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Extremal problems and results related to Gallai-colorings ☆

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ABSTRACT

A Gallai-coloring (Gallai-k-coloring) is an edge-coloring (with colors from $\{1, 2, \ldots, k\}$) of a complete graph without rainbow triangles. Given a graph H and a positive integer k, the k-colored Gallai-Ramsey number $GR_k(H)$ is the minimum integer n such that every Gallai-k-coloring of the complete graph K_n contains a monochromatic copy of H. In this paper, we consider two extremal problems related to Gallai-k-colorings. First, we determine upper and lower bounds for the maximum number of edges that are not contained in any rainbow triangle or monochromatic triangle in a k-edge-coloring of K_n. Second, for $n \geq GR_k(K_3)$, we determine upper and lower bounds for the minimum number of monochromatic triangles in a Gallai-k-coloring of K_n, yielding the exact value for $k = 3$. Furthermore, we determine the Gallai-Ramsey number $GR_k(K_4 + e)$ for the graph on five vertices consisting of a K_4 with a pendant edge.

1. Introduction

In this paper, we only consider edge-colorings of finite simple graphs. For an integer $k \geq 1$, let $c\colon E(G) \to [k]$ be a k-edge-coloring (not necessarily a proper edge-coloring) of a graph G, where $[k] := \{1, 2, \ldots, k\}$. A graph with an edge-coloring is called *rainbow* if all edges are colored differently, and *monochromatic* if all edges are colored the same. A *Gallai-k-coloring* is a k-edge-coloring of a complete graph without rainbow triangles, i.e. at most two distinct colors are assigned to the edges of every copy of K_3.

The term *Gallai-coloring* was first used by Gyárfás and Simonyi [17] in honor of Gallai's decomposition lemma for rainbow triangle-free colorings [13], but the study of Gallai-colorings has arisen in a wide range of areas, such as poset theory [13], the Erdős-Hajnal conjecture [10], rainbow Erdős-Rothschild problem [1,2], information theory [25,26], perfect graph theory [4], and Ramsey-type problems [16,18].

Given a positive integer k and graphs $H_1, H_2, \ldots, H_k$, the classical k-colored *Ramsey number* $R(H_1, H_2, \ldots, H_k)$ is the minimum integer n such that every k-edge-coloring of K_n contains a monochromatic copy of H_i in color i for some $i \in [k]$. It is well-known that determining the exact value of the Ramsey number is an extremely difficult problem, even for relatively small graphs. Many variants of Ramsey numbers concerning rainbow structures have been studied, such as rainbow-Ramsey numbers, anti-Ramsey numbers and Gallai-Ramsey numbers. We refer to two surveys [12,31] for more information on these topics.

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Given k graphs $H_1, H_2, \ldots, H_k$, the k-colored *Gallai-Ramsey number* $GR(H_1, H_2, \ldots, H_k)$ is defined to be the minimum integer n such that every Gallai-k-coloring of the complete graph on n vertices contains a monochromatic copy of H_i in color i for some $i \in [k]$. In the special case when $H_1 = H_2 = \cdots = H_k = H$, we simply write $R_k(H)$ and $GR_k(H)$ for $R(H, H, \ldots, H)$ and $GR(H, H, \ldots, H)$, respectively. Gallai-Ramsey theory has been increasingly popular over the past decade. We refer to papers [3,11,17,18,27,28,34] for more information on some related problems.

A natural problem related to Gallai-Ramsey theory is to determine the maximum number of edges that are not contained in any rainbow copy of K_3 or monochromatic copy of H. The analogous problem for Ramsey numbers was considered in [23,29,30]; in these papers the authors studied the maximum number of edges not contained in any monochromatic copy of H over all k-edge-colorings of K_n. For $k \geq 2$, let $f_k(n, H)$ denote the maximum number of edges not contained in any rainbow triangle or monochromatic copy of H, over all k-edge-colorings of K_n. The first part of this paper is devoted to this problem.

Let $ex(n, H)$ be the maximum number of edges of an H-free graph of order n, i.e., the Turán number of H. By Turán's theorem, the unique K_{r+1}-free graph on n vertices with $ex(n, K_{r+1})$ edges is the Turán graph $T_r(n)$, i.e., the complete r-partite graph on n vertices with class sizes as equal as possible. Let $t(n, r)$ be the number of edges of $T_r(n)$. Note that we have the trivial upper bound $f_k(n, H) \leq t(n, GR_k(H) - 1)$. We also have a trivial lower bound $f_k(n, H) \geq f_2(n, H) \geq ex(n, H)$. For the case $H = K_3$, we will prove the following theorem.

Theorem 1.1. *For any real number $\delta > 0$, there exists an n_0 such that for all $n \geq n_0$, we have $t(n, GR_{k-1}(K_3) - 1) \leq f_k(n, K_3) < t(n, GR_{k-1}(K_3) - 1) + \delta n^2$.*

We conjecture that the lower bound on $f_k(n, K_3)$ in Theorem 1.1 is in fact the exact value of $f_k(n, K_3)$. Moreover, we can generalize this result to a general graph H (see Theorem 3.4).

The second part of this paper is devoted to the Gallai-Ramsey multiplicity problem. By the definition of the Gallai-Ramsey number, if $n \geq GR_k(H)$, then any Gallai-k-coloring of K_n contains a monochromatic copy of H. In fact, there could be more than one monochromatic copy of H. In light of this, it is natural to consider the minimum number of monochromatic copies of H (as an unlabeled graph) in a Gallai-k-coloring of K_n. Let $g_k(H, n)$ denote the minimum number of monochromatic copies of H taken over all Gallai-k-colorings of K_n. The analogous problem for Ramsey numbers is known as the Ramsey multiplicity problem, that is, to consider the minimum number $M_k(H, n)$ of monochromatic copies of H taken over all k-edge-colorings of K_n (see [7–9,20] for some recent results). With the additional restriction imposed on Gallai-colorings, it is obvious that $g_k(H, n) \geq M_k(H, n)$. In 1959, Goodman [14] proved the following classical result concerning $M_2(K_3, n)$.

Theorem 1.2. ([14]) *For any positive integer n, we have*

$$M_2(K_3, n) = \begin{cases} n(n-2)(n-4)/24, & \text{if } n \text{ is even}, \\ n(n-1)(n-5)/24, & \text{if } n \equiv 1 \bmod 4, \\ (n+1)(n-3)(n-4)/24, & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

For the case of 3-edge-colorings, Cummings, Král', Pfender, Sperfeld, Treglown and Young [8] proved the following result, using flag algebras and a probabilistic argument.

Theorem 1.3. ([8]) *There exists an integer n_0 such that for $n \geq n_0$, if we write $n = 5m + r$ for nonnegative integers m and r with $0 \leq r \leq 4$, then*

$$M_3(K_3, n) = r\binom{m+1}{3} + (5 - r)\binom{m}{3}.$$

Our next result shows that $g_3(K_3, n) = M_3(K_3, n)$, and gives upper and lower bounds for $g_k(K_3, n)$ for other values of k.

Theorem 1.4. *For $n \geq GR_k(K_3)$, we write $n = 5^{\lfloor (k-1)/2 \rfloor} m + r$, where m and r are nonnegative integers with $0 \leq r \leq 5^{\lfloor (k-1)/2 \rfloor} - 1$. Then*

$$g_k(K_3, n) \leq \begin{cases} r\binom{m+1}{3} + \left(5^{(k-1)/2} - r\right)\binom{m}{3}, & \text{if } k \text{ is odd}, \\ rM_2(K_3, m+1) + \left(5^{(k-2)/2} - r\right)M_2(K_3, m), & \text{if } k \text{ is even}. \end{cases}$$

Moreover, let $s_0 = 1$ if k is odd, and $s_0 = 2$ if k is even. Then

$$g_k(K_3, n) \geq \frac{s_0 n(n-1)(n-2)}{GR_k(K_3)(GR_k(K_3) - 1)(GR_k(K_3) - 2)}.$$

In general, we conjecture that the above upper bound on $g_k(K_3, n)$ in Theorem 1.4 is in fact the exact value of $g_k(K_3, n)$, but we can only verify this for the following cases: (1) $k = 3$, (2) $k \geq 3$ and $n = GR_k(K_3)$, (3) k is odd and $GR_k(K_3) \leq n \leq GR_k(K_3) + 5^{(k-1)/2} - 1$.

Finally, we consider the original problem, the Gallai-Ramsey number for a graph H. In [16], Gyárfás, Sárközy, Sebő and Selkow provided the following general statement on the value of the Gallai-Ramsey number $GR_k(H)$.

Theorem 1.5. ([16]) *For any graph H and positive integer k, if H is not bipartite, then $GR_k(H)$ is exponential in k, and if H is bipartite but not a star, then $GR_k(H)$ is linear in k.*

In [10], Fox, Grinshpun and Pach posed the following conjecture on an expression for the Gallai-Ramsey numbers of complete graphs in terms of their 2-colored Ramsey numbers.

Conjecture 1.6. ([10]) *For integers $k \geq 1$ and $t \geq 3$,*

$$GR_k(K_t) = \begin{cases} (R_2(K_t) - 1)^{k/2} + 1, & \text{if } k \text{ is even}, \\ (t-1) \cdot (R_2(K_t) - 1)^{(k-1)/2} + 1, & \text{if } k \text{ is odd}. \end{cases}$$

The cases with $t = 3$ and $t = 4$ of the above conjecture were verified in [5,16] and [28], respectively. Let $K_4 + e$ denote the graph on five vertices consisting of a K_4 with a pendant edge. We prove the following related result, confirming that the expression in the above conjecture in fact also holds for $K_4 + e$ (taking $t = 5$), since $R_2(K_4 + e) = 18$ by a result in [19].

Theorem 1.7. *For integers $k \geq 1$,*

$$GR_k(K_4 + e) = \begin{cases} 17^{k/2} + 1, & \text{if } k \text{ is even}, \\ 4 \cdot 17^{(k-1)/2} + 1, & \text{if } k \text{ is odd}. \end{cases}$$

The remainder of this paper is organized as follows. In Section 2, we will introduce some additional terminology and notation, and list some known results that will be used in our proofs of the main results. In Section 3, we will prove Theorem 1.1, using a variant of the Gallai-Ramsey number. In Section 4, we will consider the Ramsey multiplicity problem for Gallai-colorings and prove Theorem 1.4. In Section 5, we will prove Theorem 1.7 in a more general form. Finally, we will conclude the paper with some remarks and open problems in Section 6.

2. Preliminaries

We begin with the following structural result on Gallai-colorings of complete graphs.

Theorem 2.1. ([13,17]) *In any Gallai-coloring of a complete graph, the vertex set can be partitioned into at least two nonempty parts such that there is only one color on the edges between every pair of parts, and there are at most two colors between the parts in total.*

We call a vertex partition as given by the statement in Theorem 2.1 a *Gallai partition*. Below we listed some known exact values of Gallai-Ramsey numbers and Ramsey numbers.

Theorem 2.2. ([5,16]) *For integers $k \geq 1$, we have*

$$GR_k(K_3) = \begin{cases} 5^{k/2} + 1, & \text{if } k \text{ is even}, \\ 2 \cdot 5^{(k-1)/2} + 1, & \text{if } k \text{ is odd}. \end{cases}$$

Theorem 2.3. *The following Ramsey numbers have been established:*

(1) ([15]) $R(K_3, K_3) = 6$, $R(K_4, K_4) = 18$.
(2) ([6]) $R(K_4 + e, K_3) = 9$.
(3) ([19]) $R(K_4 + e, K_4 + e) = 18$.

For a graph H, let $\Delta(H)$ and $\chi(H)$ be the maximum degree and chromatic number of H, respectively. Given an edge-colored graph F and an edge $e \in E(F)$, let $c_F(e)$ (or simply $c(e)$) be the color used on (i.e., assigned to) edge e. For $U, V \subseteq V(F)$ with $U \cap V = \emptyset$, we use $E(U, V)$ (resp., $C(U, V)$) to denote the set of edges between U and V (resp., the set of colors used on the edges between U and V). If all the edges in $E(U, V)$ are colored by a single color, then we use $c(U, V)$ to denote this color. Let $F[U]$ be the subgraph of F induced by $U \subseteq V(F)$, and $F - U$ be the subgraph of F induced by

$V(F) \setminus U$ (if $U \neq V(F)$). In the special case when $U = \{u\}$, we simply write $E(u, V)$, $C(u, V)$, $c(u, V)$ and $F - u$ for $E(\{u\}, V)$, $C(\{u\}, V)$, $c(\{u\}, V)$ and $F - \{u\}$, respectively. Let $C(F[U])$ (or simply, $C(U)$) and $C(F - U)$ denote the set of colors used on $E(F[U])$ and $E(F - U)$, respectively. For two graphs F_1 and F_2, let $F_1 \cup F_2$ be the disjoint union of F_1 and F_2.

Next, we define the blow-up of an edge-colored complete graph which will be used in our proofs of Theorems 1.4 and 1.7. Let G be an edge-colored complete graph with vertex set $\{v_1, v_2, \ldots, v_n\}$, and $H_1, H_2, \ldots, H_n$ be n pairwise disjoint edge-colored complete graphs. The *blow-up* $G(H_1, H_2, \ldots, H_n)$ of G is an edge-colored complete graph with vertex set $\bigcup_{i=1}^{n} V(H_i)$ such that

$$c_{G(H_1,H_2,\ldots,H_n)}(xy) = \begin{cases} c_G(v_i v_j), & \text{if } x \in V(H_i) \text{ and } y \in V(H_j) \text{ for some } 1 \le i \neq j \le n, \\ c_{H_i}(xy), & \text{if } x, y \in V(H_i) \text{ for some } i \in [n]. \end{cases}$$

If $H_1 = H_2 = \cdots = H_n = H$, we will write $G(n \cdot H)$ for $G(H, H, \ldots, H)$. If $H_1 = \cdots = H_s = H'$ and $H_{s+1} = \cdots = H_n = H''$ for some $1 \le s < n$, we will write $G(s \cdot H', (n-s) \cdot H'')$ for $G(H', \ldots, H', H'', \ldots, H'')$. Similarly, we will use the abbreviation $G(s \cdot H', t \cdot H'', (n-s-t) \cdot H''')$.

In the following, we will introduce the Regularity Lemma, Embedding Lemma and Slicing Lemma that will be used in our proof of Theorem 1.1. Given a graph F and two disjoint nonempty sets $X, Y \subseteq V(F)$, the *density* of (X, Y) is defined to be

$$d(X, Y) := \frac{|E(X, Y)|}{|X||Y|}.$$

We say that (X, Y) is ε*-regular* if for any $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \ge \varepsilon|X|$ and $|Y'| \ge \varepsilon|Y|$, we have $|d(X', Y') - d(X, Y)| \le \varepsilon$. For a positive real number d, we say that an ε-regular pair (X, Y) is (ε, d)*-regular* if $d(X, Y) \ge d$.

Lemma 2.4 *(Multicolor Regularity Lemma).* (See e.g. [24,29,33].) *For any real $\varepsilon > 0$ and positive integers k and m, there exist n' and M, such that every k-edge-colored graph F with $n \ge n'$ vertices admits a partition $V_1, V_2, \ldots, V_t$ of $V(F)$ satisfying*

(i) *$m \le t \le M$;*
(ii) *for all $1 \le i < j \le t$, we have $||V_i| - |V_j|| \le 1$; and*
(iii) *for all but at most $\varepsilon\binom{t}{2}$ pairs (i, j), the pair (V_i, V_j) is ε-regular for each color.*

We call the partition as given in Lemma 2.4 a *multicolored ε-regular partition*. Given $\varepsilon, d > 0$, a k-edge-colored graph F and a partition $V_1, V_2, \ldots, V_t$ of $V(F)$, we define the *reduced graph* $R = R(d)$ as follows: $V(R) = \{1, 2, \ldots, t\}$ and i and j are adjacent in R if (V_i, V_j) is ε-regular for each color and there exists a color with density at least d in $E(V_i, V_j)$. Moreover, we define the *multicolored reduced graph* $R^c = R^c(d)$ as follows: $V(R^c) = V(R)$, $E(R^c) = E(R)$, and for each edge $ij \in E(R^c)$, ij is assigned an arbitrary color c_0 such that (V_i, V_j) has density at least d with respect to the subgraph of F induced by the edges of color c_0.

Given two graphs G and H, we say that G is a *homomorphic copy* of H if there is a map $\varphi: V(H) \to V(G)$ such that $\varphi(u)\varphi(v) \in E(G)$ for each edge $uv \in E(H)$. Note that K_s is a homomorphic copy of H if and only if $s \ge \chi(H)$. We will use the following consequence of the Embedding Lemma. Lemma 2.5 below is in fact a corollary of Lemma 2.4 in [21].

Lemma 2.5 *(Multicolor Embedding Lemma).* (See e.g. [21,22,24].) *For every $d > 0$, any positive integer k and any graph G, there exist $\varepsilon = \varepsilon(k, d, G) > 0$ and a positive integer $n_0 = n_0(k, d, G)$ with the following property. Suppose that F is a k-edge-colored graph on $n \ge n_0$ vertices with a multicolored ε-regular partition $V_1, V_2, \ldots, V_t$ which defines the multicolored reduced graph $R^c = R^c(d)$. If R^c contains a monochromatic homomorphic copy of G, then F contains a monochromatic copy of G. If R^c contains a rainbow copy of G, then F contains a rainbow copy of G.*

Lemma 2.6 *(Slicing Lemma).* (See e.g. [24,29].) *Let $0 < \varepsilon, \alpha, d < 1$ with $\varepsilon \le \min\{d, \alpha, 1/2\}$. If a pair (X, Y) is (ε, d)-regular, then for any $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \ge \alpha|X|$ and $|Y'| \ge \alpha|Y|$, we have that (X', Y') is an $(\varepsilon', d - \varepsilon)$-regular pair, where $\varepsilon' := \max\{2\varepsilon, \varepsilon/\alpha\}$.*

Finally, we consider the Turán number. It is well-known that $ex(n, K_{r+1}) = t(n, r) = (1 - 1/r)\binom{n}{2} + o(n^2)$. In fact, if $n \equiv p \pmod{r}$ where $0 \le p \le r-1$, then $t(n, r) = (1 - 1/r)n^2/2 + (p - r)p/(2r)$. Thus $(1 - 1/r)n^2/2 - r/8 \le t(n, r) \le (1 - 1/r)n^2/2$. We will use this more precise bound in our proofs of the main results.

3. On edges not contained in a rainbow triangle or monochromatic copy of H

For the proof of Theorem 1.1, we first define the following variant of the Gallai-Ramsey number. Given a set V and an integer $k \le |V|$, let $\binom{V}{\le k}$ (resp., $\binom{V}{k}$) be the set of all nonempty subsets of V of size at most k (resp., size k).

Definition 3.1. For a graph H and an integer $k \geq 2$, let $GR_k^*(H)$ be the minimum integer n^* such that for every coloring $c : \binom{[n^*]}{\leq 2} \to [k]$, at least one of the following statements holds:

(1*) the restriction of c to $\binom{[n^*]}{2}$ contains either a rainbow triangle or a monochromatic homomorphic copy of H;
(2*) for some $1 \leq i < j \leq n^*$, we have $c(\{i, j\}) = c(\{i\})$ or $c(\{i, j\}) = c(\{j\})$.

In other words, $GR_k^*(H) - 1$ is the maximum integer n^{**} such that for the complete graph $K_{n^{**}}$ with vertex set $[n^{**}]$, there exists a coloring $c : \binom{[n^{**}]}{\leq 2} \to [k]$ satisfying

(1**) the restriction of c to $\binom{[n^{**}]}{2}$ is a Gallai-k-coloring without a monochromatic homomorphic copy of H; and
(2**) for any $1 \leq i < j \leq n^*$, we have $c(\{i, j\}) \neq c(\{i\})$ and $c(\{i, j\}) \neq c(\{j\})$.

For a set $\mathscr{H}$ of graphs, let $GR_k(\mathscr{H})$ denote the minimum integer n such that every Gallai-k-coloring of K_n contains a monochromatic copy of H for some $H \in \mathscr{H}$.

Lemma 3.2. *For a graph H, let $\mathscr{H}$ be the set of all homomorphic copies of H. Then*

(1) $GR_k^*(H) \geq GR_{k-1}(\mathscr{H})$,
(2) $f_k(n, H) \geq t(n, GR_{k-1}(\mathscr{H}) - 1)$,
(3) *if there exists a coloring c satisfying conditions (1**) and (2**) such that all elements of $\binom{[GR_k^*(H)-1]}{1}$ use a single color, then $f_k(n, H) \geq t(n, GR_k^*(H) - 1)$.*

Proof. Let $n_k^* := GR_{k-1}(\mathscr{H})$. We first prove (1). Let F be a Gallai-$(k-1)$-coloring of $K_{n_k^*-1}$ without a monochromatic copy of H' for any $H' \in \mathscr{H}$. We color the vertices of F with the kth color and then we obtain a k-coloring of $\binom{[n_k^*-1]}{\leq 2}$ satisfying conditions (1**) and (2**). Thus $GR_k^*(H) \geq n_k^* = GR_{k-1}(\mathscr{H})$.

Next, we give the proof of (2). Let G be a Gallai-$(k-1)$-coloring of $K_{n_k^*-1}$ without a monochromatic copy of H' for any $H' \in \mathscr{H}$. Let $V(G) = \{1, 2, \ldots, n_k^* - 1\}$ and let G' be the Turán graph $T_{n_k^*-1}(n)$ with parts $V_1, \ldots, V_{n_k^*-1}$. We color the edges of G' such that for any $1 \leq i < j \leq n_k^* - 1$, we have $c_{G'}(V_i, V_j) = c_G(ij)$. Let G'' be a k-edge-coloring of K_n obtained by coloring the edges within each part using color k from the above $(k-1)$-edge-coloring of G'. We claim that all the edges between the $n_k^* - 1$ parts are neither contained in a rainbow copy of K_3 nor in a monochromatic copy of H in G''. Indeed, note that there is no rainbow copy of K_3 using color k. Thus if G'' contains a rainbow copy of K_3, then G is not a Gallai-coloring, a contradiction. If there is an edge e between these $n_k^* - 1$ parts such that e is contained in a monochromatic copy of H, then G contains a monochromatic homomorphic copy of H, a contradiction. Thus $f_k(n, H) \geq \left|E\left(G'\right)\right| = t(n, n_k^* - 1)$.

Finally, we prove (3). Let $n_k := GR_k^*(H) - 1$. Let c be a coloring as in the statement of the lemma, and we may assume that all elements of $\binom{[n_k]}{1}$ are colored by color 1. Note that the restriction of c to $\binom{[n_k]}{2}$ is a Gallai-$(k-1)$-coloring without a monochromatic homomorphic copy of H. Let W be the Turán graph $T_{n_k}(n)$ with parts $V_1, \ldots, V_{n_k}$. We color the edges of W such that $c_W(V_i, V_j) = c(ij)$ for any $1 \leq i < j \leq n_k$. Let W' be a k-edge-coloring of K_n obtained by coloring the edges within each part using color 1 from the above $(k-1)$-edge-coloring of W. It is easy to check that all the edges between the n_k parts are neither contained in a rainbow copy of K_3 nor in a monochromatic copy of H in W'. Thus $f_k(n, H) \geq |E(W)| = t(n, n_k)$. □

Note that we have $GR_k^*(H) = GR_{k-1}(\mathscr{H}) = 2$ whenever H is a bipartite graph, where $\mathscr{H}$ is the set of all homomorphic copies of H. A natural question is for which non-bipartite graph H it holds that $GR_k^*(H) = GR_{k-1}(\mathscr{H})$? We can verify that K_3 is such a graph.

Lemma 3.3. *Let $\mathscr{H}(K_3)$ be the set of all homomorphic copies of K_3. For integers $k \geq 2$, we have $GR_k^*(K_3) = GR_{k-1}(\mathscr{H}(K_3)) = GR_{k-1}(K_3)$.*

Proof. For every graph $H' \in \mathscr{H}(K_3)$, we have that H' contains K_3 as a subgraph by the definition. Thus $GR_{k-1}(\mathscr{H}(K_3)) \geq GR_{k-1}(K_3)$. By Lemma 3.2 (1), we have $GR_k^*(K_3) \geq GR_{k-1}(\mathscr{H}(K_3)) \geq GR_{k-1}(K_3)$.

For $k \geq 2$, let $n_k^* := GR_{k-1}(K_3)$, and we will prove $GR_k^*(K_3) \leq n_k^*$ by induction on k. When $k = 2$, we have $GR_2^*(K_3) = 3 = n_2^*$ clearly. Suppose that for all $2 \leq k' \leq k - 1$, we have $GR_{k'}^*(K_3) \leq n_{k'}^*$. We will prove it for $k' = k$. Let n be the maximum integer such that there is a coloring $c : \binom{[n]}{\leq 2} \to [k]$ satisfying conditions (1**) and (2**). It suffices to show that $n \leq n_k^* - 1$. By Theorem 2.1, there is a Gallai partition $V_1, V_2, \ldots, V_m$ $(m \geq 2)$ of $[n]$. Note that $K_3 \in \mathscr{H}(K_3)$. For avoiding a monochromatic copy of K_3, we have $m \leq 5$. We choose such a partition so that m is minimum. Let R be an edge-coloring of a complete graph with $V(R) = \{v_1, v_2, \ldots, v_m\}$ and $c(v_i v_j) = c(V_i, V_j)$ for any $i \neq j$. If $m = 5$ (resp., $m = 4$), then R is the unique 2-edge-coloring of K_5 without a monochromatic copy of K_3, i.e., each color forms a cycle of length 5 (resp., R is one of

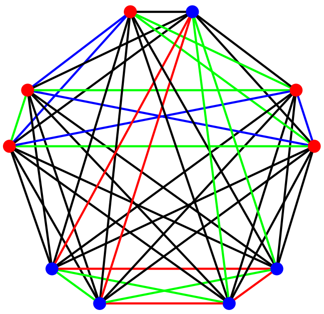

Fig. 1. An extremal coloring of $GR_4^*(K_3)$ with two colors on singletons. (For interpretation of the colors in the figure, the reader is referred to the web version of this article.)

the two 2-edge-colorings of K_4 without a monochromatic copy of K_3, i.e., each color forms a path of length 3, or one color forms a cycle of length 4 and the other color forms a matching with two edges). Then there is no edge using color 1 or 2 within each part V_i for avoiding a monochromatic copy of K_3, and there is no vertex using color 1 or 2 within each part V_i by condition (2^{**}). Thus if $k=3$, then $n \le 5 = n_3^* - 1$, and if $k \ge 4$, then $n \le 5(GR_{k-2}^*(K_3)-1) \le 5(GR_{k-3}(K_3)-1) \le n_k^* - 1$ by the induction hypothesis. If $m=3$, then at least two of the colors $c(V_1,V_2)$, $c(V_1,V_3)$ and $c(V_2,V_3)$ are the same color, say $c(V_1,V_2)=c(V_1,V_3)$. This implies that V_1 and $V_2 \cup V_3$ form a Gallai partition with exactly two parts, contradicting the minimality of m. If $m=2$, then we may assume $c(V_1,V_2)=1$. Then color 1 cannot be used on $\binom{V_1}{\le 2}$ and $\binom{V_2}{\le 2}$. Thus $n \le 2(GR_{k-1}^*(K_3)-1) \le 2(GR_{k-2}(K_3)-1) \le n_k^* - 1$ by the induction hypothesis. $\square$

By Lemma 3.3, we have $GR_k^*(K_3) = GR_{k-1}(\mathscr{H}(K_3))$. As in the proof of Lemma 3.2 (1), we can construct an extremal coloring $\binom{[GR_k^*(K_3)-1]}{\le 2} \to [k]$ satisfying conditions (1^{**}) and (2^{**}) in which we assign a single color to all elements of $\binom{[GR_k^*(K_3)-1]}{1}$. It is worth noticing that not all the extremal colorings assign a single color to all singletons. For example, Fig. 1 gives an extremal coloring of $GR_4^*(K_3)$ with two colors on singletons.

Now we have all the ingredients for our proof of Theorem 1.1.

Proof of Theorem 1.1. The lower bound follows from Lemmas 3.2 (2) and 3.3. Next, we will prove that $f_k(n,K_3) < t(n, GR_{k-1}(K_3)-1) + \delta n^2$. Let $\mathrm{nim}_k(n,K_3)$ be the maximum number of edges not contained in any monochromatic copy of K_3 over all k-edge-colorings of K_n. Note that $f_k(n,K_3) \le \mathrm{nim}_k(n,K_3)$. For sufficiently large n, since $\mathrm{nim}_2(n,K_3) = t(n,2)$ (proven in [23]) and $\mathrm{nim}_3(n,K_3) = t(n,5)$ (proven in [29]), we have $f_k(n,K_3) = t(n, GR_{k-1}(K_3)-1)$ for $k \in \{2,3\}$. In the following, we may assume $k \ge 4$.

Let $N_k := GR_{k-1}(K_3)$. We choose d such that $d \le \delta/k$. Let $\varepsilon_1 = \varepsilon_1(k, d/2, K_3)$ and $n_1 = n_1(k, d/2, K_3)$ (resp., $\varepsilon_2 = \varepsilon_2(k,d,K_3)$ and $n_2 = n_2(k,d,K_3)$) be the values obtained by applying Lemma 2.5. Let n_1' and M_1 be the values obtained by applying Lemma 2.4 with ε_1 and $1/\varepsilon_1$. Then we choose ε such that $\varepsilon \le \min\{\delta/4, \varepsilon_1/M_1, \varepsilon_2, d/2\}$. Let n' and M be the values obtained by applying Lemma 2.4 with ε and $1/\varepsilon$. Let $n_0 = \max\left\{n', n_1'M, \sqrt{(N_k-1)/(2\delta)}, MM_1n_1/3, n_2\right\}$ and $n \ge n_0$.

Let F be a k-edge-coloring of K_n, and F' be the spanning subgraph of F with $E(F') = \{e \in E(F)\colon e$ is not contained in any rainbow or monochromatic copy of $K_3\}$. For a contradiction, suppose $|E(F')| \ge t(n, N_k-1) + \delta n^2$. Let $V_1, V_2, \ldots, V_t$ be a partition of $V(F')$ obtained by applying Lemma 2.4 to F' with ε and $1/\varepsilon$, where $1/\varepsilon \le t \le M$. Let $R = R(d)$ be the reduced graph. Since there are at most $\binom{n/t}{2}$ edges within a part, at most $(n/t)^2$ edges between any two parts, and less than $kd\,(n/t)^2$ edges between a pair of parts with density less than d for each color, we have

$$
\begin{aligned}
|E(R)| &> \frac{t(n,N_k-1)+\delta n^2 - t\binom{\frac{n}{t}}{2} - \varepsilon\binom{t}{2}\left(\frac{n}{t}\right)^2 - kd\left(\frac{n}{t}\right)^2\binom{t}{2}}{\left(\frac{n}{t}\right)^2} \\
&> \frac{t^2\left(\left(1-\frac{1}{N_k-1}\right)\frac{n^2}{2} - \frac{N_k-1}{8} + \delta n^2 - \left(\frac{1}{t}+\varepsilon+kd\right)\frac{n^2}{2}\right)}{n^2} \\
&= \left(1 - \frac{1}{N_k-1} + 2\delta - \frac{N_k-1}{4n^2} - \frac{1}{t} - \varepsilon - kd\right)\frac{t^2}{2} \\
&\ge \left(1 - \frac{1}{N_k-1}\right)\frac{t^2}{2},
\end{aligned}
$$

where the last inequality is by the choices of n, d and ε. Thus $|E(R)| \ge t(t, N_k-1)+1$, so R contains a copy R' of K_{N_k}. Without loss of generality, let $V(R') = \{1,2,\ldots,N_k\}$. Then for any $1 \le i < j \le N_k$, we have that (V_i, V_j) is ε-regular for each color, and there exists a color c_{ij} with density at least d in $E(V_i,V_j)$.

For each $i \in [N_k]$, we have $|V_i| = n/t \geq (n'_1 M)/M = n'_1$. Thus we can apply Lemma 2.4 with ε_1 and $1/\varepsilon_1$ to $F[V_i]$ (note that here we consider $F[V_i]$, not only $F'[V_i]$). Then there exist two subsets $V_{i,1}, V_{i,2} \subseteq V_i$ with $|V_{i,1}| = |V_{i,2}| \geq n'_1/M_1$ such that $(V_{i,1}, V_{i,2})$ is an $(\varepsilon_1, 1/k)$-regular pair for some color $c_i \in [k]$. From the choice of d, we have $1/k \geq d/2$, so $(V_{i,1}, V_{i,2})$ is an $(\varepsilon_1, d/2)$-regular pair for color c_i. We define a coloring $\varphi : \binom{V(R')}{\leq 2} \to [k]$ such that $\varphi(\{i\}) = c_i$ and $\varphi(\{i, j\}) = c_{ij}$. Note that there might be more than one choice for $\varphi(\{i\})$ and $\varphi(\{i, j\})$, and we may choose an arbitrary one from these choices. By Lemma 3.3, we have $|V(R')| = N_k = GR_{k-1}(K_3) = GR^*_k(K_3)$. Thus at least one of the following statements holds:

(1) R' contains a rainbow copy of K_3;
(2) R' contains a monochromatic homomorphic copy of K_3;
(3) $\varphi(\{i, j\}) = \varphi(\{i\})$ for some $1 \leq i \neq j \leq N_k$.

If (1) or (2) holds, then there is a rainbow or monochromatic copy of K_3 in F' by Lemma 2.5, a contradiction. If (3) holds, then by applying Lemma 2.6 with $\alpha = 1/M_1$, we have that $(V_j, V_{i,1})$ and $(V_j, V_{i,2})$ are two $(\varepsilon M_1, d - \varepsilon)$-regular (and thus $(\varepsilon_1, d/2)$-regular) pairs for color c_i. Thus $(V_{i,1}, V_{i,2})$, $(V_j, V_{i,1})$ and $(V_j, V_{i,2})$ are three $(\varepsilon_1, d/2)$-regular pairs for color c_i. By Lemma 2.5, there is a monochromatic copy of K_3 which contains two edges of F', a contradiction. $\square$

By similar arguments as in the proof of Theorem 1.1, we can prove the following result for a general graph H. We omit the details.

Theorem 3.4. *For any $\delta > 0$, there exists an n_0 such that for all $n \geq n_0$ and any graph H, we have $t\left(n, GR_{k-1}(\mathscr{H}) - 1\right) \leq f_k(n, H) < t\left(n, GR^*_k(H) - 1\right) + \delta n^2$, where $\mathscr{H}$ is the set of all homomorphic copies of H.*

4. The Ramsey multiplicity problem for Gallai-colorings

We first prove the upper bound in Theorem 1.4, by construction. Let G_2 be a 2-edge-colored K_5 using colors 1 and 2 which contains no monochromatic copy of K_3, i.e., colors 1 and 2 induce two monochromatic copies of C_5. Suppose that $2i < k - 2$ and we have constructed a Gallai-$2i$-coloring G_{2i} of $K_{n_{2i}}$ without a monochromatic copy of K_3, where $n_{2i} := 5^i$. Let G' be a 2-edge-colored K_5 using colors $2i + 1$ and $2i + 2$ which contains no monochromatic copy of K_3. Let $G_{2i+2} = G'(5 \cdot G_{2i})$, i.e., G_{2i+2} is a blow-up of G'. This way, when k is odd (resp., k is even), we obtain a Gallai-$(k-1)$-coloring G_{k-1} of $K_{n_{k-1}}$ (resp., Gallai-$(k-2)$-coloring G_{k-2} of $K_{n_{k-2}}$) without a monochromatic copy of K_3, where $n_{k-1} = 5^{(k-1)/2}$ (resp., $n_{k-2} = 5^{(k-2)/2}$). In the following, we will construct a Gallai-k-coloring G_k from G_{k-1} or G_{k-2}.

If k is odd, then let A be a monochromatic copy of K_m using color k, and let B be a monochromatic copy of K_{m+1} using color k. Let $G_k = G_{k-1}(r \cdot B, (5^{(k-1)/2} - r) \cdot A)$. Then G_k is a Gallai-k-coloring of K_n with $r\binom{m+1}{3} + \left(5^{(k-1)/2} - r\right)\binom{m}{3}$ monochromatic copies of K_3 (here we define $\binom{1}{3} = \binom{2}{3} = 0$ for the sake of notation). If k is even, then let C be a 2-edge-coloring (using colors $k-1$ and k) of K_m with $M_2(K_3, m)$ monochromatic copies of K_3, and let D be a 2-edge-coloring (using colors $k - 1$ and k) of K_{m+1} with $M_2(K_3, m + 1)$ monochromatic copies of K_3. Let $G_k = G_{k-2}(r \cdot D, (5^{(k-2)/2} - r) \cdot C)$. Then G_k is a Gallai-k-coloring of K_n with $rM_2(K_3, m + 1) + \left(5^{(k-2)/2} - r\right) M_2(K_3, m)$ monochromatic copies of K_3. This completes the proof for the upper bound in Theorem 1.4.

It is worth noting that no matter whether k is odd or even, the above extremal coloring is a blow-up of a complete graph of order $5^{\lfloor (k-1)/2 \rfloor}$ with a special edge-coloring. Recall that we have $g_3(K_3, n) = r\binom{m+1}{3} + (5 - r)\binom{m}{3}$. An interesting fact is that the above sharpness example for $k = 3$ is the unique Gallai-3-coloring of K_n achieving the minimum number of monochromatic copies of K_3, which can be derived from a result of [8]. But when k is an even number, the extremal colorings achieving the upper bound are not unique. For example, let F be a 2-edge-coloring (using colors $k - 1$ and k) of K_{m+2} with $M_2(K_3, m + 2)$ monochromatic copies of K_3. Since $M_2(K_3, m) + M_2(K_3, m + 2) = 2M_2(K_3, m + 1)$ for any odd number m by Theorem 1.2, we can also construct G_k such that $G_k = G_{k-2}(1 \cdot F, (r - 2) \cdot D, (5^{(k-2)/2} - r + 1) \cdot C)$. However, it is still a blow-up of a complete graph of order $5^{\lfloor (k-1)/2 \rfloor}$ with a special edge-coloring.

Before presenting our proof for the lower bound in Theorem 1.4, we first provide the exact value of $g_k(K_3, GR_k(K_3))$.

Theorem 4.1. *$g_k(K_3, GR_k(K_3)) = 1$ if k is odd, and $g_k(K_3, GR_k(K_3)) = 2$ if k is even.*

Proof. By the definition of the Gallai-Ramsey number, we have $g_k(K_3, GR_k(K_3)) \geq 1$. Moreover, it follows from the above extremal coloring that $g_k(K_3, GR_k(K_3)) \leq 1$ if k is odd, and $g_k(K_3, GR_k(K_3)) \leq 2$ if k is even. Thus it suffices to prove that $g_k(K_3, GR_k(K_3)) \geq 2$ when k is even. We will prove this by induction on k. For $k = 2$, the statement is trivial since $M_2(K_3, 6) = 2$. We may assume that the statement holds for all even $k' \leq k - 2$ and we will prove it for k ($k \geq 4$).

Let F be a Gallai-k-coloring of $K_{GR_k(K_3)}$ and suppose (for a contradiction) that F contains only one monochromatic copy of K_3. Using Theorem 2.1, let $V_1, V_2, \ldots, V_t$ ($t \geq 2$) be a Gallai partition of $V(F)$. We choose such a partition so that t is minimum. We may assume that colors 1 and 2 are the two colors used between these parts. Let R be a 2-edge-coloring of K_t with $V(R) = \{v_1, v_2, \ldots, v_t\}$ and $c(v_i v_j) = c(V_i, V_j)$ for any $1 \leq i < j \leq t$. Since $M_2(K_3, 6) = 2$, we have $t \leq 5$; otherwise F contains at least two monochromatic copies of K_3.

If $2 \le t \le 3$, then we may assume that $t = 2$ by the minimality of t (since every graph admitting a Gallai partition with three parts also admits a Gallai partition with two parts). Without loss of generality, let $c(V_1, V_2) = 1$ and $|V_1| \ge |V_2|$. First, assume $1 \notin C(V_1)$. Then $F[V_1]$ is a Gallai-$(k-1)$-coloring. Note that $|V_1| \ge |V(F)|/2 \ge (5^{k/2}+1)/2 > 2 \cdot 5^{(k-2)/2} + 2$. Since k is even, we have $GR_{k-1}(K_3) = 2 \cdot 5^{(k-2)/2} + 1$. Thus there is a monochromatic copy of K_3 in $F[V_1]$. Let v be a vertex of this K_3. Since $|V_1 \setminus \{v\}| \ge 2 \cdot 5^{(k-2)/2} + 1$, there is a monochromatic copy of K_3 in $F[V_1 \setminus \{v\}]$. So there exist two monochromatic copies of K_3 in $F[V_1]$, a contradiction. We conclude that $1 \in C(V_1)$. In order to avoid two monochromatic copies of K_3, we have $|V_2| = 1$ and there is at most one edge with color 1 in $F[V_1]$. Thus there is a Gallai-$(k-1)$-coloring of $K_{|V_1|-1}$. Since $|V_1| - 1 \ge GR_{k-1}(K_3)$, there is a monochromatic copy of K_3 in $F[V_1]$. Then there exist two monochromatic copies of K_3 in F, another contradiction. This solves the case $2 \le t \le 3$.

If $t = 4$, then we first suppose that R contains a monochromatic copy of K_3, say $c(V_1, V_2) = c(V_2, V_3) = c(V_3, V_1) = 1$. Let $V' = V_1 \cup V_2 \cup V_3$. If $c(V_4, V') = 2$, then V_4 and V' form a Gallai partition with exactly two parts, contradicting the minimality of t. Thus $c(V_4, V_i) = 1$ for some $i \in \{1, 2, 3\}$. But then $c(V_i, V(G) \setminus V_i) = 1$, contradicting the minimality of t. Therefore, R is one of the two 2-edge-coloring of K_4 without a monochromatic copy of K_3, that is, each color induces a path of length three, or one color induces a cycle of length four and the other color induces a matching with two edges. In both cases we can derive that there is at most one edge with color 1 or 2 in $\bigcup_{j=1}^{4} F[V_j]$. By the induction hypothesis, we have $|V(F)| \le 4(GR_{k-2}(K_3) - 1) + 1 < GR_k(K_3)$, a contradiction.

The remaining case is $t = 5$. Then there is no edge with color 1 or 2 in $\bigcup_{j=1}^{5} F[V_j]$; otherwise F contains a 2-edge-coloring of K_6 which contains at least two monochromatic copies of K_3. Thus we have $|V(F)| \le 5(GR_{k-2}(K_3) - 1) < GR_k(K_3)$ by the induction hypothesis, a contradiction. This completes the proof of Theorem 4.1. $\square$

Now we have all ingredients to present our proof for the lower bound in Theorem 1.4. Let $s_0 = 1$ if k is odd, and $s_0 = 2$ if k is even. By Theorem 4.1, we have $g_k(K_3, GR_k(K_3)) = s_0$. This implies that if $v_1, v_2, \ldots, v_{GR_k(K_3)}$ are any $GR_k(K_3)$ vertices of K_n, then $K_n[\{v_1, v_2, \ldots, v_{GR_k(K_3)}\}]$ contains at least s_0 monochromatic copies of K_3. Since each monochromatic copy of K_3 is contained in $\binom{n-3}{GR_k(K_3)-3}$ distinct copies of $K_{GR_k(K_3)}$, there are at least

$$\left\lceil \frac{s_0\binom{n}{GR_k(K_3)}}{\binom{n-3}{GR_k(K_3)-3}} \right\rceil = \left\lceil \frac{s_0 n(n-1)(n-2)}{GR_k(K_3)(GR_k(K_3)-1)(GR_k(K_3)-2)} \right\rceil$$

monochromatic copies of K_3 in any Gallai-k-coloring of K_n. This completes the proof of Theorem 1.4.

We obtain the following corollary.

Corollary 4.2. *If k is odd and $0 \le t \le 5^{(k-1)/2} - 1$, then $g_k(K_3, GR_k(K_3) + t) = t + 1$.*

Proof. The upper bound follows from Theorem 1.4. For the proof of the lower bound, we will use induction on t. The case $t = 0$ follows from Theorem 4.1. We may assume that $g_k(K_3, GR_k(K_3) + (t-1)) = (t-1) + 1 = t$ holds and we will prove it for t $(1 \le t \le 5^{(k-1)/2} - 1)$. Let $n = GR_k(K_3) + t$. Note that each monochromatic copy of K_3 is contained in $\binom{n-3}{n-1-3} = n-3$ distinct copies of K_{n-1}, and there are $\binom{n}{n-1} = n$ distinct copies of K_{n-1} in K_n. By the induction hypothesis, there are at least $\lceil tn/(n-3) \rceil = t+1$ monochromatic copies of K_3 in any Gallai-k-coloring of K_n. $\square$

5. The Gallai-Ramsey number for $K_4 + e$

For an integer s with $0 \le s \le k$, if $H_1 = \cdots = H_s = K_4 + e$ and $H_{s+1} = \cdots = H_k = K_3$, we will write $GR_k(s \cdot K_4 + e, (k-s) \cdot K_3)$ for $GR(K_4 + e, \ldots, K_4 + e, K_3, \ldots, K_3)$. In this section, we will prove Theorem 1.7 in the following more general form. Theorem 1.7 follows from Theorem 5.1 by choosing $s = k$.

Theorem 5.1. *For integers $k \ge 1$ and $0 \le s \le k$, we have*

$$GR_k(s \cdot K_4 + e, (k-s) \cdot K_3) = \begin{cases} 17^{s/2} \cdot 5^{(k-s)/2} + 1, & \text{if } s \text{ is even and } k-s \text{ is even}, \\ 2 \cdot 17^{s/2} \cdot 5^{(k-s-1)/2} + 1, & \text{if } s \text{ is even and } k-s \text{ is odd}, \\ 8 \cdot 17^{(s-1)/2} \cdot 5^{(k-s-1)/2} + 1, & \text{if } s \text{ is odd and } k-s \text{ is odd}, \\ 4 \cdot 17^{(s-1)/2} \cdot 5^{(k-s)/2} + 1, & \text{if } s \text{ is odd and } k-s \text{ is even}. \end{cases}$$

Proof. For convenience, let

$$g(k,s) := \begin{cases} 17^{s/2} \cdot 5^{(k-s)/2}, & \text{if } s \text{ is even and } k-s \text{ is even}, \\ 2 \cdot 17^{s/2} \cdot 5^{(k-s-1)/2}, & \text{if } s \text{ is even and } k-s \text{ is odd}, \\ 8 \cdot 17^{(s-1)/2} \cdot 5^{(k-s-1)/2}, & \text{if } s \text{ is odd and } k-s \text{ is odd}, \\ 4 \cdot 17^{(s-1)/2} \cdot 5^{(k-s)/2}, & \text{if } s \text{ is odd and } k-s \text{ is even}. \end{cases}$$

We first prove $GR_k(s \cdot K_4+e, (k-s)\cdot K_3) > g(k,s)$ by construction. Let G_0 be a single vertex and G_1 be a monochromatic copy of K_4 using color 1. If s is even, then we will begin with G_0 and iteratively construct Gallai-colored graphs. If s is odd, then we will begin with G_1 and iteratively construct Gallai-colored graphs. Suppose we have constructed G_i for some $i<k$. Let G' be a 2-edge-colored K_5 using colors $i+1$ and $i+2$ which contains no monochromatic copy of K_3, and G'' be a 2-edge-colored K_{17} using colors $i+1$ and $i+2$ which contains no monochromatic copy of K_4. We construct G_{i+2} or G_{i+1} based on the following rules:

(1) If $i \le s-2$, then we construct G_{i+2} such that $G_{i+2} = G''(17 \cdot G_i)$.
(2) If $s \le i \le k-2$, then we construct G_{i+2} such that $G_{i+2} = G'(5 \cdot G_i)$.
(3) If $i = k-1$, then we construct G_{i+1} by connecting two copies of G_i with edges using color k.

Finally, we obtain a $g(k,s)$-vertex Gallai-k-colored graph G_k containing neither a monochromatic copy of K_4+e in any of the first s colors nor a monochromatic copy of K_3 in any of the last $k-s$ colors.

In the following, we will prove $GR_k(s \cdot K_4+e, (k-s)\cdot K_3) \le g(k,s)+1$ by induction on $k+s$. The case $k=1$ is trivial, the case $k=2$ follows from Theorem 2.3, and the case $s=0$ follows from Theorem 2.2. So we may assume that the result holds for all $k'+s' < k+s$ and we will prove it for $k+s$, where $k \ge 3$ and $1 \le s \le k$.

Let G be a Gallai-k-coloring of K_n, where $n = g(k,s)+1$. For a contradiction, suppose that G contains neither a monochromatic copy of K_4+e in any of the first s colors nor a monochromatic copy of K_3 in any of the last $k-s$ colors. By Theorem 2.1, let $V_1, V_2, \ldots, V_t$ $(t \ge 2)$ be a Gallai partition of $V(G)$. We choose such a partition so that t is minimum. We may assume that red and blue are the two colors used between these parts, where red and blue are two of the k colors. Note that $n = g(k,s)+1 \ge 21$ since $k \ge 3$ and $1 \le s \le k$.

Claim 5.2. *$t \ge 4$.*

Proof. If $t=3$, then at least two of the colors $c(V_1,V_2)$, $c(V_1,V_3)$ and $c(V_2,V_3)$ are the same color, say $c(V_1,V_2) = c(V_1,V_3)$. This implies that V_1 and $V(G)\setminus V_1$ form a Gallai partition with exactly two parts, contradicting the minimality of t. Hence, $t=2$, and we may assume that $c(V_1,V_2)$ is red without loss of generality.

If there is no red edge within both V_1 and V_2, then $G[V_1]$ and $G[V_2]$ are two Gallai-$(k-1)$-colorings. By the induction hypothesis, if red is one of the first s colors, then we have

$$\begin{aligned} n &= |V_1|+|V_2| \le 2\cdot g(k-1,s-1) \\ &= \begin{cases} 2\cdot 17^{(s-1)/2}\cdot 5^{(k-s)/2}, & \text{if } s-1 \text{ is even } (s \text{ is odd}) \text{ and } k-s \text{ is even},\\ 2\cdot 2\cdot 17^{(s-1)/2}\cdot 5^{(k-s-1)/2}, & \text{if } s-1 \text{ is even } (s \text{ is odd}) \text{ and } k-s \text{ is odd},\\ 2\cdot 8\cdot 17^{(s-2)/2}\cdot 5^{(k-s-1)/2}, & \text{if } s-1 \text{ is odd } (s \text{ is even}) \text{ and } k-s \text{ is odd},\\ 2\cdot 4\cdot 17^{(s-2)/2}\cdot 5^{(k-s)/2}, & \text{if } s-1 \text{ is odd } (s \text{ is even}) \text{ and } k-s \text{ is even}, \end{cases} \\ &\le g(k,s), \end{aligned}$$

a contradiction. If red is one of the last $k-s$ colors, then we have

$$\begin{aligned} n &= |V_1|+|V_2| \le 2\cdot g(k-1,s) \\ &= \begin{cases} 2\cdot 17^{s/2}\cdot 5^{(k-s-1)/2}, & \text{if } s \text{ is even and } k-s-1 \text{ is even } (k-s \text{ is odd}),\\ 2\cdot 2\cdot 17^{s/2}\cdot 5^{(k-s-2)/2}, & \text{if } s \text{ is even and } k-s-1 \text{ is odd } (k-s \text{ is even}),\\ 2\cdot 8\cdot 17^{(s-1)/2}\cdot 5^{(k-s-2)/2}, & \text{if } s \text{ is odd and } k-s-1 \text{ is odd } (k-s \text{ is even}),\\ 2\cdot 4\cdot 17^{(s-1)/2}\cdot 5^{(k-s-1)/2}, & \text{if } s \text{ is odd and } k-s-1 \text{ is even } (k-s \text{ is odd}), \end{cases} \\ &\le g(k,s), \end{aligned}$$

a contradiction.

Thus we may assume that $G[V_1]$ contains a red edge, so red is one of the first s colors. In order to avoid a red copy of K_4+e, there is no red edge within V_2 and there is no red copy of K_3 within V_1 (recall that $n \ge 21$). By the induction hypothesis, we have

$$\begin{aligned} n &= |V_1|+|V_2| \le g(k,s-1)+g(k-1,s-1) \\ &= \begin{cases} 8\cdot 17^{(s-2)/2}\cdot 5^{(k-s)/2} + 4\cdot 17^{(s-2)/2}\cdot 5^{(k-s)/2}, & \text{if } s \text{ is even and } k-s \text{ is even},\\ 4\cdot 17^{(s-2)/2}\cdot 5^{(k-s+1)/2} + 8\cdot 17^{(s-2)/2}\cdot 5^{(k-s-1)/2}, & \text{if } s \text{ is even and } k-s \text{ is odd},\\ 17^{(s-1)/2}\cdot 5^{(k-s+1)/2} + 2\cdot 17^{(s-1)/2}\cdot 5^{(k-s-1)/2}, & \text{if } s \text{ is odd and } k-s \text{ is odd},\\ 2\cdot 17^{(s-1)/2}\cdot 5^{(k-s)/2} + 17^{(s-1)/2}\cdot 5^{(k-s)/2}, & \text{if } s \text{ is odd and } k-s \text{ is even}, \end{cases} \\ &\le g(k,s), \end{aligned}$$

a contradiction. This completes the proof of Claim 5.2. □

We define R to be a 2-edge-coloring of K_t with $V(R)=\{v_1,v_2,\dots,v_t\}$ and $c(v_iv_j)=c(V_i,V_j)$ for any $1\le i<j\le t$. Note that if R contains a 2-edge-colored subgraph H, then G also contains a copy of H (in fact, G contains a blow-up of H). For each $i\in[t]$, let $N_i^r:=\{j\in[t]\setminus\{i\}: c(v_iv_j) \text{ is red}\}$, $N_i^b:=\{j\in[t]\setminus\{i\}: c(v_iv_j) \text{ is blue}\}$, $d_i^r:=\left|N_i^r\right|$ and $d_i^b:=\left|N_i^b\right|$. By Claim 5.2 and the minimality of t, we have $d_i^r\ge 1$ and $d_i^b\ge 1$ for every $i\in[t]$. We claim that at least one of red and blue is among the first s colors. Indeed, if both red and blue are among the last $k-s$ colors, then R contains no monochromatic copy of K_3. So $t\le R(K_3,K_3)-1=5$. Moreover, for every $i\in[t]$, since $d_i^r\ge 1$ and $d_i^b\ge 1$, there is no red edge and no blue edge within V_i in G. By the induction hypothesis, we have $n=\sum_{i=1}^t|V_i|\le 5\cdot g(k-2,s)\le g(k,s)$, a contradiction.

Let $\mathcal{R}:=\{i\in[t]: G[V_i] \text{ contains a red edge}\}$ and $\mathcal{B}:=\{i\in[t]: G[V_i] \text{ contains a blue edge}\}$. Let $x_0:=|[t]\setminus(\mathcal{R}\cup\mathcal{B})|$, $x_1:=|\mathcal{R}\triangle\mathcal{B}|$ and $x_2:=|\mathcal{R}\cap\mathcal{B}|$, so $t=x_0+x_1+x_2$. We have the following simple facts.

Fact 5.3.

(1) *For any $i\in\mathcal{R}$ (resp., $i\in\mathcal{B}$), we have that v_i is not contained in any red copy of K_3 (resp., blue copy of K_3) in R.*
(2) *For any $i,j\in\mathcal{R}$ (resp., $i,j\in\mathcal{B}$) with $i\ne j$, we have that $c(V_i,V_j)$ is blue (resp., red).*
(3) *For any $i\in\mathcal{R}$ (resp., $i\in\mathcal{B}$), we have $d_i^r\le 3$ (resp., $d_i^b\le 3$).*
(4) *For any $i\in[t]$, we have $d_i^r\le 8$ and $d_i^b\le 8$.*
(5) *For any $i\in[t]$, $G[V_i]$ contains neither a red copy of K_3 nor a blue copy of K_3.*
(6) $x_2\le 1$.

Proof. By the symmetry of red and blue, we will only prove the red case for (1)–(5). Note that if red is one of the last $k-s$ colors, then Fact 5.3 holds clearly. So we may assume that red is one of the first s colors.

(1) If there exists an $i\in\mathcal{R}$ such that v_i is contained in a red copy of K_3 in R, say $v_iv_jv_\ell$, then in order to avoid a red copy of K_4+e, we have that $c(V_i\cup V_j\cup V_\ell, V(G)\setminus(V_i\cup V_j\cup V_\ell))$ is blue. By the minimality of t, we have $t=2$, contradicting Claim 5.2.

(2) If there exist some $i,j\in\mathcal{R}$ with $i\ne j$ such that $c(V_i,V_j)$ is red, then for avoiding a red copy of K_4+e, we have that $c(V_i\cup V_j, V(G)\setminus(V_i\cup V_j))$ is blue. By the minimality of t, we have $t=2$, contradicting Claim 5.2.

(3) If there exists an $i\in\mathcal{R}$ such that $d_i^r\ge 4$, then $\{v_j: j\in N_i^r\}$ forms a blue copy of $K_{d_i^r}$ by (1). In order to avoid a blue copy of K_4+e, we have $d_i^r=4$ and $c(\bigcup_{j\in N_i^r}V_j, \bigcup_{\ell\in[t]\setminus N_i^r}V_\ell)$ is red. By the minimality of t, we have $t=2$, contradicting Claim 5.2.

(4) Suppose $d_i^r\ge 9$ for some $i\in[t]$. In order to avoid a red copy of K_4+e, there is no red copy of K_3 in $R[\{v_j: j\in N_i^r\}]$. Since $R(K_3,K_4+e)=9$, there is a blue copy of K_4+e (and thus a blue copy of K_3), a contradiction.

(5) Suppose that $G[V_i]$ contains a red copy of K_3 for some $i\in[t]$. Since $d_i^r\ge 1$, we may assume that $c(V_i,V_j)$ is red for some $j\in[t]\setminus\{i\}$. In order to avoid a red copy of K_4+e, we have that $c(V_i\cup V_j, V(G)\setminus(V_i\cup V_j))$ is blue. By the minimality of t, we have $t=2$, contradicting Claim 5.2.

(6) If $x_2=|\mathcal{R}\cap\mathcal{B}|\ge 2$, then we can derive a contradiction by (2). □

We divide the rest of the proof into two cases according to where red and blue are in the list of colors.

Case 1. Red is among the first s colors and blue is among the last $k-s$ colors.

In this case, there is no red copy of K_4+e and no blue copy of K_3 in G. Since $R(K_4+e,K_3)=9$, we have $4\le t\le 8$. Recall that $d_i^r\ge 1$ and $d_i^b\ge 1$ for every $i\in[t]$. So there is no blue edge within each V_i. Thus $|\mathcal{B}|=0$, $x_1=|\mathcal{R}|$, $x_2=0$ and $x_0=t-x_1$. We claim that $x_1\le 2$, since otherwise if $|\mathcal{R}|\ge 3$, then there is a blue copy of K_3 by Fact 5.3 (2).

For each $i\in\mathcal{R}$, $G[V_i]$ contains no red copy of K_3 by Fact 5.3 (5). By the induction hypothesis, we have

$$
\begin{aligned}
|V_i| &\le g(k-1,s-1)\\
&=\begin{cases}
17^{(s-1)/2}\cdot 5^{(k-s)/2}, & \text{if } s-1 \text{ is even } (s \text{ is odd}) \text{ and } k-s \text{ is even},\\
2\cdot 17^{(s-1)/2}\cdot 5^{(k-s-1)/2}, & \text{if } s-1 \text{ is even } (s \text{ is odd}) \text{ and } k-s \text{ is odd},\\
8\cdot 17^{(s-2)/2}\cdot 5^{(k-s-1)/2}, & \text{if } s-1 \text{ is odd } (s \text{ is even}) \text{ and } k-s \text{ is odd},\\
4\cdot 17^{(s-2)/2}\cdot 5^{(k-s)/2}, & \text{if } s-1 \text{ is odd } (s \text{ is even}) \text{ and } k-s \text{ is even},
\end{cases}\\
&\le \frac{1}{4}g(k,s).
\end{aligned}
$$

For each $i\in[t]\setminus(\mathcal{R}\cup\mathcal{B})$, by the induction hypothesis, we have

$$
\begin{aligned}
|V_i| \le g(k-2, s-1) &= \begin{cases} 17^{(s-1)/2} \cdot 5^{(k-s-1)/2}, & \text{if } s-1 \text{ is even and } k-s-1 \text{ is even} \\ & (s \text{ is odd and } k-s \text{ is odd}), \\ 2 \cdot 17^{(s-1)/2} \cdot 5^{(k-s-2)/2}, & \text{if } s-1 \text{ is even and } k-s-1 \text{ is odd} \\ & (s \text{ is odd and } k-s \text{ is even}), \\ 8 \cdot 17^{(s-2)/2} \cdot 5^{(k-s-2)/2}, & \text{if } s-1 \text{ is odd and } k-s-1 \text{ is odd} \\ & (s \text{ is even and } k-s \text{ is even}), \\ 4 \cdot 17^{(s-2)/2} \cdot 5^{(k-s-1)/2}, & \text{if } s-1 \text{ is odd and } k-s-1 \text{ is even} \\ & (s \text{ is even and } k-s \text{ is odd}), \end{cases} \\
&\le \frac{1}{8} g(k, s).
\end{aligned}
$$

Thus $n \le (x_1/4 + x_0/8)g(k, s)$. It suffices to prove that $x_1/4 + x_0/8 \le 1$. If $x_1 \le 8 - t$, then $x_1/4 + x_0/8 = (2x_1 + x_0)/8 = (x_1 + t)/8 \le 1$. Thus we may assume $x_1 \ge 8 - t + 1$. Recall that we have $t \le 8$ and $x_1 \le 2$ in this case. So $|\mathcal{R}| = x_1 \ge 1$ and $7 \le t \le 8$. For any $i \in \mathcal{R}$, we have $d_i^r \le 2$ for avoiding a blue copy of K_3 and by Fact 5.3 (1). Thus $d_i^b \ge 4$. Since there is no blue copy of K_3, we have that $\{v_j : j \in N_i^b\}$ forms a red copy of $K_{d_i^b}$. Then $c(\bigcup_{j \in N_i^b} V_j, \bigcup_{\ell \in [t] \setminus N_i^b} V_\ell)$ is blue. By the minimality of t, we have $t = 2$, contradicting Claim 5.2.

Case 2. Both red and blue are among the first s colors.

In this case, we have $4 \le t \le 17$ since $R(K_4 + e, K_4 + e) = 18$. Moreover, we have $s \ge 2$ and thus $g(k, s) \ge 34$ (recall that $k \ge 3$). By the induction hypothesis, for every $i \in [t] \setminus (\mathcal{R} \cup \mathcal{B})$, we have $|V_i| \le g(k-2, s-2) = \frac{1}{17} g(k, s)$. For any $i \in [t]$, $G[V_i]$ contains neither a red copy of K_3 nor a blue copy of K_3 by Fact 5.3 (5). Thus for each $i \in \mathcal{R} \cap \mathcal{B}$, by the induction hypothesis, we have $|V_i| \le g(k, s-2) = \frac{5}{17} g(k, s)$. And for each $i \in \mathcal{R} \triangle \mathcal{B}$, we have

$$
\begin{aligned}
|V_i| \le g(k-1, s-2) &= \begin{cases} 17^{(s-2)/2} \cdot 5^{(k-s+1)/2}, & \text{if } s-2 \text{ is even and } k-s+1 \text{ is even}, \\ 2 \cdot 17^{(s-2)/2} \cdot 5^{(k-s)/2}, & \text{if } s-2 \text{ is even and } k-s+1 \text{ is odd}, \\ 8 \cdot 17^{(s-3)/2} \cdot 5^{(k-s)/2}, & \text{if } s-2 \text{ is odd and } k-s+1 \text{ is odd}, \\ 4 \cdot 17^{(s-3)/2} \cdot 5^{(k-s+1)/2}, & \text{if } s-2 \text{ is odd and } k-s+1 \text{ is even}, \end{cases} \\
&\le \frac{5}{34} g(k, s).
\end{aligned}
$$

Thus $n \le (5x_2/17 + 5x_1/34 + x_0/17)g(k, s)$. It suffices to prove that $10x_2 + 5x_1 + 2x_0 = 2t + 8x_2 + 3x_1 \le 34$.

Claim 5.4. $x_2 = 0$.

Proof. By Fact 5.3 (6), we have $x_2 \le 1$. For a contradiction, suppose $\mathcal{R} \cap \mathcal{B} = \{1\}$. By Fact 5.3 (3), we have $d_1^r \le 3$ and $d_1^b \le 3$, so $t \le 7$. If $t \le 5$, then $2t + 8x_2 + 3x_1 \le 10 + 8 + 12 \le 34$. If $6 \le t \le 7$, then we may assume that $d_1^r = 3$ without loss of generality, say $N_1^r = \{2, 3, 4\}$. By Fact 5.3 (1), we have that $c(v_2v_3) = c(v_3v_4) = c(v_2v_4)$ is blue. By Fact 5.3 (1) and (2), we have $2, 3, 4 \notin \mathcal{R} \cup \mathcal{B}$. Thus $x_1 \le t - 4$, so $2t + 8x_2 + 3x_1 \le 8 + 5t - 12 \le 34$. □

Claim 5.5. *$|\mathcal{R}| \le 3$ and $|\mathcal{B}| \le 3$. If $|\mathcal{R}| = 3$ (resp., $|\mathcal{B}| = 3$), then $|\mathcal{B}| \le 1$ (resp., $|\mathcal{R}| \le 1$).*

Proof. If $|\mathcal{R}| \ge 4$ (resp., $|\mathcal{B}| \ge 4$), then G contains a blue (resp., red) $K_{2,2,2,2}$ by Fact 5.3 (2). This implies a monochromatic copy of $K_4 + e$ in G. Thus $|\mathcal{R}| \le 3$ and $|\mathcal{B}| \le 3$.

If $|\mathcal{R}| = 3$ and $2 \le |\mathcal{B}| \le 3$, then $R[\{v_i : i \in \mathcal{R}\}]$ and $R[\{v_i : i \in \mathcal{B}\}]$ form a blue clique and a red clique (by Fact 5.3 (2)), respectively. By Fact 5.3 (1), for any $i \in \mathcal{R}$ (resp., $i \in \mathcal{B}$), there is at most one red (resp., blue) edge between v_i and $\{v_j : j \in \mathcal{B}\}$ (resp., $\{v_j : j \in \mathcal{R}\}$). Thus there are at most $|\mathcal{R}| + |\mathcal{B}| < |\mathcal{R}||\mathcal{B}|$ edges between $\{v_i : i \in \mathcal{R}\}$ and $\{v_i : i \in \mathcal{B}\}$, a contradiction. Therefore, if $|\mathcal{R}| = 3$, then $|\mathcal{B}| \le 1$, and similarly, if $|\mathcal{B}| = 3$, then $|\mathcal{R}| \le 1$. □

By Claims 5.4 and 5.5, we have $x_2 = 0$ and $x_1 = |\mathcal{R}| + |\mathcal{B}| \le 4$. If $t \le 11$, then $2t + 8x_2 + 3x_1 \le 22 + 0 + 12 = 34$. If $13 \le t \le 17$, then $|\mathcal{R}| = |\mathcal{B}| = 0$ by Fact 5.3 (3) and (4), so $2t + 8x_2 + 3x_1 \le 34 + 0 + 0 = 34$. Thus $t = 12$. We have $x_1 = |\mathcal{R}| + |\mathcal{B}| = 4$; otherwise $2t + 8x_2 + 3x_1 \le 24 + 0 + 9 \le 34$. Then we further have $|\mathcal{R}| \ge 1$ and $|\mathcal{B}| \ge 1$ by Claim 5.5. Without loss of generality, let $1 \in \mathcal{R}$, $2 \in \mathcal{B}$ and let $c(V_1, V_2)$ be blue. Moreover, by Fact 5.3 (3) and (4), we have $d_1^r = 3$, $d_1^b = 8$, $d_2^b = 3$ and $d_2^r = 8$. We may further assume that $c(V_1, V_3 \cup V_4 \cup \cdots \cup V_9)$ is blue. By Fact 5.3 (1), we have $c(V_2, V_3 \cup V_4 \cup \cdots \cup V_9)$ is red. Since $R(K_3, K_3) = 6$, there is either a red copy of K_3 or a blue copy of K_3 in $R[\{v_3, v_4, \ldots, v_9\}]$. Then there is either a red copy of $K_4 + e$ or a blue copy of $K_4 + e$ in G, a contradiction. □

6. Concluding remarks

In Section 3, we studied the maximum number (denoted by $f_k(n, H)$) of edges that are not contained in any rainbow triangle or monochromatic copy of H. There we showed that $f_k(n, H) \geq t(n, GR_{k-1}(\mathscr{H}) - 1)$, where $\mathscr{H}$ is the set of homomorphic copies of H. Let $f'_k(n, H)$ be the maximum number of edges not contained in any monochromatic copy of H over all Gallai-k-colorings of K_n. Then we clearly have $f'_k(n, H) \leq f_k(n, H)$. Using the sharpness example constructed in the proof of Lemma 3.2 (2), we can also show that $f'_k(n, H) \geq t(n, GR_{k-1}(\mathscr{H}) - 1)$. Thus we have $t(n, GR_{k-1}(\mathscr{H}) - 1) \leq f'_k(n, H) \leq f_k(n, H)$. An interesting and natural question is for which graphs H the equality $f'_k(n, H) = f_k(n, H)$ holds.

Another problem related to Section 3 is to determine the maximum number $\mathrm{nim}_k(n, H)$ of edges not contained in any monochromatic copy of H over all k-edge-colorings of K_n. As remarked in [29], if the Erdős-Sós conjecture holds for a tree T (i.e., $ex(n, T) \leq (|V(T)|-2)n/2$), then for each $n \geq k^2(|V(T)|-1)^2$ with $(|V(T)|-1) \mid n$, we have $\mathrm{nim}_k(n, T) \geq (k-1)ex(n, T)$. In fact, when T is a star, we can prove the above statement for all $n \geq k^2(|V(T)|-1)^2$. Let H be an n-vertex $K_{1,h}$-free graph with $ex(n, K_{1,h})$ edges. Note that the maximum degree of H is at most $h-1$. For every $i \in [k-1]$, let $f_i\colon V(H) \to [n]$ be an arbitrary bijection and let H_i be the graph obtained by mapping H on $[n]$ via f_i. Let H^* be the graph with vertex set $[n]$ and edge set $\bigcup_{i\in[k-1]} E(H_i)$. Note that $\Delta(H^*) \leq (k-1)(h-1)$. For any vertex u, there is a vertex v that is at distance at least three from u in H^* since $n > \Delta(H^*)^2 + 1$. If there is an edge e incident with u or v such that $e \in E(H_i) \cap E(H_j)$ for some $1 \leq i \neq j \leq k-1$, then after switching u and v in f_i, we claim that there is no edge e' incident with u or v satisfying $e' \in E(H_i) \cap E(H_\ell)$ for any $\ell \in [k-1] \setminus \{i\}$. Otherwise, suppose that there is an edge $vw \in E(H_i) \cap E(H_\ell)$ after switching u and v in f_i. This implies that before switching u and v in f_i, we have $vw \in E(H_\ell)$ and $uw \in E(H_i)$. Thus uwv is a path of length two in H^*, contradicting the fact that v is at distance at least three from u. Thus we can repeat this process to obtain a graph with no edge e such that $e \in E(H_i) \cap E(H_j)$ for some $1 \leq i \neq j \leq k-1$. Hence, we can color K_n with $c(e) = i$ if $e \in E(H_i)$ for each $i \in [k-1]$ and $c(e) = k$ otherwise. Thus $\mathrm{nim}_k(n, K_{1,h}) \geq \sum_{i\in[k-1]} |E(H_i)| = (k-1)ex(n, K_{1,h})$.

Moreover, let G be a k-edge-coloring of K_n with $\mathrm{nim}_k(n, K_{1,h})$ edges not contained in any monochromatic copy of $K_{1,h}$. For $i \in [k]$, let G_i (resp., G_i^{nim}) denote the spanning subgraph of G with edge set $E(G_i) = \{e \in E(G)\colon c(e) = i\}$ (resp., $E(G_i^{nim}) = \{e \in E(G)\colon e$ is not contained in any monochromatic copy of $K_{1,h}, c(e) = i\}$) and let $V_i = \{v \in V(G)\colon d_{G_i}(v) \geq h\}$. If $n > k(h-1)$, then $\bigcup_{i\in[k]} V_i = V(G)$, and every vertex of V_i is an isolated vertex in G_i^{nim} for every $i \in [k]$. Since $ex(n, K_{1,h}) = \lfloor (h-1)n/2 \rfloor$, we have $\mathrm{nim}_k(n, K_{1,h}) = \sum_{i\in[k]} e(G_i^{nim}) \leq \sum_{i\in[k]} ex\left(n - |V_i|, K_{1,h}\right) \leq ex\left(\sum_{i\in[k]}(n - |V_i|), K_{1,h}\right) \leq ex((k-1)n, K_{1,h})$. Note that $ex((k-1)n, K_{1,h}) = (k-1)ex(n, K_{1,h}) + \eta$, where $\eta = \lfloor (k-1)/2 \rfloor$ if h is even and n is odd, and $\eta = 0$ otherwise. Therefore, for $n \geq k^2h^2$, if h is even and n is odd, then $(k-1)ex(n, K_{1,h}) \leq \mathrm{nim}_k(n\ K_{1,h}) \leq (k-1)ex(n, K_{1,h}) + \lfloor (k-1)/2 \rfloor$, and otherwise, we have $\mathrm{nim}_k(n, K_{1,h}) = (k-1)ex(n, K_{1,h})$. In particular, we have the following result in the case $k = 2$, which partly answers a problem of Keevash and Sudakov [23] in the special case when H is a star.

Proposition 6.1. *For n sufficiently large, we have* $\mathrm{nim}_2(n, K_{1,h}) = ex(n, K_{1,h})$.

In Section 4, we studied the minimum number of copies of H over all Gallai-k-colorings of K_n. Given an arbitrary k-edge-coloring G of K_n, let $r_k(K_3, n)$ and $m_k(H, n)$ be the number of rainbow triangles and monochromatic copies of H in G, respectively. It is interesting to consider the behavior of $r_k(K_3, n) + m_k(H, n)$. Clearly if $k \leq 2$, then $r_k(K_3, n) + m_k(H, n) = m_k(H, n)$, and if G is rainbow, then $r_k(K_3, n) + m_k(H, n) = \binom{n}{3}$. However, the general behavior of $r_k(K_3, n) + m_k(H, n)$ seems difficult to determine.

Finally, we pose two conjectures. Note that we have shown that Conjecture 6.2 below holds for the following cases: (1) $k = 3$, (2) $k \geq 3$ and $n = GR_k(K_3)$, (3) k is odd and $GR_k(K_3) \leq n \leq GR_k(K_3) + 5^{(k-1)/2} - 1$.

Conjecture 6.2. *For $n \geq GR_k(K_3)$, we write $n = 5^{\lfloor (k-1)/2 \rfloor} m + r$, where m and r are nonnegative integers with $0 \leq r \leq 5^{\lfloor (k-1)/2 \rfloor} - 1$. Then*

$$g_k(K_3, n) = \begin{cases} r\dbinom{m+1}{3} + \left(5^{(k-1)/2} - r\right)\dbinom{m}{3}, & \text{if } k \text{ is odd}, \\ rM_2(K_3, m+1) + \left(5^{(k-2)/2} - r\right) M_2(K_3, m), & \text{if } k \text{ is even}. \end{cases}$$

Conjecture 6.3. *For integers $k \geq 2$, we have* $f_k(n, K_3) = t(n, GR_{k-1}(K_3) - 1)$.

Note. We recently discovered that Theorem 1.7 has been proved by Su and Liu [32] and Zhao and Wei [35] independently.

Declaration of competing interest

The authors declare that they do not have a conflict of interests.

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References

[1] J. Balogh, L. Li, The typical structure of Gallai colorings and their extremal graphs, SIAM J. Discrete Math. 33 (2019) 2416–2443.
[2] J.O. Bastos, F.S. Benevides, J. Han, The number of Gallai k-colorings of complete graphs, J. Comb. Theory, Ser. B 144 (2020) 1–13.
[3] D. Bruce, Z.-X. Song, Gallai-Ramsey numbers of C_7 with multiple colors, Discrete Math. 342 (2019) 1191–1194.
[4] K. Cameron, J. Edmonds, L. Lovász, A note on perfect graphs, Period. Math. Hung. 17 (1986) 173–175.
[5] F.R.K. Chung, R.L. Graham, Edge-colored complete graphs with precisely colored subgraphs, Combinatorica 3 (1983) 315–324.
[6] M. Clancy, Some small Ramsey numbers, J. Graph Theory 1 (1977) 89–91.
[7] D. Conlon, On the Ramsey multiplicity of complete graphs, Combinatorica 32 (2012) 171–186.
[8] J. Cummings, D. Král', F. Pfender, K. Sperfeld, A. Treglown, M. Young, Monochromatic triangles in three-coloured graphs, J. Comb. Theory, Ser. B 103 (2013) 489–503.
[9] J. Fox, There exist graphs with super-exponential Ramsey multiplicity constant, J. Graph Theory 57 (2008) 89–98.
[10] J. Fox, A. Grinshpun, J. Pach, The Erdős-Hajnal conjecture for rainbow triangles, J. Comb. Theory, Ser. B 111 (2015) 75–125.
[11] S. Fujita, C. Magnant, Extensions of Gallai-Ramsey results, J. Graph Theory 70 (2012) 404–426.
[12] S. Fujita, C. Magnant, K. Ozeki, Rainbow generalizations of Ramsey theory: a survey, Graphs Comb. 26 (2010) 1–30.
[13] T. Gallai, Transitiv orientierbare Graphen, Acta Math. Acad. Sci. Hung. 18 (1967) 25–66.
[14] A.W. Goodman, On sets of acquaintances and strangers at any party, Am. Math. Mon. 66 (1959) 778–783.
[15] R.E. Greenwood, A.M. Gleason, Combinatorial relations and chromatic graphs, Can. J. Math. 7 (1955) 1–7.
[16] A. Gyárfás, G.N. Sárközy, A. Sebő, S. Selkow, Ramsey-type results for Gallai colorings, J. Graph Theory 64 (2010) 233–243.
[17] A. Gyárfás, G. Simonyi, Edge colorings of complete graphs without tricolored triangles, J. Graph Theory 46 (2004) 211–216.
[18] M. Hall, C. Magnant, K. Ozeki, M. Tsugaki, Improved upper bounds for Gallai–Ramsey numbers of paths and cycles, J. Graph Theory 75 (2014) 59–74.
[19] H. Harborth, I. Mengersen, All Ramsey numbers for five vertices and seven or eight edges, Discrete Math. 73 (1988/1989) 91–98.
[20] H. Hatami, J. Hladký, D. Král', S. Norine, A. Razborov, Non-three-colourable common graphs exist, Comb. Probab. Comput. 21 (2012) 734–742.
[21] C. Hoppen, H. Lefmann, K. Odermann, A rainbow Erdős-Rothschild problem, SIAM J. Discrete Math. 31 (2017) 2647–2674.
[22] C. Hoppen, H. Lefmann, K. Odermann, On graphs with a large number of edge-colorings avoiding a rainbow triangle, Eur. J. Comb. 66 (2017) 168–190.
[23] P. Keevash, B. Sudakov, On the number of edges not covered by monochromatic copies of a fixed graph, J. Comb. Theory, Ser. B 90 (2004) 41–53.
[24] J. Komlós, M. Simonovits, Szemerédi's Regularity Lemma and its applications in graph theory, in: Combinatorics, Paul Erdős is Eighty, vol. 2, Keszthely, 1993, in: Bolyai Soc. Math. Stud., vol. 2, János Bolyai Math. Soc., Budapest, 1996, pp. 295–352.
[25] J. Körner, G. Simonyi, Graph pairs and their entropies: modularity problems, Combinatorica 20 (2000) 227–240.
[26] J. Körner, G. Simonyi, Z. Tuza, Perfect couples of graphs, Combinatorica 12 (1992) 179–192.
[27] X.H. Li, L.G. Wang, X.X. Liu, Complete graphs and complete bipartite graphs without rainbow path, Discrete Math. 342 (2019) 2116–2126.
[28] H. Liu, C. Magnant, A. Saito, I. Schiermeyer, Y.T. Shi, Gallai-Ramsey number for K_4, J. Graph Theory 94 (2020) 192–205.
[29] H. Liu, O. Pikhurko, M. Sharifzadeh, Edges not in any monochromatic copy of a fixed graph, J. Comb. Theory, Ser. B 135 (2019) 16–43.
[30] J. Ma, On edges not in monochromatic copies of a fixed bipartite graph, J. Comb. Theory, Ser. B 123 (2017) 240–248.
[31] S.P. Radziszowski, Small Ramsey numbers, in: Dynamic Survey 1, Electron. J. Comb. 1 (2017) (electronic).
[32] X.L. Su, Y. Liu, Gallai-Ramsey numbers for monochromatic K_4^+ or K_3, arXiv:2007.02059.
[33] E. Szemerédi, Regular partitions of graphs, in: Problèmes Combinatoires et Théorie des Graphes, Orsay, in: Colloques Internationaux CNRS, vol. 260, 1976, pp. 399–401.
[34] F.F. Zhang, Z.-X. Song, Y.J. Chen, Multicolor Ramsey numbers of cycles in Gallai colorings, arXiv:1906.05263.
[35] Q.H. Zhao, B. Wei, Gallai-Ramsey numbers for graphs with five vertices of chromatic number four, arXiv:2008.00361.