European Journal of Control 54 (2020) 73–86

Contents lists available at ScienceDirect

European Journal of Control

journal homepage: www.elsevier.com/locate/ejcon

Squared-down passivity-based H_∞ and H_2 almost synchronization of homogeneous continuous-time multi-agent systems with partial-state coupling via static protocol

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ARTICLE INFO

Article history:
Received 9 April 2019
Revised 27 September 2019
Accepted 23 November 2019
Available online 27 November 2019

Recommended by Prof. T Parisini

Keywords:
Synchronization in complex networks of dynamical systems
H_∞ and H_2 almost synchronization
Multi-agent systems

ABSTRACT

This paper studies H_∞ and H_2 almost state or output synchronization of homogeneous multi-agent systems (MAS) with partial-state coupling via static protocols in the presence of external disturbances. We provide solvability conditions for designing static protocols. We characterize three classes of agents for which we can design linear static protocols for state or output synchronization of a MAS such that the impact of disturbances on the network disagreement dynamics, expressed in terms of the H_∞ and H_2 norms of the corresponding closed-loop transfer function, is reduced to arbitrarily small value. Meanwhile, the static protocol only needs rough information on the network graph, that is a lower bound for the real part and an upper bound for the modulus of the non-zero eigenvalues of the Laplacian matrix associated with the network graph. Our study focuses on three classes of agents which are squared-down passive, squared-down passifiable via output feedback and squared-down minimum-phase with relative degree 1.

1. Introduction

The problem of synchronization among agents in a multi-agent system has received substantial attention in recent years, because of its potential applications in cooperative control of autonomous vehicles, distributed sensor network, swarming and flocking and others. The objective of synchronization is to secure an asymptotic agreement on a common state or output trajectory through decentralized control protocols (see [1,17,23,35] and references therein).

State synchronization basically requires homogeneous MAS (i.e. agents have identical dynamics). State synchronization based on diffusive partial-state coupling has been considered in many papers (e.g. see [12,14,26–28,32,33]). The case where the full state is shared over the network, will be referred to as full-state coupling. If only part of the state is shared over the network, we refer to it as partial-state coupling. Historically, the problem of partial-state coupling was first addressed by using an additional communication channel between the protocols of each agent relying on the same network topology (See for instance [14,26]). This extra communication is mathematically very convenient making the solvability condition weaker and the analysis simpler. However, from a practical point of view it is not very realistic. For a MAS via partial-state coupling, basically the synchronization is achieved via a dynamic protocol. However, state synchronization via a dynamic protocol imposes restrictions on the agent dynamics. Agents are assumed to be at most weakly unstable (all poles in the closed-left half plane) in e.g. [28,34] and references therein. Alternatively, agents are assumed to be at most weakly non-minimum-phase (all invariant zeros in closed left half plane) in e.g. [4,9,30,31,39] and references therein. The main drawback of dynamic protocols is that the modes of the protocol will be added to the synchronized trajectory and in general these modes are unstable and as such the existence of these modes on the consensus trajectory leads to unbounded synchronized trajectories. Therefore, static protocols are more desirable.

There have been research efforts to determine classes of agents for which synchronization is achievable via static protocols. It has been shown that designing static protocols for MAS with partial-state coupling is doable for the class of passive or passifiable agents. For example, [36] and [6] considered linear agents which are either passive or passifiable via state/output feedback while

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https://doi.org/10.1016/j.ejcon.2019.11.008
0947-3580/

[5] dealt with agents which are strictly G-passifiable via output feedback. In [11,21,22], input feed forward passivity was considered in connection with output synchronization. Nonlinear input-affine passive agents were considered in [3,29,37,41,43] while general nonlinear passive agents were studied in [10,18,38].

Most research works have focused on the idealized case where the agents are not affected by external disturbances. In the literature where external disturbances are considered, γ-suboptimal H_∞ design is developed for MAS to achieve H_∞ norm from an external disturbance to the synchronization error among agents less to a priori given γ. In particular, [14], [42] considered the H_∞ norm from an external disturbance to the output error among agents. [25] considered the H_∞ norm from an external disturbance to the state error among agents while [15] and [16] tried to obtain an H_∞ norm from the disturbance to the average of the states in a network of single or double integrators.

In the presence of disturbances, the concept of almost synchronization was also introduced to reduce the impact of external disturbances on the disagreement dynamics to an arbitrary small level. The notion of H_∞ almost synchronization for homogeneous MAS was first introduced in [19] for homogeneous MAS, where the goal is to reduce the H_∞ norm from an external disturbance to the synchronization *error*, to any arbitrary desired level. This work was extended later in [20,39,40].

So far, the literature studied almost synchronization of MAS with partial-state coupling utilizing dynamic protocols. In contrast, in this paper, we study H_∞ and H_2 almost output or state synchronization of homogeneous multi-agent systems with partial-state coupling via *static* protocol design for passifiable agents affected by external disturbances. We see that this problem reduces to a robust stabilization problem. The impact of disturbances on the network disagreement dynamics, expressed in terms of the H_∞ and H_2 norms of the corresponding closed-loop transfer function, is reduced to any arbitrarily small value. Our contribution of this paper is twofold.

- We provide the solvability conditions of H_∞ and H_2 almost output or state synchronization via static protocols.
- We identify three classes of agents, namely, squared-down passive, squared-down passifiable via output feedback, and squared-down minimum-phase with relative degree 1, for which designing static protocols is possible. We provide static protocol design for MAS with these three classes of agents.
- We also show that in the context of almost synchronization there are distinctions between utilizing H_∞ and H_2 norm.

The organization of this paper is as follows. In Section 2, we present preliminaries and in Section 3, we formulate our problems. The connection between almost synchronization and almost disturbance decoupling is presented in Section 4. The protocol design for H_∞ and H_2 almost synchronization via static protocol is proposed in Section 5. Finally, the numerical example is presented in Section 6.

2. Preliminaries

In this section we present some notations and definitions. Also, we will introduce the concept of squared-down passivity and passifiability.

2.1. Notations and definitions

Given a matrix $A \in \mathbb{R}^{m\times n}$, A^{T} denotes the transpose of A, and $\|A\|$ denotes the induced 2-norm of A. A square matrix A is said to be Hurwitz stable if all its eigenvalues are in the open left half complex plane. $A \otimes B$ depicts the Kronecker product between A and B. I_n denotes the n-dimensional identity matrix and 0_n denotes $n\times n$ zero matrix; we will use I or 0 if the dimension is clear from the context.

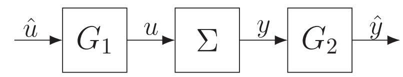

Fig. 1. A squared-down passive system.

A continuous-time system is called minimum-phase if all invariant zeros are in $\mathbb{C}^-$. A system is called weakly minimum-phase if all invariant zeros are in $\mathbb{C}^- \cap \mathbb{C}^0$ and all invariant zeros in $s \in \mathbb{C}^0$ are semi-simple. A system is called weakly non-minimum-phase if all invariant zeros are in $\mathbb{C}^- \cap \mathbb{C}^0$ and there exists an invariant zero which is not semi-simple.

A *weighted directed graph* $\mathcal{G}$ is defined by a triple $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, \dots, N\}$ is a node set, $\mathcal{E}$ is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the weighting matrix. We have $a_{ij} > 0$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. $(i, j) \in \mathcal{E}$ denotes an *edge* from node j to node i. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1, \dots, i_k\}$ such that $(i_{j+1}, i_j) \in \mathcal{E}$ for $j = 1, \dots, k-1$. A *directed tree* is a subgraph (subset of nodes and edges) in which every node has exactly one parent node except for one node, called the *root*, which has no parent node. In this case, the root has a directed path to every other node in the tree. A *directed spanning tree* is a directed tree containing all the nodes of the graph. For a weighted graph $\mathcal{G}$, a matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{\substack{k=1\\k\neq i}}^{N} a_{ik}, & i = j,\\ -a_{ij}, & i \neq j, \end{cases}$$

is called the *Laplacian matrix* associated with the graph $\mathcal{G}$. In the case where $\mathcal{G}$ has non-negative weights, L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $\mathbf{1}$ (a vector whose elements are all equal to 1). A specific class of graphs needed in this paper is presented below:

Definition 2.1. For any given $\alpha \geq \beta > 0$, let $\mathbb{G}^N_{\alpha,\beta}$ denote the set of directed graphs with N nodes that contain a directed spanning tree and for which the corresponding Laplacian matrix L satisfies $\|L\| < \alpha$ while its nonzero eigenvalues have a real part larger than or equal to β.

2.2. Squared-down passive and passifiable and squared-down minimum-phase with relative degree 1 systems

We introduce the concept of passivity and passifiability based on the idea of squaring-down in [24]. Consider a system

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu,\\ y = Cx, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. We introduce the following definitions:

Definition 2.2. A system (1) is called *squared-down passive* with a pre-compensator $G_1 \in \mathbb{R}^{m\times q}$ and a post-compensator $G_2 \in \mathbb{R}^{q\times p}$ if the interconnection in Fig. 1 with input $\hat{u}$ and output $\hat{y}$ is passive.

Remark 1. Assuming G_1 and G_2 are such that (A, BG_1) is stabilizable, (A, G_2C) is detectable while BG_1 and G_2C have full column- and row-rank, respectively, then squared-down passivity is equivalent to the existence of a positive definite matrix P, such that

$$\begin{aligned} &PA + A^{\mathrm{T}}P \leq 0,\\ &PBG_1 = C^{\mathrm{T}}G_2^{\mathrm{T}}. \end{aligned} \tag{2}$$

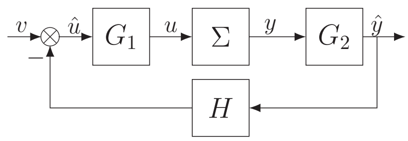

Fig. 2. A squared-down passive system via output feedback.

Remark 2. Note that when $G_1 = I$, squared-down passivity is reduced to G-passivity as used in [7]. For a square system, we can choose $G_1 = G_2 = I$ and squared-down passivity becomes conventional passivity.

Definition 2.3. A system (1) is called *squared-down passifiable via static output feedback* with a pre-compensator $G_1 \in \mathbb{R}^{m\times q}$ and a post-compensator $G_2 \in \mathbb{R}^{q\times p}$ if there exists an output feedback

$$\hat{u} = -H\hat{y} + v \tag{3}$$

which makes the system (1) squared-down passive with respect to the new input v, as shown in Fig. 2.

Remark 3. A system (1) is squared-down passifiable via static output feedback (3) if there exist a matrix H and a positive definite matrix P such that

$$\begin{aligned} P(A - BG_1HG_2C) + (A - BG_1HG_2C)^{\mathrm{T}}P \le 0,\\ PBG_1 = C^{\mathrm{T}}G_2^{\mathrm{T}}. \end{aligned} \tag{4}$$

This sufficient condition is also necessary for a system to be squared-down passifiable via static output feedback if (A, BG_1) is stabilizable, (A, G_2C) is detectable while BG_1 and G_2C have full column- and row-rank, respectively.

Now, we will define a class of agents, which are closely related to passive systems.

Definition 2.4. A system (1) is called *squared-down minimum-phase with relative degree 1* with a pre-compensator $G_1 \in \mathbb{R}^{m\times q}$ and a post-compensator $G_2 \in \mathbb{R}^{q\times p}$ if the square system (A, BG_1, G_2C) is minimum-phase with relative degree 1, i.e. $\det(G_2CBG_1) \neq 0$.

Remark 4. It is easy to show that if the system (1) is squared-down minimum-phase with relative degree 1, one can choose G_1 such that $G_2CBG_1 = I$.

Remark 5. It is known that squared-down minimum-phase with relative degree 1 agents are a subset of squared-down passifiable via output feedback agents (see [8,13] and references therein).

We introduce the following lemma which makes the structure of a system more explicit when the system is squared-down passifiable via static output feedback.

Lemma 1. *Consider system (1) and assume it is squared-down passifiable via static output feedback with compensator G_1 and G_2 and output feedback gain H as in Fig. 2, then for the system (A, BG_1, G_2C), with input $\hat{u}$, with $u = G_1\hat{u}$, and output $\hat{y} = G_2y$, there exist non-singular transformation matrices T_x, $T_{\hat{u}}$ and $T_{\hat{y}}$ with*

$$\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = T_x x, \qquad \tilde{u} = T_{\hat{u}}\hat{u}, \qquad \tilde{y} = T_{\hat{y}}\hat{y}$$

where $T_{\hat{y}} = (T_{\hat{u}}^{-1})^{\mathrm{T}}$, such that the dynamics of $\tilde{x}$ is represented by

$$\begin{aligned} \dot{\tilde{x}}_1 &= A_{11}\tilde{x}_1 + A_{12}\tilde{x}_2,\\ \dot{\tilde{x}}_2 &= A_{21}\tilde{x}_1 + A_{22}\tilde{x}_2 + \tilde{u},\\ \tilde{y} &= \tilde{x}_2, \end{aligned} \tag{5}$$

where $\tilde{x}_1 \in \mathbb{R}^{n-q}$ and $\tilde{x}_2 \in \mathbb{R}^q$. To be more specific, we have:

$$A_{11} = \begin{pmatrix} A_{11s} & 0 \\ 0 & A_{110} \end{pmatrix}, \qquad A_{12} = \begin{pmatrix} A_{121} \\ A_{122} \end{pmatrix}, \qquad A_{21} = \begin{pmatrix} A_{211} & A_{212} \end{pmatrix} \tag{6}$$

with:

$$A_{11s} + A_{11s}^{\mathrm{T}} < 0, \qquad A_{110} + A_{110}^{\mathrm{T}} = 0, \qquad A_{212} = -A_{122}^{\mathrm{T}}$$

Remark 6. If the system is squared-down passive, i.e. $H = 0$ in Figure 2, then we can additionally guarantee that

$$A_{22} + A_{22}^{\mathrm{T}} \le 0 \tag{7}$$

Proof. Obviously the system (A, BG_1, G_2C) is at most weakly non-minimum phase with relative degree 1. Note that there exists $P > 0$ such that (4) is satisfied. Choose a unitary matrix U such that

$$UP^{1/2}BG_1 = \bar{B} = \begin{pmatrix} 0 \\ \bar{B}_2 \end{pmatrix}$$

with $\bar{B}_2$ invertible which is possible since BG_1 is injective.

We first apply a state space transformation $\breve{x} = T_{x1}x$ with $T_{x1} = UP^{1/2}$ and we get:

$$\Sigma : \begin{cases} \dot{\breve{x}} = \bar{A}\breve{x} + \bar{B}\hat{u},\\ \hat{y} = \bar{C}\breve{x}, \end{cases}$$

where

$$(\bar{A} - \bar{B}H\bar{C}) + (\bar{A} - \bar{B}H\bar{C})^{\mathrm{T}} \le 0 \tag{8}$$

$$\bar{B} = \bar{C}^{\mathrm{T}}$$

We decompose $\bar{A}$ compatibly with $\bar{B}$:

$$\bar{A} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix}$$

Next, (8) implies that

$$\bar{A}_{11} + \bar{A}_{11}^{\mathrm{T}} \le 0$$

Choose a unitary matrix U_1 such that:

$$U_1\bar{A}_{11}U_1^{\mathrm{T}} = A_{11} = \begin{pmatrix} A_{11s} & 0 \\ 0 & A_{110} \end{pmatrix}$$

with $A_{11s} + A_{11s}^{\mathrm{T}} < 0$ and $A_{110} + A_{110}^{\mathrm{T}} = 0$. Then, it is easily verified that

$$T_x = \begin{pmatrix} U_1 & 0 \\ 0 & I \end{pmatrix} T_{x1}, \qquad T_u = \bar{B}_2$$

yields (5) and (6). Remains to verify that $A_{212} = -A_{122}^{\mathrm{T}}$. If we look at (8) then we get:

$$\begin{aligned} &\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} - \hat{H} \end{pmatrix} + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} - \hat{H} \end{pmatrix}^{\mathrm{T}} \\ &= \begin{pmatrix} A_{11s} + A_{11s}^{\mathrm{T}} & 0 & A_{121} + A_{211}^{\mathrm{T}} \\ 0 & 0 & A_{122} + A_{212}^{\mathrm{T}} \\ A_{121}^{\mathrm{T}} + A_{211} & A_{122}^{\mathrm{T}} + A_{212} & A_{22} + A_{22}^{\mathrm{T}} - \hat{H} - \hat{H}^{\mathrm{T}} \end{pmatrix} \le 0 \end{aligned} \tag{9}$$

where $\hat{H} = \bar{B}_2H\bar{B}_2^{\mathrm{T}}$ from which it is immediately clear that we must have $A_{212} = -A_{122}^{\mathrm{T}}$. □

3. Problem Formulation

Consider a MAS composed of N identical linear time-invariant agents of the form,

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i + E\omega_i,\\ y_i &= Cx_i, \end{aligned} \qquad (i = 1, \dots, N) \tag{10}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are respectively the state, input, and output vectors of agent i, and $\omega_i \in \mathbb{R}^r$ is the external disturbance.

The communication network provides each agent with a linear combination of its own outputs relative to that of other neighboring agents. In particular, each agent $i \in \{1, \dots, N\}$ has access to the quantity,

$$\zeta_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j) = \sum_{j=1}^{N} \ell_{ij} y_j. \tag{11}$$

The communication topology of the network can be described by a weighted and directed graph $\mathcal{G}$ with corresponding Laplacian matrix L. We will primarily focus on *partial-state coupling* where C does not have full-column rank.

If the graph $\mathcal{G}$ describing the communication topology of the network contains a directed spanning tree, then it follows from [2] that the Laplacian matrix L has a simple eigenvalue at the origin, with the corresponding right eigenvector $\mathbf{1}$ and all the other eigenvalues are in the open right-half complex plane. Let $\lambda_1, \dots, \lambda_N$ denote the eigenvalues of L such that $\lambda_1 = 0$ and $\mathrm{Re}(\lambda_i) > 0$, $i = 2, \dots, N$.

Let N be any agent and define

$$\bar{x}_i = x_N - x_i, \quad \bar{u}_i = u_N - u_i \text{ and } \bar{y}_i = y_N - y_i$$

and

$$\bar{x} = \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_{N-1} \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} \bar{u}_1 \\ \vdots \\ \bar{u}_{N-1} \end{pmatrix}, \quad \bar{y} = \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_{N-1} \end{pmatrix} \text{ and } \omega = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_N \end{pmatrix}.$$

Obviously, state synchronization is achieved if

$$\lim_{t\to\infty} (x_i(t) - x_N(t)) = 0, \quad \forall i \in \{1, \dots, N-1\}, \tag{12}$$

and output synchronization is achieved if

$$\lim_{t\to\infty} (y_i(t) - y_N(t)) = 0, \quad \forall i \in \{1, \dots, N-1\}. \tag{13}$$

Remark 7. The agent N is not necessarily a root agent. Obviously, (12) is equivalent to the condition that

$$\lim_{t\to\infty} (x_i(t) - x_j(t)) = 0$$

for all $i, j \in \{1, \dots, N\}$ and a similar connection holds for (13).

We formulate below H_∞ or H_2 almost state/output synchronization problems.

Problem 1. Consider a MAS described by (10) and (11). Let $\mathbf{G}$ be a given set of graphs such that $\mathbf{G} \subseteq \mathbb{G}^N$. The H_∞ *almost state synchronization problem via static protocol* with a set of network graphs $\mathbf{G}$ is to find, if possible, a linear static protocol parameterized in terms of a parameter ε, of the form

$$u_i = F_\varepsilon \zeta_i, \qquad i = 1, \dots N \tag{14}$$

such that, for any given real number $\delta > 0$, there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$ and for any graph $\mathcal{G} \in \mathbf{G}$, (12) is satisfied for all initial conditions in the absence of disturbances and the closed loop transfer matrix $T_{\omega\bar{x}}$ from ω to $\bar{x}$ satisfies

$$\|T_{\omega\bar{x}}\|_\infty < \delta. \tag{15}$$

Problem 2. Consider a MAS described by (10) and (11). Let $\mathbf{G}$ be a given set of graphs such that $\mathbf{G} \subseteq \mathbb{G}^N$. The H_2 *almost state synchronization problem via static protocol* with a set of network graphs $\mathbf{G}$ is to find, if possible, a linear static protocol parameterized in terms of a parameter ε, of the form

$$u_i = F_\varepsilon \zeta_i, \qquad i = 1, \dots N \tag{16}$$

such that, for any given real number $\delta > 0$, there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$ and for any graph $\mathcal{G} \in \mathbf{G}$, (12) is satisfied for all initial conditions in the absence of disturbances and the closed loop transfer matrix $T_{\omega\bar{x}}$ from ω to $\bar{x}$ satisfies

$$\|T_{\omega\bar{x}}\|_2 < \delta. \tag{17}$$

Remark 8. It is worth to note that the notion of almost state synchronization is stronger than almost output synchronization. Therefore, Problems 1 and 2 imply H_∞ and H_2 almost output synchronizations as stated in the following problems.

Note that in the case of almost output synchronization, it is very appealing to ensure that internal states do not explode if we try to achieve higher accuracy with regard to output synchronization. Hence in the following problems we imposed an upper bound on the effect of disturbances on the state.

Problem 3. Consider a MAS described by (10) and (11). Let $\mathbf{G}$ be a given set of graphs such that $\mathbf{G} \subseteq \mathbb{G}^N$. The H_∞ *almost output synchronization problem via static protocol* with a set of network graphs $\mathbf{G}$ is to find, if possible, a linear static protocol parameterized in terms of a parameter ε, of the form (16), such that, for any given real number $\delta > 0$, there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$ and for any graph $\mathcal{G} \in \mathbf{G}$, (13) is satisfied for all initial conditions in the absence of disturbances and the closed loop transfer matrix $T_{\omega\bar{y}}$ (from ω to $\bar{y}$) satisfies

$$\|T_{\omega\bar{y}}\|_\infty < \delta \tag{18}$$

We can similarly define the H_∞ *almost output synchronization problem with bounded state errors via static protocol* when

$$\|T_{\omega\bar{x}}\|_\infty < M, \qquad \textit{and} \qquad \|T_{\omega\bar{y}}\|_\infty < \delta \tag{19}$$

with M independent of ε.

Problem 4. Consider a MAS described by (10) and (11). Let $\mathbf{G}$ be a given set of graphs such that $\mathbf{G} \subseteq \mathbb{G}^N$. The H_2 *almost output synchronization problem via static protocol* with a set of network graphs $\mathbf{G}$ is to find, if possible, a linear static protocol parameterized in terms of a parameter ε, of the form (16), such that, for any given real number $\delta > 0$, there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$ and for any graph $\mathcal{G} \in \mathbf{G}$, (13) is satisfied for all initial conditions in the absence of disturbances and the closed loop transfer matrix $T_{\omega\bar{y}}$ (from ω to $\bar{y}$) satisfies

$$\|T_{\omega\bar{y}}\|_2 < \delta \tag{20}$$

We can similarly define the H_2 *almost output synchronization problem with bounded state errors via static protocol* when

$$\|T_{\omega\bar{x}}\|_2 < M, \qquad \textit{and} \qquad \|T_{\omega\bar{y}}\|_2 < \delta \tag{21}$$

with M independent of ε.

4. Connection between H_∞ and H_2 almost state/output synchronization and H_∞ and H_2 almost disturbance decoupling-solvability conditions

In this section, we establish a connection between the problem of H_∞ or H_2 almost state/output synchronization among agents in the network and a robust H_∞ or H_2 almost state/output disturbance decoupling problem via static output feedback with internal stability.

4.1. Analysis of H_∞ almost synchronization

Given $\Lambda \subset \mathbb{C}$, we introduce the following system associated to agent models as

$$\begin{aligned} \dot{x} &= Ax + \lambda Bu + Ed, \\ y &= Cx, \end{aligned} \tag{22}$$

where $\lambda \in \Lambda$, and A, B, C, E matrices are the same as agent models (10) and d is the disturbance. The *robust H_∞ almost output disturbance decoupling problem with bounded input via static output feedback* for (22) is to find, if possible, a parameterized controller

$$u = F_\varepsilon y \tag{23}$$

and $M > 0$ such that, for any given $\delta > 0$, there exists $\varepsilon^* > 0$ for which the interconnection of (23) and the system (22), has the property that for any $\lambda \in \Lambda$ and for any $0 < \varepsilon < \varepsilon^*$ we have:

1. The interconnection of the systems (23) and (22) is internally stable.
2. The resulting closed-loop transfer function
$$T_{dy}^{\lambda} = C(sI - A - \lambda BF_\varepsilon C)^{-1}E \tag{24}$$
from d to y has an H_∞ norm less than δ.
3. The resulting closed-loop transfer function
$$T_{uy}^{\lambda} = C(sI - A - \lambda BF_\varepsilon C)^{-1}B \tag{25}$$
has an H_∞ norm less than δ.
4. The resulting closed-loop transfer function
$$T_{du}^{\lambda} = F_\varepsilon C(sI - A - \lambda BF_\varepsilon C)^{-1}E \tag{26}$$
from d to u has an H_∞ norm less than M.
5. The resulting closed-loop transfer function
$$T_{uu}^{\lambda} = F_\varepsilon C(sI - A - \lambda BF_\varepsilon C)^{-1}B \tag{27}$$
has an H_∞ norm less than M.

It is important to note that M is independent of the choice for δ and independent of $\lambda \in \Lambda$.

The *robust H_∞ almost state disturbance decoupling problem with bounded input via static output feedback* for (22) is equivalent to the above with the only modification being that instead of (24) and (25), the closed-loop transfer functions

$$T_{dx}^{\lambda} = (sI - A - \lambda BF_\varepsilon C)^{-1}E \tag{28}$$

$$T_{ux}^{\lambda} = (sI - A - \lambda BF_\varepsilon C)^{-1}B \tag{29}$$

both have an H_∞ norm less than δ.

The next lemma establishes a connection between H_∞ almost output disturbance decoupling problem and H_∞ almost output synchronization problem.

Lemma 2. *Let* $\mathbf{G}$ *be a set of graphs such that the associated Laplacian matrices are uniformly bounded and let* Λ *consist of all possible nonzero eigenvalues of Laplacian matrices associated with graphs in* $\mathbf{G}$.

The H_∞ almost output synchronization problem with bounded state errors via static protocol for the MAS described by (10) and (11) given $\mathbf{G}$ is solved by a parameterized protocol $u_i = F_\varepsilon \zeta_i$ if the robust H_∞ almost output disturbance decoupling problem with bounded input via static output feedback for the system (22) with $\lambda \in \Lambda$ is solved by the parameterized controller $u = F_\varepsilon y$.

Proof. The MAS system described by (10) and (11) after implementing the linear static protocol (16) is described by

$$\begin{aligned} \dot{x}_i &= Ax_i + BF_\varepsilon \zeta_i + E\omega_i, \\ y_i &= Cx_i \end{aligned}$$

for $i = 1, \dots, N$. Let

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}, \quad \omega = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_N \end{pmatrix}.$$

Then, the overall dynamics of the N agents can be written as

$$\dot{x} = (I_N \otimes A + L \otimes BF_\varepsilon C)x + (I_N \otimes E)\omega. \tag{30}$$

Note that the Laplacian matrix L has eigenvalue 0 with associated right eigenvector $\mathbf{1}$. Let

$$L = TS_L T^{-1}, \tag{31}$$

with T unitary and $S_L = [s_{ij}]$ the upper-triangular Schur form associated to the Laplacian matrix L such that $s_{11} = 0$. Let

$$\xi := (T^{-1} \otimes I_n)x = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_N \end{pmatrix}, \qquad \bar{\omega} = (T^{-1} \otimes I_r)\omega = \begin{pmatrix} \bar{\omega}_1 \\ \vdots \\ \bar{\omega}_N \end{pmatrix}$$

where $\xi_i \in \mathbb{C}^n$ and $\bar{\omega}_i \in \mathbb{C}^r$. In the new coordinates, the dynamics of ξ can be written as

$$\dot{\xi}(t) = (I_N \otimes A + S_L \otimes BF_\varepsilon C)\xi + (T^{-1} \otimes E)\omega, \tag{32}$$

which is rewritten as

$$\begin{aligned} \dot{\xi}_1 &= A\xi_1 + \sum_{j=2}^{N} s_{1j} BF_\varepsilon C\xi_j + E\bar{\omega}_1, \\ \dot{\xi}_i &= (A + \lambda_i BF_\varepsilon C)\xi_i + \sum_{j=i+1}^{N} s_{ij} BF_\varepsilon C\xi_j + E\bar{\omega}_i, \\ \dot{\xi}_N &= (A + \lambda_N BF_\varepsilon C)\xi_N + E\bar{\omega}_N, \end{aligned} \tag{33}$$

for $i \in \{2, \dots, N-1\}$. The first column of T is an eigenvector of L associated to eigenvalue 0 with length 1, i.e. it is equal to $\mathbf{1}/\sqrt{N}$. Using this we obtain:

$$\bar{x} = \begin{pmatrix} x_N - x_1 \\ x_N - x_2 \\ \vdots \\ x_N - x_{N-1} \end{pmatrix} = \left(\begin{pmatrix} -1 & 0 & \cdots & 0 & 1 \\ 0 & -1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix} \otimes I_n \right) (T \otimes I_n)\xi$$
$$= \left(\begin{pmatrix} 0 & V \end{pmatrix} \otimes I_n \right)\xi,$$

for some suitably chosen matrix $V \in \mathbb{R}^{N \times (N-1)}$. Therefore we have

$$\bar{y} = (V \otimes I) \begin{pmatrix} \eta_2 \\ \vdots \\ \eta_N \end{pmatrix}, \qquad \bar{u} = (V \otimes I) \begin{pmatrix} \nu_2 \\ \vdots \\ \nu_N \end{pmatrix} \tag{34}$$

where

$$\eta_i = C\xi_i, \quad \nu_i = F_\varepsilon \eta_i, \ \text{for } i = 2, \dots, N.$$

Note that since T is unitary, also the matrix T^{-1} is unitary and the matrix V is uniformly bounded. Therefore the H_∞ norm of the transfer matrix from ω to $\bar{y}$ can be made arbitrarily small if and only if the H_∞ norm of the transfer matrix from $\bar{\omega}$ to η_i can be made arbitrarily small for $i = 2, \dots, N$. Similarly, the H_∞ norm of the transfer matrix from $\bar{\omega}$ to $\bar{u}$ is bounded if and only if the H_∞ norm of the transfer matrix from $\bar{\omega}$ to ν_i is bounded for $i = 2, \dots, N$.

The fact that $u = F_\varepsilon y$ solves the simultaneous H_∞ almost output disturbance decoupling problem with bounded input of (22) implies that for small ε we have that $A + \lambda BF_\varepsilon C$ is asymptotically stable for all $\lambda \in \Lambda$. In particular, $A + \lambda_i BF_\varepsilon C$ is asymptotically stable for $i = 2, \dots, N$ which guarantees that $\xi_i \to 0$ for $i = 2, \dots, N$ for zero disturbances and all initial conditions. Therefore we have H_∞ almost output synchronization.

Next, we are going to show that there exists $\bar{M} > 0$ such that for any $\bar{\delta} > 0$, we can choose ε sufficiently small such that the H_∞ norm of the transfer matrix from $\bar{\omega}$ to ξ_i is less than $\bar{\delta}$ and the H_∞ norm of the transfer matrix from $\bar{\omega}$ to ν_i is less than $\bar{M}$ for $i = 2, \dots, N$. This would guarantee that we can find $M > 0$ such that

$$\|T_{\omega\bar{x}}\|_\infty < \delta, \qquad \|T_{\omega\bar{u}}\|_\infty < M \tag{35}$$

for any $\delta > 0$ provided ε is small enough.

Since the robust H_∞ almost output disturbance decoupling problem with bounded input via static output feedback is solved by (23), there exists $\tilde{M}$ such that for any arbitrarily small $\tilde{\delta}$, we have for ε small enough that:

$$\|T_{dy}^{\lambda}\|_\infty < \tilde{\delta}, \qquad \|T_{uy}^{\lambda}\|_\infty < \tilde{\delta},$$
$$\|T_{du}^{\lambda}\|_\infty < \tilde{M}, \qquad \|T_{uu}^{\lambda}\|_\infty < \tilde{M},$$

for all $\lambda \in \Lambda$ where T_{dy}^{λ}, T_{uy}^{λ}, T_{du}^{λ} and T_{uu}^{λ}, are given by (24), (25), (26) and (27), respectively.

When $i = N$, it is easy to find that,

$$T_{\bar{\omega}\eta_N} = e_N \otimes T_{dy}^{\lambda_N}, \qquad T_{\bar{\omega}\nu_N} = e_N \otimes T_{du}^{\lambda_N},$$

where e_i is a row vector of dimension N with elements equal to zero except for the ith component which is equal to 1. Hence

$$\|T_{\bar{\omega}\eta_N}\|_\infty < \tilde{\delta}, \qquad \|T_{\bar{\omega}\nu_N}\|_\infty < \bar{M}_N$$

provided

$$\tilde{\delta} < \bar{\delta}, \qquad \tilde{M} < \bar{M}_N. \tag{36}$$

Recall that we can make $\tilde{\delta}$ arbitrarily small without affecting the bound $\tilde{M}$. Assume

$$\|T_{\bar{\omega}\eta_j}\|_\infty < \bar{\delta}, \qquad \|T_{\nu_j}\|_\infty < \bar{M}_j$$

holds for $j = i+1, \ldots, N$. We have:

$$T_{\bar{\omega}\eta_i} = e_i \otimes T_{\omega y}^{\lambda_i} + \sum_{j=i+1}^{N} s_{ij} T_{uy}^{\lambda_i} T_{\bar{\omega}\nu_j} \tag{37}$$

$$T_{\bar{\omega}\nu_i} = e_i \otimes T_{\omega u}^{\lambda_i} + \sum_{j=i+1}^{N} s_{ij} T_{uu}^{\lambda_i} T_{\bar{\omega}\nu_j}. \tag{38}$$

Since

$$\left\| e_i \otimes T_{\omega y}^{\lambda_i} + \sum_{j=i+1}^{N} s_{ij} T_{uy}^{\lambda_i} T_{\bar{\omega}\nu_j} \right\|_\infty < \tilde{\delta} + \sum_{j=i+1}^{N} |s_{ij}| \tilde{\delta} \bar{M}_j \tag{39}$$

and

$$\left\| e_i \otimes T_{\omega u}^{\lambda_i} + \sum_{j=i+1}^{N} s_{ij} T_{uu}^{\lambda_i} T_{\bar{\omega}\nu_j} \right\|_\infty < \tilde{M} + \sum_{j=i+1}^{N} |s_{ij}| \tilde{M} \bar{M}_j \tag{40}$$

we find:

$$\|T_{\bar{\omega}\eta_i}\|_\infty < \bar{\delta}, \qquad \|T_{\bar{\omega}\nu_i}\|_\infty < \bar{M}_i \tag{41}$$

provided:

$$\begin{aligned} \tilde{\delta} + \sum_{j=i+1}^{N} |s_{ij}| \tilde{\delta} \bar{M}_j &< \bar{\delta}, \\ \tilde{M} + \sum_{j=i+1}^{N} |s_{ij}| \tilde{M} \bar{M}_j &< \bar{M}_i. \end{aligned} \tag{42}$$

Note that s_{ij} depends on the graph in $\mathbb{G}$ but since the Laplacian matrices associated to graphs in $\mathbb{G}$ are uniformly bounded we find that also the s_{ij} are uniformly bounded. In this way we can recursively obtain the bounds in (41) for $i = 2, \ldots, N$ provided we choose ε sufficiently small such that the corresponding $\tilde{\delta}$ satisfies (36) and (42) for $i = 2, \ldots, N-1$.

If we define:

$$\bar{\xi} = \begin{pmatrix} \xi_2 \\ \vdots \\ \xi_N \end{pmatrix}, \qquad \bar{\eta} = \begin{pmatrix} \eta_2 \\ \vdots \\ \eta_N \end{pmatrix}, \qquad \bar{\nu} = \begin{pmatrix} 0 \\ \nu_2 \\ \vdots \\ \nu_N \end{pmatrix}$$

then we have

$$\dot{\bar{\xi}}(t) = (I_{N-1} \otimes A)\bar{\xi} + (RS_L \otimes B)\bar{\nu} + (R \otimes E)\bar{\omega},$$
$$\bar{\eta} = (I_{N-1} \otimes C)\bar{\xi}$$

where

$$R = \begin{pmatrix} 0 & I_{N-1} \end{pmatrix}$$

We obtain:

$$\dot{\bar{\xi}}(t) = [I_{N-1} \otimes (A - KC)]\bar{\xi} + (RS_L \otimes B)\bar{\nu} + (R \otimes E)\bar{\omega} + K\bar{\eta}$$

where K is an arbitrary matrix such that $A - KC$ is asymptotically stable. Clearly, the H_∞ norms from $\bar{\omega}$ to $\bar{\nu}$ and $\bar{\eta}$ are bounded (with bounds independent of graph or ε). Hence the H_∞ norm from $\bar{\omega}$ to $\bar{\xi}$ is uniformly bounded, i.e. the H_∞ norm from $\bar{\omega}$ to ξ_i is less than $\bar{M}$ for $i = 2, \ldots, N$ for some suitably chosen constant $\bar{M}$ independent of ε and the specific graph.

Hence, we can choose ε sufficiently small such that the H_∞ norm from $\bar{\omega}$ to ν_i is less than $\bar{\delta}$ and the H_∞ norm from $\bar{\omega}$ to ξ_i is less than $\bar{M}$ for $i = 2, \ldots, N$. As noted before this guarantees that we can achieve (35) for a fixed M and any arbitrarily small $\delta > 0$. □

The next lemma establishes a similar connection between the H_∞ almost state disturbance decoupling problem and the H_∞ almost state synchronization problem.

Lemma 3. *Let* **G** *be a set of graphs such that the associated Laplacian matrices are uniformly bounded and let Λ consist of all possible nonzero eigenvalues of Laplacian matrices associated with graphs in* **G**.

The H_∞ almost state synchronization problem via static protocol for the MAS described by (10) and (11) given **G** *is solved by a parameterized protocol $u_i = F_\varepsilon \zeta_i$ if the robust H_∞ almost state disturbance decoupling problem with bounded input via static output feedback for the system (22) with $\lambda \in \Lambda$ is solved by the parameterized controller $u = F_\varepsilon y$.*

Proof. The proof is completely similar to the proof of Lemma 2. □

We will also analyze whether we can keep the H_∞ norm from ω to $\bar{y}$ bounded if we cannot achieve an arbitrarily small error. We can find a similar connection as above to a robust control problem:

Lemma 4. *Let* **G** *be a set of graphs such that the associated Laplacian matrices are uniformly bounded and let Λ consist of all possible nonzero eigenvalues of Laplacian matrices associated with graphs in* **G**.

Given a MAS described by (10) and (11) and a set of graphs **G**. *Let Λ denotes all possible locations for the nonzero eigenvalues of the Laplacian matrix L when the graph varies over the set* **G**.

For a parameterized protocol $u_i = F_\varepsilon \zeta_i$, there exists $\tilde{M}$ such that when applied to the MAS the H_∞ norm from ω to $\bar{y}$ is less than $\tilde{M}$ for all $\varepsilon > 0$ and for any graph in **G** *if for the parameterized controller $u = F_\varepsilon y$, there exists M such that for any δ, we have for ε small enough that*

1. *The interconnection of the systems (23) and (22) is internally stable.*
2. *The resulting closed-loop transfer function*
$$T_{dy}^{\lambda} = C(sI - A - \lambda BF_\varepsilon C)^{-1}E \tag{43}$$
from d to y has an H_∞ norm less than M.
3. *The resulting closed-loop transfer function*
$$T_{uy}^{\lambda} = C(sI - A - \lambda BF_\varepsilon C)^{-1}B \tag{44}$$
has an H_∞ norm less than δ.
4. *The resulting closed-loop transfer function*
$$T_{du}^{\lambda} = F_\varepsilon C(sI - A - \lambda BF_\varepsilon C)^{-1}E \tag{45}$$
from d to u has an H_∞ norm less than $\frac{M}{\delta}$.

5. *The resulting closed-loop transfer function*

$$T_{uu}^{\lambda} = F_{\varepsilon}C(sI - A - \lambda BF_{\varepsilon}C)^{-1}B \tag{46}$$

has an H_∞ norm less than M.

for all $\lambda \in \Lambda$.

Proof. The proof relies on the same recursive argument as in the proof of Lemma 2. □

4.2. Analysis of H_2 Almost Synchronization

In this subsection, we consider the H_2 norm instead of the H_∞ norm. Firstly, we define the robust H_2 almost output disturbance decoupling problem via static output feedback for (22) as follows. There should exist a parameterized controller (23) and $M > 0$ such that, for any given $\delta > 0$, there exists $\varepsilon^* > 0$ for which the interconnection of (23) and the system (22) has the property that for any $\lambda \in \Lambda$ and for any $0 < \varepsilon < \varepsilon^*$ we have:

1. The interconnection of the systems (22) and (23) is internally stable;
2. The resulting closed-loop transfer function T_{dy}^{λ} from d to y has an H_2 norm less than $\sqrt{\delta}$.
3. The resulting closed-loop transfer function T_{uy}^{λ} has an H_∞ norm less than δ.
4. The resulting closed-loop transfer function T_{du}^{λ} from d to u has an H_2 norm less than $M/\sqrt{\delta}$.
5. The resulting closed-loop transfer function T_{uu}^{λ} has an H_∞ norm less than M.

It is important to note that M is independent of the choice for δ and independent of $\lambda \in \Lambda$.

The *robust H_2 almost state disturbance decoupling problem via static output feedback* for (22) is equivalent to the above with the only modification being that instead of items 2 and 3, T_{dx}^{λ} has an H_2 norm less than $\sqrt{\delta}$ and T_{ux}^{λ} has an H_∞ norm less than δ where these transfer functions are given by (28) and (29).

Remark 9. In the above problem, note that we need to consider two aspects in our controller, H_2 disturbance rejection and robust stabilization. The latter translates in the H_∞ norm constraints.

Next, we present the H_2 equivalent of Lemmas 2 and 3.

Lemma 5. *Let* **G** *be a set of graphs such that the associated Laplacian matrices are uniformly bounded and let Λ consist of all possible nonzero eigenvalues of Laplacian matrices associated with graphs in* **G***.*

The H_2 almost output synchronization problem via static protocol for the MAS described by (10) and (11) given **G** *is solved by a parameterized protocol $u_i = F_{\varepsilon}\zeta_i$ if the robust H_2 almost output disturbance decoupling problem via static output feedback for the system (22) with $\lambda \in \Lambda$ is solved by the parameterized controller $u = F_{\varepsilon}y$.*

Proof. The proof is similar to the proof of Lemma 2. The proof has the same structure. There is one step we need to be careful. The proof follows the same lines of Lemma 2 except that we require the $\bar{\omega}$ to η_j arbitrarily small while we keep the H_∞ norm from $\bar{\omega}$ to ν_j bounded. Recall that for two stable transfer matrices T_1 and T_2 with T_1 strictly proper we have:

$$\|T_1T_2\|_2 \le \|T_1\|_2\|T_2\|_\infty, \tag{47}$$

and therefore we obtain for (37) that

$$\|T_{\bar{\omega}\eta_i}\|_2 \le \|T_{dy}^{\lambda_i}\|_2 + \sum_{j=i+1}^{N} s_{ij}\|T_{uy}^{\lambda_i}\|_\infty \|T_{\bar{\omega}\nu_j}\|_2$$

and

$$\|T_{\bar{\omega}\nu_i}\|_2 \le \|T_{du}^{\lambda_i}\|_2 + \sum_{j=i+1}^{N} s_{ij}\|T_{uu}^{\lambda_i}\|_\infty \|T_{\bar{\omega}\nu_j}\|_2$$

We use these two inequalities instead of (39) and (40), respectively. Given that we have a parameterized static feedback which solves the robust H_2 almost output disturbance decoupling problem, we find constants $\bar{N}_i$ and $\bar{M}_i$ such that

$$\|T_{\bar{\omega}\eta_i}\|_2 < \bar{N}_i\sqrt{\delta}, \qquad \|T_{\bar{\omega}\nu_i}\|_2 < \frac{\bar{M}_i}{\sqrt{\delta}}$$

□

Lemma 6. *Let* **G** *be a set of graphs such that the associated Laplacian matrices are uniformly bounded and let Λ consist of all possible nonzero eigenvalues of Laplacian matrices associated with graphs in* **G***.*

The H_2 almost state synchronization problem via static protocol for the MAS described by (10) and (11) given **G** *is solved by a parameterized protocol $u_i = F_{\varepsilon}\zeta_i$ if the robust H_2 almost state disturbance decoupling problem via static output feedback for the system (22) with $\lambda \in \Lambda$ is solved by the parameterized controller $u = F_{\varepsilon}y$.*

Proof. The proof is basically identical to the proof of Lemma 5. □

5. Protocol design for H_∞ and H_2 almost synchronization

In this section, we will consider a static protocol design to achieve H_∞ and H_2 almost synchronization.

5.1. H_∞ *almost synchronization*

We consider a MAS described by (10) and (11). We split the protocol design for H_∞ almost state and output synchronization into two cases. The first case considers agents which are squared-down passifiable via output feedback given G_1, G_2 and H. Clearly this class of agents included squared-down passive agents as a special case. The second case is squared-down minimum-phase with relative degree 1 agents which as stated in Remark 5 is a subset of squared-down passifiable via output feedback agents.

5.1.1. Squared-down passifiable via output feedback agents

First by providing the following example, we will show that H_∞ almost state synchronization, in general, is not solvable via static protocol for the class of passifiable via output feedback agents. However, the H_∞ almost output synchronization problem is solvable.

Example 1. Consider the MAS with two agents

$$\dot{x}_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x_i + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_i + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \omega_i$$

$$y_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x_i \tag{48}$$

for $i = 1, 2$. The communication graph has the associated Laplacian matrix

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The system is squared-down passifiable with $G_1 = 1$ and $G_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$. If we consider an arbitrary static protocol

$$u_i = F\zeta_i = \begin{pmatrix} f_1 & f_2 \end{pmatrix}\zeta_i$$

the transfer matrix from ω_1 to $x_1 - x_2$ is given by

$$T_{\omega_1\bar{x}} = \left(sI - \begin{pmatrix} 0 & 1 \\ f_1 - 1 & f_2 \end{pmatrix}\right)^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{s(s-f_2)+1-f_1}\begin{pmatrix} s - f_2 \\ f_1 - 1 \end{pmatrix}$$

where $f_1 < 1$ and $f_2 < 0$, and

$$T_{\omega_1 \tilde{x}}(0) = \begin{pmatrix} f_2/f_1 - 1 \\ -1 \end{pmatrix}$$

Clearly H_∞ almost state synchronization is not possible.

As it is shown in the above example, the H_∞ almost state synchronization via static protocol is not possible as stated in Problem 1. Now we focus on a weaker notion of H_∞ almost synchronization, namely, H_∞ almost output synchronization of MAS with squared-down passifiable via output feedback agents.

Problem 3 formulated earlier was in terms of an arbitrary set of graphs $\mathbf{G}$. The results in this section are obtained for specific classes of graphs where:

$$\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$$

for some α, $\beta > 0$ which has been defined in Definition 2.1. For all the problems in this paper we consider the same parameterized protocol

$$u_i = -\frac{1}{\varepsilon} G_1 G_2 \zeta_i, \tag{49}$$

Our next result regarding H_∞ almost output synchronization problem via static protocol is stated as follows.

Theorem 1. *Consider a MAS described by (10) and (11). Assume (A, B, C) is squared-down passifiable with respect to G_1, G_2 and H, such that (A, BG_1) is stabilizable, (A, G_2C) is detectable while BG_1 and G_2C have full column- and row-rank, respectively. Let any real numbers α, $\beta > 0$ and a positive integer N be given, and hence a set of network graphs $\mathbb{G}^N_{\alpha,\beta}$ be defined.*

The H_∞ almost output synchronization problem with bounded state errors via static protocol as defined in Problem 3 with respect to the compensated output $\hat{y} = G_2 y$ where $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ is solvable if

$$\operatorname{Im} E \subseteq \operatorname{Im} BG_1. \tag{50}$$

In particular, for any given real number $\delta > 0$, there exists an ε^*, such that for any $\varepsilon \in (0, \varepsilon^*)$, the protocol (49) achieves state synchronization and the H_∞ norm from ω to $\hat{y}_i - \hat{y}_j$ less than δ and the H_∞ norm from ω to $x_i - x_j$ less than M for any $i, j \in 1, \ldots, N$ and for any graph $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$.

The above theorem states that for squared-down passifiable agents we can achieve H_∞ almost output synchronization with respect to the compensated output $\hat{y} = G_2 y$ if (50) is satisfied. The following remark shows that this is no longer valid if we use the original output.

Remark 10. Consider the MAS (48) with the same communication network as stated in Example 1. The transfer matrix from ω_1 to $y_1 - y_2$ is given by

$$T_{\omega_1 \tilde{y}} = \left(sI - \begin{pmatrix} 0 & 1 \\ f_1 - 1 & f_2 \end{pmatrix}\right)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{s(s-f_2)+1-f_1} \begin{pmatrix} s - f_2 \\ f_1 - 1 \end{pmatrix}$$

where $f_1 < 1$ and $f_2 < 0$, and

$$T_{\omega_1 \tilde{y}}(0) = \begin{pmatrix} f_2/f_1 - 1 \\ -1 \end{pmatrix}$$

Clearly H_∞ almost output synchronization is not possible with respect to the original output y.

Proof of Theorem 1. Given Lemma 2, we only need to verify that $u = -\rho G_1 G_2 y$ where $\rho = \frac{1}{\varepsilon}$ solves the robust H_∞ almost output disturbance decoupling problem with bounded input via static output feedback for the system (22) with $\lambda \in \Lambda$. Given $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$, we know that $\lambda \in \Lambda$ implies $\operatorname{Re} \lambda \geq \beta$.

Using Lemma 1, the dynamics of (22) with pre/post compensator G_1, G_2, i.e. (A, BG_1, G_2C), can be written as:

$$\begin{aligned} \dot{\tilde{x}}_1 &= A_{11}\tilde{x}_1 + A_{12}\tilde{x}_2 + E_1\omega, \\ \dot{\tilde{x}}_2 &= A_{21}\tilde{x}_1 + A_{22}\tilde{x}_2 + \lambda\tilde{u} + E_2\omega, \\ \tilde{y} &= \tilde{x}_2, \end{aligned} \tag{51}$$

with respect to our new basis for state, input and output. Using our output feedback we get:

$$\tilde{u} = -\rho X \tilde{y},$$

where

$$X = T_{\hat{u}} T_{\hat{u}}^{\mathrm{T}} > 0,$$

we obtain

$$\begin{aligned} \dot{\tilde{x}}_1 &= A_{11}\tilde{x}_1 + A_{12}\tilde{x}_2 + E_1\omega, \\ \dot{\tilde{x}}_2 &= A_{21}\tilde{x}_1 + (A_{22} - \lambda\rho X)\tilde{x}_2 + E_2\omega, \\ \hat{y} &= \tilde{x}_2, \end{aligned} \tag{52}$$

It is easily verified that there exists ρ^* such that for $\rho > \rho^*$,

$$S(s) = (sI - A_{22} + \rho\lambda X)^{-1}$$

satisfies:

$$\|S\|_\infty \leq \frac{\alpha_1}{\rho} \tag{53}$$

for some suitable constant α_1 independent of λ and ρ using that $\operatorname{Re} \lambda > \beta$ and $X > 0$. Consider

$$T(s) = \left(sI - A_{11} - A_{12}(sI - A_{22} + \lambda\rho X)^{-1} A_{21}\right)^{-1}$$

Given (9) for some given $\hat{H}$ it is easily seen that there exists ρ^* such that for $\rho > \rho^*$:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} - \lambda\rho X \end{pmatrix} + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} - \lambda\rho X \end{pmatrix}^* \leq 0$$

for all λ with $\operatorname{Re} \lambda \geq \beta$. It is not difficult to show that this implies that

$$T(s) = \begin{pmatrix} I & 0 \end{pmatrix} \left[sI - \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} - \lambda\rho X \end{pmatrix} \right]^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix}$$

has no poles in the open right half plane.

Next, we note that (G_2C, A) detectable implies that (A_{21}, A_{11}) is detectable and hence there exists $\eta > 0$ such that for all v and all $s \in \mathbb{C}^0$ we have:

$$\left\| \begin{pmatrix} sI - A_{11} \\ A_{21} \end{pmatrix} v \right\| > \eta \|v\| \tag{54}$$

First assume v is such that

$$\|(sI - A_{11})v\| \geq \tfrac{2\alpha_1}{\rho} \|A_{12}\| \|A_{22}\| \|v\| \tag{55}$$

In that case:

$$\begin{aligned} \|T^{-1}(s)v\| &\geq \|(sI - A_{11})v\| - \|A_{21}(sI - A_{22} + \lambda\rho X)^{-1} A_{12} v\| \\ &\geq \tfrac{\alpha_1}{\rho} \|A_{12}\| \|A_{22}\| \|v\| \end{aligned} \tag{56}$$

for ρ large enough.

Next assume (55) is not satisfied. Then clearly there exists $M > 0$ such that $|s| < M$. In that case, using the additional structure of Lemma 1, we have that:

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

and we can find μ (independent of λ and s) such that

$$\|v_1\| \leq \tfrac{\mu}{\rho} \|v\|, \qquad \|v_2\| \geq \tfrac{1}{2} \|v\| \tag{57}$$

Given (55) and (57) we find that (54) yields that we can find M_1 such that:

$$\|A_{21}v - A_{212}v_2\| \leq \tfrac{M_1}{\rho}\|v\|, \qquad \|A_{212}v_2\| \geq \tfrac{1}{2}\eta\|v\| \tag{58}$$

for ρ sufficiently large. We can also find M_2 such that:

$$\|v^*A_{12} + v_2^*A_{212}^{\mathrm{T}}\| \leq \tfrac{M_2}{\rho}\|v\| \tag{59}$$

using (57) for ρ large.

Since $s \in \mathbb{C}^0$, we have for some suitable constant M_3:

$$v^*[(sI - A_{11}) + (sI - A_{11})^*]v = v_1^*[(sI - A_{11s}) + (sI - A_{11s})^*]v_1 \leq \tfrac{M_1}{\rho^2}$$

Next, we have for some suitable constant M_4:

$$\|(sI - A_{22} + \lambda\rho X)^{-1} - (\lambda\rho X)^{-1}\| \leq \tfrac{M_4}{\rho^2}$$

for ρ large. These two bounds yield that:

$$\left|v^*\left[T^{-1}(s) + T^{-1}(s)^*\right]v\right| \geq \left|\left(\frac{1}{\lambda\rho} + \frac{1}{\bar{\lambda}\rho}\right)v^*A_{12}X^{-1}A_{21}v\right| - \tfrac{M_5}{\rho^2}\|v\|^2$$

for some suitable constant M_5. Using our earlier obtained bounds (58) and (59) we get:

$$\left|v^*\left[T^{-1}(s) + T^{-1}(s)^*\right]v\right| \geq \tfrac{\beta}{2\rho\alpha^2}v_2^*A_{212}^{\mathrm{T}}X^{-1}A_{212}v_2 > \tfrac{\beta}{\rho\alpha^2}\tfrac{\eta^2}{8\|X\|}\|v\|^2$$

for ρ large. This yields that

$$\|T^{-1}(s)v\| \geq \tfrac{\alpha_2}{\rho}\|v\| \tag{60}$$

for some constant α_2. We have (56) if (55) is not satisfied and (60) otherwise for $s \in \mathbb{C}^0$. Combining the two and using that T has no poles in the open right half plane we find there exists M_6 such that:

$$\|T\|_\infty < M_6\rho \tag{61}$$

for ρ sufficiently large.

After these preparations, we find that for the system (52), the transfer matrix from ω to $\tilde{y} = \bar{x}_2$ equals:

$$T_{\omega\tilde{y}} = SE_2 + SA_{21}TA_{12}SE_2 + SA_{21}TE_1 \tag{62}$$

while the transfer matrix from ω to $\tilde{u}$ equals:

$$T_{\omega\tilde{u}} = -\rho XT_{\omega\tilde{y}}$$

Note that, in this case, (50) implies that $E_1 = 0$. Using (62) and the above bounds we find:

$$\|T_{\omega\tilde{y}}\|_\infty \leq \tfrac{M_7}{\rho}\|E_2\|, \qquad \|T_{\omega\hat{u}}\|_\infty \leq M_7\|X\|\|E_2\|.$$

for suitable M_7. Similarly

$$\|T_{\tilde{u}\tilde{y}}\|_\infty \leq \tfrac{M_7}{\rho}, \qquad \|T_{\tilde{u}\tilde{u}}\|_\infty \leq M_7\|X\|.$$

where:

$$T_{\tilde{u}\tilde{y}} = S + SA_{21}TA_{12}S,$$
$$T_{\tilde{u}\tilde{u}} = -\rho X[S + SA_{21}TA_{12}S].$$

This clearly implies that protocol (49) solves the robust H_∞ almost disturbance decoupling problem with bounded input via static output feedback for the system (22) as required. □

Theorem 1 implies that the effect from ω on $\bar{x}$ is bounded whenever (50) is satisfied. We note from Remark 10 that the effect from ω on $\bar{x}$ can actually be unbounded if (50) is not satisfied. The following result shows that if (50) is not satisfied, the effect from ω to $G_2(y_i - y_j)$ for any $i, j \in 1, \ldots, N$, is always bounded under protocol (49).

Theorem 2. *Consider a MAS described by (10) and (11). Assume (A, B, C) is squared-down passifiable with respect to G_1, G_2 and H, such that (A, BG_1) is stabilizable, (A, G_2C) is detectable while BG_1 and G_2C have full column- and row-rank, respectively. Let any real numbers α, $\beta > 0$ and a positive integer N be given, and hence a set of network graphs $\mathbb{G}^N_{\alpha,\beta}$ be defined.*

There exists M such that the protocol (49) achieves state synchronization in the absence of disturbance and in the presence of disturbance the H_∞ norm from ω to $G_2(y_i - y_j)$ less than M for any $i, j \in 1, \ldots, N$, for any $\rho > 0$ and for any graph $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$.

Proof. Again we set $\rho = \frac{1}{\varepsilon}$. In this case, we again have (52). We find, similarly as in the proof of Theorem 1, that the transfer matrix from ω to $\hat{y} = \bar{x}_2$ is given by (62) but in this case E_1 need not be zero. We get:

$$\|T_{\omega\tilde{y}}\|_\infty \leq M_8\|E_1\| + \tfrac{M_7}{\rho}\|E_2\|$$

for suitable M_8 with M_7 as defined in the proof of Theorem 1. Similarly, we get

$$\|T_{\tilde{u}\tilde{y}}\|_\infty \leq \tfrac{M_7}{\rho},$$
$$\|T_{\omega\tilde{u}}\|_\infty \leq M_8\rho\|X\|\|E_1\| + M_7\|X\|\|E_2\|,$$
$$\|T_{\tilde{u}\tilde{u}}\|_\infty \leq M_7\|X\|.$$

Using the above and Lemma 4, we can complete the proof. □

5.1.2. Squared-down Minimum-phase with Relative Degree 1 Agents

In this subsection, we consider solvability and design of H_∞ almost state and output synchronization as stated in Problem 1 and 3, for a MAS with squared-down minimum-phase with relative degree 1 agents.

The next theorem shows that for minimum-phase agents we can achieve H_∞ almost *state* synchronization if condition (50) is satisfied. Moreover, if condition (50) is not satisfied then we still achieve H_∞ almost *output* synchronization.

Theorem 3. *Consider a MAS described by (10) and (11). Assume (A, B, C) is squared-down minimum-phase with relative degree 1 with G_1 and G_2 such that (A, BG_1) is controllable and (A, G_2C) is observable. Assume that without loss of generality G_1 is chosen such that Remark 4 is satisfied.*

Let any real numbers α, $\beta > 0$ and a positive integer N be given, and hence a set of network graphs $\mathbb{G}^N_{\alpha,\beta}$ be defined.

The H_∞ almost state synchronization problem via static protocol, as defined in Problem 1 where $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$, is solvable if (50) is satisfied.

If (50) is not satisfied then the H_∞ almost output synchronization problem with bounded state errors via static protocol, as defined in Problem 3 with respect to the compensated output $\hat{y} = G_2y$ where $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$, is solvable.

In particular, for any given real number $\delta > 0$, there exists an ε^*, such that for any $\varepsilon \in (0, \varepsilon^*)$, the protocol (49) achieves output synchronization and an H_∞ norm from ω to $\hat{y}_i - \hat{y}_j$ less than δ for any $i, j \in 1, \ldots, N$ and for any graph $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$.

If (50) is satisfied then the protocol (49) achieves state synchronization and an H_∞ norm from ω to $x_i - x_j$ less than δ for any $i, j \in 1, \ldots, N$ and for any graph $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$.

Proof. We use similar arguments as in the proof of Theorem 1. We obtain the system (52) with $X = I$. However, in this case we have that A_{11} is Hurwitz stable. This implies there exists M_9 such that

$$V(s) = (sI - A_{11})^{-1}$$

satisfies

$$\|V\|_\infty < M_9. \tag{63}$$

but then:

$$T(s) = V(s)[I - A_{21}S(s)A_{12}V(s)]^{-1}$$

which yields

$$\|T\|_\infty \leq 2M_9$$

for ρ sufficiently large, using the earlier obtain bounds (53) and (63). Note the improvement over (61) due to the fact that A_{11} is Hurwitz stable. We then use (62), and our improved bound for T in combination with the previously obtained bound for S to get:

$$\|T_{\omega\tilde{y}}\|_\infty \leq \tfrac{M_{10}}{\rho}\|E_1\| + \tfrac{M_{11}}{\rho}\|E_2\| \tag{64}$$

On the other hand, we have:

$$T_{\omega\tilde{u}} = -\rho T_{\omega\tilde{y}}$$

which yields that:

$$\|T_{\omega\tilde{u}}\|_\infty \leq M_{10}\|E_1\| + M_{11}\|E_2\|$$

for ρ sufficiently large, independent of λ.

We have

$$T_{\omega\tilde{x}_1} = TA_{21}SE_2 + TE_1$$

Hence, if (50) is satisfied, then

$$\|T_{\omega\tilde{x}_1}\| \leq \tfrac{M_{12}}{\rho}$$

for some constant M_{12} and sufficiently large ρ. The latter, in combination with (64), yields

$$\|T_{\omega\tilde{x}}\| \leq \tfrac{M_{13}}{\rho}$$

for some constant M_{13} and sufficiently large ρ.

Using these bounds in connection with either Lemma 2 or Lemma 3, yields the required results. □

5.2. H_2 *almost synchronization*

In this section, we consider the solvability and protocol design to achieve H_2 almost state and output synchronization. Primarily,we consider the class of squared-down passifiable via output feedback agents. We also consider the class of squared-down minimum-phase with relative degree 1 agents which are a subset of squared-down passifiable via output feedback agents. Later, we will show that a weaker solvability condition and stronger results can be obtained when the agents are minimum-phase with relative degree 1.

To apply either Lemma 5 or Lemma 6, we know that the graph must be uniformly bounded. Thus, the graph set $\mathbb{G}^N_{\alpha,\beta}$ with α, $\beta > 0$ is used in the following protocol designs.

To obtain our results regarding H_2 almost synchronization, we need the following classical result:

Lemma 7. *Consider an asymptotically stable system:*

$$\dot{p} = A_1 p + B_1 w$$
$$y = C_1 p$$

The H_2 norm from w to y is less than ε if there exists a matrix Q such that:

$$A_1Q + QA_1^{\mathrm{T}} + B_1B_1^{\mathrm{T}} \leq 0, \qquad C_1QC_1^{\mathrm{T}} < \varepsilon^2 I$$

or, using a dual version, the H_2 norm from w to y is less than ε if there exists a matrix P such that:

$$PA_1 + A_1^{\mathrm{T}}P + C_1^{\mathrm{T}}C_1 \leq 0, \qquad B_1^{\mathrm{T}}PB_1 < \varepsilon^2 I$$

5.2.1. Squared-down passifiable via output feedback agents

In this subsection, we consider solvability and protocol design of H_2 almost state and output synchronization as stated in Problem 2 and 4, for a MAS with squared-down passifiable via output feedback agents.

First we provide our main result for H_2 almost state synchronization via static protocol as the following theorem.

Theorem 4. *Consider a MAS described by (10) and (11). Assume (A, B, C) is squared-down passifiable with respect to G_1, G_2 and H, such that (A, BG_1) is stabilizable, (A, G_2C) is detectable while BG_1 and G_2C have full column- and row-rank, respectively. Let any real numbers α, $\beta > 0$ and a positive integer N be given, and hence a set of network graphs $\mathbb{G}^N_{\alpha,\beta}$ be defined.*

The H_2 almost state synchronization problem via static protocol stated in Problem 2 with $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ is solvable when (50) is satisfied. In particular, for any given real number $\delta > 0$, there exists an ε^*, such that for any $\varepsilon \in (0, \varepsilon^*)$, the protocol (49) achieves state synchronization and an H_2 norm from ω to $x_i - x_j$ less than δ for any $i, j \in 1, \dots, N$ and for any graph $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$.

Proof. We know that we only need to verify that the protocol (49) solves the robust H_2 almost state disturbance decoupling problem with bounded input for the system (22) with $\lambda \in \Lambda$. Given $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$, we know that $\lambda \in \Lambda$ implies $\operatorname{Re}\lambda \geq \beta$.

The agents are squared-down passifiable given G_1, G_2 and H. Using Lemma 1, the dynamics of (22) with compensator G_1, G_2 and output feedback gain H, i.e. (A, BG_1, G_2C) can be written as (51) with respect to our new basis for state, input and output. Using our output feedback we get:

$$\tilde{u} = -\rho X\tilde{y},$$

where

$$X = T_uT_u^{\mathrm{T}} > 0,$$

and we obtain (52).

In the proof of Theorem 1, we already established that the H_∞ norm of $T^\lambda_{\hat{u}\hat{y}}$ is less than $M\rho^{-1}$ for some M and the H_∞ norm of $T^\lambda_{\hat{u}\hat{u}}$ is bounded. We still need to investigate the H_2 norm from ω to x and the H_2 norm from ω to $\hat{u}$. It is clear that we can equivalently study the H_2 norm from ω to $\tilde{x}$ and $\tilde{u}$, respectively. After all, the H_2 norm from ω to x is arbitrarily small/bounded if and only if the H_2 norm from ω to $\tilde{x}$ is arbitrarily small/bounded. Similarly with using $\hat{u}$ or $\tilde{u}$.

Using the more specific structure in (6) we get:

$$P\begin{pmatrix} A_{11s} & 0 & A_{121} \\ 0 & A_{110} & A_{122} \\ A_{211} & -A_{122}^{\mathrm{T}} & A_{22} - \lambda\rho X \end{pmatrix} + \begin{pmatrix} A_{11s} & 0 & A_{121} \\ 0 & A_{110} & A_{122} \\ A_{211} & -A_{122}^{\mathrm{T}} & A_{22} - \lambda\rho X \end{pmatrix}^* P$$
$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E_2E_2^{\mathrm{T}} \end{pmatrix} \leq \frac{\alpha_3}{\rho}\begin{pmatrix} A_{11s} + A_{11s}^{\mathrm{T}} & 0 & A_{121} + A_{211}^{\mathrm{T}} \\ 0 & 0 & 0 \\ A_{121}^{\mathrm{T}} + A_{211} & 0 & A_{22} + A_{22}^{\mathrm{T}} - \beta\rho X \end{pmatrix} \leq 0$$

for ρ large enough where $P = \frac{\alpha_3}{\rho}I$ with α_3 such that

$$E_2E_2^{\mathrm{T}} \leq \alpha_3\beta X.$$

Note that we explicitly rely here on the fact that $A_{11s} + A_{11s}^{\mathrm{T}} < 0$. But then, using Lemma 7, we obtain that the H_2 norm from ω to $\tilde{x}$ is less than $\sqrt{\frac{\alpha_3}{\rho}}$ while the H_2 norm from ω to $\hat{u}$ is less than $\sqrt{\rho\alpha_3}$. The proof is completed by using Lemma 6. □

In the case (50) is not satisfied, in general we can not achieve H_2 almost state synchronization via static protocol as illustrated by the following example.

Example 2. Consider the MAS with two agents:

$$\dot{x}_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}x_i + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u_i + \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}\omega_i$$
$$y_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}x_i \tag{65}$$

for $i = 1, 2$. The communication graph has the associated Laplacian matrix

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The system is squared-down passifiable with $G_1 = 1$ and $G_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$. Hence almost synchronization with respect to the compensated output $\hat{y}_i = G_2 y_i$ is possible. However, almost state synchronization with respect to the original state is not possible unless (50) is satisfied. Consider an arbitrary static protocol

$$u_i = F\zeta_i = \begin{pmatrix} f_1 & f_2 \end{pmatrix}\zeta_i$$

Transfer matrix from ω_1 to $x_1 - x_2$ is given by

$$T_{\omega_1\bar{x}}(s) = \left(sI - \begin{pmatrix} 0 & 1 \\ f_1 - 1 & f_2 \end{pmatrix}\right)^{-1} \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$$

where $f_1 < 1$ and $f_2 < 0$.

We have

$$\begin{pmatrix} 0 & 1 \\ f_1 - 1 & f_2 \end{pmatrix}^{\mathrm{T}} P + P\begin{pmatrix} 0 & f_1 - 1 \\ 1 & f_2 \end{pmatrix} + I = 0$$

which yields

$$P = \begin{pmatrix} \frac{f_2}{2(f_1-1)} + \frac{f_1-1}{2f_2} - \frac{1}{2f_2} & -\frac{1}{2(f_1-1)} \\ -\frac{1}{2(f_1-1)} & \frac{1}{2f_2(f_1-1)} - \frac{1}{2f_2} \end{pmatrix}$$

hence

$$\|T_{\omega_1\bar{x}}\|_2 = e_1^2\left(\tfrac{f_2}{2(f_1-1)} + \tfrac{f_1-1}{2f_2} - \tfrac{1}{2f_2}\right) + e_2^2\left(\tfrac{1}{2f_2(f_1-1)} - \tfrac{1}{2f_2}\right).$$

For $f_2 < 0$ and $f_1 < 1$, we have

$$\frac{f_2}{2(f_1-1)} + \frac{f_1-1}{2f_2} - \frac{1}{2f_2} > 1.$$

Hence, H_2 almost state synchronization is not possible for $e_1 \neq 0$. For $e_1 = 0$, we have

$$\|T_{\omega_1\bar{x}}\|_2 \to 0$$

as $f_2 \to -\infty$ with $f_1 = 0$.

Now, we consider H_2 almost output synchronization in the case (50) is not satisfied.

Theorem 5. *Consider a MAS described by (10) and (11). Assume (A, B, C) is squared-down passifiable with respect to G_1, G_2 and H, such that (A, BG_1) is stabilizable, (A, G_2C) is detectable while BG_1 and G_2C have full column- and row-rank, respectively. Let any real numbers α, $\beta > 0$ and a positive integer N be given, and hence a set of network graphs $\mathbb{G}^N_{\alpha,\beta}$ be defined.*

The H_2 almost output synchronization problem stated in Problem 4 with $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ is solvable. In particular, for any given real number $\delta > 0$, there exists an ε^*, such that for any $\varepsilon \in (0, \varepsilon^*)$, the protocol (49) achieves state synchronization and the H_2 norm from ω to $G_2(y_i - y_j)$ is less than δ for any $i, j \in 1, \dots, N$ and for any graph $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$.

The above theorem states that for squared-down passifiable agents we can achieve H_2 almost output synchronization with respect to the compensated output $\hat{y} = G_2 y$. The following remark shows that this is no longer valid if we use the original output.

Remark 11. Consider the MAS (48) with the same communication network as stated in Example 1.

We have

$$\begin{pmatrix} 0 & 1 \\ f_1 - 1 & f_2 \end{pmatrix}^{T} P + P\begin{pmatrix} 0 & f_1 - 1 \\ 1 & f_2 \end{pmatrix} + I = 0$$

which yields

$$P = \begin{pmatrix} \frac{f_2}{2(f_1-1)} + \frac{f_1-1}{2f_2} - \frac{1}{2f_2} & -\frac{1}{2(f_1-1)} \\ -\frac{1}{2(f_1-1)} & \frac{1}{2f_2(f_1-1)} - \frac{1}{2f_2} \end{pmatrix}$$

hence

$$\|T_{\omega_1\bar{y}}\|_2 = \frac{f_2}{2(f_1-1)} + \frac{f_1-1}{2f_2} - \frac{1}{2f_2}$$

since $f_1 < 1$ and $f_2 < 0$, it is easily verified that

$$\|T_{\omega_1\bar{y}}\|_2 \geq 1.$$

hence H_2 almost synchronization is not possible.

Proof of Theorem 5. We use similar arguments as in the proof of Theorem 4.

This time, we obtain, using our output feedback,

$$\begin{aligned} \dot{\tilde{x}}_1 &= A_{11}\tilde{x}_1 + A_{12}\tilde{x}_2 + E_1\omega, \\ \dot{\tilde{x}}_2 &= A_{21}\tilde{x}_1 + (A_{22} - \lambda\rho)\tilde{x}_2 + E_2\omega, \\ \tilde{y} &= \tilde{x}_2, \end{aligned} \tag{66}$$

In the proof of Theorem 1, we already established that the H_∞ norm of $T^\lambda_{\hat{u}\hat{y}}$ is less than $M\rho^{-1}$ for some M and the H_∞ norm of $T^\lambda_{\hat{u}\hat{u}}$ is bounded for ρ sufficiently large. We still need to study the H_2 norm from ω to $\hat{y}$ and the H_2 norm from ω to $\hat{u}$. We have

$$\begin{aligned} &\begin{pmatrix} A_{11s} & 0 & A_{121} \\ [3pt]0 & A_{110} & A_{122} \\ A_{211} & -A_{122}^{\mathrm{T}} & A_{22} - \lambda\rho X \end{pmatrix} Q + Q\begin{pmatrix} A_{11s} & 0 & A_{121} \\ 0 & A_{110} & A_{122} \\ A_{211} & -A_{122}^{\mathrm{T}} & A_{22} - \lambda\rho X \end{pmatrix}^{*} \\ &\quad + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix} \leq \frac{\alpha_4}{\rho}\begin{pmatrix} A_{11s} + A_{11s}^{\mathrm{T}} & 0 & A_{121} + A_{211}^{\mathrm{T}} \\ 0 & 0 & 0 \\ A_{121}^{\mathrm{T}} + A_{211} & 0 & A_{22} + A_{22}^{\mathrm{T}} - \beta\rho X \end{pmatrix} \leq 0 \end{aligned} \tag{67}$$

for ρ large enough with $Q = \frac{\alpha_4}{\rho} I$ while α_4 is such that

$$I \leq \alpha_4 \beta X$$

But then, using Lemma 7, we obtain that the H_2 norm from ω to $\tilde{y}$ is less than

$$\sqrt{\tfrac{\alpha_4}{\rho}}\left\| E_1 E_1^{\mathrm{T}} + E_2 E_2^{\mathrm{T}} \right\|^{1/2}.$$

while the H_2 norm from ω to $\hat{u}$ is less than

$$\sqrt{\alpha_4\rho}\|X\|\|E_1E_1^{\mathrm{T}} + E_2E_2^{\mathrm{T}}\|^{1/2}.$$

The proof is completed by using Lemma 5. □

5.2.2. Squared-down minimum-phase with relative degree 1 agents

Note that for squared-down minimum-phase with relative degree 1 agents we still cannot achieve H_2 almost state synchronization in the case that (50) is not satisfied. The only additional property we can obtain for this class of agents is that the H_2 norm from the disturbance ω to $x_i - x_j$ is uniformly bounded, i.e.

Theorem 6. *Consider a MAS described by (10) and (11). Assume (A, B, C) is squared-down passifiable with respect to G_1, G_2 and H, such that (A, BG_1) is stabilizable, (A, G_2C) is detectable while BG_1 and G_2C have full column- and row-rank, respectively. Let any real numbers α, $\beta > 0$ and a positive integer N be given, and hence a set of network graphs $\mathbb{G}^N_{\alpha,\beta}$ be defined.*

The H_2 almost output synchronization problem with bounded state errors via static protocol stated in Problem 4 with $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ is solvable. In particular, there exists $M > 0$ such that for any given real number $\delta > 0$, we have an $\varepsilon^* > 0$, with the property that for any $\varepsilon \in (0, \varepsilon^*)$, the protocol (49) achieves state synchronization and an H_2 norm from ω to $y_i - y_j$ less than δ while the H_2 norm from ω to $x_i - x_j$ less than M for any $i, j \in 1, \dots, N$ and for any graph $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$.

Proof. Similar to the proof of Theorem 5. But since

$$A_{11} + A_{11}^{\mathrm{T}} < 0$$

there exists μ such that

$$A_{11} + A_{11}^{\mathrm{T}} + \mu I \leq 0$$

We can then obtain instead of (67)

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} - \lambda\rho X \end{pmatrix} Q + Q \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} - \lambda\rho X \end{pmatrix}^{\mathrm{T}} + \frac{\mu}{\rho}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq 0$$

for ρ sufficiently large provided $Q = \frac{1}{\rho} I$. But then, using Lemma 7, we obtain that the H_2 norm from ω to $\tilde{x}$ is less than

$$\frac{1}{\sqrt{\mu}} \left\| E_1 E_1^{\mathrm{T}} + E_2 E_2^{\mathrm{T}} \right\|^{1/2}.$$

which is exactly the extra property that we needed to establish. □

6. Numerical example

The effectiveness and performance of the H_∞ almost output synchronization design is demonstrated through the following numerical example. We illustrate our results for a homogeneous MAS of $N = 6$ agents. We consider H_∞ almost output synchronization problem via static protocol for squared-down passive agents.

The agent model is written as

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

According to condition (2), we verify that this agent model is squared-down passive by choosing $P = I$, and G_1, G_2 and E given by

$$G_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 3 & 1 \end{pmatrix}, \text{ and } E = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

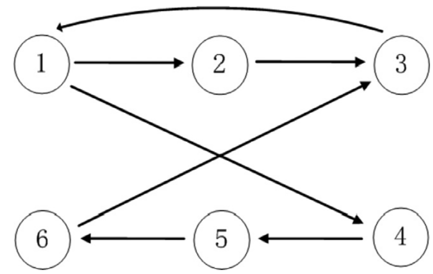

Fig. 3. The directed communication topology.

with disturbances

$$\omega_1 = 0, \omega_2 = \cos(t), \omega_3 = 0.5,$$
$$\omega_4 = \sin(2t), \omega_5 = \sin(t), \omega_6 = \cos(2t).$$

The communication topology is shown in Fig. 3 with associated Laplacian matrix

$$L = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Now by choosing $\rho = 4$ and $\rho = 20$, we obtain the results as Fig. 4 and 5, respectively. The simulations confirm the results of the paper that by increasing the value of ρ we achieve higher degree of accuracy.

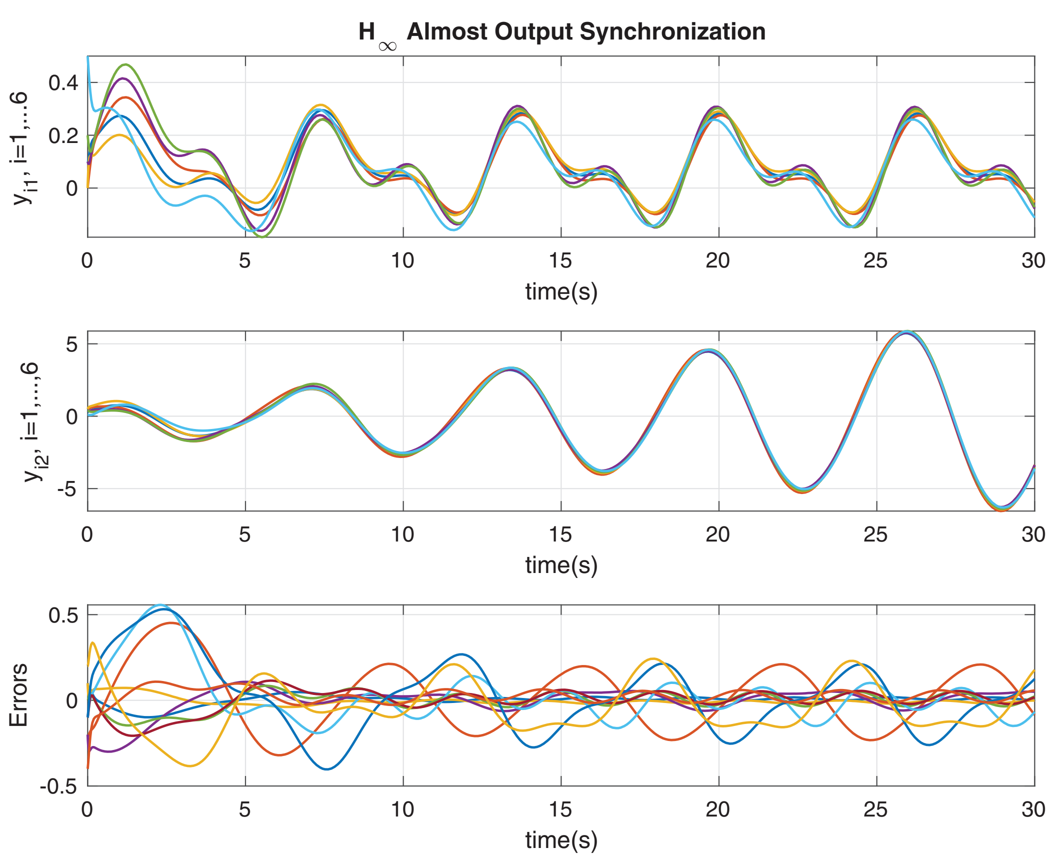

Fig. 4. H_∞ almost output synchronization with $\rho = 4$.

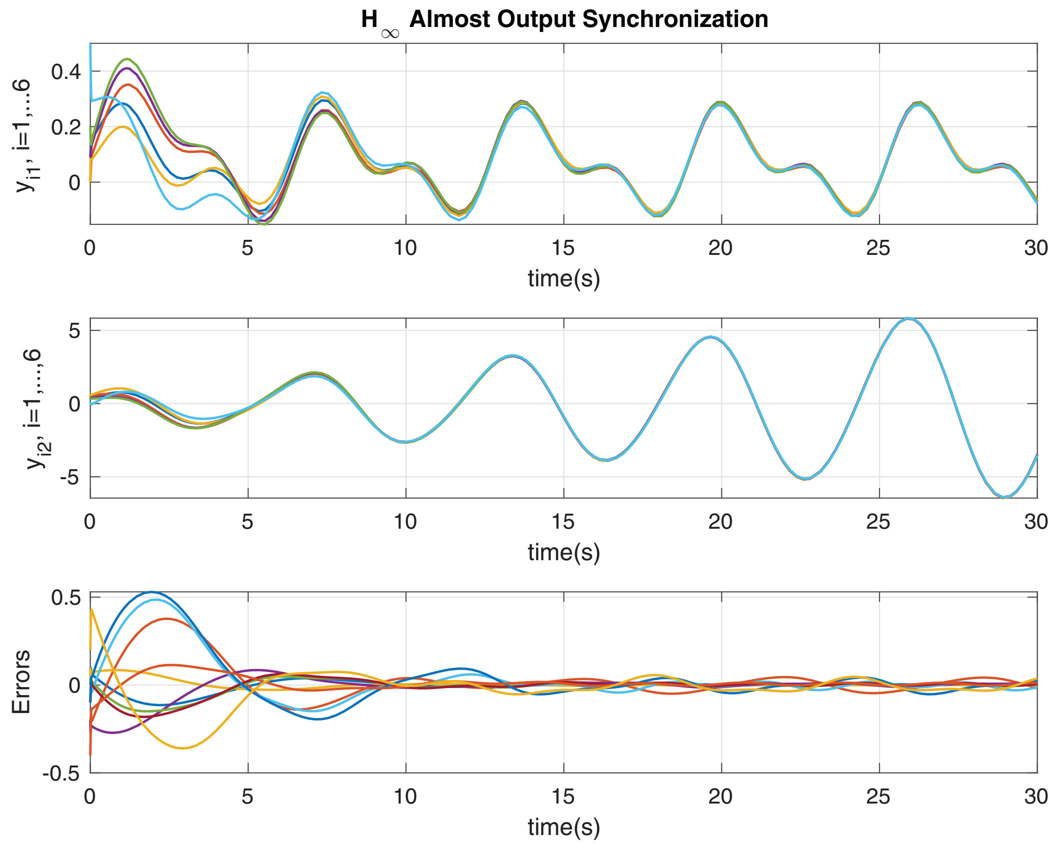

Fig. 5. H_∞ almost output synchronization with $\rho = 20$.

7. Conclusion

In this paper, we studied H_∞ and H_2 almost state or output synchronization problems for homogeneous MAS with partial-state coupling via static protocols in the presence of external disturbances. We provided solvability conditions for designing static protocols for three classes of agents, namely squared-down passive, squared-down passifiable via output feedback, and squared-down minimum-phase with relative degree 1. Finally, we concluded our paper by a numerical example to show that we can decrease the impact of disturbances on the network disagreement dynamics to an arbitrary small value. As an extension of this work, in the future, we will work on almost synchronization of MAS with nonlinear passive agents.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.ejcon.2019.11.008.

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