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ORIGINAL PAPER

Toughness, Forbidden Subgraphs and Pancyclicity

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Abstract

Motivated by several conjectures due to Nikoghosyan, in a recent article due to Li et al., the aim was to characterize all possible graphs H such that every 1-tough H-free graph is hamiltonian. The almost complete answer was given there by the conclusion that every proper induced subgraph H of $K_1 \cup P_4$ can act as a forbidden subgraph to ensure that every 1-tough H-free graph is hamiltonian, and that there is no other forbidden subgraph with this property, except possibly for the graph $K_1 \cup P_4$ itself. The hamiltonicity of 1-tough $K_1 \cup P_4$-free graphs, as conjectured by Nikoghosyan, was left there as an open case. In this paper, we consider the stronger property of pancyclicity under the same condition. We find that the results are completely analogous to the hamiltonian case: every graph H such that any 1-tough H-free graph is hamiltonian also ensures that every 1-tough H-free graph is pancyclic, except for a few specific classes of graphs. Moreover, there is no other forbidden subgraph having this property. With respect to the open case for hamiltonicity of 1-tough $K_1 \cup P_4$-free graphs we give infinite families of graphs that are not pancyclic.

Keywords Toughness · Forbidden subgraph · Pancyclic graph · Hamiltonian graph

Mathematics Subject Classification 05C38 · 05C42 · 05C45

1 Introduction

In this paper, we consider only undirected, finite and simple graphs. The terminology and notation not defined here can be found in [6].

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Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $u \in V(G)$ and a subgraph H of G, the *neighborhood* of u in H is denoted by $N_H(u) = \{v \in V(H) \mid uv \in E(G)\}$, and the *degree* of u in H is denoted by $d_H(u) = |N_H(u)|$. The *length* of a path or cycle is its number of edges. For two vertices u and v in a connected graph H, the *distance* between u and v in H, denoted by $d_H(u, v)$, is the length of a shortest (u, v)-path in H, which is a path connecting u and v. When there is no danger of ambiguity, we use $N(u)$, $d(u)$ and $d(u, v)$ instead of $N_G(u)$, $d_G(u)$ and $d_G(u, v)$, respectively. The *girth* of a graph G is the length of a shortest cycle of G, and the *circumference* is the length of a longest cycle of G. The *complete bipartite graph* on $m + n$ vertices, denoted by $K_{m,n}$, consists of a vertex set $A \cup B$ with $|A| = m > 0$, $|B| = n > 0$, and $A \cap B = \emptyset$, and the edge set $\{uv \mid u \in A, v \in B\}$. We say that $K_{m,n}$ is *balanced* if $m = n$. For a subset M of $E(G)$, we say M is a *matching* of G if no two edges of M share an end vertex, and we say a matching M is *perfect* if $V(M) = V(G)$. We use $|M|$ to denote the number of edges of M.

Let $\omega(G)$ denote the number of components of the graph G. As introduced in [11], a connected graph G is said to be *t-tough* if $t \cdot \omega(G - X) \leq |X|$ for all $X \subseteq V(G)$ with $\omega(G - X) > 1$. The *toughness* of G, denoted $\tau(G)$, is the maximum value of t such that G is t-tough (taking $\tau(K_n) = \infty$ for all $n \geq 1$). For a subset S of $V(G)$, we use $\langle S \rangle$ to denote the subgraph of G induced by S. If the subgraph induced by S is isomorphic to a graph H, we also write $S \cong H$. For a given graph H, we say G is *H-free* if G does not contain an induced copy of H.

A cycle (path) in a graph G is called a *Hamilton cycle* (*Hamilton path*) if it contains all vertices of G, and G is called *hamiltonian* if it contains a Hamilton cycle. G is called *hamiltonian-connected* if there is a Hamilton path between any two distinct vertices of G. A graph of order n is called *pancyclic* if it contains cycles of any length from 3 up to n. Obviously, both hamiltonian-connected graphs (on $n \geq 3$ vertices) and pancyclic graphs are hamiltonian graphs. A lot of research has been devoted to these hamiltonian properties, and it has been observed that various sufficient conditions for a graph to be hamiltonian are so strong that they imply considerably more about the cycle structure of the graph. Based on this observation, Bondy [5] presented a metaconjecture in 1971 in which he stated that almost any nontrivial condition on a graph which implies that the graph is hamiltonian also implies that it is pancyclic (except possibly for a simple family of well-characterized exceptional graphs). For hamiltonicity, Chvátal's Conjecture [11] states that there exists a constant t_0 such that every t_0-tough graph on $n \geq 3$ vertices is hamiltonian, and it is proved in [2] that $t_0 \geq 9/4$. For pancyclicity, the following theorem shows that there exists no such constant.

Theorem 1 (Brandt [7]) *There are t-tough graphs with t arbitrarily large which are not weakly pancyclic.*

Here a graph is called *weakly pancyclic* if it contains cycles of every length between the girth and the circumference. Hence, if a graph is not weakly pancyclic, then it is also not pancyclic. For more results about toughness and hamiltonian properties, we refer the interested reader to [1, 3, 9].

Forbidden subgraph conditions are an important type of sufficient conditions for the existence of Hamilton cycles and other hamiltonian properties of graphs. These structural conditions have a direct effect on the cycle and path properties of graphs. We refer to [8, 19–22] for more research about the structural implications and the cycle and path properties related to forbidden subgraph conditions. Over the years, researchers have established full characterizations of all possible single forbidden graphs and pairs of forbidden subgraphs ensuring that every 2-connected graph is hamiltonian. Some of these forbidden subgraph results give support for Bondy's metaconjecture, as shown by the following two theorems. We refer to Fig. 1 for some of the graphs that appear in the below statements.

Theorem 2 (Bedrossian [4]; Faudree and Gould [12]) *Let R and S be connected graphs with $R,\ S \neq P_3$, and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W.*

Theorem 3 (Bedrossian [4]) *Let R and S be connected graphs with $R,\ S \neq P_3$, and let G be a 2-connected graph which is not a cycle. Then G being $\{R, S\}$-free implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, Z_1$ or Z_2.*

It can be observed that many of the nonhamiltonian graph families that show the necessity of forbidding certain subgraphs are not 1-tough. This fact caused researchers to consider using the necessary condition of being 1-tough instead of 2-connected. In [18], Nikoghosyan posed several conjectures relating toughness and forbidden subgraph conditions to hamiltonicity. Motivated by one of these conjectures, Li et al. [16] considered single forbidden subgraphs under the condition of 1-toughness, and came up with the following results. Here, $G_1 \cup G_2$ denotes the disjoint union of two vertex-disjoint graphs G_1 and G_2, and kG denotes the disjoint union of k copies of the graph G.

Theorem 4 (Li et al. [16]) *Let R be an induced subgraph of P_4, $K_1 \cup P_3$ or $2K_1 \cup K_2$. Then every R-free 1-tough graph on at least three vertices is hamiltonian.*

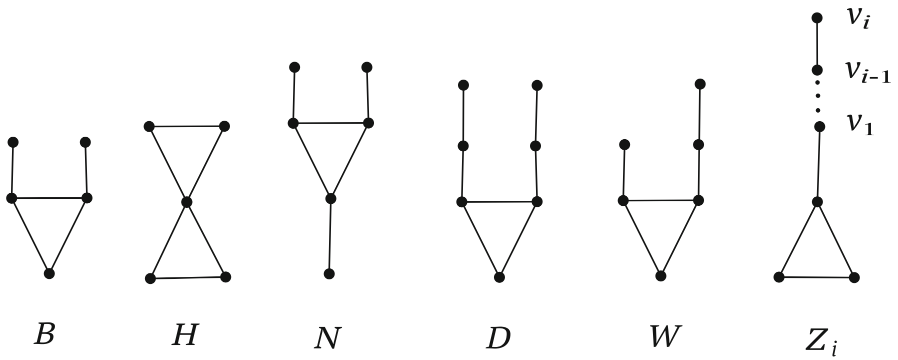


Fig. 1 Graphs B, H, N, D, W and Z_i

Theorem 5 (Li et al. [16]) *Let R be a graph on at least three vertices. If every R-free 1-tough graph on at least three vertices is hamiltonian, then R is an induced subgraph of $K_1 \cup P_4$.*

The question whether every 1-tough $K_1 \cup P_4$-free graph is hamiltonian is still open, and seems very hard to answer. In fact, this was the conjecture of Nikoghosyan in [18] that motivated the work in [16]. Regarding hamiltonian-connectivity, one easily checks that every hamiltonian-connected graph has toughness strictly larger than one. In 1978, Jung [14] showed that for P_4-free graphs, the toughness condition $\tau(G) > 1$ is a necessary and sufficient condition for hamiltonian-connectivity.

Theorem 6 (Jung [14]) *Let G be a P_4-free graph. Then G is hamiltonian-connected if and only if $\tau(G) > 1$.*

Recently, this work was followed up by studying more cases of forbidden subgraphs for the property of hamiltonian-connectivity in [23]. By combining their results with Theorem 6, in [23] the authors concluded that the results on hamiltonian-connectivity are completely analogous to the hamiltonian case: every graph H such that any 1-tough H-free graph is hamiltonian also ensures that every H-free graph with toughness larger than one is hamiltonian-connected. And similarly, there is no other forbidden subgraph having this property, except possibly for the graph $K_1 \cup P_4$ itself.

Theorem 7 (Zheng et al. [23]) *Let R be an induced subgraph of $K_1 \cup P_3$ or $2K_1 \cup K_2$. Then every R-free graph G with $\tau(G) > 1$ on at least three vertices is hamiltonian-connected.*

Theorem 8 (Zheng et al. [23]) *Let R be a graph on at least three vertices. If every R-free graph G with $\tau(G) > 1$ on at least three vertices is hamiltonian-connected, then R is an induced subgraph of $K_1 \cup P_4$.*

Turning to pancyclicity, inspired by Bondy's metaconjecture, we examined whether the condition in Theorem 4 in fact implies pancyclicity, and we obtained the following three results. We postpone the proofs of these results to Sects. 3, 4 and 5, respectively.

Theorem 9 *Let G be a $K_1 \cup P_3$-free 1-tough graph on $n \geq 3$ vertices. Then G is pancyclic or $G \in \{C_5, K_{\frac{n}{2},\frac{n}{2}}\}$.*

Clearly, the latter case can only occur when n is even. For the next result we first define the graph C_6^+ and the class of graphs $\mathcal{K}^-$. The graph C_6^+ is obtained from C_6 by adding an edge between two vertices at distance 2 in C_6. The class $\mathcal{K}^-$ consists of all balanced bipartite graphs $K_{s,s} - M$ $(s \geq 2)$, where M is a matching of $K_{s,s}$ with $0 \leq |M| \leq s$.

Theorem 10 *Let G be a $2K_1 \cup K_2$-free 1-tough graph on at least three vertices. Then G is pancyclic or $G \in \mathcal{K}^- \cup \{C_5, C_6^+\}$.*

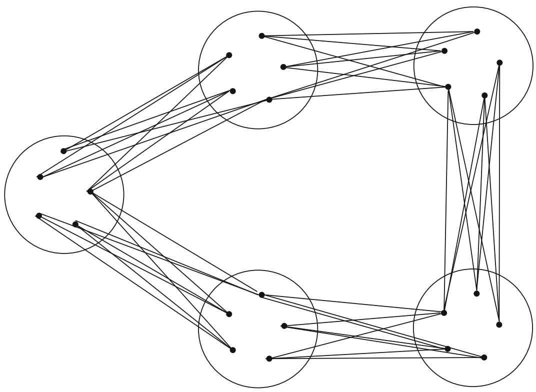
Theorem 11 *Let G be a P_4-free 1-tough graph on $n \geq 3$ vertices. Then G is pancyclic or $G = K_{\frac{n}{2},\frac{n}{2}}$.*

By Theorem 5, there is no graph H other than the induced subgraphs of $K_1 \cup P_4$ that can ensure every 1-tough H-free graph is hamiltonian. Hence, we obtain the following conclusion.

Theorem 12 *Let R be a graph on at least three vertices. If every R-free 1-tough graph G on at least three vertices is pancyclic, then R is an induced subgraph of $K_1 \cup P_4$.*

For the open case with $K_1 \cup P_4$, it is natural to ask whether every $K_1 \cup P_4$-free 1-tough graph G on at least three vertices is pancyclic (except possibly for some well-defined classes of graphs). Note that the exceptional graphs in the statements of Theorems 9–11 are also exceptions to $K_1 \cup P_4$-free 1-tough graphs being pancyclic, and they are C_5, C_6^+, $K_{\frac{n}{2},\frac{n}{2}}$ and $\mathcal{K}^-$. In fact, there are many other infinite classes of exceptional graphs for this statement. One of these classes is illustrated in Fig. 2. This class $\mathcal{C}_5^s$ consists of graphs that are obtained from a C_5 by replacing each vertex v of the C_5 by an independent set I_v of cardinality $s \geq 1$, and adding all edges between I_u and I_v whenever uv is an edge of the C_5. These graphs clearly contain no C_3, so they are not pancyclic. They are hamiltonian (even if the sets I_v have different cardinalities, as long as the graphs are 1-tough. We refer to [10], where these graphs are called C_5^*-type graphs and treated as special cases of triangle-free $2K_2$-free graphs). Using that C_5 is $K_1 \cup P_4$-free, it is easy to check that all these graphs are $K_1 \cup P_4$-free. There are basically two choices for cut sets that should be considered for determining the toughness. One option is to delete two nonconsecutive sets I_v and I_w, resulting in $s + 1$ components; the other option is to delete an additional set I_z, resulting in $2s$ components. The latter option determines the toughness if $s \geq 3$, i.e., the toughness of these graphs is $\frac{3}{2}$ if $s \geq 3$. Hence, the class of graphs $\mathcal{C}_5^s$ shows that even with a toughness strictly larger than one, there exist infinitely many exceptional graphs to the above statement. In a similar way, one can define the classes $\mathcal{C}_4^s$ (balanced complete bipartite graphs $K_{2s,2s}$) and $\mathcal{C}_6^s$, based on a C_4 and C_6,

Fig. 2 Graphs $\mathcal{C}_5^s$



respectively. Graphs from these classes are also not pancyclic, $K_1 \cup P_4$-free, and they have toughness equal to one. Here, we pose the following problem.

Problem 1 Except for C_6^+ and the graphs from $K_{\frac{n}{2},\frac{n}{2}}$, $\mathcal{K}^-$, $\mathcal{C}_4^s$, $\mathcal{C}_5^s$, and $\mathcal{C}_6^s$, are there any other 1-tough $K_1 \cup P_4$-free graphs on at least three vertices that are not pancyclic?

In Theorems 9–11, all the exceptional graphs have toughness exactly one. Hence, recalling the above remarks on the class $\mathcal{C}_5^s$, we know that forbidding any proper induced subgraph of $K_1 \cup P_4$ can ensure that a graph on at least three vertices with toughness larger than 1 is pancyclic, and $K_1 \cup P_4$ itself does not have this property. In fact, we can prove that only the proper induced subgraphs of $K_1 \cup P_4$ have this property.

Theorem 13 *Let R be a graph, and let G be a graph on at least three vertices with $\tau(G) > 1$. Then G being R-free implies G is pancyclic if and only if R is a proper induced subgraph of $K_1 \cup P_4$.*

The remainder of the paper is devoted to the proofs of our main results, but we start with a short section containing some preliminaries.

2 Preliminaries

We call a cycle with m vertices an *m-cycle*. Let C be an m-cycle of G with a given orientation, and denoted as $C = x_1x_2\ldots x_mx_1$. For a vertex $x_i \in V(C)$ $(1 \le i \le m)$, let x_i^{-l}, x_i^{+l} $(1 \le i - l < i + l \le m)$ denote the vertices x_{i-l} and x_{i+l} on C, respectively. Instead of x_i^{-1} and x_i^{+1}, we simply use x_i^- and x_i^+ to denote the immediate predecessor and successor of x_i on C, respectively. For two vertices $x_i, x_j \in V(C)$, x_iCx_j denotes the subpath of C from x_i to x_j, and $x_j\overline{C}x_i$ denotes the path from x_j to x_i in the reverse direction. For any $I \subseteq V(C)$, let $I^- = \{x_i^- \mid x_i \in I\}$ and $I^+ = \{x_i^+ \mid x_i \in I\}$. A similar notation is used for paths. In the proofs, we often use $\{u, v, w, x\} \cong H$ as shorthand for $\{u, v, w, x\}$ induces a copy of H in G.

The main idea of our proofs of Theorems 9–11 is as follows. First we consider a shortest cycle of the graph G. In case G does not contain some specific short cycles, we characterize G as one of the exceptional graphs. For the other case, we prove by contradiction that if G contains a k-cycle, then it also contains a $(k+1)$-cycle for any integer $k \in \{3, 4, \ldots, n-1\}$. By induction, this is sufficient to show that G is pancyclic. All our proofs are modelled along these lines and look similar, but contain different argumentations. In particular, our proofs considerably differ in length.

3 Proof of Theorem 9

Suppose that G is a graph satisfying the conditions of Theorem 9. Since G is 1-tough, G contains a cycle. Clearly, the shortest (induced) cycle of G is C_3, C_4 or C_5; otherwise, G has an induced subgraph isomorphic to $K_1 \cup P_3$. We first prove the following claim to characterize the exceptional graphs.

Claim 1 *If the shortest cycle of G is C_5, then G is C_5; if the shortest cycle of G is C_4, then G is $K_{s,s}$ $(s \geq 2)$.*

Proof Suppose that the shortest cycle of G is C_5. Let $C = v_1v_2...v_5v_1$ be a shortest cycle. Then C is an induced cycle. If $|V(G)| = 5$, then $G = C_5$, and the claim holds. Now we assume that $|V(G)| \geq 6$ and x is a vertex of $G - V(C)$. To avoid $\{x, v_1, v_2, v_3\}$ inducing a $K_1 \cup P_3$, we have that x is adjacent to v_1, v_2 or v_3. Without loss of generality, we assume $xv_1 \in E(G)$. If $N(x) \cap \{v_2, v_3, v_4\} \neq \emptyset$, then G has a C_3 or C_4, contradicting the fact that C_5 is a shortest cycle. If $N(x) \cap \{v_2, v_3, v_4\} = \emptyset$, then $\{x, v_2, v_3, v_4\}$ induces a copy of $K_1 \cup P_3$, a contradiction. Hence, G has 5 vertices and G is C_5.

Suppose that the shortest cycle of G is C_4. Let $C = v_1v_2...v_4v_1$ be a shortest cycle. Hence C is an induced cycle. If G has 4 vertices, then $G = C_4 = K_{2,2}$, and the claim holds. If G is not C_4, then there is a vertex x_0 in $G - V(C)$. Similar to the above case, without loss of generality, we assume $x_0v_1 \in E(G)$. To avoid inducing $K_1 \cup P_3$ and C_3, we have that $N(x_0) \cap \{v_2, v_3, v_4\} = \{v_3\}$. Using the same arguments, we have that every vertex x_i of $G - V(C)$ has two neighbors on C, which are $\{v_1, v_3\}$ or $\{v_2, v_4\}$. For two vertices $x_i, x_i \in G - V(C)$, if x_i and x_j have the same neighbors on C, then $x_ix_j \notin E(G)$; otherwise x_i, x_j and one of their neighbors on C induce a copy of C_3, a contradiction. If x_i and x_j have different neighbors on C, without loss of generality assume that $N_C(x_i) = \{v_1, v_3\}$ and $N_C(x_j) = \{v_2, v_4\}$. Then $x_ix_j \in E(G)$; otherwise $\{x_i, x_j, v_2, v_4\} \cong K_1 \cup P_3$, a contradiction. Now denote $A = N_G(v_1, v_3)$, $B = N_G(v_2, v_4)$. Then $V(G) = A \cup B$. Moreover, A, B are two independent vertex sets and every vertex of A is adjacent to every vertex of B. Hence, G is a complete bipartite graph, and according to the toughness, G is a balanced complete bipartite graph $K_{s,s}$. □

By Claim 1, we see that if G has no C_3, then G is either C_5 or $K_{s,s}$. Next, we suppose that G is neither C_5 nor $K_{s,s}$. This implies that G contains a C_3. We show that G is pancyclic by proving the following fact.

Fact If G has a k-cycle $(k = 3, 4, ..., n - 1)$, then G has a $(k + 1)$-cycle.

Proof Suppose, by contradiction, that G has a k-cycle, but no $(k + 1)$-cycle, for some $k \in \{3, 4, ..., k - 1\}$. Let $C = v_1v_2...v_kv_1$ be a k-cycle, and let H be a component of $G - V(C)$. Since G is 1-tough, H has at least two neighbors on C. We distinguish two cases.

Case 1. H has two neighbors that are consecutive on C.

Clearly, H is not trivial in this case. We prove a number of claims before we complete the proof for this case.

Claim 2 *H contains an edge ab such that* $av_i \in E(G)$ *and* $bv_{i+1} \in E(G)$, $i \in \{1, 2, \ldots, k\}$.

Proof Choose two consecutive neighbors of H on C such that their distance in H is shortest. Without loss of generality, assume that v_1, v_2 are such vertices, and $a, b \in V(H)$ are neighbors of v_1 and v_2, respectively. We have that $a \neq b$; otherwise G has a $(k+1)$-cycle $v_1av_2Cv_1$, contradicting the assumption that G has no $(k+1)$-cycle. Let $P = ax_1x_2\ldots x_sb$ be a shortest (a, b)-path in H. Clearly, $V(P) \cap N(v_1) = \{a\}$ and $V(P) \cap N(v_2) = \{b\}$. Then we have $s \leq 1$; otherwise $\{v_2, a, x_1, x_2\} \cong K_1 \cup P_3$, a contradiction. Suppose that $s = 1$ and $P = ax_1b$. By the choice of v_1, v_2 and the assumption that G has no $(k+1)$-cycle, we have that $\{x_1, b\} \cap N(v_3) = \emptyset$ and $\{a, x_1\} \cap N(v_k) = \emptyset$. If $k = 3$, i.e., $v_k = v_3$, then $\{v_k, a, x_1, b\} \cong K_1 \cup P_3$, a contradiction. Assume that $k \geq 4$. To avoid $\{v_k, a, x_1, b\}$ and $\{v_3, a, x_1, b\}$ inducing $K_1 \cup P_3$, we have $bv_k \in E(G)$ and $av_3 \in E(G)$. To avoid $\{v_3, x_1, b, v_k\}$ inducing $K_1 \cup P_3$, we have $v_3v_k \in E(G)$. If $k = 4$, then $v_1ax_1bv_2v_1$ is a $(k+1)$-cycle, a contradiction. If $k \geq 5$, then according to the choice of v_1, v_2 and the assumption that G has no $(k+1)$-cycle we have that $v_4a, v_4x_1 \notin E(G)$. To avoid $\{v_4, a, x_1, b\}$ inducing $K_1 \cup P_3$, we have $v_4b \in E(G)$, but then $v_3ax_1bv_4Cv_kv_3$ is a $(k+1)$-cycle, a contradiction. Hence, $s = 0$ and the claim holds. □

By Claim 2 and the assumptions, we have that $k \geq 4$. Without loss of generality, assume that $ab \in E(H)$ and $av_1 \in E(G)$, $bv_2 \in E(G)$. We next prove the following claim.

Claim 3 $av_3 \in E(G)$.

Proof Suppose that $av_3 \notin E(G)$. To avoid $\{v_3, b, a, v_1\}$ inducing $K_1 \cup P_3$, we have $v_1v_3 \in E(G)$. We also have that $av_4 \notin E(G)$ and $bv_4 \notin E(G)$; otherwise, $v_2bav_4Cv_2$ or $v_2bv_4Cv_1v_3v_2$ is a $(k+1)$-cycle, respectively. To avoid $\{v_4, a, b, v_2\}$ inducing $K_1 \cup P_3$, we have $v_2v_4 \in E(G)$. Then $v_1abv_2v_4Cv_1$ is a $(k+1)$-cycle, a contradiction. □

We now divide $V(C)$ into two sets. Let $A = \{v_i \mid i \text{ is odd}, 1 \leq i \leq k\}$, and $B = \{v_i \mid i \text{ is even}, 1 \leq i \leq k\}$. Then clearly $V(C) = A \cup B$. We prove the following claim on the structure of A and B.

Claim 4 *k is even, and* $A \subseteq N(a) \backslash N(b)$, $B \subseteq N(b) \backslash N(a)$. *Moreover, A and B are independent sets, and each vertex of A is adjacent to each vertex of B.*

Proof We use induction to prove that $A \subseteq N(a) \backslash N(b)$ and $B \subseteq N(b) \backslash N(a)$. First, we show that $v_4 \in N(b) \backslash N(a)$. Since $av_3 \in E(G)$, $av_4 \notin E(G)$. We also have $v_2v_4 \notin E(G)$; otherwise, $v_1abv_2v_4Cv_1$ is a $(k+1)$-cycle, a contradiction. Hence $v_4 \in N(b)$; otherwise $\{v_4, a, b, v_2\} \cong K_1 \cup P_3$, a contradiction. Next, we show that if $v_{s-1} \in N(a), v_s \in N(b)$ $(s \geq 4)$, then $v_{s+1} \in N(a) \backslash N(b)$. First, $v_{s+1} \notin N(b)$ since $v_s \in N(b)$. We have $v_{s-1}v_{s+1} \notin E(G)$; otherwise, $v_1abv_2Cv_{s-1}v_{s+1}Cv_1$ is a $(k+1)$-cycle, a contradiction. To avoid $\{v_{s+1}, b, a, v_{s-1}\}$ inducing a $K_1 \cup P_3$, we have $v_{s+1} \in N(a)$. By a similar analysis, we get that if $v_{s-1} \in N(b), v_s \in N(a)$ $(s \geq 4)$, then $v_{s+1} \in N(b) \backslash N(a)$. Thus, $V(C) \subseteq N(a) \cup N(b)$, and $N_C(a)$ and $N_C(b)$ occur alternately on C. Since $v_1 \in N(a)$, $v_k \in N(b)$. Hence, k is even, and $A \subseteq N(a) \backslash N(b)$,

$B \subseteq N(b) \backslash N(a)$.

Suppose that $v_i, v_j \in A$ and $v_i v_j \in E(G)$. Then $v_{i+1}, v_{j+1} \in B \subseteq N(b)$, and there is a $(k+1)$-cycle $v_{i+1} b v_{j+1} C v_i v_j \overline{C} v_{i+1}$, a contradiction. Hence, A is an independent set. Similarly, B is also an independent set. Suppose that there is a pair of vertices $v_i \in A, v_j \in B$ such that $v_i v_j \notin E(G)$. We have that $v_j \neq v_{i+1}$, and since $\{v_j, v_{i+1}\} \in B$, we have $v_j v_{i+1} \notin E(G)$. Then $\{v_j, a, v_i, v_{i+1}\} \cong K_1 \cup P_3$, a contradiction. Hence, $v_i v_j \in E(G)$ for any $v_i \in A$, $v_j \in B$, and the claim holds. □

We need two more claims before we can complete our proof for this case.

Claim 5 $G - V(C) = H$.

Proof Suppose, by contradiction, that H' is another component distinct from H of $G - V(C)$. Then there is a vertex $y \in V(H')$ such that y has a neighbor on C. By symmetry, assume $y v_1 \in E(G)$. If y has another neighbor v_i on C distinct from v_1, then $v_i \notin A$; otherwise, $\{b, v_1, y, v_i\} \cong K_1 \cup P_3$, a contradiction. Thus, $v_i \in B \backslash \{v_2, v_k\}$ and $k \neq 4$. By Claim 4, $v_{i+1} v_2 \in E(G)$. Then $v_i y v_1 \overline{C} v_{i+1} v_2 C v_i$ is a $(k+1)$-cycle, a contradiction. Hence y has only one neighbor on C. Since G is 1-tough, there is another vertex y' in $V(H')$, and y' has a neighbor on C distinct from v_1. By the same arguments, y' has only one neighbor on C, say v_i. If $yy' \in E(G)$, then $\{b, y', y, v_1\} \cong K_1 \cup P_3$, a contradiction. If $yy' \notin E(G)$, then there is an induced P_3 in H', and this induced P_3 together with the vertex a in H will induce a $K_1 \cup P_3$, a contradiction. Hence, $G - V(C) = H$. □

By Claim 5, we know that H is the only component of $G - V(C)$, and ab is an edge of H. We denote $A' = N_H(a)$ and $B' = N_H(b)$.

Claim 6 *The following properties hold*:

(1) $V(H) = A' \cup B'$ *and* $A' \cap B' = \emptyset$.
(2) *each vertex of* A' *is adjacent to each vertex of* B, *and each vertex of* B' *is adjacent to each vertex of* A.
(3) $A \cup A'$ *and* $B \cup B'$ *are independent sets.*
(4) *each vertex of* A' *is adjacent to each vertex of* B'.

Proof We prove the properties in the same order.

(1) If $V(H) = \{a, b\}$, then the claim holds. Now we suppose that $V(H) \neq \{a, b\}$ and by contradiction, we suppose that $V(H) \neq A' \cup B'$. There is a vertex $x \in V(H) \backslash (A' \cup B')$ such that $x a_1 \in E(G)$ or $x b_1 \in E(G)$, where $a_1 \in A'$ and $b_1 \in B'$. Without loss of generality, assume that $x a_1 \in E(G)$. If x is adjacent to a vertex v_i of B, then by Claim 4 we have that $v_i v_{i+3} \in E(G)$, and there is a $(k+1)$-cycle $v_{i-1} a a_1 x v_i v_{i+3} C v_{i-1}$ (if $k = 4$, $v_{i+3} = v_{i-1}$), a contradiction. Thus, x has no neighbor in B. For a vertex v_i of B, $\{x, a, b, v_i\}$ induces a $K_1 \cup P_3$, a contradiction. Hence, $V(H) = A' \cup B'$. Suppose that $x' \in A' \cap B'$. Then there is a $(k+1)$-cycle $v_1 a x' b v_2 v_5 C v_1$ (possible $v_5 = v_1$), a contradiction. Hence, $A' \cap B' = \emptyset$.

(2) Let a_1 be a vertex of A'. From (1), we have $ba_1 \notin E(G)$. Since $k \geq 4$, $|A| = |B| \geq 2$. If there are two vertices $v_i, v_j \in B$ such that $a_1 v_i \notin E(G)$ and $a_1 v_j \notin E(G)$, then $\{a_1, v_i, b, v_j\} \cong K_1 \cup P_3$, a contradiction. Thus, there is at most one vertex of B that is not adjacent to a_1. Suppose that $a_1 v_i \notin E(G)$ and $a_1 v_j \in E(G)$ $(v_i, v_j \in B)$. Then $\{v_i, a, a_1, v_j\} \cong K_1 \cup P_3$, a contradiction. Hence, a_1 is adjacent to each vertex of B. By the arbitrary selection of a_1, each vertex of A' is adjacent to each vertex of B. By symmetry, each vertex of B' is adjacent to all the vertices of A.

(3) First, A' is independent set; otherwise, suppose $a_1, a_2 \in A'$ and $a_1 a_2 \in E(G)$. Then by (2), there is a $(k+1)$-cycle $v_1 a a_1 a_2 v_4 C v_1$, a contradiction. Similarly, B' is also an independent set. Next, $N(A') \cap A = \emptyset$; otherwise, suppose that $a_1 \in A'$ and $v_i \in A$ such that $a_1 v_i \in E(G)$. Then there is a $(k+1)$-cycle $v_{i-2} a a_1 v_i C v_{i-2}$, a contradiction. Hence, $A \cup A'$ is an independent set. Similarly, $B \cup B'$ is also an independent set.

(4) Suppose that $a_1 \in A', b_1 \in B'$ and $a_1 b_1 \notin E(G)$. By (2) and (3), for any pair of vertices $v_i, v_j \in A$, $\{a_1, v_i, b_1, v_j\} \cong K_1 \cup P_3$, a contradiction. Hence, each vertex of A' is adjacent to each vertex of B'.

□

From Claims 4–6, we have that G is a complete bipartite graph with two independent sets $A \cup A'$ and $B \cup B'$. Since G is 1-tough, G is a balanced complete bipartite graph $K_{s,s}$, contradicting the assumption. This completes the proof for Case 1.

Case 2. For every component H of $G - V(C)$, any two neighbors of H on C are not consecutive.

Let $N_C(H) = \{u_1, u_2, \ldots, u_s\}$ $(s \geq 2$, and with all u_i chosen in this order according to the orientation of $C)$. The s neighbors of H on C divide the cycle C into s segments, denoted by $S_i = u_i^+ C u_{i+1}$ $(i = 1, 2, \ldots, s$, and with $u_{s+1} = u_1)$, so with $|S_i| \geq 2$ for any $i \in \{1, 2, \ldots, s\}$. Since G is 1-tough, there are at least two segments that are connected by a path internally-disjoint with $V(C) \cup V(H)$. As we will see, the choice of these two segments is irrelevant for the remainder of the proof. So we ignore the indices, and assume without loss of generality that S_1 and S_2 are connected by such a path. We choose a path P with two end vertices $y \in S_1, y' \in S_2$ such that the path $u_1^+ C y P y' \overline{C} u_2^+$ is as short as possible. Then we have that $|V(P)| = 2$ and $y = u_1^+, y' = u_2^+$; otherwise, the path $u_1^+ C y P y' \overline{C} u_2^+$ contains an induced P_3. Combining that induced P_3 with one vertex of H we get an induced $K_1 \cup P_3$, a contradiction. Thus, $u_1^+ u_2^+ \in E(G)$. Suppose that $au_1 \in E(G), bu_2 \in E(G)$ for $a, b \in V(H)$. We have $a \neq b$; otherwise, $u_1 a u_2 \overline{C} u_1^+ u_2^+ C u_1$ is a $(k+1)$-cycle. We also have that $ab \in E(G)$; otherwise, H contains an induced P_3, and combining that induced P_3 with u_1^+ we get an induced $K_1 \cup P_3$, a contradiction. If $u_1^+ = u_2^-$, then $u_1 a b u_2 C u_1$ is a $(k+1)$-cycle, a contradiction. Hence, $u_1^+ \neq u_2^-$. If $u_1^{++} u_2^+ \in E(G)$, then $u_1 a b u_2 \overline{C} u_1^{++} u_2^+ C u_1$ is a $(k+1)$-cycle, a contradiction. Hence, $u_1^{++} u_2^+ \notin E(G)$. Then $\{a, u_1^{++}, u_1^+, u_2^+\} \cong K_1 \cup P_3$, our final contradiction. □

This completes the proof of Theorem 9. □

4 Proof of Theorem 10

Suppose that G is a graph satisfying the conditions of Theorem 10. Since G is 1-tough, G contains a cycle. The shortest (induced) cycle of G is C_3, C_4, C_5 or C_6; otherwise, G clearly has an induced subgraph isomorphic to $2K_1 \cup K_2$. We again start by proving a number of claims.

Claim 1 *If the shortest cycle of G is C_5 or C_6, then G is C_5 or C_6, respectively.*

Proof Suppose that the shortest cycle of G is C_5. Let $C = v_1v_2\ldots v_5v_1$ be an induced 5-cycle. If G is not C_5, then there is a vertex $u \in G - V(C)$ such that u is adjacent to a vertex of C, say v_1. Since G has neither a C_3 nor a C_4, u has no other neighbor on C distinct from v_1. Now $\{u, v_2, v_4, v_5\} \cong 2K_1 \cup K_2$, a contradiction. In the same way, we can prove that if the shortest cycle of G is C_6, then G is C_6. □

Claim 2 *If the shortest cycle of G is C_4, then G is C_4 or $K_{s,s} - M$, where $s \geq 3$ and M is a matching of $K_{s,s}$ with $0 \leq |M| \leq s$.*

Proof Suppose that w is a vertex of G with the maximum degree. If $d(w) = 2$, then G is C_4, and the claim holds. Assume that $d(w) \geq 3$. Now we draw the graph G arranged as a rooted tree: w is the root and denoted as the first layer L_1, all the neighbors of w are arranged as the second layer L_2, all the new neighbors of vertices of L_2 (where new means that the vertices do not appear in the existing layers) are arranged as the third layer L_3, etc., until all layers together cover $V(G)$. By this labeling method, we know that there is no edge between L_i and L_j if $i \neq j - 1$ and $i \neq j + 1$. Since G has no C_3, L_2 is an independent set. Next we prove three subclaims on the structure of the layers.

Claim 2.1 *G has at most 4 layers.*

Proof Suppose that G has 5 or more layers. Let u_1, u_2 be two vertices of L_2, and let x and y be two vertices of L_4 and L_5, respectively, such that $xy \in E(G)$. Then $\{u_1, u_2, x, y\} \cong 2K_1 \cup K_2$, a contradiction. □

Let $l_i = |L_i|$ for $i = 1, 2, 3, 4$. Then we denote $L_1 = \{w\}$, $L_2 = \{u_1, u_2, \ldots, u_{l_2}\}$, $L_3 = \{v_1, v_2, \ldots, v_{l_3}\}$, $L_4 = \{z_1, z_2, \ldots, z_{l_4}\}$.

Claim 2.2 $L_4 = \{z_1\}$ *or* $L_4 = \emptyset$.

Proof If L_4 has 2 or more vertices, then $\{z_1, z_2, w, u_1\} \cong 2K_1 \cup K_2$ (if $z_1z_2 \notin E(G)$) or $\{u_1, u_2, z_1, z_2\} \cong 2K_1 \cup K_2$ (if $z_1z_2 \in E(G)$), a contradiction. □

Claim 2.3 *L_3 is independent set.*

Proof By contradiction, suppose that v_iv_j is an edge of L_3. Since G has no C_3, v_i and v_j have no common neighbor in L_2. Assume that $u_iv_i, u_jv_j \in E(G)$ and $u_i \neq u_j$. We have that u_i, u_j have no common neighbor in L_3. Otherwise, suppose that v_k is a common neighbor of u_i and u_j in L_3. Obviously, $v_iv_k, v_jv_k \notin E(G)$; otherwise G contains a C_3. Then $\{w, v_k, v_i, v_j\} \cong 2K_1 \cup K_2$, a contradiction. Moreover, $N_{L_3}(u_i) = \{v_i\}$ and $N_{L_3}(u_j) = \{v_j\}$. Otherwise, suppose that $v_k \in N_{L_3}(u_i) \backslash \{v_i\}$. Clearly,

$v_iv_k \notin E(G)$; otherwise G contains a C_3. Then $\{v_i, v_k, w, u_j\} \cong 2K_1 \cup K_2$, a contradiction. For any vertex $u_k \in L_2 \backslash \{u_i, u_j\}$, we have that $N_{L_3}(u_k) = \{v_i\}$ or $N_{L_3}(u_k) = \{v_j\}$. Otherwise, suppose that v_k is a vertex of L_3 different from v_i, v_j that is adjacent to u_k. Then $\{u_i, u_j, u_k, v_k\} \cong 2K_1 \cup K_2$, a contradiction. Thus, $L_3 = \{v_1, v_2\}$ and every vertex of L_2 has degree 2. Since G is 1-tough and L_2 is independent, $l_2 \leq 3$. Since $l_2 = d(w) \geq 3$, $l_2 = 3$. Without loss of generality, assume that $u_1v_1, u_2v_2, u_3v_2 \in E(G)$. Then $\{u_2, u_3, u_1, v_1\} \cong 2K_1 \cup K_2$, a contradiction. $\square$

To complete the proof of Claim 2, we consider two cases.

Case A. $L_4 = \{z_1\}$. Suppose that v_1 is a neighbor of z_1 in L_3. Since G is 1-tough, the two independent sets L_2 and L_3 obviously have the same order, i.e., $l_2 = l_3 \geq 3$. If there is a vertex $v_i \in L_3 \backslash \{v_1\}$ such that $v_iz_1 \notin E(G)$, then $\{w, v_i, v_1, z_1\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $N(z_1) = L_3$. If there are two vertices $u_i, u_j \in L_2$ that are not adjacent to a vertex $v_k \in L_3$, then $\{u_i, u_j, v_k, z_1\} \cong 2K_1 \cup K_2$, a contradiction. If all the vertices of L_2 are neighbors of v_k, then $d(v_k) = l_2 + 1 > d(w)$, contradicting the assumption that w is a vertex with maximum degree. Thus, every vertex of L_3 has exactly $l_2 - 1$ neighbors in L_2. Similarly, every vertex of L_2 has exactly $l_3 - 1$ neighbors in L_3. Hence, G is a balanced bipartite graph $K_{s,s} - M^*$ with two independent vertex sets $L_2 \cup \{z_1\}$ and $L_3 \cup \{w\}$, where $s = l_2 + 1 = l_3 + 1$ and M^* is a perfect matching of $K_{s,s}$.

Case B. $L_4 = \emptyset$. Since G is 1-tough, $l_2 = l_3 + 1$. For a vertex $u_k \in L_2$, if there are two vertices $v_i, v_j \in L_3$ that are not neighbors of u_k, then $\{v_i, v_j, u_k, w\} \cong 2K_1 \cup K_2$, a contradiction. Thus, every vertex of L_2 has at least $l_3 - 1$ neighbors in L_3, and has degree at least l_3 in the graph G. For a vertex $v_k \in L_3$, suppose that $v_ku_k \in E(G)$. If there are two vertices $u_i, u_j \in L_2$ that are not neighbors of v_k, then $\{u_i, u_j, v_k, u_k\} \cong 2K_1 \cup K_2$, a contradiction. Thus, every vertex of L_3 has at least $l_2 - 1 = l_3$ neighbors in L_3. Hence, G is a balanced bipartite graph $K_{s,s} - M$ with two independent vertex sets L_2 and $\{w\} \cup L_3$, where $s = l_2 = l_3 + 1$ and M is a matching of $K_{s,s}$ with $0 \leq |M| \leq s - 1$.

In both cases, we conclude that G is either C_4 or $K_{s,s} - M$, where $s \geq 3$ and M is a matching of $K_{s,s}$ with $0 \leq |M| \leq s$. This completes the proof of Claim 2. $\square$

Claim 3 *If the minimum cycle of G is C_3 and $G \neq C_3$, then G contains C_4, unless $G = G_1$, where G_1 is the graph obtained by adding an edge to C_6 between two vertices at distance 2 in C_6.*

Proof Suppose that $C = abca$ is a 3-cycle of G. For a component H of $G - V(C)$, H has at least two neighbors on C. If one vertex of H has two neighbors on C, then G contains C_4, and the claim holds. Assume that each vertex of H is adjacent to at most one vertex of C. Suppose that $a_1, b_1 \in V(H)$ and $aa_1, bb_1 \in E(G)$. If $a_1b_1 \in E(G)$, then G contains C_4, and the claim holds. Assume that $a_1b_1 \notin E(G)$. Let $P = a_1x_1x_2...x_sb_1$ be a shortest (a_1, b_1)-path in H. We prove the following claims.

Claim 3.1 $s = 1$, *i.e.*, $P = a_1x_1b_1$.

Proof Suppose that $s \geq 2$. Since P is a shortest path, $x_1b_1 \notin E(G)$. If $x_1c \in E(G)$, then G contains C_4, and the claim holds. Assume $x_1c \notin E(G)$. Then $\{b_1, c, a_1, x_1\} \cong 2K_1 \cup K_2$, a contradiction. □

Claim 3.2 $H = P$.

Proof Suppose, by contradiction, that $w \in V(H)\backslash\{a_1, x_1, b_1\}$. If $wa_1 \in E(G)$ and $wb_1 \in E(G)$, then $a_1wb_1x_1a_1$ is a 4-cycle, and the claim holds. Without loss of generality, we assume that $wb_1 \notin E(G)$. If $wa_1 \in E(G)$ and $wc \in E(G)$, then a_1wcaa_1 is a 4-cycle, and the claim holds. If $wa_1 \in E(G)$ and $wc \notin E(G)$, then $\{c, b_1, a_1, w\} \cong 2K_1 \cup K_2$, a contradiction. Thus, $wa_1 \notin E(G)$, and $N_H(a) \cup N_H(b) = \{x_1\}$. There must be a vertex $w' \in V(H)\backslash\{a_1, x_1, b_1\}$ such that $w'x_1 \in E(G)$ (possibly $w' = w$). If $w'b \in E(G)$, then $w'bb_1x_1w'$ is a 4-cycle, and the claim holds. Assume that $w'b \notin E(G)$. If $w'c \in E(G)$, then $\{a_1, b_1, w', c\} \cong 2K_1 \cup K_2$, a contradiction. If $w'c \notin E(G)$, then $\{a_1, w', b, c\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $H = P$. □

By Claim 3.2, $H = a_1x_1b_1$. If x_1 is adjacent to a vertex of C, then G contains C_4, and the claim holds. Assume that x_1 has no neighbor on C. If $G - V(C)$ has no other component than H, then G is G_1, and the claim holds. If H' is a component of $G - V(C)$ different from H, then H' is trivial; otherwise, an edge of H' with a_1, b_1 will induce a $2K_1 \cup K_2$, a contradiction. Since G is 1-tough, the vertex of H' has two neighbors on C. Thus, G contains C_4, and the claim holds. □

By Claims 1 and 2, if G has no 3-cycle, then G is C_4, C_5, C_6 or $K_{s,s} - M$, where $s \geq 3$ and M is a matching of $K_{s,s}$ with $0 \leq |M| \leq s$. By Claim 3, if G has a 3-cycle and G is not C_3, then G has a 4-cycle, unless G is G_1. Now we assume that $G \notin \{C_4, C_5, C_6, G_1, K_{s,s} - M\}$, hence that G has a C_3 and a C_4. Next we will show that G is pancyclic by proving the following fact.

Fact If G has a k-cycle ($k = 4, 5, \ldots, n-1$), then G has a $(k+1)$-cycle.

Proof Suppose, by contradiction, that G has a k-cycle but no $(k+1)$-cycle for some $k \in \{4, 5, \ldots, k-1\}$. Let $C = v_1v_2\ldots v_kv_1$ be a k-cycle, and let H be a component of $G - V(C)$. Since G is 1-tough, H has at least two neighbors on C. We distinguish the cases that H is trivial, i.e., $|V(H)| = 1$, and that all components of $G - V(C)$ contain at least one edge.

Case 1. H is trivial.

Suppose that $H = \{w\}$. Denote $N_C(w) = \{u_1, u_2, \ldots, u_s\}$, with $s \geq 2$, the vertices u_i chosen in this order according to the orientation of C, and taking $u_{s+1} = u_1$. Clearly, $u_{i+1} \neq u_i^+$ for any $i \in \{1, 2, \ldots, s\}$; otherwise G contains a $(k+1)$-cycle, a contradiction. Now, the s neighbors of H on C divide the cycle C into s segments. Let S_i be the segment of C from u_i^+ to u_{i+1}^-, denoted as $S_i = x_{i_1}x_{i_2}\ldots x_{i_{r_i}}$. We again prove a number of claims.

Claim 4 *For any $i \in \{1, 2, \ldots, s\}$, r_i is odd, and $x_{1_1}x_{i_j} \notin E(G)$ for every odd j, and $x_{1_1}x_{i_j} \in E(G)$ for every even j.*

Proof We divide the proof into two cases according to the length of the segment S_1.
Case A. $|S_1| = 1$. In this case, $S_1 = x_{1_1}$, and the claim holds for segment S_1 itself. For any segment S_i $(i = 2, 3, \ldots, s)$, we have that if $x_{1_1}x_{i_j} \in E(G)$, then $x_{1_1}x_{i_{j+1}} \notin E(G)$; otherwise, suppose that $x_{1_1}x_{i_j} \in E(G)$ and $x_{1_1}x_{i_{j+1}} \in E(G)$. Then there is a $(k+1)$-cycle $u_1wu_2Cx_{i_j}x_{1_1}x_{i_{j+1}}Cu_1$, a contradiction. If $x_{1_1}x_{i_j} \notin E(G)$, then $x_{1_1}x_{i_{j+1}} \in E(G)$; otherwise, suppose that $x_{1_1}x_{i_j} \notin E(G)$ and $x_{1_1}x_{i_{j+1}} \notin E(G)$. Then $\{w, x_{1_1}, x_{i_j}, x_{i_{j+1}}\} \cong 2K_1 \cup K_2$, a contradiction. For the first vertex x_{i_1} and the last vertex $x_{i_{r_i}}$ of a segment, we have $x_{1_1}x_{i_1} \notin E(G)$ and $x_{1_1}x_{i_{r_i}} \notin E(G)$; otherwise, $u_1wu_i\overline{C}x_{1_1}x_{i_1}Cu_1$ or $u_2wu_{i+1}Cx_{1_1}x_{i_{r_i}}\overline{C}u_2$ is a $(k+1)$-cycle, a contradiction. Thus, the neighbors of x_{1_1} occur alternately along the cycle on every segment S_i, and the two end vertices of S_i are not its neighbor. Hence, r_i is odd, and $x_{1_1}x_{i_j} \notin E(G)$ for every odd j, and $x_{1_1}x_{i_j} \in E(G)$ for every even j.
Case B. $|S_1| \geq 2$. First, we deal with the segment S_1. We have that $x_{1_1}x_{2_1} \notin E(G)$; otherwise, $u_1wu_2\overline{C}x_{1_1}x_{2_1}Cu_1$ is a $(k+1)$-cycle, a contradiction. We also have that $x_{1_2}x_{2_1} \in E(G)$; otherwise, $\{w, x_{2_1}, x_{1_1}, x_{1_2}\} \cong 2K_1 \cup K_2$, a contradiction. If $x_{1_1}x_{1_j} \in E(G)$ and $x_{1_1}x_{1_{j+1}} \in E(G)$, then we have a $(k+1)$-cycle $u_1wu_2\overline{C}x_{1_{j+1}}x_{1_1}x_{1_j}\overline{C}x_{1_2}x_{2_1}Cu_1$, a contradiction. If $x_{1_1}x_{1_j} \notin E(G)$ and $x_{1_1}x_{1_{j+1}} \notin E(G)$, then $\{w, x_{1_1}, x_{1_j}, x_{1_{j+1}}\} \cong 2K_1 \cup K_2$, a contradiction. For the last vertex $x_{1_{r_1}}$ of S_1, we will show that it is not a neighbor of x_{1_1}. Suppose that $|S_2| \geq 2$. We have that $x_{1_1}x_{2_2} \in E(G)$; otherwise, $\{w, x_{1_1}, x_{2_1}, x_{2_2}\} \cong 2K_1 \cup K_2$, a contradiction. If $x_{1_1}x_{1_{r_1}} \in E(G)$, then $u_1wu_2x_{2_1}x_{1_2}Cx_{1_{r_1}}x_{1_1}x_{2_2}Cu_1$ is a $(k+1)$-cycle, a contradiction. Suppose that $|S_2| = 1$. If $x_{1_1}x_{1_{r_1}} \in E(G)$, then $u_2wu_3Cx_{1_1}x_{1_{r_1}}\overline{C}x_{1_2}x_{2_1}u_2$ is a $(k+1)$-cycle, a contradiction. Hence, $x_{1_1}x_{1_{r_1}} \notin E(G)$. Then the neighbors of x_{1_1} on segment S_1 occur alternately along the cycle, and the first vertex and the last vertex of S_1 are not its neighbor. Therefor, r_1 is odd and the claim holds for segment S_1.

Next, we consider the other segments S_i $(i = 2, 3, \ldots, s)$. Similar with x_{2_1}, for x_{i_1} we have that $x_{1_1}x_{i_1} \notin E(G)$. Also, we have that $x_{1_2}x_{i_1} \in E(G)$; otherwise, $\{w, x_{i_1}, x_{1_1}, x_{1_2}\} \cong 2K_1 \cup K_2$, a contradiction. If $x_{1_1}x_{i_j} \in E(G)$ and $x_{1_1}x_{i_{j+1}} \in E(G)$, then we have a $(k+1)$-cycle $u_1wu_i\overline{C}x_{1_2}x_{i_1}Cx_{i_j}x_{1_1}x_{i_{j+1}}Cu_1$, a contradiction. If $x_{1_1}x_{i_j} \notin E(G)$ and $x_{1_1}x_{i_{j+1}} \notin E(G)$, then $\{w, x_{1_1}, x_{i_j}, x_{i_{j+1}}\} \cong 2K_1 \cup K_2$, a contradiction. Thus the neighbors of x_{1_1} occur alternately on every segment S_i along the cycle. Moreover, $x_{1_1}x_{i_{r_i}} \notin E(G)$; otherwise, we have a $(k+1)$-cycle $u_iwu_{i+1}Cx_{1_1}x_{i_{r_i}}\overline{C}x_{i_1}x_{1_2}Cu_i$, a contradiction. Hence, r_i is odd, and $x_{1_1}x_{i_j} \notin E(G)$ for every odd j, and $x_{1_1}x_{i_j} \in E(G)$ for every even j. □

Denote $W = N_C(w)$ and $A = N_C(x_{1_1}) \backslash W$. By Claim 4, $|W| + |A| = \frac{|V(C)|}{2}$. If $N(x_{1_1}) = N_C(x_{1_1})$, then $A \cup W$ is a cut set and $G - (A \cup W)$ generates at least three components, including two trivial components with vertex sets $\{w\}$ and $\{x_{1_1}\}$. Since G is $2K_1 \cup K_2$-free, all the other components are also trivial. Then we get $\frac{|V(C)|}{2} + 1$ components by deleting $\frac{|V(C)|}{2}$ vertices, contradicting the fact that G is 1-tough. Hence, we next assume that x_{1_1} has a neighbor outside the cycle C. Suppose that y is a neighbor of x_{1_1} in $G - V(C)$. Then $y \neq w$ and $yw \notin E(G)$. Denote

$B = V(C)\backslash(W \cup A)$. Before continuing the proof for Case 1, we first prove three more claims.

Claim 5 *$B \subseteq N(y)$, $A \cap N(y) = \emptyset$, $W \cap N(y) = \emptyset$ and A, B, and W are independent sets.*

Proof By the definition of B, we have $B \cap N(x_{1_1}) = \emptyset$. If there is a vertex $v_i \in B$ such that $by \notin E(G)$, then $\{w, v_i, x_{1_1}, y\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $B \subseteq N(y)$. Suppose that $v_j \in A$ and $v_j y \in E(G)$. Since $v_j^+ \in B$, $v_j^+ y \in E(G)$, and there is a $(k+1)$-cycle, a contradiction. For any vertex $v_i \in W$, $v_i^-, v_i^+ \in B$, hence $v_i^- y, v_i^+ y \in E(G)$. If $v_i y \in E(G)$, then there is a $(k+1)$-cycle, a contradiction. Hence, $A \cap N(y) = \emptyset$ and $W \cap N(y) = \emptyset$.

If there is an edge $v_i v_j$ in A, then $\{w, y, v_i, v_j\} \cong 2K_1 \cup K_2$, a contradiction. If there is an edge $v_i v_j$ in B, then $\{w, x_{1_1}, v_i, v_j\} \cong 2K_1 \cup K_2$, a contradiction. Suppose $v_i, v_j \in W$ and $v_i v_j \in E(G)$. Since $v_i^-, v_j^- \in B$, $v_i^- y, v_j^- y \in E(G)$. Then $v_i^- y v_j^- \overline{C} v_i v_j C v_i^-$ is a $(k+1)$-cycle, a contradiction. Hence, A, B, and W are independent sets. □

Claim 6 *$A = \emptyset$.*

Proof Suppose, by contradiction, that $A \neq \emptyset$. We claim that every vertex of A is adjacent to every vertex of W; otherwise, suppose that $v_i \in A, v_j \in W$ and $v_i v_j \notin E(G)$. Then, using Claim 5 we have that $\{y, v_i, v_j, w\} \cong 2K_1 \cup K_2$, a contradiction. Suppose that $v_i \in A$ and $v_j \in W$, with $v_i v_j \in E(G)$. Since $v_i^+, v_j^+ \in B$, by Claim 5, $v_i^+ y \in E(G), v_j^+ y \in E(G)$. Then there is a $(k+1)$-cycle $v_i^+ y v_j^+ C v_i v_j \overline{C} v_i^+$, a contradiction. □

By Claim 6, $S_i = x_{i_1}$ for every $i \in \{1, 2, \ldots, s\}$, and $V(C) = B \cup W$, $|B| = |W|$. By Claim 5, we have that B and W are independent sets and $B \subseteq N(y), W = N(w), W \cap N(y) = \emptyset$.

Claim 7 *$N_{G-V(C)}(C) = \{w, y\}$.*

Proof Suppose, by contradiction, that $z \in N_{G-V(C)}(C)\backslash\{w, y\}$. If $zv_i \in E(G)$ and $v_i \in B$, then $yz \notin E(G)$; otherwise, $v_i z y v_i^{+2} C v_i$ is a $(k+1)$-cycle, a contradiction. Then, $\{y, z, v_i^+, w\} \cong 2K_1 \cup K_2$, a contradiction. Now suppose that $zv_i \in E(G)$ and $v_i \in W$. We have that $yz \in E(G)$; otherwise, $\{w, z, y, v_i^+\} \cong 2K_1 \cup K_2$, a contradiction. If there is another vertex $z' \in N_{G-V(C)}(C)\backslash\{w, y, z\}$, using the same arguments, we have that $N_C(z') \subseteq W$ and $z'y \in E(G), z'w \notin E(G)$. That means, if $N_{G-V(C)}(C) \neq \{w, y\}$, then every vertex $a \in N_{G-V(C)}(C)\backslash\{w, y\}$ has the properties: $ay \in E(G), aw \notin E(G)$ and $N_C(a) \subseteq W$. By deleting the vertices of $W \cup \{y\}$ we obtain at least $|B| + 2$ components, contradicting the fact that G is 1-tough. □

Now, we are ready to complete the proof for Case 1. By Claim 7, the component that contains y is trivial; otherwise y is a cut vertex. Thus, $G - V(C)$ has precisely two trivial components $\{w\}$ and $\{y\}$. Since $k \geq 4$, $|B| = |W| \geq 2$. If there are two vertices $v_i, v_j \in W$ that are not adjacent to a vertex $v_k \in B$, then

$\{v_i, v_j, v_k, y\} \cong 2K_1 \cup K_2$, a contradiction. If there are two vertices $v_i, v_j \in B$ that are not adjacent to a vertex $v_k \in W$, then $\{v_i, v_j, v_k, w\} \cong 2K_1 \cup K_2$, a contradiction. Thus, each vertex of W has at least $|B| - 1$ neighbors in B and each vertex of B has at least $|W| - 1$ neighbors in W. Hence, G is a balanced bipartite graph $K_{s,s} - M$ with two vertex sets $\{w\} \cup B$ and $\{y\} \cup W$, where $s \geq 3$ and M is a matching of $K_{s,s}$. Since $wy \notin E(G)$, $1 \leq |M| \leq s$. This contradicts the assumption and completes the proof for Case 1.

Case 2. All the components of $G - V(C)$ contain at least one edge.

Suppose that H is a component of $G - V(C)$. We distinguish three subcases according to the distribution of $N_C(H)$, in particular whether there are nontrivial segments (containing at least one nonneighbor of H between two subsequent neighbors of H on C) or not. We start with the subcase that there are at least two such segments.

Case 2.1. There are at least two nontrivial segments on C.

For a vertex $v_i \in V(C)$, if $v_i \notin N_C(H)$ and $v_i^- \in N_C(H)$, then we say v_i is a *break vertex*. By the assumption, there are at least two break vertices on C. We call any two break vertices a *break pair*. Let S denote the set of all break vertices, and call S the *break set* (of C). We use the shorthand S is complete to indicate that S induces a complete graph in G. Suppose ab is an edge of H. We next prove four claims.

Claim 8 *S is complete, $V(H) = \{a, b\}$, and $V(C) \backslash S \subseteq N(a) \cup N(b)$, hence $S = V(C) \backslash N_C(H)$. Moreover, for any two subsequent break vertices $v_i, v_j \in S$, $|v_i^+ C v_j^-|$ is even, and the vertices of $v_i^+ C v_j^-$ are alternately neighbors of a and b.*

Proof First, we have that $V(C) \backslash N_C(H)$ induces a complete graph; otherwise, suppose that v_i, v_j are two nonadjacent vertices of $V(C) \backslash N_C(H)$. Then $\{v_i, v_j, a, b\} \cong 2K_1 \cup K_2$, a contradiction. Hence, S is complete, since $S \subseteq V(C) \backslash N_C(H)$. Also, H is a complete graph; otherwise, suppose that w_1, w_2 are two nonadjacent vertices of H, and v_i, v_j are two vertices of $V(C) \backslash N_C(H)$. Then $\{w_1, w_2, v_i, v_j\} \cong 2K_1 \cup K_2$, a contradiction.

Suppose that $\{v_i, v_j\}$ is a break pair and in the segment $v_i^+ C v_j^-$ there is no other break vertex. Since $v_i, v_j \in S$, $v_i v_j \in E(G)$. We have that $N_H(v_i^-) \cap N_H(v_j^-) = \emptyset$; otherwise, let $w \in N_H(v_i^-) \cap N_H(v_j^-)$. Then $v_i^- w v_j^- \overline{C} v_i v_j C v_i^-$ is a $(k+1)$-cycle, a contradiction. Since H is complete, without loss of generality, we assume that $v_i^- a \in E(G)$ and $v_j^- b \in E(G)$. Then $v_i^+ \neq v_j^-$; otherwise, $v_i^- a b v_j^- C v_i^-$ is a $(k+1)$-cycle, a contradiction. If $v_i^+ \notin N_C(H)$, then $v_i^+ v_j \in E(C)$, and $v_i^- a b v_j^- \overline{C} v_i^+ v_j C v_i^-$ is a $(k+1)$-cycle, a contradiction. Hence, $v_i^+ \in N_C(H)$. We have that $N_H(v_i^+) = \{a\}$; otherwise, suppose that $v_i^+ c \in E(G)$ and $c \in V(H) \backslash \{a\}$. Then $v_i^- a c v_i^+ C v_i^-$ is a $(k+1)$-cycle, a contradiction. Since v_i^{+2} is not a break vertex and $v_i^-, v_i^+ \in N(a)$, we have $v_i^{+2} \in N(b)$; otherwise, $\{v_i, v_i^{+2}, a, b\} \cong 2K_1 \cup K_2$, a contradiction. If $V(H) \neq \{a, b\}$, suppose $c \in V(H) \backslash \{a, b\}$. Then $v_i^- a c b v_i^{+2} C v_i^-$ is a $(k+1)$-cycle, a contradiction. Hence, $V(H) = \{a, b\}$.

Since $v_i^+ C v_j^-$ contains no other break vertex, $v_i^+ C v_j^- \subseteq N_C(H) = N(a) \cup N(b)$. Since G has no $(k+1)$-cycle, the vertices of $v_i^+ C v_j^-$ are alternately neighbors of a

and b. And since the two end vertices of $v_i^+Cv_j^-$ belong to different neighbor sets of a and b, $|v_i^+Cv_j^-|$ is even.

By the definition of break vertex, any two break vertices are not consecutive vertices on C. Thus the break vertices divide the cycle into segments, and from the above analysis we know that each segment (between two break vertices, and not containing any break vertices) belongs to the union of the neighbor sets of a and b. Hence, $V(C)\backslash S \subseteq N(a) \cup N(b)$, and therefore, $S = V(C)\backslash N_C(H)$. □

Claim 9 $|S| = 2$.

Proof Suppose, by contradiction, that $|S| \geq 3$. Assume that $v_i, v_j, v_k \in S$. By Claim 8, S is complete. Thus, $v_iv_j, v_iv_k, v_jv_k \in E(G)$. Since $V(H) = \{a, b\}$, two of the three vertices v_i^-, v_j^-, v_k^- have a common neighbor in $\{a, b\}$. Without loss of generality, assume that $v_i^-, v_j^- \in N(a)$. Then $v_i^-av_j^-\overline{C}v_iv_jCv_i^-$ is a $(k+1)$-cycle, a contradiction. □

Suppose that $S = \{v_s, v_t\}$. By Claim 8, all vertices of $V(C)\backslash\{v_s, v_t\}$ are adjacent to a or b alternately, v_s^-, v_s^+ share the same neighbor in $\{a, b\}$, while v_s^+, v_t^- have different neighbors in $\{a, b\}$. Thus, we have that either $v_s^-, v_s^+ \in N(a)$ and $v_t^-, v_t^+ \in N(b)$, or $v_s^-, v_s^+ \in N(b)$ and $v_t^-, v_t^+ \in N(a)$. Without loss of generality, we assume that $v_s^-, v_s^+ \in N(a)$ and $v_t^-, v_t^+ \in N(b)$. Let $A = N_C(a)$, $B = N_C(b)$. Clearly, $A \cap B = \emptyset$, $|A| = |B|$ and $V(C) = A \cup B \cup S$.

Claim 10 *$A \cup \{v_t\}$ and $B \cup \{v_s\}$ are independent sets.*

Proof First, we prove that A and B are independent sets. Suppose that $v_i, v_j \in A$ and $v_iv_j \in E(G)$. If $\{v_i, v_j\} = \{v_s^-, v_s^+\}$, then $v_s^+abv_s^{+2}Cv_s^-v_s^+$ is a $(k+1)$-cycle, a contradiction. If $\{v_i, v_j\} \neq \{v_s^-, v_s^+\}$, then either $v_i^+, v_j^+ \in B$ or $v_i^-, v_j^- \in B$. Without loss of generality, assume that $v_i^+, v_j^+ \in B$. Then $v_i^+bv_j^+Cv_iv_j\overline{C}v_i^+$ is a $(k+1)$-cycle, a contradiction. Hence, A is independent. Similarly, B is also independent.

Next, we prove that $N(v_t) \cap A = \emptyset$ and $N(v_s) \cap B = \emptyset$. Suppose that $v_i \in A$ and $v_iv_t \in E(G)$. Clearly, either $v_i^- \in B$ or $v_i^+ \in B$. Without loss of generality, assume $v_i^- \in B$. Then we have a $(k+1)$-cycle $v_t^-bv_i^-\overline{C}v_tv_iCv_t^-$, a contradiction. Hence, $N(v_t) \cap A = \emptyset$. Similarly, $N(v_s) \cap B = \emptyset$. □

Claim 11 $G - V(C) = H$.

Proof Suppose that $w \in V(G)\backslash(V(C) \cup V(H))$. We have that $wv_s, wv_t \in E(G)$; otherwise, suppose that $wv_s \notin E(G)$. Then $\{w, v_s, a, b\} \cong 2K_1 \cup K_2$, a contradiction. Since G has no $(k+1)$-cycle, $v_s^-, v_s^+, v_t^-, v_t^+ \notin N(w)$. Using Claim 10, $\{v_t^-, v_t^+, v_s, w\} \cong 2K_1 \cup K_2$, a contradiction. □

For a vertex $v_i \in A$, if $v_iv_s \notin E(G)$, then $\{v_i, b, v_s, v_t\} \cong 2K_1 \cup K_2$, a contradiction. Hence, for each vertex $v_i \in A$, $v_iv_s \in E(G)$. Similarly, for each vertex $v_j \in B$, $v_jv_t \in E(G)$. If there are two vertices $v_i, v_j \in B$ that are not adjacent to a vertex $v_k \in A$, then $\{v_i, v_j, v_k, a\} \cong 2K_1 \cup K_2$, a contradiction. Hence, every vertex of A has at least $|B| - 1$ neighbors in B. Similarly, every vertex of B has at least $|A| - 1$

neighbors in A. Let $A' = A \cup \{v_t\} \cup \{b\}$ and $B' = B \cup \{v_s\} \cup \{a\}$. From the above analysis, we conclude that $G = K_{p,p} - M$ for the two independent vertex set A' and B', and a matching M of $K_{p,p}$. Since $v_sb, v_ta \notin E(G)$, $2 \leq |M| \leq p$. This final contradiction to the assumption completes the proof for Case 2.1.

Case 2.2. There is precisely one nontrivial segment on C.

In this subcase, there exist two vertices v_i, v_j on C such that $v_iCv_j \subseteq N_C(H)$ and $v_j^+Cv_i^- \cap N_C(H) = \emptyset$. Let $P = v_j^+Cv_i^-$, and let ab be an edge of H. We again start by proving several claims and considering some subcases.

Claim 12 $|V(P)| \leq 2$.

Proof Suppose, by contradiction, that $|V(P)| \geq 3$. We have that $\langle V(P) \rangle$ is complete; otherwise, any two nonadjacent vertices of P together with the edge ab will induce a $2K_1 \cup K_2$. Similarly, H is complete; otherwise, two nonadjacent vertices of H together with an edge of P will induce a $2K_1 \cup K_2$, a contradiction. Let v_k, v_k^+ be two consecutive vertices of $N_C(H)$, and w_1, w_2 be their neighbors in H, respectively. Then $w_1w_2 \in E(G)$ and we get a $(k+1)$-cycle $v_kw_1w_2v_k^+Cv_j^+v_j^{+3}Cv_k$, a contradiction. □

We distinguish the subcases that $|V(P)| = 2$ and $|V(P)| = 1$.

Case 2.2.1. $|V(P)| = 2$, i.e., $P = v_j^+v_i^-$.

For convenience, we denote $C = v_1v_2 \ldots v_kv_1$, and $N_C(H) = v_1Cv_{k-2}$. Then $P = v_{k-1}v_k$. In this case, H is complete; otherwise, any two nonadjacent vertices of H together with the edge of P will induce a $2K_1 \cup K_2$, a contradiction.

Claim 13 *$V(H) = \{a, b\}$ and k is even.*

Proof If v_1 and v_{k-2} have two different neighbors in H, without loss of generality, assume that $v_1a, v_{k-2}b \in E(G)$. Then $V(H) = \{a, b\}$; otherwise, suppose $c \in V(H) \backslash \{a, b\}$. Then $v_1acbv_{k-2}\overline{C}v_1$ is a $(k+1)$-cycle, a contradiction. Suppose that $N_H(v_1) = N_H(v_{k-2}) = \{a\}$. If $v_1v_{k-2} \in E(G)$, then $V(H) = \{a, b\}$; otherwise, suppose that $c \in V(H) \backslash \{a, b\}$, and without loss of generality, suppose $v_2b \in E(G)$. Then $v_1acbv_2Cv_{k-2}v_1$ is a $(k+1)$-cycle, a contradiction. If $v_1v_{k-2} \notin E(G)$, then $V(H) = \{a, b\}$; otherwise, for any vertex $c \in V(H) \backslash \{a, b\}$ we have $\{v_1, v_{k-2}, b, c\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $V(H) = \{a, b\}$ and $V(C) \backslash V(P) \subseteq N(a) \cup N(b)$. Since G has no $(k+1)$-cycle, each vertex of $V(C) \backslash V(P)$ is adjacent to one and only one of $\{a, b\}$, alternately. If we can prove that v_1, v_{k-2} are adjacent to different vertices of $\{a, b\}$, then we get that k is even. First, we have that $v_kv_2 \notin E(G)$ and $v_{k-3}v_{k-1} \notin E(G)$; otherwise, suppose $v_kv_2 \in E(G)$. Then $v_2bav_3Cv_kv_2$ (if $k \geq 5$) or $v_kv_2bav_1v_k$ (if $k = 4$) is a $(k+1)$-cycle, a contradiction. Suppose that v_1, v_{k-2} are adjacent to the same vertex of $\{a, b\}$, say a without loss of generality. Clearly, $v_2v_{k-1} \notin E(G)$. Then $\{a, v_2, v_{k-1}, v_k\} \cong 2K_1 \cup K_2$, a contradiction. Hence, v_1, v_{k-2} are adjacent to different vertices of $\{a, b\}$, and $|V(C) \backslash V(P)|$ is even, so k is even. □

Let $A = N_C(a) = \{v_1, v_3, \ldots, v_{k-3}\}$, $B = N_C(b) = \{v_2, v_4, \ldots, v_{k-2}\}$. Then $V(C) = A \cup B \cup V(P)$, $A \cap B = \emptyset$ and $|A| = |B|$.

Claim 14 $G - V(C) = H$.

Proof Suppose, by contradiction, that $w \in V(G) \backslash (V(C) \cup V(H))$. To avoid $\{w, v_{k-1}, a, b\}$ or $\{w, v_k, a, b\}$ inducing $2K_1 \cup K_2$, we have that $wv_{k-1}, wv_k \in E(G)$. Then there clearly is a $(k+1)$-cycle, a contradiction. □

Claim 15 *$A \cup \{v_{k-1}\}$ and $B \cup \{v_k\}$ are independent sets.*

Proof First, we prove that A and B are independent sets. If $|A| = |B| = 1$, then the claim holds. Now we suppose that $|A| = |B| \geq 2$. If $v_i, v_j \in A$ and $v_i v_j \in E(G)$, then either $v_i^-, v_j^- \in B$ or $v_i^+, v_j^+ \in B$. Without loss of generality, assume $v_i^-, v_j^- \in B$. Then there is a $(k+1)$-cycle $v_i^- b v_j^- \overline{C} v_i v_j C v_i^-$, a contradiction. Hence, A is independent set. Similarly, B is also independent set.

Next, we prove that $N(v_{k-1}) \cap A = \emptyset$ and $N(v_k) \cap B = \emptyset$. Suppose that $v_i \in A$ and $v_i v_{k-1} \in E(G)$. If $v_i = v_1$, then $v_{k-1} v_1 a b v_2 C v_{k-1}$ is a $(k+1)$-cycle. If $v_i \neq v_1$, then $v_i^- \in B$ and $v_i^- b v_{k-2} \overline{C} v_i v_{k-1} C v_i^-$ is a $(k+1)$-cycle, a contradiction. Hence, $N(v_{k-1}) \cap A = \emptyset$. Similarly, $N(v_k) \cap B = \emptyset$. □

If there is a vertex $v_i \in A$ such that $v_i v_k \notin E(G)$, then $v_i \neq v_1$. Since $v_1 \in A$, $v_i v_1 \notin E(G)$. Then $\{b, v_i, v_k, v_1\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $A \subseteq N(v_k)$. Similarly, $B \subseteq N(v_{k-1})$. If there are two vertices $v_i, v_j \in B$ that are not adjacent to a vertex $v_s \in A$, then $\{v_i, v_j, v_s, a\} \cong 2K_1 \cup K_2$, a contradiction. Hence, each vertex of A has at least $|B| - 1$ neighbors in B. Similarly, each vertex of A has at least $|B| - 1$ neighbors in B. Thus, $G = K_{s,s} - M$ for the two independent vertex sets $A \cup \{b\} \cup \{v_{k-1}\}$ and $B \cup \{a\} \cup \{v_k\}$, and a matching M of $K_{s,s}$. Since $av_{k-1} \notin E(G)$ and $bv_k \notin E(G)$, $2 \leq |M| \leq s$. This contradiction to the assumption completes the proof for Case 2.2.1.

Case 2.2.2. $|V(P)| = 1$.

Without loss of generality, we assume that v_k is the only vertex that has no neighbor in H.

Claim 16 $|V(H)| \geq 3$.

Proof Suppose, by contradiction, that $V(H) = \{a, b\}$. All vertices except v_k of C are adjacent to a or b. Without loss of generality, assume $v_1 \in N(a)$. Since G has no $(k+1)$-cycle, k is even, and $N_C(a) = \{v_1, v_3, v_5, \ldots, k-1\}, N_C(b) = \{v_2, v_4, v_6, \ldots, k-2\}$. Denote $A = N_C(a)$ and $B = N_C(b)$. Then $V(C) = A \cup B \cup \{v_k\}$, and $A \cap B = \emptyset$.

Claim 16.1 $G - V(C) = H$.

Proof Suppose that H' is another component of $G - V(C)$. We have that $V(H') \subseteq N(v_k)$; otherwise, suppose that $w \in V(H')$ and $wv_k \notin E(G)$. Then $\{w, v_k, a, b\} \cong 2K_1 \cup K_2$, a contradiction. Now $v_1, v_{k-1} \notin N_C(H')$; otherwise there is a $(k+1)$-cycle. Since $k \geq 4$ and $|N_C(H')| \geq 2$, the neighbors of H' on C are not consecutive, and we are in Case 2.1, a contradiction. □

Claim 16.2 A and B are independent sets.

Proof Suppose that $v_i, v_j \in A$ and $v_iv_j \in E(G)$. If $\{v_i, v_j\} = \{v_1, v_{k-1}\}$, then $v_1abv_2Cv_{k-1}v_1$ is a $(k+1)$-cycle, a contradiction. If $\{v_i, v_j\} \neq \{v_1, v_{k-1}\}$, then either $v_i^+, v_j^+ \in B$ or $v_i^-, v_j^- \in B$, say $v_i^+, v_j^+ \in B$ without loss of generality. Then we have a $(k+1)$-cycle $v_i^+bv_j^+Cv_iv_j\overline{C}v_i^+$, a contradiction. Hence, A is an independent set. If $v_i, v_j \in B$ and $v_iv_j \in E(G)$, then $v_i^+, v_j^+ \in A$ and we get a $(k+1)$-cycle $v_i^+av_j^+Cv_iv_j\overline{C}v_i^+$, a contradiction. Hence, B is an independent set. □

Claim 16.3 $A \subseteq N(v_k)$ and $N(v_k) \cap B = \emptyset$.

Proof Suppose there is a vertex $v_i \in A$ such that $v_iv_k \notin E(G)$. Clearly, $v_i \neq v_1$, and $v_iv_1 \notin E(G)$ by Claim 16.2. Then $\{b, v_i, v_k, v_1\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $A \subseteq N(v_k)$. If $v_j \in B$ such that $v_jv_k \in E(G)$, then $v_j^- \in A$ and there is a $(k+1)$-cycle $v_j^-av_{k-1}\overline{C}v_jv_kCv_j^-$, a contradiction. Hence, $N(v_k) \cap B = \emptyset$. □

If there are two vertices $v_i, v_j \in B$ that are not adjacent to a vertex $v_k \in A$, then $\{v_i, v_j, v_k, a\} \cong 2K_1 \cup K_2$, a contradiction. Similarly, if there are two vertices $v_i, v_j \in A$ that are not adjacent to a vertex $v_k \in B$, then $\{v_i, v_j, v_k, b\} \cong 2K_1 \cup K_2$, a contradiction. Hence, every vertex of A has at least $|B| - 1$ neighbors in B, and every vertex of B has at least $|A| - 1$ neighbors in A. Thus, $G = K_{s,s} - M$ for the two independent sets $A \cup \{b\}$ and $B \cup \{a\} \cup \{v_k\}$, and a matching M of $K_{s,s}$. Since $bv_k \notin E(G)$ and $A \cup \{b\} \subseteq N(a)$, $1 \leq |M| \leq s - 1$. This contradicts the assumption. □

Claim 17 *H is not complete.*

Proof Suppose, by contradiction, that H is complete. Using Claim 16, if $v_1w_1 \in E(G), v_{k-1}w_2 \in E(G)$ for two distinct vertices $w_1, w_2 \in V(H)$, then $v_1w_1w_2v_{k-1}\overline{C}v_1$ is a $(k+1)$-cycle, a contradiction. Hence v_1 and v_{k-1} have only one common neighbor, say w_1. We have that $v_kv_{k-2} \notin E(G)$; otherwise, $v_1Cv_{k-2}v_kv_{k-1}w_1v_1$ is a $(k+1)$-cycle, a contradiction. Suppose that $w_1w_2 \in E(H)$. To avoid $\{v_k, v_{k-2}, w_1, w_2\}$ inducing $2K_1 \cup K_2$, we have $w_2v_{k-2} \in E(G)$. Since $|V(H)| \geq 3$ and H is complete, there is another vertex $w_3 \in V(H) \backslash \{w_1, w_2\}$ and $w_1w_3, w_2w_3 \in E(G)$. Then $v_1Cv_{k-2}w_2w_3w_1v_1$ is a $(k+1)$-cycle, a contradiction. Hence, H is not complete. □

Denote $A = \{v_i \in V(C) \mid 1 \leq i < k \text{ and } i \text{ is odd}\}$, $B = \{v_i \in V(C) \mid 2 \leq i < k \text{ and } i$ is even$\}$, and denote $A_H = N_H(A)$ and $B_H = N_H(B)$.

Claim 18 *A_H and B_H are independent sets.*

Proof It is sufficient if we can prove that for any two vertices v_i, v_j $(1 \leq i < j < k)$ such that $|v_iCv_j|$ is odd, either they share a common neighbor in H or their neighbors in H are nonadjacent. We use induction to prove that fact. First, suppose that $|v_iCv_j| = 3$. If v_i, v_j have different neighbors in H, say w_1, w_2, respectively, then using Claim 17, $w_1w_2 \in E(H)$. But then $v_iw_1w_2v_jCv_i$ is a $(k+1)$-cycle, a contradiction. Hence the claim holds for the case $|v_iCv_j| = 3$. Now suppose that the claim holds for $|v_iCv_j| \leq 2m - 1$ $(m \geq 2)$. Then it is sufficient to deal with the case $|v_iCv_j| = 2m+1$. Suppose that $v_iw_1 \in E(G), v_jw_2 \in E(G)$ and $w_1w_2 \in E(H)$. If $v_i^{+2}w_1 \in E(G)$, then $|v_i^{+2}Cv_j| = 2m-1$ and $w_1w_2 \in E(H)$, contradicting the assumption. If $v_i^{+2}w_2 \in E(G)$, then $|v_iCv_i^{+2}| = 3$ and $w_1w_2 \in E(H)$, contradicting the assumption. Hence, v_i^{+2} has a neighbor different from w_1, w_2 in H, say w_3, and $w_1w_3, w_2w_3 \notin E(G)$. To avoid $\{w_3, v_i^+, w_1, w_2\}$ inducing $2K_1 \cup K_2$, we have $v_i^+w_2 \in E(G)$. For the vertex v_j^-, by the same arguments we get that $v_j^-w_1 \in E(G)$. Then $|v_i^+Cv_j^-| = 2m-1$ and w_2, w_1 are their neighbors, respectively. Since $w_1w_2 \in E(H)$, that contradicts the induction hypothesis. Hence, the claim holds for the case $|v_iCv_j| = 2m+1$ and the proof is complete. □

Claim 19 *$A_H \cap B_H = \emptyset$ and $V(H) = A_H \cup B_H$.*

Proof Suppose that there is a vertex $w \in A_H \cap B_H$. Then w has two neighbors v_i, v_j on C such that i is odd and j is even. Clearly, $j \neq i+1, j \neq i-1$ and $v_i^+v_j^+ \notin E(G)$; otherwise there is a $(k+1)$-cycle. By Claim 18, w has no neighbor in $A_H \cup B_H$. Since H is connected and nontrivial, there is a vertex $w' \in V(H)\backslash(A_H \cup B_H)$ such that $ww' \in E(G)$. Since $N_H(C) = A_H \cup B_H$, $w'v_i^+ \notin E(G)$ and $w'v_j^+ \notin E(G)$. Then $\{v_i^+, v_j^+, w, w'\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $A_H \cap B_H = \emptyset$.

Suppose that $w \in V(H)\backslash(A_H \cup B_H)$. Since $N_H(C) = A_H \cup B_H$, $N(w) \cap V(C) = \emptyset$. Assume that $w_1 \in B_H$ and $w_1v_2 \in E(G)$. We have that $ww_1 \in E(G)$; otherwise, $\{w, w_1, v_1, v_k\} \cong 2K_1 \cup K_2$, a contradiction. Assume that $w_2v_{k-1} \in E(G)$. We have that $w_2 \neq w_1$; otherwise, $v_kv_3 \notin E(G)$ and $\{v_k, v_3, w_1, w\} \cong 2K_1 \cup K_2$, a contradiction. If $ww_2 \in E(G)$, then we get a $(k+1)$-cycle $v_2Cv_{k-1}w_2ww_1v_2$, a contradiction. Thus, $ww_2 \notin E(G)$. To avoid $\{w, w_2, v_1, v_k\}$ or $\{w_2, v_k, w, w_1\}$ inducing $2K_1 \cup K_2$, we have that $w_2v_1 \in E(G)$ and $w_2w_1 \in E(G)$. Thus, $w_2 \in A_H$ and $v_{k-1} \in A$. We have $v_1v_{k-1} \notin E(G)$; otherwise, $v_1w_2w_1v_2Cv_{k-1}v_1$ is a $(k+1)$-cycle, a contradiction. Since $A_H \cap B_H = \emptyset$, $w_1v_{k-1} \notin E(G)$. Then $\{v_1, v_{k-1}, w, w_1\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $V(H) = A_H \cup B_H$. □

Claim 20 $|B_H| \geq 2$.

Proof Suppose, by contradiction, that $B_H = \{w\}$. Since $V(H) = A_H \cup B_H$ and $|V(H)| \geq 3$, $|A_H| \geq 2$. Suppose first that k is odd. Then $v_{k-1}w \in E(G)$. Suppose that $v_1w_1 \in E(G)$ and $w_1 \in A_H$. We have that $ww_1 \notin E(G)$; otherwise, $v_1Cv_{k-1}ww_1v_1$ is a $(k+1)$-cycle, a contradiction. By Claims 18 and 19, w_1 has no neighbor in H, contradicting the fact that H is connected. Hence, k is even. Moreover, we have that $B \cup \{v_k\}$ is an independent set; otherwise, suppose that $v_i, v_j \in B \cup \{v_k\}$ and $v_iv_j \in E(G)$. Then $\{w_1, w_2, v_i, v_j\} \cong 2K_1 \cup K_2$, where $w_1, w_2 \in A_H$, a contradiction. Now, if we delete all the vertices of $A \cup \{w\}$, then we will get $|B \cup \{v_k\}| + |A_H|$ trivial components. Since $|B \cup \{v_k\}| = |A|$ and $|A_H| \geq 2$, $|B \cup \{v_k\}| + |A_H| > |A \cup \{w\}|$. This contradicts the fact that G is 1-tough. □

By Claims 18–20, there are two vertices $w_1, w_2 \in B_H$ such that $w_1w_2 \notin E(G)$ and $w_1, w_2 \notin N(v_1)$. Then $\{w_1, w_2, v_1, v_k\} \cong 2K_1 \cup K_2$, our final contradiction that completes the proof for Case 2.2.

Case 2.3. There is no nontrivial segment on C.

In this case, all the vertices of C are neighbors of a component H of $G-V(C)$. We first claim that H has at least three vertices; otherwise G belongs to the class of graphs that we have excluded.

Claim 21 $|V(H)| \geq 3$.

Proof Suppose, by contradiction, that $V(H)=\{a,b\}$. Then all the vertices of C are adjacent to a or b. Without loss of generality, assume that $v_1a \in E(G)$. Then we have that k is even and $N_C(a)=\{v_i \mid 1 \leq i \leq k$ and i is odd$\}$, $N_C(b)=\{v_j \mid 1 \leq j \leq k$ and i is even$\}$. Denote $A=N_C(a)$, $B=N_C(b)$. Then $V(C)=A \cup B$.

Claim 21.1 *A and B are independent sets.*

Proof Suppose that $v_i, v_j \in A$. Clearly, $v_i^+, v_j^+ \in B$. If $v_iv_j \in E(G)$, then we get a $(k+1)$-cycle $v_i^+bv_j^+Cv_iv_j\overline{C}v_i^+$, a contradiction. Hence, A is independent. Similarly, B is also independent. □

Claim 21.2 $G-V(C)=H$ *or* $V(G)\backslash(V(C)\cup V(H))=\{a',b'\}$. *Moreover, in the latter case* $A \subseteq N(a')$ *and* $B \subseteq N(b')$.

Proof Since G is $2K_1 \cup K_2$-free and all components of $G-V(C)$ are nontrivial, $G-V(C)$ has either one component or two complete components. If $G-V(C)$ has one component, then the claim holds. Suppose that there is another component $H'=G-(V(C)\cup V(H))$. Then there is a vertex $a' \in H'$ such that a' has a neighbor on C. Without loss of generality, assume $a'v_1 \in E(G)$. Then $A \subseteq N(a')$; otherwise, suppose $v_ia' \notin E(G)$ and $v_i \in A$. Then $\{b, v_i, a', v_1\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $A \subseteq N(a')$ and $B \cap N(a') = \emptyset$. Suppose that $a'b' \in E(H')$. Then $B \subseteq N(b')$; otherwise, suppose $v_ib' \notin E(G)$ and $v_i \in B$. Then $\{a, v_i, a', b'\} \cong 2K_1 \cup K_2$, a contradiction. If $V(H') \neq \{a',b'\}$, assume $c' \in V(H')\backslash\{a',b'\}$. Then there is a $(k+1)$-cycle $v_1a'c'b'v_4Cv_1$, a contradiction. Hence, the claim holds. □

If there are two vertices $v_i, v_j \in B$ that are not adjacent to a vertex $v_k \in A$, then $\{v_i, v_j, v_k, a\} \cong 2K_1 \cup K_2$, a contradiction. Similarly, if there are two vertices $v_i, v_j \in A$ that are not adjacent to a vertex $v_k \in B$, then $\{v_i, v_j, v_k, b\} \cong 2K_1 \cup K_2$, a contradiction. Hence, every vertex of A has at least $|B|-1$ neighbors in B, and every vertex of B has at least $|A|-1$ neighbors in A. Thus, if $G-V(C)=H$, then $G=K_{s,s}-M$ for the two independent sets $A \cup \{b\}$ and $B \cup \{a\}$, and a matching M of $K_{s,s}$. Since $N(a)=A\cup\{b\}$ and $N(b)=B\cup\{a\}$, $0 \leq |M| \leq s-2$. If $V(G)\backslash(V(C)\cup V(H))=\{a',b'\}$, then $G=K_{s,s}-M$ for the two independent sets $A\cup\{b,b'\}$ and $B\cup\{a,a'\}$, and a matching M of $K_{s,s}$. Since $ab' \notin E(G)$ and $ba' \notin E(G)$, $2 \leq |M| \leq s$. This contradicts the assumption. □

Claim 22 *H is not complete.*

Proof Suppose, by contradiction, that H is complete. If $|N_H(C)|=2$ and $N_H(C)=\{a,b\}$, then all the vertices of C are neighbors of a or b alternately. Denote $A=N_C(a)$ and $B=N_C(b)$. Then $|A|=|B|\geq 2$ and A, B are independent sets. Taking two vertices v_i, v_j from A and a vertex c from $V(H)\backslash\{a,b\}$, we get that

$\{v_i, v_j, b, c\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $|N_H(C)| \geq 3$. Then there are three vertices $a, b, c \in V(H)$ and a vertex $v_i \in V(C)$ such that $av_i, bv_i^+, cv_i^{+2} \in E(G)$. Then we get a $(k+1)$-cycle $v_i acv_i^{+2} Cv_i$, a contradiction. Hence, H is not complete. □

Using Claim 22, we get that $G - V(C) = H$; otherwise, suppose H' is another component of $G - V(C)$. Then two nonadjacent vertices of H with an edge of H' will induce a $2K_1 \cup K_2$, a contradiction. Denote $A = \{v_i \in V(C) \mid 1 \leq i \leq k$ and i is odd$\}$, $B = \{v_i \in V(C) \mid 2 \leq i \leq k$ and i is even$\}$, and denote $A_H = N_H(A)$ and $B_H = N_H(B)$.

Claim 23 *A_H and B_H are independent sets.*

Proof The only difference between the conditions of Claims 18 and 23 is that v_k is a neighbor of H in the latter one but not in the former one. In the induction proof of Claim 18, the absence of v_k does not affect the result. Hence, the proof of Claim 18 is also valid here. □

Claim 24 *$V(H) = A_H \cup B_H$ and $A_H \cap B_H = \emptyset$.*

Proof Suppose, by contradiction, that $V(H) \neq A_H \cup B_H$. There is a vertex $w \in V(G) \backslash (A_H \cup B_H)$ such that $ww_1 \in E(H)$, where $w_1 \in A_H \cup B_H$. Since $N_H(C) = A_H \cup B_H$, w has no neighbor on C. Without loss of generality, assume that $w_1 \in A_H$ and $w_1 v_1 \in E(G)$. If $v_k v_2 \notin E(G)$, then $\{v_k, v_2, w_1, w\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $v_k v_2 \in E(G)$. Suppose that $v_2 w_2, v_3 w_3 \in E(G)$ with $w_2, w_3 \in V(H)$. Obviously, $w_2 \neq w_1$, $w_2 \neq w_3$, and since $v_k v_2 \in E(G)$ we have $w_1 \neq w_3$ and $w_1 v_3, w_3 v_1 \notin E(G)$. By Claim 23, $w_1 w_3 \notin E(G)$. Moreover, we have $w_2 w_3 \notin E(G)$; otherwise, $v_2 w_2 w_3 v_3 C v_k v_2$ is a $(k+1)$-cycle, a contradiction. If $w_1 w_2 \notin E(G)$, then $\{w_2, w_3, w_1, v_1\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $w_1 w_2 \in E(G)$. If $k = 4$, then we get a $(k+1)$-cycle $v_1 w_1 w_2 v_2 v_k v_1$, a contradiction. Thus, $k \geq 5$ and $v_4 \neq v_k$. We have $v_1 v_4 \notin E(G)$ and $w_1 v_4 \notin E(G)$; otherwise, $v_k v_2 w_2 w_1 v_1 v_4 C v_k$ or $v_2 w_2 w_1 v_4 C v_2$ is a $(k+1)$-cycle, respectively, a contradiction. Then $\{w_3, v_4, w_1, v_1\} \cong 2K_1 \cup K_2$, a contradiction. Hence, $V(H) = A_H \cup B_H$.

Suppose that $w \in A_H \cap B_H$. By Claim 23, w has no neighbor in $A_H \cup B_H$. Since $V(H) = A_H \cup B_H$, w is a independent vertex of H, contradicting the fact that H is connected. Hence, $A_H \cap B_H = \emptyset$. □

Claim 25 *k is even.*

Proof For two vertices $v_i, v_j \in V(C)$, if i and j are both odd or both even, then we say that v_i, v_j are in the same group, and by Claim 23 we know that their neighbors on H are independent. Suppose that k is odd, and $v_k w \in E(G)$ with $w \in V(H)$. Clearly, $w \in A_H$. Since $v_k^{+2} = v_2$, $v_k^{+4} = v_4$, ..., if we relabel the vertices on C by increasing the subscript of every vertex by one, then v_k becomes v_1, v_2 becomes v_3, ... Then the original v_k and the original $v_2, v_4, \ldots$ are in the same group. Then w is independent with every vertex of B_H. By Claims 23 and 24, w has no neighbor in H, contradicting that H is connected. Hence, k is even. □

Claim 26 *A and B are independent sets.*

Proof Suppose that $v_i, v_j \in A$ and $v_i v_j \in E(G)$. If $|B_H| = 1$, assume $B_H = \{w\}$. Then $v_i^+ w, v_j^+ w \in E(G)$ and we get a $(k+1)$-cycle $v_i^+ w v_j^+ C v_i v_j \overline{C} v_i^+$, a contradiction. If $|B_H| \geq 2$, assume $w_1, w_2 \in B_H$. Then $\{w_1, w_2, v_i, v_j\} \cong 2K_1 \cup K_2$, a contradiction. Hence, A is an independent set. Similarly, B is also and independent set. □

By Claims 23–26, $A \cup B_H$ and $B \cup A_H$ are independent. Since G is 1-tough and $V(G) = A \cup B \cup A_H \cup B_H$, we have $|A \cup B_H| = |B \cup A_H|$.

Claim 27 *For each vertex $x \in A \cup B_H$, x has at least $|B \cup A_H| - 1$ neighbors in $B \cup A_H$, and for each vertex $y \in B \cup A_H$, y has at least $|A \cup B_H| - 1$ neighbors in $A \cup B_H$.*

Proof Suppose that y_1, y_2 are two vertices of $B \cup A_H$ such that they are not adjacent to a vertex $x \in A \cup B_H$. If $x \in A$ and $y_1, y_2 \in B$, then $\{y_1, y_2, x, w\} \cong 2K_1 \cup K_2$, where w is a neighbor of x in A_H, a contradiction. If $x \in A$ and $y_1, y_2 \in A_H$, then $\{y_1, y_2, x, x^+\} \cong 2K_1 \cup K_2$, a contradiction. If $x \in A$ and $y_1 \in B$, $y_2 \in A_H$, then there is another vertex $y_3 \in A_H \backslash \{y_2\}$ such that $xy_3 \in E(G)$, and $\{y_1, y_2, x, y_3\} \cong 2K_1 \cup K_2$, a contradiction.

If $x \in B_H$ and $y_1, y_2 \in B$, then $\{y_1, y_2, x, z\} \cong 2K_1 \cup K_2$, where z is a neighbor of x in A_H, a contradiction. If $x \in B_H$ and $y_1, y_2 \in A_H$, then $\{y_1, y_2, x, z'\} \cong 2K_1 \cup K_2$, where z' is a neighbor of x in B, a contradiction. If $x \in B_H$ and $y_1 \in B$, $y_2 \in A_H$, then there is another vertex $y_3' \in B \backslash \{y_1\}$ such that $xy_3' \in E(G)$, and $\{y_1, y_2, x, y_3'\} \cong 2K_1 \cup K_2$, a contradiction. Hence, for each vertex $x \in A \cup B_H$, x has at least $|B \cup A_H| - 1$ neighbors in $B \cup A_H$. By symmetry, for each vertex $y \in B \cup A_H$, y has at least $|A \cup B_H| - 1$ neighbors in $A \cup B_H$. □

By Claim 27, we have that $G = K_{s,s} - M$ for the two independent sets $A \cup B_H$ and $B \cup A_H$, and a matching M of $K_{s,s}$ with $0 \leq |M| \leq s$. This final contradiction to the assumption completes the proof for Case 2.3, and also completes the proof of Theorem 10. □

5 Proof of Theorem 11

Before we present our proof of Theorem 11, we state the following lemma to narrow down the category of graphs that we need to consider.

Lemma 1 (Hendry [13]) *If G is a graph of order n with $\delta(G) \geq (n+1)/2$, then G is fully cycle extendable.*

Here a graph G is called *fully cycle extendable* if every vertex of G lies on a triangle of G and furthermore every nonhamiltonian cycle C of G can be extended to another cycle C' such that $V(C) \subseteq V(C')$ and $|V(C')| = |V(C)| + 1$. Hence, if a graph is fully cycle extendable then it is surely pancyclic. By Lemma 1, if $\delta(G) \geq (n+1)/2$, then G is pancyclic. So we only need to consider graphs whose minimum degree is less than $(n+1)/2$. Suppose that G is a graph of order n satisfying the conditions of Theorem 11 and with $\delta(G) < (n+1)/2$, i.e.,

$\delta(G) \leq n/2$. Let S be a minimal cut set of G. Since $\delta(G) \leq n/2$, $|S| \leq n/2$. The following claim was given by Li et al. in [16]. We add its proof for convenience.

Claim 1 (Li et al. [16]). *Every vertex of S is adjacent to every vertex of $V(G) \backslash S$.*

Proof Clearly, the choice of S implies that for every vertex $x \in S$ and every component H of $G - S$, x is adjacent to at least one vertex of H; otherwise $S \backslash \{x\}$ is a vertex cut, contradicting the choice of S. Suppose that $xy \notin E(G)$ for some $y \in V(G) \backslash S$. Let H be the component of $G - S$ containing y, let P be a shortest path from y to x with all internal vertices in H, and let y' be a neighbor of x in a component of $G - S$ other than H. Then $yPxy'$ is an induced path on at least 4 vertices, contradicting that G is P_4-free. □

We now proceed with the proof of Theorem 11 and consider two cases.

Case 1. $|S| = n/2$.

Clearly, n is even and $|V(G) \backslash S)| = n/2$. Denote $S = \{x_1, x_2, \ldots, x_{n/2}\}$, $V(G) \backslash S = \{y_1, y_2, \ldots, y_{n/2}\}$. If S and $V(G) \backslash S$ are independent sets, then $G = K_{n/2,n/2}$, and we get the result. Without loss of generality, assume that S is not independent and $x_1x_2 \in E(S)$. By Claim 1, $K_{n/2,n/2}$ is a spanning subgraph of G. Then from $K_{n/2,n/2}$ we can get all cycles of even length from 4 up to $n-2$ containing x_1 but not x_2. By inserting x_2 after vertex x_1 in every even cycle, we get all cycles of odd length from 5 up to $n-1$. Since $x_1x_2y_1x_1$ is a 3-cycle and $K_{n/2,n/2}$ contains an n-cycle, G is pancyclic.

Case 2. $|S| < n/2$.

Let $s = |S|$, hence $|V(G) \backslash S| = n - s$, and clearly $n - s > n/2$. Since G is 1-tough, $G - S$ is not independent. Let H be the subgraph of G induced by S and s vertices of $G - S$ that contain at least one adjacent pair. Then $K_{s,s}$ is a spanning subgraph of H. By the same method as in Case 1, we can prove that H contains cycles of length 3 up to $2s$, and hence G contains cycles of length 3 up to $2s$.

We know from earlier results that every 1-tough P_4-free graph on at least three vertices is hamiltonian, so G contains a Hamilton cycle C. Let the vertices of $S = \{x_1, x_2, \ldots, x_s\}$ be arranged in this order around C according to a fixed orientation of C, and denote every segment of C from x_i to x_{i+1} by $S_i = x_iy_{i_1}y_{i_2}\ldots y_{i_{r_i}}x_{i+1}$ (possible with $r_i = 0$, i.e., no vertex y_i in between for some $1 \leq i \leq s$). By Claim 1, x_i is adjacent to every vertex of $S_i \backslash \{x_{i+1}\}$. Hence, if $r_i \neq 0$, then we can get cycles of length from n down to $n - r_i + 1$ using $x_iy_{i_k}Cx_i$ $(1 \leq k \leq r_i)$. In this way, if $r_i \neq 0$ for each $1 \leq i \leq s$, then we can delete the vertices within every segment one by one, until we are left with only two end vertices and one inside. Thus, we get cycles with length from n down to $2s$. Hence, G is pancyclic. If $r_i = 0$ for some $1 \leq i \leq s$, we can use similar arguments for the segments with $r_i \neq 0$ to get cycles with all possible missing lengths. Hence, G is pancyclic. This completes the proof of Theorem 11. □

6 Proof of Theorem 13

One direction of the equivalence statement in Theorem 13 follows directly from Theorems 9–11, whereas the graphs $\mathcal{C}_5^s$ show that forbidding the induced subgraph $K_1 \cup P_4$ does not imply the graphs are pancyclic. Hence, it suffices to prove that there is no graph H, apart from the proper induced subgraphs of $K_1 \cup P_4$, that can ensure every H-free graph with toughness larger than one on at least three vertices is pancyclic. We use the following lemma to complete our proof.

Lemma 2 (Li et al. [16]) *Let R be a graph on at least three vertices. If R is not an induced subgraph of* $K_1 \cup P_4$, *then R contains one of the graphs in* $\mathcal{H} = \{C_3, C_4, C_5, K_{1,3}, 2K_2, 4K_1\}$ *as an induced subgraph.*

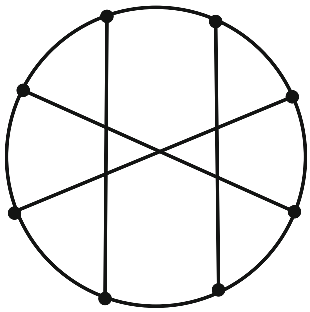
Using Lemma 2, it is sufficient to prove that for each of the graphs $R \in \mathcal{H}$ not every R-free graph on at least three vertices with toughness larger than one is pancyclic. To show this, we give suitable counterexamples for each case. Most of the counterexamples we present here also appear in [23], in which they serve as counterexamples for hamiltonian-connectivity. For the class of $4K_1$-free graphs, the small graph sketched in Fig. 3 is $4K_1$-free and has toughness larger than one, but is not pancyclic. For counterexamples related to the other members of the class of graphs $\mathcal{H}$ we refer to some useful known results.

For $R = C_3$, the Petersen graph is a suitable counterexample, since it is C_3-free, has toughness 4/3 and is nonhamiltonian.

For $R = C_4$, C_5 or $2K_2$, we can find suitable split graphs as counterexamples. Split graphs are known to be $\{C_4, C_5, 2K_2\}$-free. It was proved in [15] that every $\frac{3}{2}$-tough split graph is hamiltonian, and that there is a sequence $\{G_n\}_{n=1}^{\infty}$ of split graphs with no 2-factor (a 2-regular spanning subgraph, not necessarily connected) and $\tau(G_n) \to 3/2$. Here, we select the latter graphs to serve as examples for our purposes.

For $R = K_{1,3}$, every claw-free noncomplete graph G has the property that $2\tau(G) = \kappa(G)$, where $\kappa(G)$ denotes the (vertex) connectivity of G. In [17], it is conjectured that every 4-connected claw-free graph is hamiltonian, and the authors give examples of 3-connected (hence 3/2-tough) claw-free graphs that are not hamiltonian. These examples clearly serve our purposes and complete our proof of Theorem 13.

Fig. 3 $4K_1$-free non-pancyclic graph



Compliance with Ethical Standards

Conflict of Interest The authors declare that they have no conflict of interest.

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