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Semi-global state synchronization for discrete-time multi-agent systems subject to actuator saturation and unknown nonuniform input delay

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ABSTRACT

This paper studies semi-global state synchronization of discrete-time homogeneous networks with diffusive full-state coupling or partial-state coupling subject to actuator saturation and unknown nonuniform input delay. We assume that agents are at most critical unstable, that is the agents have all its eigenvalues in the closed unit disc. The communication network is associated with an undirected and weighted graph, which is represented by a row stochastic matrix. In this paper, we derive an upper bound for the input delay tolerance, which explicitly depends only on the agent dynamics. Moreover, for any unknown delay less than the upper bound, we propose a linear static protocol for MAS with full-state coupling and a linear dynamic protocol for MAS with partial-state coupling based on a low-gain methodology such that state synchronization is achieved among agents for any initial conditions in a priori given compact set.

1. Introduction

The problem of synchronization among agents in a multi-agent system has received substantial attention, because of its potential applications in cooperative control of autonomous vehicles, distributed sensor network, swarming and flocking and others. The objective of synchronization is to secure an asymptotic agreement on a common state or output trajectory through decentralized control protocols (see [1,16,23,42] and references therein). Most work has focused on state synchronization based on full-state/partial-state coupling in a homogenous network (i.e. agents have identical dynamics), where the agent dynamics progress from single- and double-integrator dynamics to more general dynamics (e.g., [11,17,20,21,24,30,34–36,41,46]). In this case of full-state coupling, universally, static protocols are considered. While, in the case of partial-state coupling, the standard approach leads to dynamic, observer-based protocols. The counterpart of state synchronization is output synchronization, which is mostly done in heterogeneous networks (i.e., agents are non-identical).

In engineering applications, the network model is always imperfect. In particular, time-delay effects are ubiquitous in any communication scheme. As clarified in [3], we can identify two kinds of time delay: input delay and communication delay. Input delay results from processing time to generate an input for each agent while communication delay refers to the time consumed during the transfer of information between agents. Most effort has been put into input delay problems (see [2,9,12–14,21,32,33] and [44] for example). These references, although including results on linear and non-linear agents, are mostly restricted to simple agent models such as first/second-order dynamics. Recently, in [38] and [39], the synchronization problem under unknown uniform constant input delay is solved for both discrete- and continuous-time high-order linear agents that are critically unstable. This work has been recently extended to unknown nonuniform input delay in [49]. In the case of communication delay, some results can be found. Tian et al. [32] and [43] consider single-integrator dynamics in the network and it is demonstrated that the communication delay does not affect the synchronizability of the network. Munz et al. [18] and [19] give the consensus conditions for networks with higher-order but SISO dynamics. In [13], second-order dynamics are investigated, but the communication delays are assumed known. Recently, [4] and [5] dealt

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with nonlinear heterogeneous MAS with unknown non-uniform constant communication delay where they solved a delayed synchronization problem. Synchronization for an homogeneous time-varying network with non-uniform time-varying communication delay is achieved in [25] (general system) and [7] (single-integrator system). Time-varying communication delay is also considered in [8] for second-order uncertain Euler–Lagrange systems.

It should be also noted that actuator saturation is pretty common and indeed is ubiquitous in engineering applications. For MAS in the presence of input saturation, usually two problems are addressed: global synchronization and semi-global synchronization. Global stabilization for MAS with full-state coupling has been studied by Meng et al. [15] (continuous) and [45] (discrete) for neutrally stable agents. Chu et al. [6] has considered the global case of partial-state coupling, using an adaptive approach. However the observer-based protocol requires extra communication and the agents are introspective (i.e., agents have access to part of their states). Semi-global synchronization has been studied in [27] and [28] in the case of full-state coupling. For partial state coupling, there are [26,29] and [40]. All of these papers actually require extra communication and agents to be introspective. Zhang et al. [47] considers non-introspective agents but still requires the extra communication. So far we only find [31] that deals with non-introspective agents and requires no extra communication. However, that paper requires the solution of a nonconvex optimization problem as part of the design of a dynamic protocol. Moreover, an underlying assumption basically requires the agents to be passifiable via input feedforward. We notice that all these papers assume that the network is either undirected or is so-called detailed balanced (a slightly weaker condition than undirected). One paper dealing with networks that are not detailed balanced is in [10], which intrinsically requires the agents to be single integrator. In [37] semi-global stabilization with full-state coupling has been studied for networks which only need to contain a directed spanning tree. Moreover, the agents are not introspective. Recently, Zhang et al. [48] addressed the semiglobal state synchronization for both general continuous/discrete-time MASs with full-state coupling or partial-state coupling, where the network is directed and contains a spanning tree.

The objective of this paper is to extend the works in [49] to the case in the presence of saturation using ideas from [48]. The idea is to make sure the system input can be squeezed enough such that the saturation does not get activated. However, this paper is not a combination of the works in [49] and [48]. The techniques and arguments regarding how to squeezing the system input are completely different. Therefore, in this paper, we investigate the semiglobal state synchronization problem for discrete-time MAS subject to actuator saturation and unknown nonuniform input delay. Both full-state coupling and partial-state coupling are considered. The agents in the MAS are general and at most critically unstable. The network graph is undirected and connected. We derive an upper bound for the input delay tolerance, which is only dependent on the agent dynamics. Then, for any unknown input delay satisfying the upper bound, we design a linear static protocol in the full-state coupling case and a linear dynamic protocol in the partial-state coupling case based on a low-gain methodology, such that state synchronization is achieved among agents for any initial conditions in a priori given compact set. In particular, the saturation can be avoided by tuning a low-gain parameter in the protocols. Moreover, the protocols are designed not only for a specific network, but for a set of networks. Only the upper bound and lower bound of associated Laplacian matrices are needed for the protocol design. The additional communication of controller states is also dispensed in this paper.

1.1. Notations and definitions

Given a matrix $A \in \mathbb{C}^{m\times n}$, A' denotes its conjugate transpose, $\|A\|$ is the induced 2-norm, and $\lambda_i(A)$ denotes its i'th eigenvalue when $m = n$. A square matrix A is said to be Schur stable if all its eigenvalues are in the open unit disc. We denote by $\mathrm{diag}\{a_1,\dots,a_N\}$ or $\mathrm{diag}\{a_i\}$, a diagonal matrix with a_i $(i = 1,\dots,N)$ as the diagonal elements, and by $\mathrm{col}\{x_1,\dots,x_N\}$ or $\mathrm{col}\{x_i\}$, a column vector with x_i $(i = 1,\dots,N)$ stacked together. $A\otimes B$ depicts the Kronecker product between A and B. I_n denotes the n-dimensional identity matrix, and $\mathbf{0}_n$ (or $\mathbf{1}_n$) denotes zero (or one) column or row vector. Sometimes we drop the subscript if the dimension is clear from the context. Given a transfer matrix $G(j\omega)$, $\|G\|_\infty$ denotes the H_∞ norm of the system. Suppose two functions f: $X\to Y$ and g: $Y\to Z$. Then, $(g\circ f)$: $X\to Z$ is the composite function, meaning $(g\circ f)(x) = g(f(x))$. Moreover, let $\mathfrak{L}^n_\infty(\bar\kappa)$ denote the Banach space of finite sequences $\{y_1,\dots,y_{\bar\kappa}\}\subset\mathbb{C}^n$ with norm $\|\cdot\|_\infty = \max_i\{\|y_i\|\}$.

A matrix $D = \{d_{ij}\}_{N\times N}$ is called a row stochastic matrix if $d_{ij}\ge 0$ for any i,j and $\sum_{j=1}^n d_{ij} = 1$ for $i = 1,\dots,N$. A row stochastic matrix D has at least one eigenvalue at 1 with right eigenvector $\mathbf{1}$. D can be associated with a graph $G = (\mathcal{V},\mathcal{E},D)$, where $\mathcal{V} = \{1,\dots,N\}$ is a node set, $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is a set of pairs of nodes indicating connections among nodes, and $D = [d_{ij}]\in\mathbb{R}^{N\times N}$ is the weighting matrix, with $d_{ij} > 0$ iff $(j,i)\in\mathcal{E}$ and $d_{ii} > 0$. If $d_{ij} = d_{ji}$ for all $i,j\in\{1,\dots,N\}$, the graph is called *undirected*; otherwise *directed*. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1,\dots,i_k\}$ such that $(i_j,i_{j+1})\in\mathcal{E}$ for $j = 1,\dots,k-1$. An undirected graph is *connected* if there exists a path between every pair of nodes. A *directed tree* is a subgraph (subset of nodes and edges) in which every node has exactly one parent node except for one node, called the *root*, which has no parent node. In this case, the root has a directed path to every other node in the tree. A *directed spanning tree* is a subgraph which is a directed tree containing all the nodes of the original graph. Let G be the graph associated with D. It is shown in [22] that 1 is a simple eigenvalue of D if and only if G contains a directed spanning tree. Moreover, the other eigenvalues are in the open unit disc if $d_{ii} > 0$ for all i. For a weighted graph $\mathcal{G}$, the Laplacian matrix $\bar L = [\ell_{ij}]$ is defined as $\bar L = I - D$ with

$$\bar\ell_{ij} = \begin{cases} 1 - d_{ii}, & i = j,\\ -d_{ij}, & i \ne j.\end{cases}$$

2. Problem formulation

Consider a discrete-time multiagent system (MAS) composed of N identical linear time-invariant agents subject to actuator saturation and unknown nonuniform input delay,

$$\begin{cases} x_i(k+1) = Ax_i(k) + B\sigma(u_i(k-\kappa_i)),\\ y_i(k) = Cx_i(k),\\ x_i(\varsigma) = \phi_{i,\varsigma}, \quad \varsigma\in[-\bar\kappa, 0]\end{cases} \tag{1}$$

for $i = 1,\dots,N$, where $x_i\in\mathbb{R}^n$, $u_i\in\mathbb{R}^m$, $y_i\in\mathbb{R}^p$ are respectively the state, input, and output vectors of agent i, while $\kappa_i\in[0,\bar\kappa]$ is an unknown constant, $\bar\kappa$ is a known upper bound, and $\phi_i\in\mathfrak{L}^n_\infty(\bar\kappa)$. Moreover,

$$\sigma(u_i(k-\kappa_i)) = \begin{pmatrix}\mathrm{sat}(u_{i,1}(k-\kappa_i))\\ \vdots\\ \mathrm{sat}(u_{i,m}(k-\kappa_i))\end{pmatrix} \quad \text{with} \quad u_i = \begin{pmatrix}u_{i,1}\\ \vdots\\ u_{i,m}\end{pmatrix} \tag{2}$$

with $\mathrm{sat}(u)$ being the standard saturation function,

$$\mathrm{sat}(u) = \mathrm{sgn}(u)\min\{1, |u|\}.$$

The communication network provides each agent with a linear combination of its own outputs relative to that of other neighboring agents. In particular, each agent $i \in \{1, \dots, N\}$ has access to the quantity,

$$\zeta_i = \sum_{j=1}^{N} d_{ij}(y_i - y_j), \tag{3}$$

where $d_{ij} \geq 0$ indicates the communication topology among agents and $D = [d_{ij}]$ is a row stochastic matrix that satisfies $d_{ii} > 0$. This communication topology of the network can be described by an undirected weighted graph $\mathcal{G}$ with nodes corresponding to the agents in the network and the weight of edges given by the coefficient d_{ij}. We refer to this network as with *partial-state coupling*. Note that if C has full column rank then, without loss of generality, we can assume that $C = I$, and the quantity ζ_i becomes

$$\zeta_i = \sum_{j=1}^{N} d_{ij}(x_i - x_j). \tag{4}$$

We refer to this network as with *full-state coupling*.

We make the following standard assumption for the agent dynamics.

Assumption 1. We assume that

- (A, B) is stabilizable, (A, C) is detectable;
- The agents are at most critically unstable, that is A has all its eigenvalues in the closed unit disc.

Definition 1. We define the following network graph sets.

- Let $\mathbb{G}^N$ denote the set of undirected, weighted, and connected graphs with N nodes,
- For any given $\beta \in (0, 1)$, let $\mathbb{G}^N_\beta$ denote the set of undirected, weighted and connected graphs with N nodes and for which the corresponding row-stochastic matrix has the property that its eigenvalues inside the unit disc, denoted by $\lambda_2, \dots, \lambda_N$, satisfy $|\lambda_i| < \beta$.

Definition 2. We also define $\omega_{\max}$ as

$$\omega_{\max} = \begin{cases} 0, & A \text{ is Schur stable}, \\ \max\{\omega \in [0, \pi] \mid \\ \quad \det(e^{j\omega} I - A) = 0\}, & \text{otherwise} \end{cases}$$

We formulate below two state synchronization problems, one for a network with full-state coupling and the other for partial-state coupling.

Problem 1 (Full-state coupling)**.** Consider a MAS described by (1) and (4) with a given upper bound $\bar{\kappa}$ for the input delay. Let $\mathbf{G}$ be a given set of graphs such that $\mathbf{G} \subseteq \mathbb{G}^N$. The *semi-global state synchronization* problem with a set of network graphs $\mathbf{G}$ is to find, if possible, for any a priori given bounded set of initial conditions $\mathcal{W} \subset \mathfrak{L}^n_\infty(\bar{\kappa})$, a parameterized family of linear protocols of the form,

$$u_i = F_\delta \zeta_i, \quad (i = 1, \dots, N) \tag{5}$$

where there exists a δ^* such that for all $\delta < \delta^*$, state synchronization among agents is achieved for any graph $\mathcal{G} \in \mathbf{G}$ and for any input delay $\kappa_i \in [0, \bar{\kappa}]$ and any initial conditions $\phi_i \in \mathcal{W}$ for $i = 1, \dots, N$.

Problem 2 (Partial-state coupling)**.** Consider a MAS described by (1) and (3) with a given upper bound $\bar{\kappa}$ for the input delay. Let $\mathbf{G}$ be a given set of graphs such that $\mathbf{G} \subseteq \mathbb{G}^N$. The *semi-global state synchronization* problem with a set of network graphs $\mathbf{G}$ is to find, if possible, a positive integer q and for any a priori given bounded set of initial conditions $\mathcal{W} \subset \mathfrak{L}^n_\infty(\bar{\kappa}) \times \mathbb{R}^q$, a parameterized family of linear dynamic protocols of the form,

$$\begin{cases} \chi_i(k+1) = A_{c,\delta}\chi_i(k) + B_{c,\delta}\zeta_i(k), \\ u_i(k) = C_{c,\delta}\chi_i(k) + D_{c,\delta}\zeta_i(k), \\ \chi_i(0) = \psi_i, \end{cases} \tag{6}$$

for $i = 1, \dots, N$ with $\chi_i \in \mathbb{R}^q$, where there exists a δ^* such that for all $\delta < \delta^*$, state synchronization among agents is achieved for any graph $\mathcal{G} \in \mathbf{G}$, for any input delay $\kappa_i \in [0, \bar{\kappa}]$ and any initial conditions $(\phi_i, \psi_i) \in \mathcal{W}$ for $i = 1, \dots, N$.

3. Protocol design

In this section, we design a protocol for discrete-time MAS subject to input saturation and unknown nonuniform input delay. Both full-state coupling and partial-state coupling are considered. The protocol design is based on the low-gain method.

3.1. Full-state coupling

For a discrete-time MAS with full-state coupling, we design the following parameterized family of protocols,

$$u_i = \gamma F_\delta \zeta_i, \tag{7}$$

where

$$F_\delta = -\frac{1}{1-\beta}(B'P_\delta B + I)^{-1}B'P_\delta A \tag{8}$$

with $P_\delta > 0$ being the unique solution of the discrete-time algebraic Riccati equation

$$P_\delta = A'P_\delta A + \delta I - A'P_\delta B(B'P_\delta B + I)^{-1}B'P_\delta A, \tag{9}$$

while δ is sufficiently small such that

$$B'P_\delta B < \frac{1-\beta}{2\beta} I,$$

where $0 < \beta < 1$ is the upper bound of the eigenvalues inside the unit disc for some row stochastic matrix D associated with a graph in a set of graphs $\mathbb{G}^N_\beta$.

Before the main result, we first need the following technical lemmas.

Lemma 1. *Suppose (A, B) is stabilizable and all the eigenvalues of A are within the closed unit disc. Let F_δ be designed in (8). Then, we have the following properties:*

1. *The closed-loop system matrix $A + (1-\lambda)BF_\delta$ is Schur stable for all $\delta > 0$ and for all λ with $|\lambda| < \beta$.*
2. *For any $\beta > 0$, there exists a $\delta^* > 0$ such that for all $\delta \in (0, \delta^*]$ there exist $r_{1,\delta} > 0$ and $0 < \eta_\delta < 1$ with $r_{1,\delta} \to 0$ as $\delta \to 0$ such that*
$$\|F_\delta(A + (1-\lambda)BF_\delta)^k\| \leq r_{1,\delta}\eta_\delta^k, \tag{10}$$
for all $k \geq 0$ and for all $\lambda \in \mathbb{R}$ with $|\lambda| < \beta$.
3. *Let $G(z) = (1-\lambda)F_\delta(zI - A - (1-\lambda)BF_\delta)^{-1}B$. Then, for any $\mu > 0$, there exists a $\delta^* > 0$ such that for all $\delta \in (0, \delta^*]$*
$$\|I + G(z)\|_\infty \leq 1 + \mu \tag{11}$$
for all $\lambda \in \mathbb{R}$ with $|\lambda| < \beta$.

Proof. Consider

$$x(k+1) = (A + (1-\lambda)BF_\delta)x(k),$$

and let $A_f = A + (1-\lambda)BF_\delta$.

It is found that

$$\begin{aligned} &A'P_\delta A - A'P_\delta B(I + B'P_\delta B)^{-1}B'P_\delta A \\ &\quad = A'P_\delta A - (1-\beta)^2 F_\delta'(I + B'P_\delta B)F_\delta \\ &\quad = A_f'P_\delta A_f + [2(1-\lambda)(1-\beta) - (1-\beta)^2]F_\delta'(I + B'P_\delta B)F_\delta \\ &\qquad - (1-\lambda)^2 F_\delta' B'P_\delta BF_\delta \\ &\quad = A_f'P_\delta A_f + (1-\lambda)(1-\beta)F_\delta'F_\delta + [(1-\lambda)(1-\beta) - (1-\beta)^2]F_\delta'F_\delta \\ &\qquad + [2(1-\lambda)(1-\beta) - (1-\beta)^2 - (1-\lambda)^2]F_\delta'B'P_\delta BF_\delta \end{aligned}$$

$$= A_f' P_\delta A_f + (1-\lambda)(1-\beta) F_\delta' F_\delta + (1-\beta)(\beta-\lambda) F_\delta' F_\delta - (\beta-\lambda)^2 F_\delta' B' P_\delta B F_\delta$$
$$= A_f' P_\delta A_f + (1-\lambda)(1-\beta) F_\delta' F_\delta + (\beta-\lambda) F_\delta' [(1-\beta) - (\beta-\lambda) B' P_\delta B] F_\delta$$
$$\geq A_f' P_\delta A_f + (1-\lambda)(1-\beta) F_\delta' F_\delta,$$

where the last inequality holds because

$$(\beta-\lambda) B' P_\delta B \leq 2\beta B' P_\delta B \leq (1-\beta) I.$$

Therefore, we obtain

$$P_\delta \geq A_f' P_\delta A_f + \delta I + (1-\lambda)(1-\beta) F_\delta' F_\delta, \tag{12}$$

which implies that

$$x'(k+1) P_\delta x(k+1) \leq x'(k) P_\delta x(k) - \delta x'(k) x(k) \leq \eta_\delta^2 x'(k) P_\delta x(k),$$

where $\eta_\delta^2 = 1 - \delta \|P_1\|^{-1}$. Hence

$$\|P_\delta^{1/2} x(k)\| \leq \eta_\delta^k \|P_\delta^{1/2} x(0)\|. \tag{13}$$

From (12), we get

$$(1-\beta)^2 F_\delta' F_\delta \leq P_\delta.$$

Hence,

$$\|F_\delta x(k)\| \leq (1-\beta)^{-1} \|P_\delta^{1/2} x(k)\| \leq (1-\beta)^{-1} \eta_\delta^k \|P_\delta^{1/2} x(0)\|,$$

and then,

$$\|F_\delta (A + (1-\lambda) B F_\delta)^k x(0)\| = \|F_\delta x(k)\| \leq (1-\beta)^{-1} \eta_\delta^k \|P_\delta^{1/2} x(0)\|. \tag{14}$$

Since (14) is true for all $x(0) \in \mathbb{R}^n$, it follows trivially that

$$\|F_\delta (A + (1-\lambda) B F_\delta)^k\| \leq (1-\beta)^{-1} \|P_\delta^{1/2}\| \eta_\delta^k. \tag{15}$$

The proof of the inequality (10) is then completed by taking $r_{1,\delta} = (1-\beta)^{-1} \|P_\delta^{1/2}\|$.

The inequality (12) yields

$$(z^{-1} I - A_f) P_\delta (zI - A_f) + A_f' P_\delta (zI - A_f) + (z^{-1} I - A_f) P_\delta A_f \geq \delta I + (1-\lambda)(1-\beta) F_\delta' F_\delta.$$

Premultiplying it with $\frac{1-\lambda}{1-\beta} B' (z^{-1} I - A_f)^{-1}$ and post multiplying it with $(zI - A_f)^{-1} B$ yields

$$\tfrac{1-\lambda}{1-\beta} B' P_\delta B + \tfrac{1-\lambda}{1-\beta} B' (z^{-1} I - A_f)^{-1} A_f' P_\delta B + \tfrac{1-\lambda}{1-\beta} B' P_\delta A_f (zI - A_f)^{-1} B$$
$$\geq \delta \tfrac{1-\lambda}{1-\beta} B' (z^{-1} I - A_f')^{-1} (zI - A_f)^{-1} B + (1-\lambda)^2 B' (z^{-1} I - A_f)^{-1} F_\delta' F_\delta (zI - A_f)^{-1} B. \tag{16}$$

We have

$$\tfrac{1-\lambda}{1-\beta} B' P_\delta A_f = \tfrac{1-\lambda}{1-\beta} B' P_\delta A - \tfrac{(1-\lambda)^2}{1-\beta} B' P_\delta B F_\delta = -\Big[(I + B' P_\delta B) + \tfrac{1-\lambda}{1-\beta} B' P_\delta B\Big](1-\lambda) F_\delta = -V_\delta (1-\lambda) F_\delta,$$

where

$$V_\delta = (I + B' P_\delta B) + \tfrac{1-\lambda}{1-\beta} B' P_\delta B.$$

Then, (16) yields

$$\tfrac{1-\lambda}{1-\beta} B' P_\delta B - G'(z^{-1}) V_\delta - V_\delta G(z) \geq G'(z^{-1}) G(z),$$

which is equivalent to

$$[V_\delta + G'(z^{-1})][V_\delta + G(z)] \leq V_\delta^2 + \tfrac{1-\lambda}{1-\beta} B' P_\delta B.$$

Since $V_\delta \to I$ and $B' P_\delta B \to 0$ as $\delta \to 0$, we find that, for any $\mu > 0$, there exists δ^* such that for any $\delta \in (0, \delta^*]$

$$[I + G'(z^{-1})][I + G(z)] \leq (1+\mu) I.$$

This implies

$$\|I + G(z)\|_\infty \leq 1 + \mu. \qquad \square$$

Lemma 2. *Assume D is associated with a strongly connected undirected graph and $\bar{L} = I - D$. If $\bar{L} = R_e J_e R_e'$ with R_e unitary and $J_e = \operatorname{diag}\{J, 0\}$ with J is diagonal, then we have*

$$T_1 \bar{L} T_2 = R J R^{-1}, \tag{17}$$

with $R = T_1 R_e T_2$ and $R^{-1} = T_2' R_e' T_2$, where $T_1 \in \mathbb{R}^{(N-1)\times N}$ and $T_2 \in \mathbb{R}^{N\times(N-1)}$ are given by

$$T_1 = \begin{pmatrix} I & -\mathbf{1}_{N-1} \end{pmatrix}, \quad T_2 = \begin{pmatrix} I \\ \mathbf{0} \end{pmatrix}.$$

Proof. Since $\bar{L}\mathbf{1}_N = 0$, we have

$$\bar{L} T_2 T_1 = \bar{L}\begin{pmatrix} I & -\mathbf{1}_{N-1} \\ 0 & 0 \end{pmatrix} = \bar{L}\begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix} - \bar{L}\begin{pmatrix} 0 & \mathbf{1}_N \end{pmatrix} = \bar{L}.$$

Therefore, $\bar{L} T_2 T_1$ has $N-1$ nonzero eigenvalues, denoted by, $\bar{\lambda}_2 = 1 - \lambda_2, \dots, \bar{\lambda}_N = 1 - \lambda_N$. Then $T_1 \bar{L} T_2$ has the same $N-1$ nonzero eigenvalues and hence $T_1 \bar{L} T_2$ is invertible. Define $R = T_1 R_e T_2$.

We have

$$R_e = \begin{pmatrix} R_{11} & \frac{1}{\sqrt{N}} \mathbf{1}_{N-1} \\ R_{21} & \frac{1}{\sqrt{N}} \end{pmatrix}$$

given that $\bar{L}\mathbf{1} = 0$. Then, it is found that

$$R T_2' R_e' = T_1 R_e T_2 T_2' R_e' = T_1 \begin{pmatrix} R_{11} \\ R_{21} \end{pmatrix} \begin{pmatrix} R_{11}' & R_{21}' \end{pmatrix} = T_1 \begin{pmatrix} R_{11} & \frac{1}{\sqrt{N}} \mathbf{1}_{N-1} \\ R_{21} & \frac{1}{\sqrt{N}} \end{pmatrix} \begin{pmatrix} R_{11}' & R_{21}' \\ \frac{1}{\sqrt{N}} \mathbf{1}_{N-1}' & \frac{1}{\sqrt{N}} \end{pmatrix} = T_1 R_e R_e' = T_1.$$

The third equality holds because $T_1 \begin{pmatrix} \mathbf{1}_{N-1} \\ 1 \end{pmatrix} = 0$. Therefore,

$$R^{-1} T_1 = T_2' R_e',$$

which yields that $R^{-1} = T_2' R_e' T_2$ and moreover

$$T_1 \bar{L} T_2 R = T_1 \bar{L} T_2 T_1 R_e T_2 = T_1 \bar{L} R_e T_2 = T_1 R_e J_e T_2 = T_1 R_e T_2 J = R J.$$

Hence, (17) is satisfied. $\square$

Lemma 3. *Suppose (A, B) is stabilizable and all the eigenvalues of A are within the closed unit disc. Let F_δ be designed in (8). Then, we have,*

$$\left\| (J \otimes F_\delta)(I_{N-1} \otimes A + (J \otimes B F_\delta))^k \right\| \leq r_{2,\delta}, \tag{18}$$

where J is the Jordan form of matrix $T_1 \bar{L} T_2$ and $\bar{L} = I - D$ with D the row stochastic matrix with its associated graph in $\mathbb{G}_\beta^N$, and moreover, $r_{2,\delta} \to 0$ as $\delta \to 0$.

Proof. We have $J = \operatorname{diag}\{\bar{\lambda}_2, \dots, \bar{\lambda}_N\}$. Consider the dynamics of η

$$\eta(k+1) = (I_{N-1} \otimes A + J \otimes B F_\delta)\eta(k),$$

where

$$\eta = \begin{pmatrix} \eta_2 \\ \vdots \\ \eta_N \end{pmatrix}.$$

In that case,

$$\eta_i(k+1) = (A + \bar{\lambda}_i B F_\delta)\eta_i(k)$$

for $i = 2, \dots, N$. Then, following the results of the above Lemma 1, we can achieve

$$\|F_\delta \eta_i(t)\| \le r_{1,\delta}$$

with $r_{1,\delta} \to 0$ as $\delta \to 0$. Then, we have

$$\left\| (J \otimes F_\delta)(I_{N-1} \otimes A + (J \otimes BF_\delta))^k \right\| \le r_{1,\delta},$$

since J is diagonal. This completes the proof. □

The main result for discrete-time MAS with full-state coupling is stated as follows.

Theorem 1. *Consider a MAS described by (1) and (4) with an input delay upper bound $\bar{\kappa}$ and input saturation. Let any $0 < \beta < 1$ be given, and hence a set of network graphs $\mathbb{G}_\beta^N$ be defined.*

If (A, B) is stabilizable and the agents are at most weakly unstable, then the semi-global state synchronization problem stated in Problem 1 with $\mathbf{G} = \mathbb{G}_\beta^N$ is solvable if

$$\bar{\kappa}\omega_{\max} < \tfrac{\pi}{2}. \tag{19}$$

Moreover, for any a priori given compact set of initial conditions $\mathcal{W} \subset \mathfrak{L}_\infty^n(\bar{\kappa})$, there exist $\gamma > 0$ and $\delta^* > 0$ such that for this γ and any $\delta \in (0, \delta^*]$, the protocol (7) achieves state synchronization for any graph $\mathcal{G} \in \mathbb{G}_\beta^N$, for any input delay $\kappa_i \in [0, \bar{\kappa}]$, and for any initial condition $\phi_{i,\varsigma} \in \mathcal{W}$ for $i = 1, \dots, N$.

Proof of Theorem 1. Let $\mathbf{D}_i$ $(i = 1, \dots, N)$ be a delay operator for agent i such that $(\mathbf{D}_i u_i)(k) = u_i(k - \kappa_i)$. In the frequency domain, $\hat{\mathbf{D}}_i(\omega) = e^{-j\omega\kappa_i}$. Define $x = \operatorname{col}\{x_i\}$, $u = \operatorname{col}\{u_i\}$ and $\mathbf{D} = \operatorname{diag}\{\mathbf{D}_i\}$ and $\hat{\mathbf{D}}(\omega) = \operatorname{diag}\{\hat{\mathbf{D}}_i(\omega)\}$, the overall dynamics of multiagent system described by (1) and (4) can be represented by

$$\begin{cases} x(k+1) = (I_N \otimes A)x(k) + (I_N \otimes B)\sigma(\mathbf{D}u(k)), \\ u(k) = (\bar{L} \otimes \gamma F_\delta)x(k), \end{cases} \tag{20}$$

If the input $u(k) = (\bar{L} \otimes \gamma F_\delta)x(t)$ can be squeezed small enough, i.e. the input can avoid triggering saturation, the overall dynamics (20) becomes a system without saturation

$$\begin{cases} x(k+1) = (I_N \otimes A)x(k) + (I_N \otimes B)\mathbf{D}u(k), \\ u(k) = (\bar{L} \otimes \gamma F_\delta)x(k). \end{cases} \tag{21}$$

The synchronization of (21) has been proved in [49, Theorem 1 with $\gamma = \alpha(1-\beta)$], and we can show that synchronization of (20) by establishing that the system (21) does not saturate provided δ is small enough.

Now define $\bar{x}_i = x_i - x_N$ and $\bar{x} = \operatorname{col}\{\bar{x}_1, \dots, \bar{x}_{N-1}\}$. Since

$$\begin{aligned} u_i &= \gamma F_\delta \sum d_{ij}((x_i - x_N) - (x_j - x_N)) \\ &= \gamma F_\delta \sum \bar{\ell}_{ij}(x_j - x_N) = \gamma F_\delta \sum \bar{\ell}_{ij}\bar{x}_j, \end{aligned}$$

we have

$$u = (\bar{L}T_2 \otimes \gamma F_\delta)\bar{x} \tag{22}$$

and

$$\bar{x}(k+1) = (I_{N-1} \otimes A)\bar{x}(k) + \gamma(T_1\mathbf{D}\bar{L}T_2 \otimes BF_\delta)\bar{x}(k). \tag{23}$$

The following step is to show that we can avoid the saturation. Since $\operatorname{Im}\bar{L} \perp \ker T_1$, we only need to show that $(T_1\bar{L}T_2 \otimes \gamma F_\delta)\bar{x}(k)$ is sufficiently small. We have

$$\left\| (T_1\bar{L}T_2 \otimes \gamma F_\delta)\bar{x} \right\| \le \left\| (T_1\bar{L}T_2 \otimes \gamma F_\delta)\bar{x} \right\|_2,$$

and hence it is sufficient to prove that

$$\left\| (T_1\bar{L}T_2 \otimes \gamma F_\delta)\bar{x} \right\|_2 \le r_{3,\delta}, \tag{24}$$

with $r_{3,\delta} \to 0$ as $\delta \to 0$.

Next, we define the linear time-invariant operator $g_\delta : v_\delta \to w_\delta$ with the state space representation:

$$\begin{cases} \xi(k+1) = (I_{N-1} \otimes A + (J \otimes BF_\delta))\xi(k) + (I_{N-1} \otimes B)v_\delta(k), \\ w_\delta(k) = (J \otimes F_\delta)\xi(k). \end{cases} \tag{25}$$

We also define another linear time-invariant operator ϑ by:

$$g(k) = \vartheta(f)(k) = R^{-1}T_1 \begin{pmatrix} h_1 \\ \vdots \\ h_N \end{pmatrix}$$

with

$$h_i(k) = \begin{cases} \left[(\gamma\mathbf{D}_i - I)e_i'R_eT_2 \otimes I_m\right]f(k) & k \ge \kappa_i \\ \left[-e_i'R_eT_2 \otimes I_m\right]f(k) & k < \kappa_i \end{cases}$$

where e_i is a zero row vector, but with the ith element being 1. We can see that the Z-transform of these two operators are given by

$$G_\delta(j\omega) = (J \otimes F_\delta)(e^{j\omega}I - (I_{N-1} \otimes A) - (J \otimes BF_\delta))^{-1}(I_{N-1} \otimes B),$$
$$\Delta(j\omega) = R^{-1}T_1(\gamma\hat{\mathbf{D}}(\omega) - I)R_eT_2 \otimes I_m.$$

Next, define $\tilde{x} = (R^{-1} \otimes I_n)\bar{x}$. Then, the dynamics of $\tilde{x}$ can be written as, for $k \ge 0$

$$\begin{aligned} \tilde{x}(k+1) = {} & (I_{N-1} \otimes A + (J \otimes BF_\delta))\tilde{x}(k) + (I_{N-1} \otimes B)\vartheta((J \otimes F_\delta)\tilde{x}(k)) \\ & + (I_{N-1} \otimes B)v_\delta(k), \end{aligned} \tag{26}$$

where

$$v_\delta(k) = R^{-1}T_1 \begin{pmatrix} \tilde{h}_1 \\ \vdots \\ \tilde{h}_N \end{pmatrix}$$

with

$$\tilde{h}_i(k) = \begin{cases} \left[\gamma\mathbf{D}_i e_i'R_eT_2 \otimes I_m\right](J \otimes F_\delta)\tilde{x}(k) & k < \kappa_i, \\ 0 & k \ge \kappa_i. \end{cases}$$

Note that v_δ vanishes for $k \ge \bar{\kappa}$ and $\tilde{x}(k)$ is bounded for $k < \bar{\kappa}$ since the initial conditions are in the bounded set $\mathcal{W}$. Moreover, since $F_\delta \to 0$, we have $\|v_\delta\|_\infty \to 0$ and $\|v_\delta\|_2 \to 0$ as $\delta \to 0$.

From (26), we obtain

$$\begin{aligned} (J \otimes F_\delta)\tilde{x}(k) = {} & (J \otimes F_\delta)(I_{N-1} \otimes A + (J \otimes BF_\delta))^k\tilde{x}(0) \\ & + (g_\delta \circ \vartheta)((J \otimes F_\delta)\tilde{x}(k)) + g_\delta(v_\delta)(k) \end{aligned}$$

and hence

$$\begin{aligned} (J \otimes F_\delta)\tilde{x}(k) = {} & (1 - g_\delta \circ \vartheta)^{-1} \\ & \times\left[(J \otimes F_\delta)(I_{N-1} \otimes A + (J \otimes BF_\delta))^k\tilde{x}(0) + g_\delta(v_\delta)(k)\right]. \end{aligned} \tag{27}$$

According to the definition of operator g_δ, we have $w_\delta = g_\delta(v_\delta)(k)$. Then, $\|w_\delta\|_2 \le \|G_\delta\|_\infty\|v_\delta\|_2 \le 2\|v_\delta\|_2$ by choosing $\mu = 1$ in Lemma 1. Therefore, for any given initial condition $\phi_i \in \mathcal{W}$ $(i = 1, \dots, N)$, $\|w_\delta\|_2 \to 0$ as $\delta \to 0$.

From (27), we get

$$\begin{aligned} \|(J \otimes F_\delta)\tilde{x}(k)\|_2 \le {} & \|(I - \Delta G_\delta)^{-1}\|_\infty \\ & \times\left\|(J \otimes F_\delta)(I_{N-1} \otimes A + (J \otimes BF_\delta))^k\tilde{x}(0)\right\|_2 \\ & + \|(I - \Delta G_\delta)^{-1}\|_\infty\|w_\delta\|_2 \end{aligned}$$

Therefore, we can obtain (24) provided that we have $\|(I-\Delta G_\delta)^{-1}\|_\infty$ is bounded independent of δ, using that $(T_1\bar{L}T_2\otimes\gamma F_\delta)\bar{x}$ is small if and only if $(J\otimes\gamma F_\delta)\tilde{x}$ is small.

Since $\bar{\kappa}\omega_{\max}<\frac{\pi}{2}$, we can choose γ such that

$$\gamma\cos(\bar{\kappa}\omega_{\max})>1. \tag{28}$$

Note that this γ is independent of low-gain parameter δ and condition (29) implies that $\gamma>1$. Let this γ be fixed during the remaining proof.

Given (28), there exists a $\varpi>0$ such that

$$\gamma\cos(\bar{\kappa}(\omega_{\max}+\varpi))>1. \tag{29}$$

For $|\omega|<\omega_{\max}+\varpi$, we find that

$$\begin{aligned}\Delta(j\omega)+\Delta(j\omega)' &= T_2'R_e'(\gamma\hat{\mathbf{D}}(\omega)-I)R_eT_2\otimes I_m\\ &\quad+T_2'R_e'(\gamma\hat{\mathbf{D}}(\omega)'-I)R_eT_2\otimes I_m\\ &= T_2'R_e'(\gamma\hat{\mathbf{D}}(\omega)+\gamma\hat{\mathbf{D}}(\omega)'-2I)R_eT_2\otimes I_m\\ &\geq T_2'R_e'(2\gamma\cos(\omega\bar{\tau})-2)R_eT_2\otimes I_m\\ &\geq 0,\end{aligned}$$

because γ is chosen to satisfy (29). Furthermore, we obtain that

$$\begin{aligned}\|\Delta(j\omega)\|_\infty &= \left\|T_2'R_e'(\gamma\hat{\mathbf{D}}(\omega)-I)R_eT_2\otimes I_m\right\|_\infty\\ &= \left\|(\gamma\hat{\mathbf{D}}(\omega)-I)R_eT_2T_2'R_e'\otimes I_m\right\|_\infty\\ &\leq \left\|(\gamma\hat{\mathbf{D}}(\omega)-I)\otimes I_m\right\|_\infty\\ &\leq 1+\gamma,\end{aligned}$$

since $\|\hat{\mathbf{D}}(\omega)\|\leq 1$. Then, we have that, for $|\omega|<\omega_{\max}+\varpi$

$$\begin{aligned}\Delta(j\omega)'\Delta(j\omega) &\leq \Delta(j\omega)'\Delta(j\omega)+\Delta(j\omega)+\Delta(j\omega)'\\ &\quad+\left[I-(2+\gamma)^{-2}(I+\Delta(j\omega)')(I+\Delta(j\omega))\right]\\ &\leq \left[1-(2+\gamma)^{-2}\right](I+\Delta(j\omega)')(I+\Delta(j\omega)),\end{aligned}$$

which leads to

$$(I+\Delta(j\omega)')^{-1}\Delta(j\omega)'\Delta(j\omega)(I+\Delta(j\omega))^{-1}\leq\left[1-(2+\gamma)^{-2}\right]I.$$

Hence, there exists a $\rho>0$ that is independent of parameter δ, such that

$$\|\Delta(j\omega)(I+\Delta(j\omega))^{-1}\|\leq 1-\rho.$$

Moreover, we get

$$\begin{aligned}\|(I+\Delta(j\omega))^{-1}\| &= \|I-\Delta(j\omega)(I+\Delta(j\omega))^{-1}\|\\ &\leq 1-\|\Delta(j\omega)(I+\Delta(j\omega))^{-1}\|\\ &\leq\rho.\end{aligned}$$

Hence,

$$\underline{\sigma}(I+\Delta(j\omega))\geq\frac{1}{\rho}.$$

On the other hand, from Property 3 in Lemma 1 with $\mu=\frac{\rho}{2-2\rho}$, we can immediately obtain that

$$\|I+G_\delta(j\omega)\|<1+\frac{\rho}{2-2\rho}.$$

Then, it follows that, for $|\omega|<\omega_{\max}+\varpi$

$$\begin{aligned}&\underline{\sigma}[I-\Delta(j\omega)G_\delta(j\omega)]\\ &\quad=\underline{\sigma}[I+\Delta(j\omega)-\Delta(j\omega)(I+G_\delta(j\omega))]\\ &\quad=\underline{\sigma}(I+\Delta(j\omega))\underline{\sigma}\left[I-(I+\Delta(j\omega))^{-1}\Delta(j\omega)(I+G_\delta(j\omega))\right]\\ &\quad\geq\frac{1}{\rho}\left(1-(1-\rho)\left(1+\frac{\rho}{2-2\rho}\right)\right)=\frac{1}{2},\end{aligned}$$

for all $\kappa_i\in[0,\bar{\kappa}]$ and all possible D (note that $\bar{L}=I-D$) associated with a network graph in $\mathbb{G}_\beta^N$. Therefore, we have

$$\left\|(I-\Delta(j\omega)G_\delta(j\omega))^{-1}\right\|\leq 2, \tag{30}$$

for all $|\omega|<\omega_{\max}+\varpi$.

For all $|\omega|\geq\omega_{\max}+\varpi$, we know that $\Delta(j\omega)G_\delta(j\omega)\to 0$ as $\delta\to 0$ uniformly in ω. Therefore, we can conclude that for small enough δ, we also have (30) for $|\omega|\geq\omega_{\max}+\varpi$. This completes the proof of Theorem 1. □

3.2. Partial-state coupling

We still design a low-gain based protocol for MAS with partial-state coupling. Choose an observer gain K such that $A+KC$ is Schur stable. Next, we consider a feedback gain F_δ in (8), which results in the protocol,

$$\begin{cases}\chi_i(k+1)=(A+KC)\chi_i(k)-K\zeta_i(k),\\ u_i(k)=\gamma F_\delta\chi_i(k).\end{cases} \tag{31}$$

Again, we need the following technical lemmas before we proceed to the main result for MAS with partial-state coupling.

Lemma 4. *Consider*

$$\hat{A}=\begin{pmatrix}A&0\\-KC&A+KC\end{pmatrix},\quad\hat{B}=\begin{pmatrix}B\\0\end{pmatrix},\quad\hat{F}_\delta=\begin{pmatrix}0&F_\delta\end{pmatrix}.$$

Then, for any $0<\beta<1$, there exists a δ^* such that we have the following.

1. *The closed-loop system matrix $\hat{A}+(1-\lambda)\hat{B}\hat{F}_\delta$ is Schur stable for all $\delta\in[0,\delta^*]$ and all $|\lambda|<\beta$.*
2. *There exists a $\hat{r}_{1,\delta}$ with $\hat{r}_{1,\delta}\to 0$ as $\delta\to 0$ such that*
$$\left\|\hat{F}_\delta(\hat{A}+(1-\lambda)\hat{B}\hat{F}_\delta)^k\right\|\leq\hat{r}_{1,\delta} \tag{32}$$
for all $\delta\in[0,\delta^*]$, for all $k\geq 0$, and all $\lambda\in\mathbb{R}$ with $|\lambda|<\beta$.
3. *Let $\hat{G}(z)=(1-\lambda)\hat{F}_\delta(zI-\hat{A}-(1-\lambda)\hat{B}\hat{F}_\delta)^{-1}\hat{B}$. Then, for any $\hat{\mu}>0$, there exists a $\delta^*>0$ such that for all $\delta\in(0,\delta^*]$*
$$\left\|I+\hat{G}(z)\right\|_\infty\leq 1+\hat{\mu} \tag{33}$$
for all $\delta\in[0,\delta^*]$ and all $\lambda\in\mathbb{R}$ with $|\lambda|<\beta$.

Proof. Since

$$(1-\lambda)\frac{1}{1-\beta}>(1-\beta)\frac{1}{1-\beta}=1$$

for any $|\lambda|<\beta$, $\hat{A}+(1-\lambda)\hat{B}\hat{F}_\delta$ is Schur stable, according to [49, Lemma 5]. Note that a coefficient $\frac{1}{1-\beta}$ is added because it does not show up in the F_δ in [49, Lemma 5].

Next, the system with realization $(\hat{A},\hat{B},\hat{C})$ with input v and output z is given by,

$$\begin{cases}\tilde{x}_1(k+1)=A\tilde{x}_1(k)+(1-\lambda)BF_\delta\tilde{x}_2(k)+Bv(k),\\ \tilde{x}_2(k+1)=(A+KC)\tilde{x}_2(k)-KC\tilde{x}_1(k),\\ z(k)=(1-\lambda)\begin{pmatrix}0&F_\delta\end{pmatrix}\tilde{x}_2(k).\end{cases} \tag{34}$$

Now, let $x_1=\tilde{x}_1$ and $x_2=\tilde{x}_2-\tilde{x}_1$. Then, we have

$$\begin{cases}x_1(k+1)=(A+(1-\lambda)BF_\delta)x_1(k)+(1-\lambda)BF_\delta x_2(k)+Bv(k),\\ x_2(k+1)=-(1-\lambda)BF_\delta x_1(k)+(A+KC-(1-\lambda)BF_\delta)x_2(k)-Bv(k),\\ z(k)=F_\delta x_1(k)+F_\delta x_2(k).\end{cases} \tag{35}$$

Then, the remaining proof follows the arguments from [48] to show that $\|\hat{G}-G\|_\infty$ converges to zero as $\delta\to 0$ with G as defined in Lemma 1. Hence, (33) can then be obtained from the result of Lemma 1. □

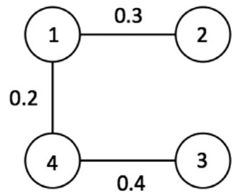

Fig. 1. The network topology.

The main result for discrete-time MAS with partial-state coupling is stated as follows.

Theorem 2. *Consider a MAS described by (1) and (3) with an input delay upper bound $\bar{\kappa}$ and input saturation. Let any $0<\beta<1$ be given, and hence a set of network graphs $\mathbb{G}_\beta^N$ be defined.*

If (A, B) is stabilizable, (A, C) is detectable and A is at most weakly unstable, then the semi-global state synchronization problem stated in Problem 2 with $\mathbf{G}=\mathbb{G}_\beta^N$ is solvable if condition (19) is satisfied. In particular, there exists an integer n and for a priori given compact set of initial conditions $\mathcal{W}\subset\mathfrak{L}_\infty^n(\bar{\kappa})\times\mathbb{R}^n$, there exist $\gamma>0$ and $\delta^*>0$, such that for this γ and any $\delta\in(0, \delta^*]$, the protocol (31) achieves state synchronization for any graph $\mathcal{G}\in\mathbb{G}_\beta^N$, for any input delay $\kappa_i\in[0,\bar{\kappa}]$, and for any initial condition $(\phi_i, \psi)\in\mathcal{W}$ for $i=1,\dots,N$.

Proof of Theorem 2. Define

$$\begin{cases}\tilde{\chi}_i(k+1)=(A+KC)\tilde{\chi}_i(k)-Ky_i(k),\\ \nu(k+1)=(A+KC)\nu(k)\end{cases}$$

with $\nu(0)=\frac{1}{N}\sum_{j=1}\chi_j(0)$ and

$$\begin{pmatrix}\chi_1(0)\\ \vdots\\ \chi_N(0)\end{pmatrix}=\bar{L}\begin{pmatrix}\tilde{\chi}_1(0)\\ \vdots\\ \tilde{\chi}_N(0)\end{pmatrix}+\begin{pmatrix}\nu(0)\\ \vdots\\ \nu(0)\end{pmatrix}.$$

Then, it is easily verified that

$$\chi_i=\sum_{j=1}^N\bar{\ell}_{ij}\tilde{\chi}_j+\nu.$$

Since $A+KC$ is asymptotically stable and $F_\delta\to 0$ as $\delta\to 0$, we have that

$$\varsigma_\delta(k)=\gamma F_\delta\nu(k)$$

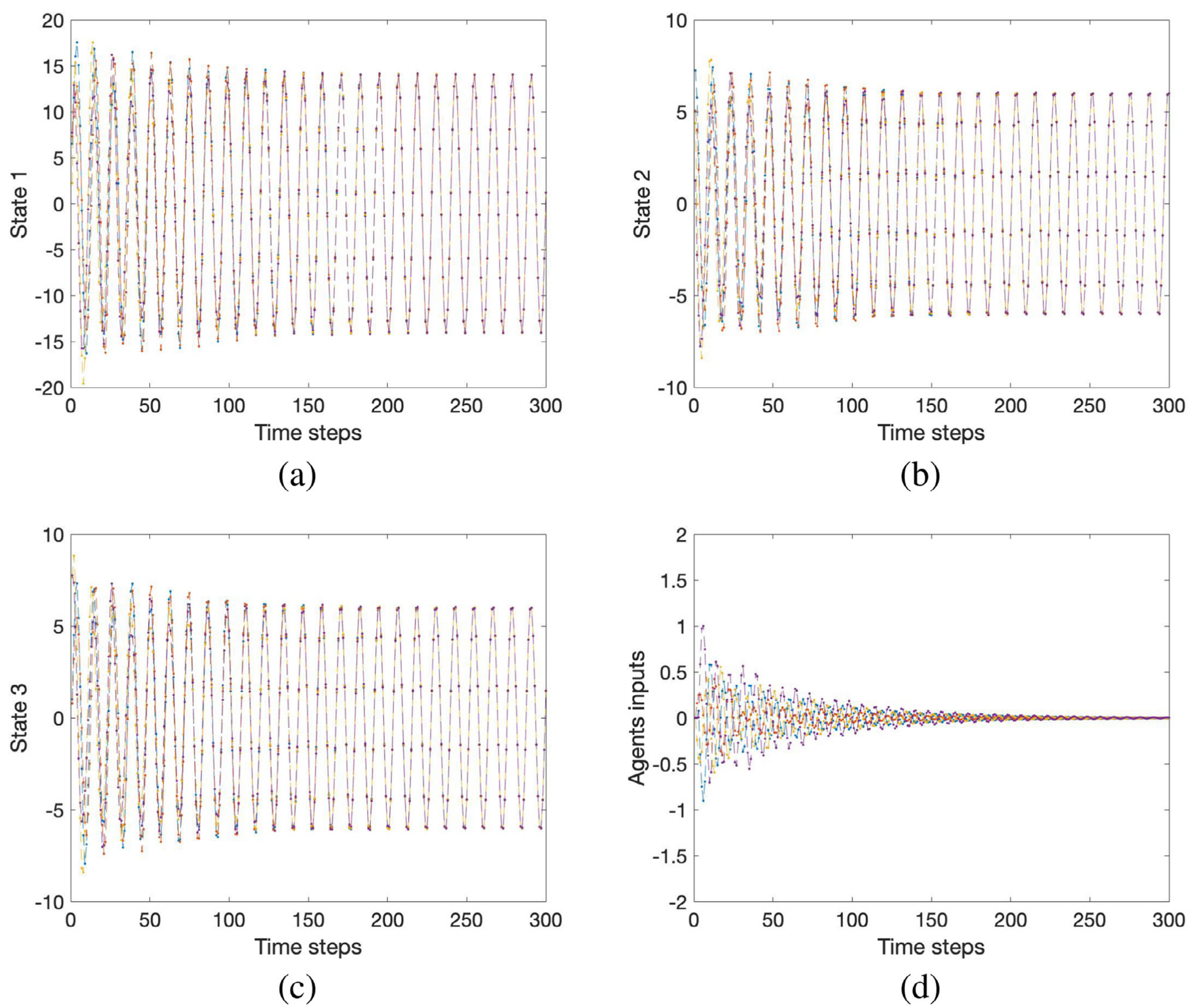

Fig. 2. Trajectories of the states and input signals in the case of full-state coupling.

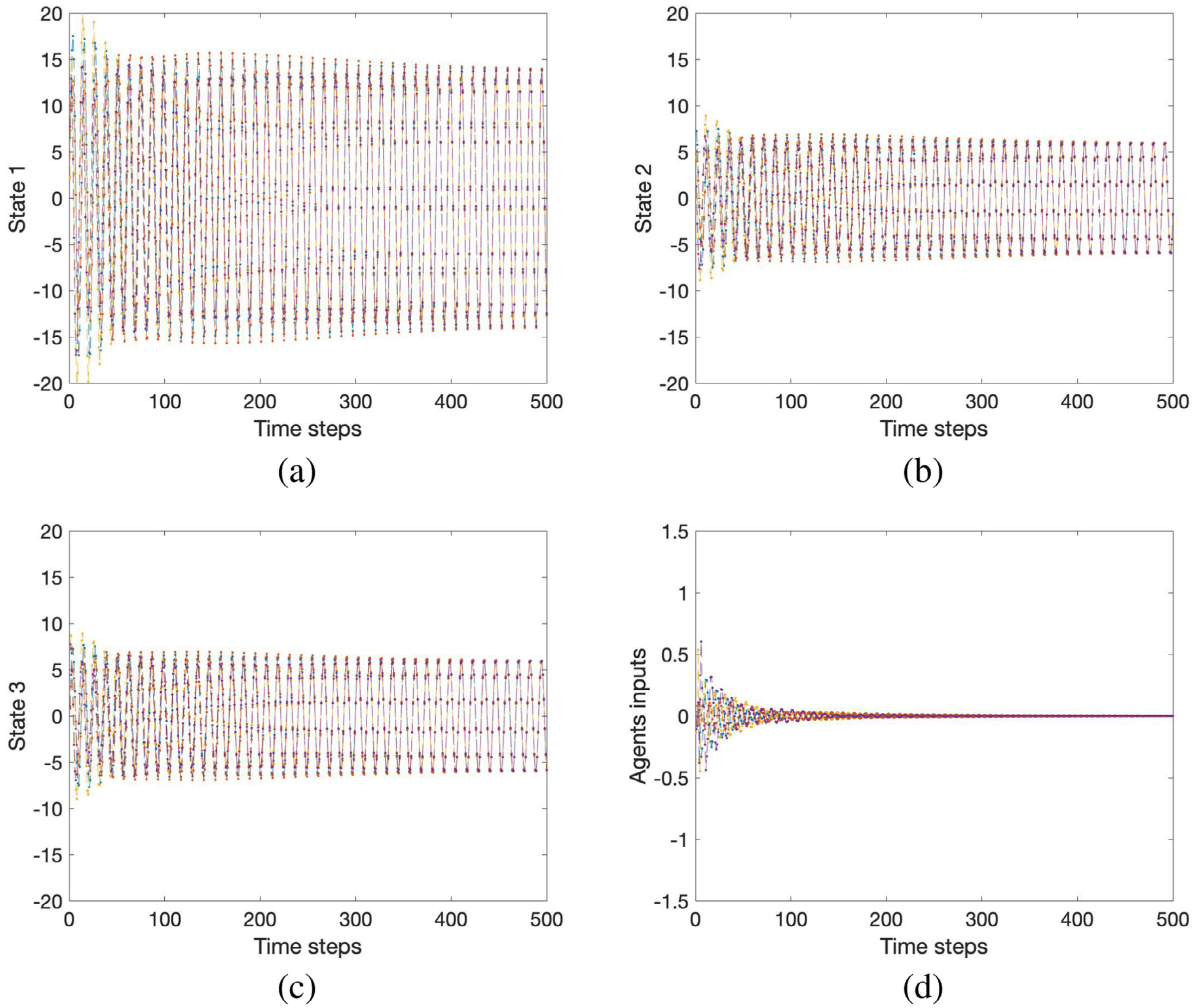

Fig. 3. Trajectories of the states and input signals in the case of partial-state coupling.

is such that $\|\varsigma_\delta\|_2 \to 0$, $\|\varsigma_\delta\|_\infty \to 0$ as $\delta \to 0$. The interconnection of (1) and the protocol (31) yields

$$\begin{cases} x_i(k+1) = Ax_i(k) + B(\mathbf{D}_i\tilde{u}_i)(k) + B(\mathbf{D}_i\varsigma_\delta)(k), \\ \tilde{\chi}_i(k+1) = (A+KC)\tilde{\chi}_i(k) - Ky_i(k), \\ \tilde{u}_i(k) = \sum_{j=1}^N \gamma F_\delta \bar{\ell}_{ij}\tilde{\chi}_j, \end{cases} \tag{36}$$

provided the saturation does not get activated. Note that

$$u_i = \tilde{u}_i + \varsigma_\delta. \tag{37}$$

Therefore the input does not get saturated if we show that $\tilde{u}_i$ is arbitrarily small for sufficiently small δ. Let

$$\hat{x}_i = \begin{pmatrix} \hat{x}_{i,x} \\ \hat{x}_{i,\tilde{\chi}} \end{pmatrix} = \begin{pmatrix} x_i \\ \tilde{\chi}_i \end{pmatrix} - \begin{pmatrix} x_N \\ \tilde{\chi}_N \end{pmatrix},$$

and

$$\hat{A} = \begin{pmatrix} A & 0 \\ -KC & A+KC \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}, \quad \hat{F}_\delta = \begin{pmatrix} 0 & F_\delta \end{pmatrix}.$$

Then, the overall dynamics of $\hat{x} = \text{col}\{\hat{x}_1, \ldots, \hat{x}_{N-1}\}$ is given by

$$\hat{x}(k+1) = (I_N \otimes \hat{A})\hat{x}(k) + \gamma(T_1\mathbf{D}\bar{L}T_2 \otimes \hat{B}\hat{F}_\delta)\hat{x}(k) + \hat{B}\tilde{\varsigma}_\delta(k), \tag{38}$$

with

$$\tilde{\varsigma}_\delta(k) = (T_1\mathbf{D1} \otimes I)\varsigma_\delta(k),$$

assuming the saturation is not active. Moreover,

$$\tilde{u}(k) = (\bar{L}T_2 \otimes \gamma\hat{F}_\delta)\hat{x}(k) \tag{39}$$

Since $\tau\omega_{\max} < \frac{\pi}{2}$, we can choose γ such that

$$\gamma\cos(\tau\omega_{\max}) > 1. \tag{40}$$

Clearly γ is independent of low-gain parameter δ. Let this γ be fixed. We can now use similar arguments as we did in the proof of Theorem 1 with the bounds from Lemma 4 and the fact that $\tilde{\varsigma}_\delta$ is such that $\|\tilde{\varsigma}_\delta\|_2 \to 0$, $\|\tilde{\varsigma}_\delta\|_\infty \to 0$ as $\delta \to 0$. We obtain that $\hat{x} \to 0$ and $\|\tilde{u}\|_\infty$ is arbitrarily small for sufficiently small δ. Given (37), $\tilde{u}$ small guarantees that the saturation does not get activated for δ sufficiently small while $\hat{x}(k) \to 0$ guarantees synchronization. □

4. Example

We will illustrate our result on a network of $N = 4$ identical discrete-time agents. The agent dynamics is as follows,

$$\begin{cases} x_i(k+1) = \begin{pmatrix} 0.5 & 1 & 1 \\ 0 & \sqrt{3}/2 & -0.5 \\ 0 & 0.5 & \sqrt{3}/2 \end{pmatrix} x_i(k) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \sigma(u_i(k-\kappa_i)), \\ y_i(k) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x_i(k). \end{cases}$$

Eigenvalues of A are 0.5, $\sqrt{3}/2 \pm 0.5j$, which implies that $\omega_{\max} = \pi/6$. According to the condition $\bar{\kappa}\omega_{\max} < \frac{\pi}{2}$, $\bar{\kappa}$ should be less than 3. Choosing $\bar{\kappa} = 2$, we allow input delays $\kappa_1 = 2$, $\kappa_2 = 1$, $\kappa_3 = 1$, $\kappa_4 = 2$. The initial conditions are $x(0) = [6.52; 7.25; 1.02; 7.31; 5.06; 0.78; 2.23; 4.38; 7.66; 7.72; 1.26; 7.76]$

The network topology is given by Fig. 1 with the row stochastic matrix

$$D = \begin{pmatrix} 0.5 & 0.3 & 0 & 0.2 \\ 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0.2 & 0 & 0.4 & 0.4 \end{pmatrix}. \tag{41}$$

The D matrix has eigenvalues 0.0141, 0.3316, 0.8543, 1.0. Hence, it associated graph is in the graph set $\mathbb{G}_{\beta}^{4}$ with $\beta = 0.9$. We select $\gamma = 15$ and condition (29)

$$\gamma \cos\left(2 * \frac{\pi}{6}\right) > 1.$$

Full-state coupling By choosing $\delta = 10^{-6}$, the low-gain feedback F_δ

$$F_\delta = \begin{pmatrix} -0.00000274 & -0.01338 & -0.02323 \end{pmatrix}.$$

The low-gain feedback protocol is

$$u_i = \begin{pmatrix} -0.00004111 & -0.20065 & -0.34848 \end{pmatrix} z_i, \qquad i = 1, \dots, 4. \tag{42}$$

Fig. 2a–c show that state synchronization is achieved for the network with D in (41). And Fig. 2d shows that the input saturation is not activated.

Partial-state coupling By choosing $K = \begin{pmatrix} -2 & -1 & -1 \end{pmatrix}'$, we find that $A + KC$ has eigenvalues of 0.4803 and $0.5759 \pm 0.7362j$, and choosing $\delta = 9 \times 10^{-8}$, $\alpha = 15$, the protocol for the partial-state coupling is designed as

$$\begin{cases} \chi_i(t+1) = \begin{pmatrix} -1.5 & 1 & 1 \\ -1 & 0.866 & -0.5 \\ -1 & 0.5 & 0.866 \end{pmatrix} \chi_i(t) - \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} z_i(t), \\ u_i(k) = \begin{pmatrix} -0.0000037105 & -0.06035 & -0.10461 \end{pmatrix} \chi_i(t). \end{cases} \tag{43}$$

Fig. 3a–c show that state synchronization is achieved and Fig. 3d shows that the input saturation is not activated.

5. Conclusion

In this paper we have studied semi-global state synchronization of homogeneous discrete-time MASs with full-state or partial-state coupling and subject to actuation saturation and unknown nonuniform input delay. We derived an upper bound for the input delay tolerance. And, for any input delay within the upper bound, wee proposed a low-gain based controller design such that the actuator saturation is not activated after a transient phase. Because of the use of low-gain methodology, the agents are assumed to be at most critically unstable. In this paper, the network graph is undirected and connected and the input delay is constant. In the future, we will need to extend these results to more complicated MASs, such as with time-varying input delay, directed network graph, communication delays and disturbances.

Declaration of Competing Interest

The authors whose names are listed immediately below certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

References

[1] H. Bai, M. Arcak, J. Wen, Cooperative control design: a systematic, passivity-based approach, Communications and Control Engineering, Springer Verlag, 2011.
[2] P. Bliman, G. Ferrari-Trecate, Average consensus problems in networks of agents with delayed communications, Automatica 44 (8) (2008) 1985–1995.
[3] Y. Cao, W. Yu, W. Ren, G. Chen, An overview of recent progress in the study of distributed multi-agent coordination, IEEE Trans. Ind. Inform. 9 (1) (2013) 427–438.
[4] N. Chopra, Output synchronization on strongly connected graphs, IEEE Trans. Autom. Control 57 (1) (2012) 2896–2901.
[5] N. Chopra, M.K. Spong, Output synchronization of nonlinear systems with time delay in communication, in: Proceedings of the 45th CDC, San Diego, CA, 2006, pp. 4986–4992.
[6] H. Chu, J. Yuan, W. Zhang, Observer-based consensus tracking for linear multi-agent systems with input saturation, IET Control Theory Appl. 9 (14) (2015) 2124–2131.
[7] R. Ghabcheloo, A.P. Aguiar, A. Pascoal, C. Silvestre, Synchronization in multi-agent systems with switching topologies and non-homogeneous communication delays, in: Proceedings of the 46th CDC, New Orleans, LA, 2007, pp. 2327–2332.
[8] J. Klotz, S. Obuz, Z. Kan, W.E. Dixon, Synchronization of uncertain euler-lagrange systems with unknown time-varying communication delays, in: Proceedings of the American Control Conference, Chicago, IL, 2015, pp. 683–688.
[9] I. Lestas, G. Vinnicombe, Heterogeneity and scalability in group agreement protocols: Beyond small gain and passivity approaches, Automatica 46 (7) (2010) 1141–1151.
[10] Y. Li, J. Xiang, W. Wei, Consensus problems for linear time-invariant multi-agent systems with saturation constraints, IET Control Theory Appl. 5 (6) (2011) 823–829.
[11] Z. Li, Z. Duan, G. Chen, L. Huang, Consensus of multi-agent systems and synchronization of complex networks: a unified viewpoint, IEEE Trans. Circ. & Syst.-I Reg. Pap. 57 (1) (2010) 213–224.
[12] P. Lin, Y. Jia, Average consensus in networks of multi-agents with both switching topology and coupling time-delay, Phys. A: Stat. Mech. Appl. 387 (1) (2008) 303–313.
[13] P. Lin, Y. Jia, Consensus of second-order discrete-time multi-agent systems with nonuniform time-delays and dynamically changing topologies, Automatica 45 (9) (2009) 2154–2158.
[14] P. Lin, Y. Jia, L. Li, Distributed robust H_∞ consensus control in directed networks of agents with time-delay, Syst. Control Lett. 57 (8) (2008) 643–653.
[15] Z. Meng, Z. Zhao, Z. Lin, On global leader-following consensus of identical linear dynamic systems subject to actuator saturation, Syst. Control Lett. 62 (2) (2013) 132–142.
[16] M. Mesbahi, M. Egerstedt, Graph Theoretic Methods in Multiagent Networks, Princeton University Press, Princeton, 2010.
[17] J.M. Montenbruck, M. Bürger, F. Allgöwer, Practical synchronization with diffusive couplings, Automatica 53 (2015) 235–243.
[18] U. Münz, A. Papachristodoulou, F. Allgöwer, Delay robustness in consensus problems, Automatica 46 (8) (2010) 1252–1265.
[19] U. Münz, A. Papachristodoulou, F. Allgöwer, Delay robustness in non-identical multi-agent systems, IEEE Trans. Autom. Control 57 (6) (2012) 1597–1603.
[20] R. Olfati-Saber, R. Murray, Agreement problems in networks with direct graphs and switching topology, in: Proceedings of the 42nd CDC, Maui, Hawaii, 2003, pp. 4126–4132.
[21] R. Olfati-Saber, R. Murray, Consensus problems in networks of agents with switching topology and time-delays, IEEE Trans. Autom. Control 49 (9) (2004) 1520–1533.
[22] W. Ren, R. Beard, Consensus seeking in multiagent systems under dynamically changing interaction topologies, IEEE Trans. Auomt. Control 50 (5) (2005) 655–661.
[23] W. Ren, Y. Cao, Distributed coordination of multi-agent networks, Communications and Control Engineering, Springer-Verlag, London, 2011.
[24] J. Seo, H. Shim, J. Back, Consensus of high-order linear systems using dynamic output feedback compensator: low gain approach, Automatica 45 (11) (2009) 2659–2664.
[25] A. Stoorvogel, A. Saberi, Synchronization in an homogeneous, time-varying network with nonuniform time-varying communication delays, in: Proceedings of the 55th CDC, Las Vegas, NV, 2016, pp. 910–915.
[26] H. Su, M. Chen, Multi-agent containment control with input saturation on switching topologies, IET Control Theory Appl. 9 (3) (2015) 399–409.
[27] H. Su, M. Chen, G. Chen, Robust semi-global coordinated tracking of linear multi-agent systems with input saturation, Int. J. Robust Nonlinear Control 25 (14) (2015) 2375–2390.
[28] H. Su, M. Chen, J. Lam, Z. Lin, Semi-global leader-following consensus of linear multi-agent systems with input saturation via low gain feedback, IEEE Trans. Circ. Syst. I Reg. Pap. 60 (7) (2013) 1881–1889.
[29] H. Su, G. Jia, M. Chen, Semi-global containment control of multi-agent systems with intermittent input saturation, J. Frankl. Inst. 352 (9) (2015) 3504–3525.

[30] Y. Su, J. Huang, Stability of a class of linear switching systems with applications to two consensus problem, IEEE Trans. Autom. Control 57 (6) (2012) 1420–1430.
[31] K. Takaba, A dynamic protocol for local synchronization of linear multi-agent systems subject to input saturation, in: Proceedings of the 54th CDC, Osaka, Japan, 2015, pp. 4923–4927.
[32] Y.-P. Tian, C.-L. Liu, Consensus of multi-agent systems with diverse input and communication delays, IEEE Trans. Autom. Control 53 (9) (2008) 2122–2128.
[33] Y.-P. Tian, C.-L. Liu, Robust consensus of multi-agent systems with diverse input delays and asymmetric interconnection perturbations, Automatica 45 (5) (2009) 1347–1353.
[34] S. Tuna, LQR-based coupling gain for synchronization of linear systems, 2008. Available: arXiv:0801.3390v1.
[35] S. Tuna, Synchronizing linear systems via partial-state coupling, Automatica 44 (8) (2008) 2179–2184.
[36] S. Tuna, Conditions for synchronizability in arrays of coupled linear systems, IEEE Trans. Autom. Control 55 (10) (2009) 2416–2420.
[37] Q. Wang, C. Yu, H. Gao, Semiglobal synchronization of multiple generic linear agents with input saturation, Int. J. Robust Nonlinear Control 24 (18) (2014) 3239–3254.
[38] X. Wang, A. Saberi, A. Stoorvogel, H. Grip, T. Yang, Consensus in the network with uniform constant communication delay, Automatica 49 (8) (2013) 2461–2467.
[39] X. Wang, A. Saberi, A. Stoorvogel, H. Grip, T. Yang, Synchronization in a network of identical discrete-time agents with uniform constant communication delay, Int. J. Robust Nonlinear Control 24 (18) (2014) 3076–3091.
[40] X. Wang, H. Su, X. Wang, G. Chen, An overview of coordinated control for multi-agent systems subject to input saturation, Perspect. Sci. 7 (2016) 133–139.
[41] P. Wieland, J. Kim, F. Allgöwer, On topology and dynamics of consensus among linear high-order agents, Int. J. Syst. Sci. 42 (10) (2011) 1831–1842.
[42] C. Wu, Synchronization in Complex Networks of Nonlinear Dynamical Systems, World Scientific Publishing Company, Singapore, 2007.
[43] F. Xiao, L. Wang, Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays, IEEE Trans. Autom. Control 53 (8) (2008) 1804–1816.
[44] F. Xiao, L. Wang, Consensus protocols for discrete-time multi-agent systems with time-varying delays, Automatica 44 (10) (2008) 2577–2582.
[45] T. Yang, Z. Meng, D. Dimarogonas, K. Johansson, Global consensus for discrete–time multi-agent systems with input saturation constraints, Automatica 50 (2) (2014) 499–506.
[46] T. Yang, S. Roy, Y. Wan, A. Saberi, Constructing consensus controllers for networks with identical general linear agents, Int. J. Robust Nonlinear Control 21 (11) (2011) 1237–1256.
[47] L. Zhang, M. Chen, H. Su, Observer-based semi-global consensus of discrete–time multi-agent systems with input saturation, Trans. Inst. Meas. Control 38 (6) (2016) 665–674.
[48] M. Zhang, A. Saberi, A. Stoorvogel, Semiglobal state synchronization for continuous- or discrete-time multiagent systems subject to actuator saturation, Int. J. Robust Nonlinear Control 28 (16) (2018) 4966–4980.
[49] M. Zhang, A. Saberi, A. Stoorvogel, Synchronization in a network of identical continuous- or discrete-time agents with unknown nonuniform constant input delay, Int. J. Robust Nonlinear Control 28 (13) (2018) 3959–3973.