will consider signed edges by associating either a "+" or a "−" with each edge. We denote by $\{\pm\}$ the set $\{+, -\}$ and by $-p$ a sign change for $p \in \{\pm\}$. In order to introduce simple odd β-cycle inequalities, we first present some more definitions.

Definition 2. *A* closed walk *in G of length $k \geq 3$ is a sequence $C = v_1\text{-}e_1\text{-}v_2\text{-}e_2\text{-}v_3\text{-}\cdots\text{-}v_{k-1}\text{-}e_{k-1}\text{-}v_k\text{-}e_k\text{-}v_1$, where we have $e_i \in E$ as well as $v_i \in e_{i-1} \cap e_i$ and $e_{i-1} \cap e_i \cap e_{i+1} = \varnothing$ for each $i \in [k]$, where we denote $e_0 \coloneqq e_k$ and $e_{k+1} \coloneqq e_1$ for convenience. A* signature *of C is a map $\sigma : [k] \to \{\pm\}$. A* signed closed walk in G *is a pair (C, σ) for a closed walk C and a signature σ of C. Similarly, we denote $v_0 \coloneqq v_k$, $v_{k+1} \coloneqq v_1$, $\sigma(0) \coloneqq \sigma(k)$ and $\sigma(k+1) \coloneqq \sigma(1)$. We say that (C, σ) is* odd *if there is an odd number of indices $i \in [k]$ with $\sigma(i) = -$; otherwise we say that (C, σ) is* even. *Finally, for any signed closed walk (C, σ) in G, its* length function *is the map $\ell_{(C,\sigma)} : \mathrm{FR}(G) \to \mathbb{R}$ defined by*

$$\begin{aligned}\ell_{(C,\sigma)}(z) \coloneqq &\sum_{i \in I_{(+,+,+)}} \left(s^{\text{inc}}_{e_i,v_i}(z) + s^{\text{inc}}_{e_i,v_{i+1}}(z)\right) + \sum_{i \in I_{(-,-,-)}} s^{\text{odd}}_{e_i,e_i\cap e_{i-1},e_i\cap e_{i+1}}(z) + \sum_{i \in I_{(+,+,-)}} s^{\text{inc}}_{e_i,v_i}(z) \\ &+ \sum_{i \in I_{(-,-,+)}} s^{\text{one}}_{e_i,e_i\cap e_{i-1},e_{i+1}}(z) + \sum_{i \in I_{(-,+,+)}} s^{\text{inc}}_{e_i,v_{i+1}}(z) + \sum_{i \in I_{(+,-,-)}} s^{\text{one}}_{e_i,e_i\cap e_{i+1},e_{i-1}}(z) + \sum_{i \in I_{(+,-,+)}} s^{\text{two}}_{e_i,e_{i-1},e_{i+1}}(z),\end{aligned}$$

where $I_{(a,b,c)}$ is the set of edge indices i for which e_{i-1}, e_i and e_{i+1} have sign pattern $(a,b,c) \in \{\pm\}^3$, i.e., $I_{(a,b,c)} \coloneqq \{i \in [k] : \sigma(i-1) = a,\ \sigma(i) = b,\ \sigma(i+1) = c\}$.

We remark that the definition of $\ell_{(C,\sigma)}(z)$ is independent of where the closed walk starts and ends. Namely, if instead of C we consider $C' = v_i\text{-}e_i\text{-}\cdots\text{-}v_k\text{-}e_k\text{-}v_1\text{-}e_1\text{-}\cdots\text{-}v_{i-1}\text{-}e_{i-1}\text{-}v_i$, and we define σ' accordingly, then we have $\ell_{(C,\sigma)}(z) = \ell_{(C',\sigma')}(z)$. Moreover, if $\sigma(i-1) = -$ or $\sigma(i) = -$, then $\ell_{(C,\sigma)}(z)$ is independent of the choice of $v_i \in e_{i-1} \cap e_i$.

By Lemma 1, the length function of a signed closed walk is nonnegative. We will show that for odd signed closed walks, the length function evaluated in each integer solution is at least 1. Hence, we define the *simple odd β-cycle inequality* corresponding to the odd signed closed walk (C, σ) as

$$\ell_{(C,\sigma)}(z) \geq 1. \tag{4}$$

We first establish that this inequality is indeed valid for $\mathrm{ML}(G)$.

Theorem 3. *Simple odd β-cycle inequalities (4) are valid for $\mathrm{ML}(G)$.*

Proof. Let $z \in \mathrm{ML}(G) \cap \{0,1\}^{V \cup E}$ and assume, for the sake of contradiction, that z violates inequality (4) for some odd signed closed walk (C, σ). Since the coefficients of $\ell_{(C,\sigma)}$ are integer, we obtain $\ell_{(C,\sigma)} \leq 0$. From Lemma 1, we have that $s(z) = 0$ holds for all involved functions $s(z)$. Moreover, edge variables z_{e_i} for all edges e_i with $\sigma(i) = +$, node variables z_{v_i} for all nodes v_i with $\sigma(i-1) = \sigma(i) = +$, and the expressions $\prod_{v \in e_{i-1} \cap e_i} z_v$ for all nodes i with $\sigma(i-1) = \sigma(i) = -$ are either equal or complementary (see Table 1), where the latter happens if and only if the corresponding edge e_i satisfies $\sigma(i) = -1$. Since the signed closed walk C is odd, this yields a contradiction $z_e = 1 - z_e$ for some edge e of C or $z_v = 1 - z_v$ for some node v of C or $\prod_{v \in e \cap f} z_v = 1 - \prod_{v \in e \cap f} z_v$ for a pair e, f of subsequent edges of C. $\square$

Next, we provide an example of a simple odd β-cycle inequality.

Example 4. *We consider the closed walk of length 5 given by the sequence $C = v_1\text{-}e_1\text{-}v_2\text{-}e_2\text{-}v_3\text{-}\cdots\text{-}v_5\text{-}e_5\text{-}v_1$ with signature $(\sigma(1), \sigma(2), \ldots, \sigma(5)) = (-,+,+,-,-)$ depicted in Figure 1. We have $1 \in I_{(-,-,+)}$, $2 \in I_{(-,+,+)}$, $3 \in I_{(+,+,-)}$, $4 \in I_{(+,-,-)}$, $5 \in I_{(-,-,-)}$. The corresponding simple odd β-cycle inequality is $\ell_{(C,\sigma)}(z) \geq 1$. Using Definition 2, we write $\ell_{(C,\sigma)}(z)$ in terms of the building blocks as*

$$\ell_{(C,\sigma)}(z) = s^{\text{one}}_{e_1,e_1\cap e_5,e_2}(z) + s^{\text{inc}}_{e_2,v_3}(z) + s^{\text{inc}}_{e_3,v_3}(z) + s^{\text{one}}_{e_4,e_4\cap e_5,e_3}(z) + s^{\text{odd}}_{e_5,e_5\cap e_4,e_5\cap e_1}(z).$$

Using the definition of the building blocks, we obtain

$$\begin{aligned}\ell_{(C,\sigma)}(z) = &+ 2z_{e_1} - 1 + \sum_{u \in e_1 \cap e_5} (1 - z_u) + (1 - z_{e_2}) + \sum_{v \in e_1 \setminus (e_1 \cap e_5 \cup e_2)} (2 - 2z_v) \\ &+ (z_{v_3} - z_{e_2}) + (z_{v_3} - z_{e_3}) \\ &+ 2z_{e_4} - 1 + \sum_{u \in e_4 \cap e_5} (1 - z_u) + (1 - z_{e_3}) + \sum_{v \in e_4 \setminus (e_4 \cap e_5 \cup e_3)} (2 - 2z_v) \\ &+ 2z_{e_5} - 1 + \sum_{u \in e_5 \cap e_4} (1 - z_u) + \sum_{w \in e_5 \cap e_1} (1 - z_w) + \sum_{v \in e_5 \setminus (e_5 \cap e_4 \cup (e_5 \cap e_1))} (2 - 2z_v).\end{aligned}$$

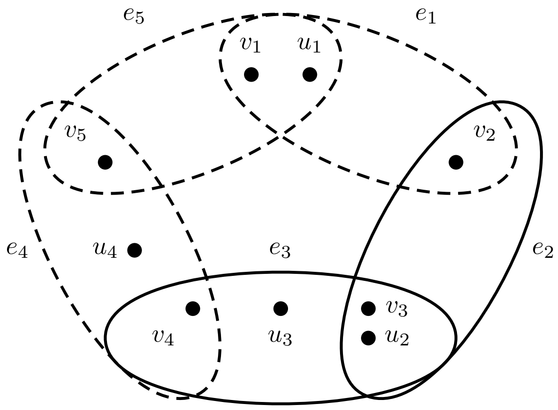

Figure 1: Figure of the closed walk considered in Example 4. The solid edges have sign $+$ and the dashed edges have sign $-$.

We write the sums explicitly and obtain

$$\begin{aligned}\ell_{(C,\sigma)}(z) = &+2z_{e_1} - 1 + (1 - z_{v_1}) + (1 - z_{u_1}) + (1 - z_{e_2})\\ &+ (z_{v_3} - z_{e_2}) + (z_{v_3} - z_{e_3})\\ &+ 2z_{e_4} - 1 + (1 - z_{v_5}) + (1 - z_{e_3}) + (2 - 2z_{u_4})\\ &+ 2z_{e_5} - 1 + (1 - z_{v_5}) + (1 - z_{v_1}) + (1 - z_{u_1})\\ = &\ 2z_{e_1} - 2z_{e_2} - 2z_{e_3} + 2z_{e_4} + 2z_{e_5} - 2z_{v_1} - 2z_{u_1} + 2z_{v_3} - 2z_{u_4} - 2z_{v_5} + 7.\end{aligned}$$

$\diamond$

Example 4 suggests that, when the function is written explicitly, the coefficients in the function $\ell_{(C,\sigma)}(z)$ exhibit a certain pattern. This different expression of $\ell_{(C,\sigma)}(z)$ is formalized in the next lemma. The proof of the lemma can be obtained directly from the definition of $\ell_{(C,\sigma)}(z)$ by summing up each variable that appears in more than one building block.

Lemma 5. *Given a signed closed walk (C,σ) in G with $k \geq 3$, we have*

$$\begin{aligned}\ell_{(C,\sigma)}(z) = \sum_{\substack{i\in[k]\\ \sigma(i)=-}} (2z_{e_i} + 1) - \sum_{\substack{i\in[k]\\ \sigma(i)=+}} 2z_{e_i} + \sum_{\substack{i\in[k]\\ \sigma(i-1)=\sigma(i)=+}} 2z_{v_i} + \sum_{\substack{i\in[k]\\ \sigma(i-1)=\sigma(i)=-\\ v\in e_{i-1}\cap e_i}} 2(1 - z_v) + \sum_{\substack{i\in[k]:\sigma(i)=-\\ v\in e_i\setminus(e_{i-1}\cup e_{i+1})}} 2(1 - z_v)\\ - 2|\{i \in [k] : \sigma(i-1) = \sigma(i) = -\}|. \quad (5)\end{aligned}$$

Using Lemma 5, we obtain the following result.

Proposition 6. *Simple odd β-cycle inequalities are Chvátal-Gomory inequalities for* $\mathrm{FR}(G)$ *and can be written in the form*

$$\begin{aligned}\sum_{\substack{i\in[k]\\ \sigma(i)=-}} z_{e_i} - \sum_{\substack{i\in[k]\\ \sigma(i)=+}} z_{e_i} + \sum_{\substack{i\in[k]\\ \sigma(i-1)=\sigma(i)=+}} z_{v_i} - \sum_{\substack{i\in[k]\\ \sigma(i-1)=\sigma(i)=-\\ v\in e_{i-1}\cap e_i}} (z_v - 1) - \sum_{\substack{i\in[k]:\sigma(i)=-\\ v\in e_i\setminus(e_{i-1}\cup e_{i+1})}} (z_v - 1)\\ \geq \frac{1 - |\{i \in [k] : \sigma(i) = -\}|}{2} - |\{i \in [k] : \sigma(i-1) = \sigma(i) = -\}|.\end{aligned} \quad (6)$$

Proof. Let (C,σ) be an odd signed closed walk in a hypergraph G. From Lemma 1 we obtain that $\ell_{(C,\sigma)}(z) \geq 0$ holds for each $z \in \mathrm{FR}(G)$. Lemma 5 reveals that in the inequality $\ell_{(C,\sigma)}(z) \geq 0$, all variables' coefficients are even integers, while the constant term is an odd integer. Hence, the inequality divided by 2 has integral variable coefficients, and we can obtain the corresponding Chvátal-Gomory inequality by rounding the constant term up. The resulting inequality is the simple odd β-cycle inequality (4) scaled by $1/2$ and has the form (6). This shows that simple odd β-cycle inequalities are Chvátal-Gomory inequalities for $\mathrm{FR}(G)$. $\square$

It follows from Proposition 6 that, under some conditions on (C,σ), simple odd β-cycle inequalities are in fact $\{0,1/2\}$-cuts (see [3]) with respect to $\mathrm{FR}(G)$. Some classes of such cutting planes can be separated in polynomial time, in particular if the involved inequalities only have two odd coefficients. In such a case, these inequalities are patched together such that odd coefficients cancel out and eventually all coefficients are even. We want to emphasize that this generic separation approach does not work in our case since our building block inequalities may have more than 2 odd-degree coefficients. Nevertheless, the separation algorithm presented in the next section is closely related to the idea of cancellation of odd-degree coefficients.

4 Separation algorithm

The main goal of this section is to show that the separation problem over simple odd β-cycle inequalities can be solved in strongly polynomial time (Theorem 10). This will be achieved by means of an auxiliary undirected graph in which several shortest-path computations must be carried out. The auxiliary graph is inspired by the one for the separation problem of odd-cycle inequalities for the maximum cut problem [1]. However, to deal with our different problem and the more general hypergraphs we will extend it significantly.

Let $G=(V,E)$ be a hypergraph and let $\hat{z}\in \mathrm{FR}(G)$. Define $\mathcal{T} \coloneqq \{(e,f,g)\in E : e\cap f\neq\varnothing,\ f\cap g\neq\varnothing,\ e\cap f\cap g=\varnothing\}$ to be the set of potential subsequent edge triples. We define the auxiliary graph

$$\bar{G}=(\bar{V},\bar{E})=(\bar{V}_+\cup\bar{V}_-\cup\bar{V}_{\mathrm{E}},\bar{E}^{-,-,-}\cup\bar{E}^{+,-,+}\cup\bar{E}^{+,-,-}\cup\bar{E}^{+,+,\pm})$$

and length function $\bar{\ell}:\bar{E}\to\mathbb{R}$ as follows.

$$\begin{aligned}
\bar{V}_+ &\coloneqq V\times\{\pm\}\\
\bar{V}_- &\coloneqq \{e\cap f : e,f\in E,\ e\neq f,\ e\cap f\neq\varnothing\}\times\{\pm\}\\
\bar{V}_{\mathrm{E}} &\coloneqq E\times\{\pm\}\\
\bar{E}^{-,-,-} &\coloneqq \{\{(e\cap f,p),(f\cap g,-p)\} : (e,f,g)\in\mathcal{T},\ p\in\{\pm\}\}\\
\bar{\ell}_{\{(U,p),(W,-p)\}} &\coloneqq \min_{e,f,g}\{s^{\text{odd}}_{f,U,W}(\hat{z}) : U=e\cap f,\ W=f\cap g \text{ for some } (e,f,g)\in\mathcal{T}\}\\
\bar{E}^{+,-,+} &\coloneqq \{\{(e,p),(g,-p)\} : e,g\in E, e\cap f\neq\varnothing \text{ and } f\cap g\neq\varnothing \text{ for some } f\in E\\
&\qquad \text{with } e\cap f\cap g=\varnothing,\ p\in\{\pm\}\}\\
\bar{\ell}_{\{(e,p),(g,-p)\}} &\coloneqq \min_{f}\{s^{\text{two}}_{e,f,g}(\hat{z}) : f\in E,\ e\cap f\neq\varnothing,\ f\cap g\neq\varnothing,\ e\cap f\cap g=\varnothing\}\\
\bar{E}^{+,-,-} &\coloneqq \{\{(e,p),(f\cap g,-p)\} : (e,f,g)\in\mathcal{T},\ p\in\{\pm\}\}\\
\bar{\ell}_{\{(e,p),(U,-p)\}} &\coloneqq \min_{f,g}\{s^{\text{one}}_{f,U,e}(\hat{z}) : (e,f,g)\in\mathcal{T},\ U=f\cap g\}\\
\bar{E}^{+,+,\pm} &\coloneqq \{\{(v,p),(e,p)\} : v\in e\in E,\ p\in\{\pm\}\}\\
\bar{\ell}_{\{(v,p),(e,p)\}} &\coloneqq s^{\text{inc}}_{e,v}(\hat{z})
\end{aligned}$$

We point out that the graph $\bar{G}$ can have parallel edges, possibly with different lengths. We immediately obtain the following corollary from Lemma 1.

Corollary 7. *The edge lengths $\bar{\ell}:\bar{E}\to\mathbb{R}$ are nonnegative.*

We say that two nodes $\bar{u},\bar{v}\in\bar{V}$ are *twins* if they only differ in the second component, i.e., the sign. We call a walk $\bar{W}$ in the graph $\bar{G}$ a *twin walk* if its end nodes are twin nodes. For a walk $\bar{W}$ in $\bar{G}$, we denote by $\bar{\ell}(\bar{W})$ the total *length*, i.e., the sum of the edge lengths $\bar{\ell}_e$ along the edges e in $\bar{W}$. In the next two lemmas we study the relationship between odd signed closed walks in G and twin walks in $\bar{G}$.

Lemma 8. *For each odd signed closed walk (C,σ) in G there exists a twin walk $\bar{W}$ in $\bar{G}$ of length $\bar{\ell}(\bar{W})\leq 1+s$, where s is the slack of the simple odd β-cycle inequality (4) induced by (C,σ) with respect to $\hat{z}$. In particular, if the inequality is violated by $\hat{z}$, then we have $\bar{\ell}(\bar{W})<1$.*

Proof. Let (C, σ) be an odd signed closed walk with $C = v_1\text{-}e_1\text{-}v_2\text{-}e_2\text{-}v_3\text{-}\cdots\text{-}v_{k-1}\text{-}v_{k-1}\text{-}v_k\text{-}e_k\text{-}v_1$. For $i \in [k]$, let $p_i := \prod_{j=1}^{i} \sigma(j)$ be the product of signs of all edges up to e_i. Moreover, define $p_0 := \sigma(0) = \sigma(k)$. For each $i \in [k]$, we determine a walk $\bar{W}_i$ in $\bar{G}$ of length at most 2, and construct $\bar{W}$ by going along all these walks in their respective order. The walk $\bar{W}_i$ depends on $\sigma(i-1)$, $\sigma(i)$ and $\sigma(i+1)$:

$$\bar{W}_i := \begin{cases} (v_i, p_{i-1}) \to (e_i, p_i) \to (v_{i+1}, p_i) & \text{if } i \in I_{+,+,+} \\ (v_i, p_{i-1}) \to (e_i, p_i) & \text{if } i \in I_{+,+,-} \\ (e_i, p_i) \to (v_{i+1}, p_i) & \text{if } i \in I_{-,+,+} \\ (e_i, p_i) \qquad\qquad\qquad \text{(length 0)} & \text{if } i \in I_{-,+,-} \\ (e_{i-1}, p_{i-1}) \to (e_i \cap e_{i+1}, p_i) & \text{if } i \in I_{+,-,-} \\ (e_{i-1} \cap e_i, p_{i-1}) \to (e_{i+1}, p_i) & \text{if } i \in I_{-,-,+} \\ (e_{i-1} \cap e_i, p_{i-1}) \to (e_i \cap e_{i+1}, p_i) & \text{if } i \in I_{-,-,-} \\ (e_{i-1}, p_{i-1}) \to (e_{i+1}, p_i) & \text{if } i \in I_{+,-,+}. \end{cases}$$

The walks $\bar{W}_i$ help to understand the meaning of the different node types: the walk $\bar{W}_i$ starts at a node from $\bar{V}_+$ if $\sigma(i-1) = \sigma(i) = +$, it starts at a node from $\bar{V}_-$ if $\sigma(i-1) = \sigma(i) = -$, and it starts at a node from $\bar{V}_{\mathrm{E}}$ if $\sigma(i-1) \neq \sigma(i)$ holds. Similarly, the walk $\bar{W}_i$ ends at a node from $\bar{V}_+$ if $\sigma(i) = \sigma(i+1) = +$, it ends at a node from $\bar{V}_-$ if $\sigma(i) = \sigma(i+1) = -$, and it ends at a node from $\bar{V}_{\mathrm{E}}$ if $\sigma(i) \neq \sigma(i+1)$ holds.

Note that all edges traversed by each $\bar{W}_i$ are indeed in $\bar{E}$. It is easily verified that, for each $i \in [k-1]$, the walk $\bar{W}_i$ ends at the same node at which the walk $\bar{W}_{i+1}$ starts. Hence $\bar{W}$ is indeed a walk in $\bar{G}$. Since $v_{k+1} = v_1$ holds, C is closed and (C, σ) is odd, it can be checked that $\bar{W}$ is a twin walk. Finally, by construction, $\bar{\ell}(\bar{W}) \leq \ell_{(C,\sigma)}(\hat{z})$ holds, where the inequality comes from the fact that the minima in the definition of $\bar{\ell}$ need not be attained by the edges from C. By definition of s we have $\ell_{(C,\sigma)}(\hat{z}) = 1 + s$, thus $\bar{\ell}(\bar{W}) \leq 1 + s$. □

Lemma 9. *For each twin walk $\bar{W}$ in $\bar{G}$ there exists an odd signed closed walk (C, σ) in G whose induced simple odd β-cycle inequality (4) has slack $\bar{\ell}(\bar{W}) - 1$ with respect to $\hat{z}$. In particular, if $\bar{\ell}(\bar{W}) < 1$ holds, then the inequality is violated by $\hat{z}$.*

Proof. Let $\bar{W}$ be a twin walk in $\bar{G}$. We first construct the signed closed walk (C, σ) by processing the edges of $\bar{W}$ in their order. Throughout the construction we maintain the index i of the next edge to be constructed, which initially is $i := 1$. Since the construction depends on the type of the current edge $\bar{e} = \{\bar{u}, \bar{v}\} \in \bar{W}$ (where $\bar{W}$ visits $\bar{u}$ first), we distinguish the relevant cases:

Case 1: $\bar{e} \in \bar{E}^{+,+,\pm}$ and $\bar{u} \in \bar{V}_{\mathrm{E}}$. Hence, $\bar{u} = (e, p)$ and $\bar{v} = (v, p)$ for some $v \in e \in E$ and some $p \in \{\pm\}$. We define $v_i := v$ and continue.

Case 2: $\bar{e} \in \bar{E}^{+,+,\pm}$ and $\bar{u} \in \bar{V}_+$. Hence, $\bar{u} = (v, p)$ and $\bar{v} = (e, p)$ for some $v \in e \in E$ and some $p \in \{\pm\}$ as well as $\ell_{\bar{e}} = s^{\mathrm{inc}}_{e,v}$. We define $e_i := e$ and $\sigma(i) := +$. We then increase i by 1 and continue.

Case 3: $\bar{e} \in \bar{E}^{+,-,-}$ and $\bar{u} \in V_{\mathrm{E}}$. Hence, $\bar{u} = (e, p)$ and $\bar{v} = (f \cap g, -p)$ for some $(e, f, g) \in \mathcal{T}$ as well as $\ell_{\bar{e}} = s^{\mathrm{one}}_{f, f\cap g, e}(\hat{z})$. We define v_i (resp. v_{i+1}) to be any node in $e \cap f$ (resp. $f \cap g$), $e_i := f$ and $\sigma(i) := -$. We then increase i by 1 and continue.

Case 4: $\bar{e} \in \bar{E}^{-,-,-}$. Hence, $\bar{u} = (e \cap f, p)$ and $\bar{v} = (f \cap g, -p)$ for some $(e, f, g) \in \mathcal{T}$ as well as $\ell_{\bar{e}} = s^{\mathrm{odd}}_{f, e\cap f, f\cap g}(\hat{z})$. We define $e_i := f$, $\sigma(i) := -$ and v_{i+1} to be any node in $f \cap g$. We then increase i by 1 and continue.

Case 5: $\bar{e} \in \bar{E}^{+,-,-}$ and $\bar{u} \in V_-$. Hence, $\bar{u} = (e \cap f, p, -)$ and $\bar{v} = (g, -p)$ for some $(e, f, g) \in \mathcal{T}$ with $\ell_{\bar{e}} = s^{\mathrm{one}}_{f, e\cap f, g}(\hat{z})$. We define $e_i := f$, $\sigma(i) := -$ and v_{i+1} to be any node in $f \cap g$. We then increase i by 1 and continue.

Case 6: $\bar{e} \in \bar{E}^{+,-,+}$. Hence, $\bar{u} = (e, p)$ and $\bar{v} = (g, -p)$ for some $(e, f, g) \in \mathcal{T}$ as well as $\ell_{\bar{e}} = s^{\mathrm{two}}_{f,e,g}(\hat{z})$. We define v_i (resp. v_{i+1}) to be any node in $e \cap f$ (resp. $f \cap g$), $e_i := f$, $\sigma(i) := -$, $e_{i+1} := g$ and $\sigma(i+1) := +$. We then increase i by 2 and continue.

After processing all edges of $\bar{W}$, the last defined edge is e_{i-1} and thus we define $k := i - 1$ and $C := v_1\text{-}e_1\text{-}v_2\text{-}e_2\text{-}v_3\text{-}\cdots\text{-}v_{k-1}\text{-}v_{k-1}\text{-}v_k\text{-}e_k\text{-}v_1$. By checking pairs of edges of $\bar{W}$ that arise consecutively, one verifies that for each $i \in [k]$, we also have $v_i \in e_{i-1} \cap e_i$.

To see that (C, σ) is odd, we use the fact that the endnodes of $\bar{W}$ are twin nodes. When traversing an edge $\bar{e}$ from $\bar{u}$ to $\bar{v}$, the second entries of $\bar{u}$ and $\bar{v}$ differ if and only if we set a σ-entry to $-$. Note that in Case 6 we set two such entries, but only one to $-$. We conclude that $\sigma(i) = -$ holds for an odd number of indices $i \in [k]$.

By construction we have $\bar{\ell}(\bar{W}) = \ell_{(C,\sigma)}(\hat{z})$. The slack of the simple odd β-cycle inequality induced by (C, σ) with respect to $\hat{z}$ is then $\ell_{(C,\sigma)}(\hat{z}) - 1 = \bar{\ell}(\bar{W}) - 1$. □

Theorem 10. *Let $G = (V, E)$ be a hypergraph and let $\hat{z} \in \mathrm{FR}(G)$. The separation problem for simple odd β-cycle inequalities (4) can be solved in time $\mathcal{O}(|E|^5 + |V|^2 \cdot |E|)$.*

Proof. Let $n := |V|$ and $m := |E|$ and assume $m \geq \log(n)$ since otherwise we can merge nodes that are incident to exactly the same edges. First note that, regarding the size of the auxiliary graph $\bar{G}$, we have $|\bar{V}| = \mathcal{O}(m^2 + n)$ and $|\bar{E}| = \mathcal{O}(mn + m^3)$. For the construction of $\bar{G}$ and the computation of $\bar{\ell}$ we need to inspect all triples $(e, f, g) \in \mathcal{T}$ of edges. This can be done in time $\mathcal{O}(m^3 n)$ since for each of the m^3 edge triples (e, f, g) we have to inspect at most n nodes to check the requirements on the intersections of e, f and g.

According to Lemmas 8 and 9 we only need to check for the existence of a twin walk $\bar{W}$ in $\bar{G}$ with $\ell(\bar{W}) < 1$. This can be accomplished with $|\bar{V}|/2 = \mathcal{O}(m^2 + n)$ runs of Dijkstra's algorithm [11] on $\bar{G}$, each of which takes

$$\mathcal{O}(|\bar{E}| + |\bar{V}| \cdot \log(|\bar{V}|)) = \mathcal{O}((mn + m^3) + (m^2 + n) \cdot \log(m^2 + n))$$

time when implemented with Fibonacci heaps [13]. If $m^2 \geq n$, then the total running time simplifies to $\mathcal{O}(m^5)$, and otherwise we obtain $\mathcal{O}(n^2 m)$.

□

The main reason for this large running time bound is the fact that $|\bar{V}_-|$ can be quadratic in $|E|$.

Clearly, our separation algorithm requires that the edge lengths $\bar{\ell}$ of the auxiliary graph $\bar{G}$ are nonnegative. This in turn requires $\hat{z} \in \mathrm{FR}(G)$, i.e., that the flower inequalities with at most two neighbors are satisfied. As we already mentioned, the number of these flower inequalities is bounded by a polynomial in $|V|$ and $|E|$. We like to point out that one can combine the separation of these flower inequalities with the construction of $\bar{G}$, i.e., one can determine violated inequalities while constructing the auxiliary graph.

5 Redundancy

Denote by $\mathrm{CR}(G)$ the set of points in $\mathrm{FR}(G)$ that satisfy all simple odd β-cycle inequalities. It turns out that many such inequalities are redundant for $\mathrm{CR}(G)$. However, the redundancy proofs do not provide much insight and often require many case distinctions (on the involved sign patterns and the way edges intersect). Hence, we restrict ourselves to providing an example of such a redundancy result.

Since $\mathrm{ML}(G)$ is full-dimensional (provided that G has no loops or parallel edges) and contained in $\mathrm{CR}(G)$, then also $\mathrm{CR}(G)$ is full-dimensional. As a consequence, an inequality is redundant for $\mathrm{CR}(G)$ if and only if it is not facet-defining for $\mathrm{CR}(G)$ (see Chapter 8 of Schrijver's book [19]).

Proposition 11. *Let (C, σ) be an odd signed closed walk in G such that there exist distinct indices $i, j \in [k]$ such that $e_i = e_j$ and $\sigma(i) = \sigma(j) = +$. Then the simple odd β-cycle inequality corresponding to (C, σ) is redundant for $\mathrm{CR}(G)$.*

Proof. Without loss of generality we can assume $i = 1$. Note that, from the definition of closed walk, we have and $4 \leq j \leq k - 2$. We define the following two closed walks in G: $C_1 := v_j\text{-}e_1\text{-}v_2\text{-}e_2\text{-}\cdots\text{-}v_{j-1}\text{-}e_{j-1}\text{-}v_j$, and $C_2 := v_1\text{-}e_j\text{-}v_{j+1}\text{-}e_{j+1}\text{-}\ldots\text{-}v_k\text{-}e_k\text{-}v_1$. Since $4 \leq j \leq k - 2$, both C_1 and C_2 consist of at least three edges. Let σ_1 and σ_2 be the signatures of C_1 and C_2, respectively, obtained from σ. Since (C, σ) is odd, exactly one among (C_1, σ_1) and (C_2, σ_2) is odd, while the other is even. Without loss of generality we assume that (C_1, σ_1) is odd and (C_2, σ_2) is even. To prove that the simple odd β-cycle inequality $\ell_{(C,\sigma)}(z) \geq 1$ corresponding to (C, σ) is redundant for $\mathrm{CR}(G)$, it suffices to show that it is the sum of the simple odd β-cycle inequality $\ell_{(C_1,\sigma_1)}(z) \geq 1$ corresponding to (C_1, σ_1) and of the inequality $\ell_{(C_2,\sigma_2)}(z) \geq 0$, which is valid for $\mathrm{FR}(G)$ by Lemma 1. To see this, it suffices to consider the following cases, where each case not explicitly discussed below is symmetric to one of the written ones:

Case 1: $1 \in I_{+,+,+}$ and $j \in I_{+,+,+}$. In this case we have $1 \in I_{+,+,+}$ in C_1 and $j \in I_{+,+,+}$ in C_2. It follows that the contribution $\left(s^{\mathrm{inc}}_{e_1,v_1}(z) + s^{\mathrm{inc}}_{e_1,v_2}(z)\right) + \left(s^{\mathrm{inc}}_{e_j,v_j}(z) + s^{\mathrm{inc}}_{e_j,v_{j+1}}\right)$ from C is equal to the sum of the contribution $\left(s^{\mathrm{inc}}_{e_1,v_j}(z) + s^{\mathrm{inc}}_{e_1,v_2}(z)\right)$ from C_1 and the contribution $\left(s^{\mathrm{inc}}_{e_j,v_1}(z) + s^{\mathrm{inc}}_{e_j,v_{j+1}}(z)\right)$ from C_2.

Case 2: $1 \in I_{+,+,+}$ and $j \in I_{+,+,-}$. In this case we have $1 \in I_{+,+,+}$ in C_1 and $j \in I_{+,+,-}$ in C_2. It follows that the contribution $\left(s^{\mathrm{inc}}_{e_1,v_1}(z) + s^{\mathrm{inc}}_{e_1,v_2}(z)\right) + s^{\mathrm{inc}}_{e_j,v_j}(z)$ from C is equal to the sum of the contribution $\left(s^{\mathrm{inc}}_{e_1,v_j}(z) + s^{\mathrm{inc}}_{e_1,v_2}(z)\right)$ from C_1 and the contribution $s^{\mathrm{inc}}_{e_j,v_1}(z)$ from C_2.

Case 3: $1 \in I_{+,+,+}$ and $j \in I_{-,+,-}$. In this case we have $1 \in I_{-,+,+}$ in C_1 and $j \in I_{+,+,-}$ in C_2. It follows that the contribution $\left(s^{\mathrm{inc}}_{e_1,v_1}(z) + s^{\mathrm{inc}}_{e_1,v_2}(z)\right)$ from C is equal to the sum of the contribution $s^{\mathrm{inc}}_{e_1,v_2}(z)$ from C_1 and the contribution $s^{\mathrm{inc}}_{e_j,v_1}(z)$ from C_2.

Case 4: $1 \in I_{+,+,-}$ and $j \in I_{+,+,-}$. In this case we have $1 \in I_{+,+,-}$ in C_1 and $j \in I_{+,+,-}$ in C_2. It follows that the contribution $s^{\mathrm{inc}}_{e_1,v_1}(z) + s^{\mathrm{inc}}_{e_j,v_j}(z)$ from C is equal to the sum of the contribution $s^{\mathrm{inc}}_{e_1,v_j}(z)$ from C_1 and the contribution $s^{\mathrm{inc}}_{e_j,v_1}(z)$ from C_2.

Case 5: $1 \in I_{+,+,-}$ and $j \in I_{-,+,+}$. In this case we have $1 \in I_{-,+,-}$ in C_1 and $j \in I_{+,+,+}$ in C_2. It follows that the contribution $s^{\mathrm{inc}}_{e_1,v_1}(z) + s^{\mathrm{inc}}_{e_j,v_{j+1}}(z)$ from C is equal to the sum of the contribution 0 from C_1 and the contribution $s^{\mathrm{inc}}_{e_j,v_1}(z) + s^{\mathrm{inc}}_{e_j,v_{j+1}}(z)$ from C_2.

Case 6: $1 \in I_{+,+,-}$ and $j \in I_{-,+,-}$. In this case we have $1 \in I_{-,+,-}$ in C_1 and $j \in I_{+,+,-}$ in C_2. It follows that the contribution $s^{\mathrm{inc}}_{e_1,v_1}(z)$ from C is equal to the sum of the contribution 0 from C_1 and the contribution $s^{\mathrm{inc}}_{e_j,v_1}(z)$ from C_2.

Case 7: $1 \in I_{-,+,-}$ and $j \in I_{-,+,-}$. In this case we have $1 \in I_{-,+,-}$ in C_1 and $j \in I_{-,+,-}$ in C_2. It follows that the contribution 0 from C is equal to the sum of the contribution 0 from C_1 and the contribution 0 from C_2. □

Note that Proposition 11 is only stated for repetition of edges whose signs are both $+$. However, we have evidence (based on computations for small instances) that also other types of closed walks yield redundant inequalities. Examples are those with repetitions of edges of arbitrary sign, those with two subsequent equal nodes, those in which two nodes are repeated and the four involved edges all have the same sign. The strongest redundancy statement that we can think of is captured in the following conjecture.

Conjecture 12. *Let (C, σ) be an odd signed closed walk in G for which a proper subsequence C' of C is a β-cycle. Then the simple odd β-cycle inequality corresponding to (C, σ) is redundant for* $\mathrm{CR}(G)$.

We recall that a *β-cycle* of length k, for some $k \geq 3$, is a sequence v_1-e_1-v_2-e_2-$\cdots$-v_k-e_k-v_1 such that $v_1, v_2, \ldots, v_k$ are distinct nodes, $e_1, e_2, \ldots, e_k$ are distinct edges, and v_i belongs to e_{i-1}, e_i and no other e_j for all $i = 1, \ldots, k$, where $e_0 := e_k$. Conjecture 12 justifies that although inequalities (4) are defined for closed walks, we call them simple odd *β-cycle* inequalities. The main reason for considering closed walks instead of β-cycles is the separation algorithm described in Section 4.

6 Relation to non-simple odd β-cycle inequalities

In this section we relate our simple odd β-cycle inequalities to the odd β-cycle inequalities in [5].

A *cycle hypergraph* is a hypergraph $G = (V, E)$, with $E = \{e_1, \ldots, e_m\}$, where $m \geq 3$, and every edge e_i has nonempty intersection only with e_{i-1} and e_{i+1} for every $i \in \{1, \ldots, m\}$, where, for convenience, we define $e_{m+1} := e_1$ and $e_0 := e_m$. If $m = 3$, it is also required that $e_1 \cap e_2 \cap e_3 = \emptyset$. Given a closed walk $C = v_1\text{-}e_1\text{-}v_2\text{-}e_2\text{-}\cdots\text{-}v_k\text{-}e_k\text{-}v_1$ in a hypergraph $G = (V, E)$, the *support hypergraph* of C is the hypergraph $G(C) = (V(C), E(C))$, where $E(C) := \{e_1, e_2, \ldots, e_k\}$ and $V(C) := e_1 \cup e_2 \cup \cdots \cup e_k$.

Lemma 13. *Let (C, σ) be a signed closed walk in a hypergraph G and assume that the support hypergraph of C is a cycle hypergraph. Let $E^- := \{e_i : i \in [k],\ \sigma(i) = -\}$, $E^+ := \{e_i : i \in [k],\ \sigma(i) = +\}$, $S_1 := (\bigcup_{e \in E^-} e) \setminus \bigcup_{e \in E^+} e$, and $S_2 := \{v_1, \ldots, v_k\} \setminus \bigcup_{e \in E^-} e$. Then*

$$\ell_{(C,\sigma)}(z) = -\sum_{v \in S_1} 2z_v + \sum_{e \in E^-} 2z_e + \sum_{v \in S_2} 2z_v - \sum_{e \in E^+} 2z_e + 2|S_1| - 2|\{i \in [k] : e_{i-1}, e_i \in E^-\}| + |E^-|.$$

In particular, the simple odd β-cycle inequality corresponding to (C, σ) coincides with the odd β-cycle inequality corresponding to (C, σ). Furthermore, in a cycle hypergraph, every odd β-cycle inequality is a simple odd β-cycle inequality.

Proof. It suffices to observe that

$$\sum_{\substack{i \in [k] \\ \sigma(i-1) = \sigma(i) = +}} 2z_{v_i} = \sum_{v \in S_2} 2z_v, \qquad \sum_{\substack{i \in [k] \\ \sigma(i-1) = \sigma(i) = - \\ v \in e_{i-1} \cap e_i}} 2z_v + \sum_{\substack{i \in [k] : \sigma(i) = - \\ v \in e_i \setminus (e_{i-1} \cup e_{i+1})}} 2z_v = \sum_{v \in S_1} 2z_v,$$

$$\sum_{\substack{i \in [k] \\ \sigma(i-1) = \sigma(i) = - \\ v \in e_{i-1} \cap e_i}} 2 + \sum_{\substack{i \in [k] : \sigma(i) = - \\ v \in e_i \setminus (e_{i-1} \cup e_{i+1})}} 2 = 2|S_1| \quad \text{and} \quad \sum_{\substack{i \in [k] \\ \sigma(i) = -}} 1 = |E^-|.$$

The statement for cycle hypergraphs G follows by inspecting the definition of the odd β-cycle inequalities. ☐

As a consequence, we can use the two following known results in order to gain insights about simple odd β-cycle inequalities.

Proposition 14 (Example 2 in [5])**.** *There exists a cycle hypergraph for which the Chvátal rank of odd β-cycle inequalities can be equal to 2.*

Proposition 15 (Implied by Theorem 1 in [5])**.** *Flower inequalities are Chvátal-Gomory cuts for* $\mathrm{SR}(G)$.

Theorem 16. *Simple odd β-cycle inequalities can have Chvátal rank 2 with respect to* $\mathrm{SR}(G)$.

Proof. Combining Proposition 15 with Proposition 6 shows that simple odd β-cycle inequalities have Chvátal rank at most 2. Lemma 13 and Proposition 14 show that the Chvátal rank of simple odd β-cycle inequalities for cycle hypergraphs can be equal to 2. ☐

For the second insight, we consider a strengthened form of Theorem 5 in [5].

Proposition 17 (Theorem 5 in [5], strengthened)**.** *Let $G = (V, E)$ be a cycle hypergraph. Then* $\mathrm{ML}(G)$ *is described by all odd β-cycle inequalities and all inequalities from* $\mathrm{FR}(G)$.

The strengthening lies in the fact that in the original statement of Theorem 5 in [5] all flower inequalities are used rather than only those with at most two neighbors. This strengthening of the original statement can be seen by inspecting its proof in [5]. By applying Lemma 13 to Proposition 17 we immediately obtain the following result.

Theorem 18. *Let $G = (V, E)$ be a cycle hypergraph. Then* $\mathrm{ML}(G) = \mathrm{CR}(G)$.

Future research. We would like to conclude this paper with a couple of open questions that could be investigated. An interesting research direction is a computational investigation of simple odd β-cycle inequalities, especially in relation to the applications discussed in Section 1, i.e., the image restoration problem in computer vision and the low auto-correlation binary sequence problem in theoretical physics.

The next research direction has a more theoretical flavor. The LP relaxations defined by odd-cycle inequalities [1] for the cut polytope and the affinely isomorphic correlation polytope (see [10]) have the following property: when maximizing a specific objective vector, then one can remove a subset of the odd-cycle inequalities upfront without changing the optimum. More precisely, the removal is based only on the sign pattern of the objective vector (see Theorem 2 in [16]). Since the simple odd β-cycle inequalities can be seen as an extension of the odd cycle inequalities for the cut polytope, the research question is whether a similar property can be proven for simple odd β-cycle inequalities.

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