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On hamiltonicity of 1-tough triangle-free graphs

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Abstract

Let $\omega(G)$ denote the number of components of a graph G. A connected graph G is said to be 1-*tough* if $\omega(G-X) \leq |X|$ for all $X \subseteq V(G)$ with $\omega(G-X) > 1$. It is well-known that every hamiltonian graph is 1-tough, but that the reverse statement is not true in general, and even not for triangle-free graphs. We present two classes of triangle-free graphs for which the reverse statement holds, i.e., for which hamiltonicity and 1-toughness are equivalent. Our two main results give partial answers to two conjectures due to Nikoghosyan.

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1. Introduction

All graphs we consider are finite, simple and undirected graphs. For terminology, notation and concepts not defined here, we refer the reader to [2]. A cycle in a graph G is called a *Hamilton cycle* if it contains all vertices of G, and G is called *hamiltonian* if it contains a Hamilton cycle.

Hamiltonicity has been a central topic in structural graph theory since the 1950s, and has regained more popularity since the development of algorithmic graph theory and the discovery that

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the associated decision problem is NP-complete. It also has many and sometimes very surprising applications, as the recent publication [5] in this journal shows. Forbidden subgraph conditions for guaranteeing hamiltonian properties have been studied intensively since the PhD thesis of Bedrossian [1] appeared in 1991. For more recent work on forbidden subgraph conditions we refer to the publication [6] in the first issue of this journal and the references therein.

The research we report here is inspired by several conjectures on the hamiltonicity of graphs presented by Nikoghosyan in the paper [8]. A central concept in that paper is the toughness of graphs, so we first recall the essential definitions related to toughness.

Let $\omega(G)$ denote the number of components of a graph G. As it is defined in [4], a connected graph G is said to be *t-tough* if $t \cdot \omega(G - X) \leq |X|$ for all $X \subseteq V(G)$ with $\omega(G - X) > 1$. The *toughness* of G, denoted by $\tau(G)$, is the maximum value of t such that G is t-tough (taking $\tau(K_n) = \infty$ for all $n \geq 1$).

It is an easy exercise to show that every hamiltonian graph is 1-tough, but that the reverse statement does not hold. Nikoghosyan [8] investigated the hamiltonicity of 1-tough graphs by considering disconnected single forbidden subgraphs, and he presented the following conjectures. Here we use $\triangle$ to denote a complete graph on 3 vertices, $G_1 \cup G_2$ to denote the disjoint union of two vertex-disjoint graphs G_1 and G_2, and kG to denote the graph consisting of k disjoint copies of the graph G. For a fixed graph H, a graph G is called *H-free* if G does not contain an induced copy of H, i.e., a set $S \subseteq V(G)$ such that the graph on vertex set S containing all edges of G between pairs of vertices in S is isomorphic to H. The latter is also called an *induced subgraph* and denoted by $\langle S \rangle$.

Conjecture 1 (Nikoghosyan [8]). *Every 1-tough $K_1 \cup P_4$-free graph on at least three vertices is hamiltonian.*

In an attempt to prove this conjecture, Li et al. [7] left one open case, as expressed in the following two results.

Theorem 1.1 (B. Li et al. [7]). *Let R be an induced subgraph of P_4, $K_1 \cup P_3$ or $2K_1 \cup K_2$. Then every R-free 1-tough graph on at least three vertices is hamiltonian.*

Note that every induced subgraph of $K_1 \cup P_4$ is also an induced subgraph of P_4, $K_1 \cup P_3$ or $2K_1 \cup K_2$, except for $K_1 \cup P_4$ itself.

Theorem 1.2 (B. Li et al. [7]). *Let R be a graph on at least three vertices. If every R-free 1-tough graph on at least three vertices is hamiltonian, then R is an induced subgraph of $K_1 \cup P_4$.*

These two results together imply the following. While forbidding any proper subgraph of $K_1 \cup P_4$ can guarantee 1-tough graphs (on at least three vertices) to be hamiltonian, the case with the graph $K_1 \cup P_4$ itself is still open. Since Conjecture 1 seems to be very hard to resolve, we considered partial solutions. In particular, if we impose the additional condition that the graphs under consideration are triangle-free, we can prove the following partial result.

Theorem 1.3. *Every 1-tough $\{\triangle, K_1 \cup P_4\}$-free graph on at least three vertices is hamiltonian.*

We postpone the proof of Theorem 1.3 to Section 3.

Another conjecture in [8] deals with the hamiltonicity of $K_1 \cup K_{1,3}$-free graphs. Clearly, Theorem 1.2 implies that the toughness of these graphs must be strictly larger than one.

Conjecture 2 (Nikoghosyan [8])**.** *Every $K_1 \cup K_{1,3}$-free graph G on at least three vertices with $\tau(G) > 4/3$ is hamiltonian.*

In [8], the Petersen graph was used to show that the condition $\tau(G) > 4/3$ in Conjecture 2 cannot be relaxed to $\tau(G) = 4/3$. Similarly as in Theorem 1.3, we involved the triangle as a second forbidden subgraph, and obtained the following result related to Conjecture 2.

Theorem 1.4. *If G is a 1-tough $\{\triangle, K_1 \cup K_{1,3}\}$-free graph on at least three vertices, then G is hamiltonian or the Petersen graph.*

The proof of Theorem 1.4 is postponed to Section 4. In [8], Nikoghosyan also raised the following conjecture.

Conjecture 3 (Nikoghosyan [8])**.** *Every $2K_2$-free graph G on at least three vertices with $\tau(G) > 1$ is hamiltonian.*

As far as we know, this conjecture is still open, but it is known from a recent paper due to Shan [9] that 3-tough $2K_2$-free graphs on at least three vertices are hamiltonian. This result considerably improves the result due to Broersma et al. [3] that 25-tough $2K_2$-free graphs on at least three vertices are hamiltonian. A partial result in [3] deals with triangle-free $2K_2$-free graphs, and supplements our results, as follows.

Theorem 1.5 (Broersma et al. [3])**.** *If G is a 1-tough $\{\triangle, 2K_2\}$-free graph on at least three vertices, then G is hamiltonian.*

Another open conjecture from [8] states that every $K_1 \cup P_5$-free graph G on at least three vertices with $\tau(G) > 1$ is hamiltonian. By involving triangle-freeness, we propose the following conjecture for future work.

Conjecture 4. *If G is a 1-tough $\{\triangle, K_1 \cup P_5\}$-free graph on at least three vertices, then G is hamiltonian.*

Before we give our proofs of Theorems 1.3 and 1.4, we first provide some additional notation and terminology.

2. Some notation and terminology

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $u \in V(G)$ and subgraphs H and R of G, let $N_R(u)$ and $N_R(H)$ denote the set of neighbors of the vertex u and the subgraph H in R, respectively, that is

$$N_R(u) = \{v \in V(R) \mid uv \in E(G)\},$$

$$N_R(H) = \big(\bigcup_{u \in V(H)} N_R(u) \big) \setminus V(H).$$

The numbers $|N_R(u)|$ and $|N_R(H)|$ are respectively called the *degree* of the vertex u and the *degree* of the subgraph H in R, and denoted as $d_R(u)$ and $d_R(H)$, respectively. If $R = G$, then we use $N(u)$ and $N(H)$ instead of $N_R(u)$ and $N_R(H)$, and $d(u)$ and $d(H)$ instead of $d_R(u)$ and $d_R(H)$, respectively.

Let $C = x_1x_2 \cdots x_tx_1$ be a cycle of length $t \geq 3$ in G with a given orientation. For a vertex $x_i \in V(C)$ $(1 \leq i \leq t)$, we let x_i^{-l}, x_i^{+l} $(1 \leq i - l < i + l \leq t)$ denote the vertices x_{i-l} and x_{i+l} on C, respectively. Instead of x_i^{-1} and x_i^{+1}, we simply use x_i^- and x_i^+ to denote the immediate predecessor and successor of x_i on C, respectively. For two vertices $x_i, x_j \in V(C)$, x_iCx_j denotes the subpath of C from x_i to x_j, and $x_j\overline{C}x_i$ denotes the path from x_j to x_i in the reverse direction. For any $I \subseteq V(C)$, let $I^- = \{x_i^- \mid x_i \in I\}$ and $I^+ = \{x_i^+ \mid x_i \in I\}$. A similar notation is used for paths.

Recall that for a subset S of $V(G)$, we use $\langle S \rangle$ to denote the subgraph of G induced by S. If the induced subgraph $\langle S \rangle$ is isomorphic to a graph H, then we slightly abuse the notation by writing $S \cong H$. We use $\{u; v, w, x\}$ to denote the graph isomorphic to the *claw* $K_{1,3}$ induced by $\{u, v, w, x\}$ with edge set $\{uv, uw, ux\}$, and we call u the *center*, and v, w, x the *end vertices* of this claw.

3. Proof of Theorem 1.3

Suppose, to the contrary, that G is a 1-tough $\{\triangle, K_1 \cup P_4\}$-free graph on at least three vertices, but not hamiltonian. Then G contains a cycle. We choose a longest cycle C of G. Since G is not hamiltonian, we use H to denote a component of $G - V(C)$. We denote all neighbors of H on C as $N_C(H) = \{u_1, u_2, \ldots, u_t\}$ with $t \geq 2$, in this order around C according to a fixed orientation of C, and we denote the (vertex set of the) segment of C from u_i^+ to u_{i+1}^- as $S_i = u_i^+Cu_{i+1}^-$ for $i = 1, 2, \ldots, t$. The next result is obvious, but we give it and its proof for later reference.

Claim 1. $N_C(H)^+$ and $N_C(H)^-$ are independent, and there is no path outside C connecting any two vertices of any one of these two sets.

Proof. Without loss of generality, assume that u_i^+ and u_j^+ are connected by a path P outside C (possibly P is an edge). We find a cycle longer than C: $u_iHu_j\overline{C}u_i^+Pu_j^+Cu_i$, a contradiction. □

We continue with proving another set of claims that are more specific for this graph class. We say that two sets A and B are connected by a path outside C if there is a path between a vertex $x \in A$ and $y \in B$ with all internal vertices not on C.

Let S_i and S_j be two distinct segments.

Claim 2. If S_i and S_j are connected by a path outside C, then $\{u_j^{+2}, u_j^{+4}, \ldots, u_{j+1}^{-2}\} \subseteq N(u_i^+)$ and $\{u_i^{+2}, u_i^{+4}, \ldots, u_{i+1}^{-2}\} \subseteq N(u_j^+)$, and $|S_i|$ and $|S_j|$ are odd.

Proof. Suppose S_i and S_j are connected by a path outside C. Then we can find a shortest path from u_i^+ to u_j^+ along $u_i^+CxPy\overline{C}u_j^+$, where $x \in S_i$, $y \in S_j$ and P is a path such that $V(P) \cap (V(C) \cup V(H)) = \{x, y\}$. We use P_{ij} to denote this shortest path for segments S_i and S_j. Then P_{ij} is an induced path. If $|V(P_{ij})| \geq 4$, then any vertex of H together with a subpath of P_{ij} of length 3 induces a copy of $K_1 \cup P_4$, contradicting the hypothesis. Using Claim 1, $|V(P_{ij})| = 3$. Thus, P is a path with $x = u_i^+$ or $y = u_j^+$. Denote $P_{ij} = p_1p_2p_3$, with $p_1 = u_i^+$ and $p_3 = u_j^+$. Since P is a path outside C connecting S_i and S_j, and by Claim 1, we have that $p_2 \in S_i \cup S_j$, and $|S_i| > 1$ or $|S_j| > 1$. Without loss of generality, we assume $|S_i| > 1$. If $p_2 \neq u_i^{+2}$, then to avoid any vertex $w \in V(H)$ with the path $u_i^{+2}u_i^+p_2u_j^+$ inducing $K_1 \cup P_4$, we have $p_2u_i^{+2} \in E(G)$. Then $\{u_i^+, u_i^{+2}, p_2\} \cong \triangle$, a contradiction. Hence, $u_j^+u_i^{+2} \in E(G)$. If u_i^{+4} exists, then to avoid inducing triangles and to avoid any vertex $w \in V(H)$ with the path $u_j^+u_i^{+2}u_i^{+3}u_i^{+4}$ inducing $K_1 \cup P_4$, we have $u_j^+u_i^{+4} \in E(G)$. Similarly, we have $u_i^{+6}, u_i^{+8}, \ldots \in N(u_j^+)$ if these vertices exist. For the last vertex of S_i, if $u_j^+u_{i+1}^- \in E(G)$, then $u_j^+ \neq u_{j+1}^-$ by Claim 1. To avoid any vertex $w \in V(H)$ with the path $u_i^+u_i^{+2}u_j^+u_j^{+2}$ inducing $K_1 \cup P_4$, we have $u_i^+u_j^{+2} \in E(G)$. Then there is a cycle longer than C: $u_iHu_{i+1}Cu_j^+u_{i+1}^-\overline{C}u_i^+u_j^{+2}Cu_i$, a contradiction. Hence, $u_j^+u_{i+1}^- \notin E(G)$ and $|S_i|$ is odd. If $u_j^{+2}, u_j^{+4}, u_j^{+6}, \ldots$ exist, then by symmetry, we have that $u_j^{+2}, u_j^{+4}, u_j^{+6}, \ldots \in N(u_i^+)$ and $|S_j|$ is odd. □

For any segment S_i that is connected to another segment by a path outside C, by Claim 2 we know that $|S_i|$ is odd. We divide S_i into two sets: $S_i^o = \{u_i^+, u_i^{+3}, \ldots, u_{i+1}^-\}$ and $S_i^e = \{u_i^{+2}, u_i^{+4}, \ldots, u_{i+1}^{-2}\}$. By Claim 2, if S_i is connected to S_j by a path outside C, then $S_i^e \subseteq N(u_j^+)$, and since G is $\triangle$-free, $S_i^o \cap N(u_j^+) = \emptyset$.

Claim 3. If S_i and S_j are connected by a path outside C, then $S_i^o \cup S_j^o$ is independent.

Proof. Suppose that $u_i^{+s}, u_i^{+t} \in S_i^o$ ($t > s$) and $u_i^{+s}u_i^{+t} \in E(G)$. Since G is $\triangle$-free, $t > s + 2$, and since $u_i^{+(s+1)}, u_i^{+(t-1)} \in N(u_j^+)$, $u_i^{+s+1}u_i^{t-1} \notin E(G)$. Then any vertex $w \in V(H)$ with the path $u_i^{s+1}u_i^{+s}u_i^{+t}u_i^{+(t-1)}$ induces a copy of $K_1 \cup P_4$, contradicting the hypothesis. Hence, S_i^o is independent. Similarly, S_j^o is also independent.

Suppose that $u_i^{+l} \in S_i^o, u_j^{+m} \in S_j^o$ and $u_i^{+l}u_j^{+m} \in E(G)$. By Claim 2, $u_i^{+l} \neq u_i^+$, $u_j^{+m} \neq u_j^+$ and $u_j^{+(m-1)}u_i^+ \in E(G)$. Also we know that $u_i^{+(l+1)} = u_{i+1}$ or $u_i^{+(l+1)}u_j^+ \in E(G)$. Then the cycle $u_iHu_j\overline{C}u_i^{+(l+1)}u_j^+Cu_j^{+(m-1)}u_i^+Cu_i^{+l}u_j^{+m}Cu_i$ is longer than C, a contradiction. □

Claim 4. If S_i is connected to S_j by a path outside C, and S_i has a neighbor $w' \in V(G) \setminus (V(C) \cup V(H))$, then $S_i^e \subseteq N(w')$ and $S_i^o \cap N(w') = \emptyset$.

Proof. First, $u_i^+ \notin N(w')$; otherwise, to avoid any vertex $w \in V(H)$ with a path $w'u_i^+u_j^{+2}u_j^+$ (if $|S_j| \neq 1$) or with a path $w'u_i^+u_i^{+2}u_j^+$ (if $|S_i| \neq 1$) inducing $K_1 \cup P_4$, we have $w'u_j^+ \in E(G)$. That contradicts Claim 1. Suppose that $u_i^{+k} \in N(w')$ ($k \neq 1$). To avoid any vertex $w \in V(H)$ with a path $w'u_i^{+k}u_i^{+(k+1)}u_i^{+(k+2)}$ or with a path $w'u_i^{+k}u_i^{+(k-1)}u_i^{+(k-2)}$ inducing $K_1 \cup P_4$, we have $w'u_i^{+(k+2)} \in E(G)$ and $w'u_i^{+(k-2)} \in E(G)$ if these vertices exist. By this argumentation, we know that every vertex on S_i having even distance to u_i^{+k} on C is a neighbor of w'. Since $u_i^+ \notin N(w')$ and G is $\triangle$-free, $S_i^e \subseteq N(w')$ and $S_i^o \cap N(w') = \emptyset$. □

We use S^o to denote the union of all S_i^o, and S^e to denote the union of all S_i^e, for all segments S_i ($1 \leq i \leq t$) that are connected to some other segment by a path outside C. By Claim 3 and Claim 4, there is no path outside C connecting any two vertices of S^o. Hence, if we remove all the vertices of $N_C(H) \cup S^e$, we get at least $|N_C(H) \cup S^e| + 1$ components, contradicting the hypothesis that G is 1-tough. This completes the proof of Theorem 1.3. □

4. Proof of Theorem 1.4

Let G be a 1-tough $\{\triangle, K_1 \cup K_{1,3}\}$-free graph on at least three vertices. For any vertex $u \in V(G)$ of degree larger than 2, since G is $\triangle$-free, we have that u and any three of its neighbors together induce a $K_{1,3}$. Next we distinguish two cases according to the connectivity of G in order to complete the proof of Theorem 1.4.

Case 1. G is 3-connected.
Suppose that G is not hamiltonian. Using a number of claims, we are going to prove that G is the Petersen graph. Here we use the same notations as in the proof of Theorem 1.3. Let C be a longest cycle of G, let H be a component of $G - V(C)$, and let $N_C(H) = \{u_1, u_2, \dots, u_t\}$ be all the neighbors of H on C, in this order according to a fixed orientation of C. Since G is 3-connected, $t \geq 3$. We denote the segment of C from u_i^+ to u_{i+1}^- as $S_i = u_i^+ C u_{i+1}^-$ for $i = 1, 2, \dots, t$. Claim 1 in the proof of Theorem 1.3 clearly also holds here, but we recall it here without proof for later reference.

Claim 5. $N_C(H)^+$ and $N_C(H)^-$ are independent, and there is no path outside C connecting any two vertices of any one of these two sets.

We present several other claims, each followed by a proof.

Claim 6. If S_i and S_j are connected by a path P_{ij} outside C, then $P_{ij} = u_i^+ u_{j+1}^-$ or $P_{ij} = u_{i+1}^- u_j^+$.

Proof. Let $P_{ij} = p_1 p_2 \dots p_s$ ($s \geq 2$) be such a path with $p_1 \in S_i$ and $p_s \in S_j$. If $p_1 \notin \{u_i^+, u_{i+1}^-\}$ or $p_s \notin \{u_j^+, u_{j+1}^-\}$, then $\{w, p_1, p_1^-, p_1^+, p_2\} \cong K_1 \cup K_{1,3}$ and $\{w, p_s, p_s^-, p_s^+, p_{s-1}\} \cong K_1 \cup K_{1,3}$ for any vertex $w \in V(H)$, a contradiction. By Claim 5, if $p_1 = u_i^+$, then $p_s = u_{j+1}^-$; if $p_1 = u_{i+1}^-$, then $p_s = u_j^+$ ($j \neq i+1$). In both cases, $|S_i| \geq 2$ and $|S_j| \geq 2$. To avoid any vertex of $V(H)$ with $\{u_i^+; u_i, u_i^{+2}, p_2\}$ or $\{u_j^+; u_j, u_j^{+2}, p_{s-1}\}$ inducing $K_1 \cup K_{1,3}$, we have $V(H) \subseteq N(u_i)$ or $V(H) \subseteq N(u_j)$. Since G is $\triangle$-free, $|V(H)| = 1$ and $N_C(H)$ is independent. Denote $H = \{w\}$.

Suppose $s \geq 3$ and $p_2 \in V(H')$, where H' is another component of $G - V(C)$ different from H. If P_{ij} is connecting u_i^+ to u_{j+1}^-, then $u_i \neq u_{j+1}$, and $p_2 u_i \notin E(G)$, $p_2 u_{j+1} \notin E(G)$; otherwise there clearly is a longer cycle. Then to avoid p_2 with $\{w; u_i, u_j, u_{j+1}\}$ inducing $K_1 \cup K_{1,3}$, we have $p_2 u_j \in E(G)$. Since $\{u_i^{+2}, u_j^+\} \in N_C(H')^+$, $u_i^{+2} u_j^+ \notin E(G)$ by Claim 5. To avoid u_j^+ with $\{u_i^+; u_i, u_i^{+2}, p_2\}$ inducing $K_1 \cup K_{1,3}$, we have $u_j^+ u_i \in E(G)$. Then $u_i^- u_j^+ \notin E(G)$; otherwise $\{u_i, u_i^-, u_j^+\} \cong \triangle$. To avoid u_i^- with $\{u_j; u_j^-, u_j^+, w\}$ inducing $K_1 \cup K_{1,3}$, we have $u_i^- u_j \in E(G)$. Then we find a cycle longer than C: $u_i w u_{j+1} C u_i^- u_j \overline{C} u_i^+ P_{ij} u_{j+1}^- \overline{C} u_j^+ u_i$, a contradiction. If P_{ij} is connecting u_{i+1}^- to u_j^+, then $u_{i+1} \neq u_j$ and $p_2 u_{i+1} \notin E(G)$, $p_2 u_j \notin E(G)$. We have $p_2 u_i \in E(G)$; otherwise, p_2 with $\{w; u_i, u_{i+1}, u_j\}$ induces $K_1 \cup K_{1,3}$. By Claim 5, $u_i^+ u_{i+1} \notin E(G)$ since $\{u_i^+, u_{i+1}\} \in N_C(H')^+$. To avoid u_{i+1} with $\{u_i; u_i^+, u_i^-, p_2\}$ inducing $K_1 \cup K_{1,3}$, we have $u_{i+1} u_i^- \in E(G)$. Then $u_{i+1}^+ u_i^- \notin E(G)$; otherwise $\{u_i^-, u_{i+1}, u_{i+1}^+\} \cong \triangle$.

To avoid u_{i+1}^+ with $\{u_i; u_i^-, u_i^+, w\}$ inducing $K_1 \cup K_{1,3}$, we have $u_{i+1}^+ u_i \in E(G)$. Then the cycle $u_{i+1} w u_j \overline{C} u_{i+1}^+ u_i C u_{i+1}^- P_{ij} u_j^+ C u_i^- u_{i+1}$ is longer than C, a contradiction. Hence, $s = 2$ and $P_{ij} = u_i^+ u_{j+1}^-$ or $P_{ij} = u_{i+1}^- u_j^+$. □

Since G is 1-tough, there are two distinct segments S_i and S_j that are connected by a path outside C. By Claim 6, $u_i^+ u_{j+1}^- \in E(G)$ or $u_{i+1}^- u_j^+ \in E(G)$. To avoid any vertex of $V(H)$ with $\{u_i^+; u_i, u_i^{+2}, u_{j+1}^-\}$ or with $\{u_j^+; u_j, u_j^{+2}, u_{i+1}^-\}$ inducing $K_1 \cup K_{1,3}$, we have $V(H) \subseteq N(u_i)$ or $V(H) \subseteq N(u_j)$. Since G is $\triangle$-free, $|V(H)| = 1$ and $N_C(H)$ is independent. Denote $H = \{w\}$.

Claim 7. $|S_i| = 2$ for all $i \in \{1, 2, \ldots, t\}$.

Proof. Suppose that there is a segment S_i with $i \in \{1, 2, \ldots, t\}$ such that $|S_i| \geq 3$. Then $u_i^+ \neq u_{i+1}^{-2}$. By Claim 6, u_{i+1}^{-2} has no neighbor in $V(C) \setminus (S_i \cup N_C(H))$. To avoid u_{i+1}^{-2} with $\{u_{i+2}; u_{i+2}^-, u_{i+2}^+, w\}$ inducing $K_1 \cup K_{1,3}$, we have $u_{i+1}^{-2} u_{i+2} \in E(G)$. To avoid u_i^- with $\{u_{i+1}^{-2}; u_{i+1}^-, u_{i+1}^{-3}, u_{i+2}\}$ inducing $K_1 \cup K_{1,3}$, we have $u_i^- u_{i+2} \in E(G)$. To avoid u_{i+1}^- with $\{u_{i+2}; u_{i+2}^-, u_i^-, w\}$ inducing $K_1 \cup K_{1,3}$, we have $u_{i+1}^- u_{i+2} \in E(G)$. But now $\{u_{i+1}^{-2}, u_{i+1}^-, u_{i+2}\} \cong \triangle$, a contradiction. Hence, $|S_i| \leq 2$ for all $i \in \{1, 2, \ldots, t\}$.

Suppose that there is a segment S_i with $i \in \{1, 2, \ldots, t\}$ such that $|S_i| = 1$. Then $u_i^+ = u_{i+1}^-$. By Claim 5 and Claim 6, u_i^+ has no neighbor in $V(C) \setminus N_C(H)$. To avoid u_i^+ with $\{u_j; u_j^-, u_j^+, w\}$ inducing $K_1 \cup K_{1,3}$ ($j \neq i+1, j \neq i-1$), we have $u_i^+ u_j \in E(G)$ for all $j \in \{1, 2, \ldots, t\}$. Since G is 1-tough, there are two segments S_j and S_k ($k > j$) that are connected by an edge $u_j^+ u_{k+1}^-$ or $u_{j+1}^- u_k^+$. Since $|S_i| \leq 2$ for all $i \in \{1, 2, \ldots, t\}$ and by Claim 5, $|S_j| = |S_k| = 2$ in both cases. If $u_j^+ u_{k+1}^- \in E(G)$, then to avoid u_j^+ with $\{u_k; u_k^+, u_i^+, w\}$ inducing $K_1 \cup K_{1,3}$, we have $u_j^+ u_k \in E(G)$. Since G is $\triangle$-free, $k \neq j+1$ and $u_{j+1}^- u_k \notin E(G)$. Then u_{j+1}^- with $\{u_k; u_k^-, u_i^+, w\}$ induces a $K_1 \cup K_{1,3}$, a contradiction. If $u_{j+1}^- u_k^+ \in E(G)$, then $k > j+1$. To avoid u_{j+1}^- with $\{u_k; u_k^-, u_i^+, w\}$ inducing $K_1 \cup K_{1,3}$, we have $u_{j+1}^- u_k \in E(G)$, but then $\{u_{j+1}^-, u_k, u_k^+\} \cong \triangle$, a contradiction. Hence, $|S_i| = 2$ for all $i \in \{1, 2, \ldots, t\}$. □

Claim 8. For any component $H' \neq H$ of $G - V(C)$, $N_C(H') = N_C(H)$.

Proof. Suppose that H' has a neighbor in $V(C) \setminus N_C(H)$. By Claim 7, this neighbor of H' is either u_i^+ or u_i^- for some $i \in \{1, 2, \ldots, t\}$. Without loss of generality, we assume that $w' u_1^+ \in E(G)$, where $w' \in V(H')$. By Claim 5, $w' u_2^+ \notin E(G)$. To avoid u_2^+ with $\{u_1^+; u_1, u_1^{+2}, w'\}$ inducing $K_1 \cup K_{1,3}$, we have $u_1 u_2^+ \in E(G)$. To avoid u_3^+ with $\{u_1; u_1^+, u_2^+, w\}$ inducing $K_1 \cup K_{1,3}$, we have $u_1 u_3^+ \in E(G)$. To avoid u_2^{+2} with $\{u_1; u_1^+, u_3^+, w\}$ inducing $K_1 \cup K_{1,3}$, we have $u_1^+ u_2^{+2} \in E(G)$. We also have $w' u_2^{+2} \notin E(G)$; otherwise, $\{w', u_1^+, u_2^{+2}\} \cong \triangle$, a contradiction. But now w with $\{u_1^+; u_1^{+2}, u_2^{+2}, w'\}$ induces $K_1 \cup K_{1,3}$, a contradiction. Hence, $N_C(H') \subseteq N_C(H)$. Since we have chosen H arbitrarily, by symmetry we have $N_C(H) \subseteq N_C(H')$. Hence, $N_C(H') = N_C(H)$. □

Claim 9. $t = 3$.

Proof. Since G is 1-tough, there are at least two distinct segments that are connected by a path outside C. By Claim 6 and Claim 7, without loss of generality, we can assume $u_1^+ u_i^- \in E(G)$ ($i \geq 3$). To avoid u_i^+ with $\{u_1^+; u_1, u_2^-, u_i^-\}$ inducing $K_1 \cup K_{1,3}$, we have $u_1 u_i^+ \in E(G)$ or $u_2^- u_i^+ \in E(G)$. If $u_1 u_i^+ \in E(G)$, then $u_{i+1} \neq u_1$ and $u_{i+1}^- u_1 \notin E(G)$. To avoid u_{i+1}^- with

$\{u_1^+; u_1, u_2^-, u_i^-\}$ inducing $K_1 \cup K_{1,3}$, we have $u_{i+1}^- u_1^+ \in E(G)$. But then w with $\{u_1^+; u_2^-, u_i^-, u_{i+1}^-\}$ induces a $K_1 \cup K_{1,3}$, a contradiction. Hence, $u_1 u_i^+ \notin E(G)$. Suppose that $u_2^- u_i^+ \in E(G)$. If $i \neq t$, then $u_{i+1}^- \neq u_1^-$. To avoid u_i^+ with $\{u_1; u_1^+, u_1^-, w\}$ inducing $K_1 \cup K_{1,3}$, we have $u_i^+ u_1^- \in E(G)$. But then w with $\{u_i^+; u_1^-, u_2^-, u_{i+1}^-\}$ induces a $K_1 \cup K_{1,3}$, a contradiction. Hence, $i = t$.

If $i \neq 3$, then $u_{i-1} \neq u_2$. To avoid u_2^+ with $\{u_1^+; u_1, u_2^-, u_i^-\}$ inducing $K_1 \cup K_{1,3}$, we have $u_2^+ u_i^- \in E(G)$ or $u_2^+ u_1 \in E(G)$. If $u_2^+ u_i^- \in E(G)$, then w with $\{u_i^-; u_1^+, u_2^+, u_{i-1}^+\}$ induces a $K_1 \cup K_{1,3}$, a contradiction. If $u_2^+ u_1 \in E(G)$, then $u_3^- u_1 \notin E(G)$. To avoid u_3^- with $\{u_1^+; u_1, u_2^-, u_i^-\}$ inducing $K_1 \cup K_{1,3}$, we have $u_3^- u_1^+ \in E(G)$. Then w with $\{u_1^+; u_2^-, u_3^-, u_i^-\}$ induces a $K_1 \cup K_{1,3}$, a contradiction. Hence, $i = t = 3$. □

Since G is 1-tough and by Claims 5–9, without loss of generality, we can assume that $u_1^+ u_3^- \in E(G)$. To avoid u_3^+ with $\{u_1^+; u_1, u_2^-, u_3^-\}$ inducing $K_1 \cup K_{1,3}$, we have $u_2^- u_3^+ \in E(G)$. To avoid u_1^- with $\{u_3^-; u_3, u_2^+, u_1^+\}$ inducing $K_1 \cup K_{1,3}$, we have $u_1^- u_2^+ \in E(G)$. By Claims 5–9, there is no other edge joining a pair of nonadjacent vertices on C. Suppose that $H' \neq H$ is another component of $G - V(C)$. By Claim 8, $N_C(H') = \{u_1, u_2, u_3\}$. Assume that $w' \in V(H')$ and $w' u_1 \in E(G)$. Then u_3^+ with $\{u_1; u_1^+, w, w'\}$ induces a $K_1 \cup K_{1,3}$, a contradiction. Hence, H is the only component of $G - V(C)$. Recalling that $|V(H)| = 1$, it is clear that G is the Petersen graph. This completes the proof for Case 1.

Case 2. $\kappa(G) = 2$.
Suppose that $\{u_1, u_2\}$ is a cut set of G. Since G is 1-tough, $G - \{u_1, u_2\}$ has exactly two components, say H_1 and H_2. Since G is $\{\triangle, K_1 \cup K_{1,3}\}$-free, each of H_1 and H_2 is an induced path or an induced cycle. We again prove a number of claims in order to complete the proof.

Claim 10. If H_i is an induced cycle C' for $i \in \{1, 2\}$, then there are two consecutive vertices on C' such that one vertex is adjacent to u_1 and the other one is adjacent to u_2.

Proof. Suppose, by contradiction, that for any neighbor of u_1 on C', its predecessor and successor on C' are not adjacent to u_2. Since G is $\triangle$-free, any two vertices of $N_{H_i}(u_1) \cup N_{H_i}(u_2)$ are not consecutive on C'. Since C' is induced, by removing all the vertices of $N_{H_i}(u_1) \cup N_{H_i}(u_2)$ from G we get $|N_{H_i}(u_1) \cup N_{H_i}(u_2)| + 1$ components, contradicting the hypothesis that G is 1-tough. □

Claim 11. If H_i is an induced path P for $i \in \{1, 2\}$, then one end vertex of P is adjacent to u_1 and the other one is adjacent to u_2.

Proof. Suppose that $H_1 = P = p_1 p_2 \dots p_t$. If $t \leq 2$, then the claim clearly holds. Suppose that $t \geq 3$ and $\{p_1, p_t\} \in N(u_1) \setminus N(u_2)$. Let $p_i \in V(H_1)$ $(1 < i < t)$ be a vertex adjacent to u_2. If $u_1 u_2 \in E(G)$, then there is a vertex $x \in V(H_2)$ such that $x u_2 \notin E(G)$. Then x with $\{p_i; p_{i-1}, p_{i+1}, u_2\}$ induces a $K_1 \cup K_{1,3}$, a contradiction. Thus, $u_1 u_2 \notin E(G)$. If $|V(H_2)| \geq 2$, then there will also be a vertex in H_2 not adjacent to u_2, and similarly we can get an induced $K_1 \cup K_{1,3}$. Hence, $|V(H_2)| = 1$. Denote $V(H_2) = \{w\}$. If $t \geq 6$, then $p_3 u_1 \in E(G)$ and $p_4 u_1 \in E(G)$; otherwise, p_3 or p_4 with $\{u_1; p_1, p_t, w\}$ induces $K_1 \cup K_{1,3}$, a contradiction. Now $\{u_1, p_3, p_4\} \cong \triangle$, a contradiction. If $t = 5$, then to avoid p_3 with $\{u_1; p_1, p_t, w\}$ inducing $K_1 \cup K_{1,3}$, we have $p_3 u_1 \in E(G)$. To avoid u_2 with $\{u_1; p_1, p_3, p_t\}$ inducing $K_1 \cup K_{1,3}$, we have $p_3 u_2 \in E(G)$. To avoid inducing a triangle, $\{p_2, p_4\} \cap (N(u_1) \cup N(u_2)) = \emptyset$. Obviously, now $\{u_1, p_3\}$ is a cut set that induces three components consisting of p_1p_2, p_4p_5 and u_2w. This contradicts the hypothesis that G is 1-tough. If $t = 4$, then precisely one vertex of $\{p_2, p_3\}$ is adjacent to u_2. Without loss of generality, assume $p_2u_2 \in E(G)$. To avoid inducing a triangle, $p_2u_1 \notin E(G)$ and $p_3u_1 \notin E(G)$. Then $\{u_1, p_2\}$ is a cut set that induces three components consisting of p_1, p_3p_4 and u_2w. This contradicts the hypothesis that G is 1-tough. If $t = 3$, then $\{u_1, p_2\}$ is a cut set that induces three components consisting of p_1, p_3 and u_2w, contradicting the hypothesis that G is 1-tough. □

Using Claim 10 and Claim 11, it is clear that there is a cycle in G containing all the vertices of G. Hence, G is hamiltonian. This completes the proof of Theorem 1.4. □

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