Investigating the subline frequency setting problem for autonomous minibusses under demand uncertainty

Dr Konstantinos Gkiotsalitis
University of Twente

Department of Civil Engineering
P.O. Box 217, 7500 AE Enschede, Netherlands
Email: k.gkiotsalitis@utwente.nl

Dr Marie Schmidt
Erasmus University Rotterdam

Department of Technology and Operations Management
Postbus 1738, 3000 DR Rotterdam, Netherlands
Email: schmidt2@rsm.nl

Dr Evelien van der Hurk
Technical University of Denmark

Department of Management Science
2800 Kgs. Lyngby, Denmark
Email: evdh@dtu.dk

100th Annual Meeting of the Transportation Research Board, Washington D.C.

Paper Number: 21-00355

January 2021

ABSTRACT

Over the last years, several pilots with autonomous minibusses operating in urban environments have been initiated. Unlike regular bus services, autonomous minibusses serve a limited number of stops and have more flexible schedules since they do not require drivers. This allows the operation of a line through a combination of *sublines*, where a subline serves a subset of consecutive stops in the same order as the original line. This paper studies the subline frequency setting problem under uncertain passenger demand. We present a frequency setting model that assigns autonomous minibusses to sublines in order to minimize operational costs and passenger waiting time costs. Passenger waiting time costs may depend on the combination of several lines whose frequencies cannot be perfectly aligned for each passenger journey. We present a new estimation of the expected waiting for passengers to improve the accuracy of the passenger waiting time costs. The model is originally formulated as a MINLP and it is reformulated as a MILP that can be easily solved to global optimality. Further, we explicitly consider the uncertainty of passenger demand in the optimization process by formulating stochastic and robust optimization models, respectively. The performance of the stochastic and robust optimization models is tested under various passenger demand scenarios in a bi-directional autonomous minibus line operating in Frankfurt, Germany. Our analysis shows that a stochastic optimization design has the best on-average performance and a robust design the most stable performance.

Keywords: autonomous minibusses; vehicle scheduling; subline frequency setting; stochastic optimization; robust optimization.

INTRODUCTION

Autonomous minibusses are gaining momentum as they are deployed in several pilots across Europe to offer last-mile solutions to travelers in urban areas. Recently, five autonomous minibus trials were launched in five European cities (Helsinki, Gjesdal, Tallinn, Lamia, and Helmond) under the EU project Fabulos Fabulos (*1*). Autonomous minibusses have been operating in several EU trials in Frankfurt, Luxembourg, Lyon, Paris, Berlin under speeds than can be up to 40 km/h Muezner, Duss, Stein and Goebel, Modijefsky (*2, 3, 4, 5*). They do not need a driver or steward on board as they are fully autonomous and they typically serve a small number of stops while providing first/last-mile services.

Operational planning for minibusses follows to a large extend that of traditional buslines Ceder (*6*): frequency setting, timetabling and vehicle scheduling – while the last step of crew scheduling can be ommitted. At the frequency settings stage, the frequency of each service line is planned considering the trade-off between the operational and the passenger-related costs Yu et al., Szeto and Wu, Gkiotsalitis and Cats (*7, 8, 9*). This frequency provides also a first indication of the number of resources (vehicles) required to operate the service line Ceder, Hassold and Ceder (*10, 11*). The dispatching times of the assigned vehicles are determined at a subsequent step, known as timetable scheduling Ceder, Gkiotsalitis and Alesiani (*12, 13*).

This paper focuses on frequency setting for autonomous vehicle bus lines in the context of uncertain passenger demand and the use of *sublines*. A subline serves a consecutive subset of stops from the main line, and can be obtained from the mainline by short-turning. Sublines thus enable to provide a higher passenger service level at lower operating costs in case of heterogeneous demand among the line. The subline frequency setting problem (SFS) minimizes operating costs in terms of vehicle fleet size and operating time as well as passenger waiting time through the assignment of frequencies to all possible sublines. Our model includes a novel estimate for passenger waiting time under the assumption that multiple sublines may serve a single origin-destination pair. To evaluate the impact of uncertainty in passenger demand, we present a stochastic and robust optimization SFS model, and compare results of these against the deterministic model that does not consider sublines in a realistic case study based on the autonomous vehicle line in Frankfurt.

To summarize, the main contributions of our work to the state-of-the-art are: (a) the development of an easy-to-solve mixed-integer linear programming model for the autonomous minibus planning problem that stretches the available resources by tailoring them to OD-pairs with higher demand, and (b) the incorporation of the passenger demand uncertainties in the problem formulation with the development of a robust and a stochastic optimization model for the planning of autonomous minibusses.

The remainder of this paper is structured as follows: in section 2 we provide the literature review on bus frequency setting problems that allocate the available vehicle resources to bus lines or sublines. In section 3, we introduce our SFS model. In section 4, we develop a stochastic and a robust formulation of the SFS. Our case study is detailed in section 5 where we test the performance of our robust and stochastic optimization solutions in a simulation study of an autonomous minibus line operating in Frankfurt. Finally, section 6 provides the concluding remarks of our study and discusses future research directions.

LITERATURE REVIEW AND CONTRIBUTION

Frequency setting models determine the required number of trips to optimally operate a service line and the required number of vehicles to operate those trips Ibarra-Rojas et al., Gkiotsalitis and

Cats, Gkiotsalitis and Cats (*14, 15, 16*). Ceder (*17*) proposed closed-form expressions that do not need to solve complex mathematical programs when determining the frequency of a single line. Namely, in many practical applications the frequency of a bus line is set based on *policy headways* or the *maximum loading point* Ceder (*6*). Policy headways determine a lower bound of the line frequency and are used by operators that operate low-frequency services in suburban areas. The maximum load point method determines the frequency of a line based on the ratio of the number of passengers on board at the peak-load point to the desired passenger load of the vehicle.

Apart from closed-form expressions that determine the service frequency in a crude manner, there are several methods that try to find an optimal trade-off between passenger and operational-related costs (see Yu et al., dell'Olio et al., Cipriani et al., Cats and Glück (*7, 18, 19, 20*)). There are also several works that consider short-turning and interlining lines when setting frequencies (see Table 1). These works, however, do not consider the uncertainty of passenger demand when determining the service frequencies of sublines and their non-convex model formulations do not allow to find globally optimal solutions resulting in the employment of heuristics that compromise the solution quality.

TABLE 1 : Research studies that consider sublines for exploiting the allocation of vehicles to OD-pairs with higher demand

Study	Key performance indicators	Line flexibility	Demand uncertainty	Solution method
Delle Site and Filippi (*21*)	Waiting times, running costs and personnel costs	Short-turning	Not considered	Locally optimal by splitting the problem into tractable subproblems
Cortés et al. (*22*)	Waiting time, in-vehicle time, personnel costs and running costs	Short-turning and deadheading	Not considered	Locally optimal with applying an integrated deadheading-short-turning strategy
Verbas and Mahmassani (*23*)	Ridership and waiting time savings	Service patterns that only use a subset of the entire stops of a route	Not considered	Locally optimal solution with KNITRO solver
Verbas and Mahmassani (*24*)	maximize wait time savings subject to budget, fleet, vehicle load, and policy headway constraints	service patterns	Not considered	Locally optimal solution by solving an upper and a lower level problem with KNITRO
Gkiotsalitis et al. (*25*)	Passenger waiting costs and vehicle running and depreciation costs	Short-turning and interlining	Not considered	Locally optimal solution with Genetic Algorithm
This study	Waiting times, running costs and operating cost of each extra minibus	Short-turning	Considered	Globally optimal with LINDO solver

To consider the vehicle productivity and the operational costs, in this study we use as a baseline frequency model a modified version of the well-established frequency setting model of Furth and Wilson (*26*). Furth and Wilson (*26*) developed a frequency setting model for a bus line considering the trade-off between the operational costs and the passenger waiting times. Similarly to our study, the work of Furth and Wilson (*26*) focuses on urban, high-frequency services and it is based on the assumption that the mean passenger waiting time at a stop is half the mean headway. Our specific contributions to the baseline model of Furth and Wilson (*26*) are (i) the inclusion of sublines in our model formulation, and (ii) the explicit consideration of passenger demand uncertainty in the optimization process with the development of stochastic and robust optimization models.

SUBLINE FREQUENCY SETTING MODEL

Proposed Model

In our proposed model, we expand the classical bus frequency settings model of Furth and Wilson (*26*) to our flexible minibus planning problem where autonomous minibusses can operate in sub-lines that serve a fraction of stops of the originally planned line. We build a model to answer the questions:

1. which sublines should we establish?
2. at which frequencies should the established sublines operate?

Our approach focuses on symmetric, bi-directional lines because this is the most typical structure of autonomous minibus lines operating in several cities (e.g., Frankfurt, Lyon, Luxembourg).

Eligible sublines

We require that all sublines start at the depot. However, they do not need to operate until the end of the line. Instead, they can short-turn at some pre-selected bus stop(s). We call the bus stops where short-turning can take place *control stops*. That is, the generated sublines serve a fraction of the stops operated by the originally planned line. For instance, Fig.1 presents an originally planned bi-directional line with three control stops where short-turning is permitted. A subline can be generated by traveling from a control stop to the symmetric bus stop in the opposite direction. This results in two potential sublines for the case of Fig.1.

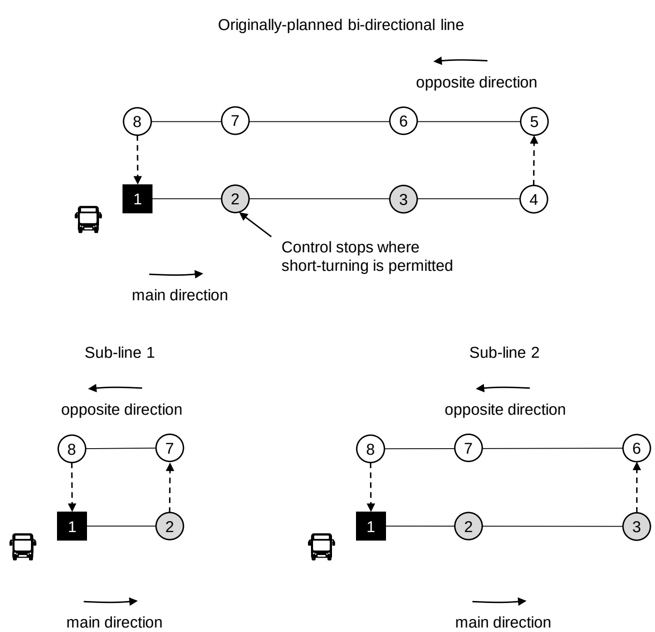

FIGURE 1 : Generation of sublines from an originally planned bi-directional line.

We should note here that the number of potential sublines is equal to the number of control point stops. If the number of control stops increases, then the number of generated sublines will also increase. Importantly, because we visit all stops from the origin to the control stop and do not allow to skip intermediate stops, the number of generated sublines increases linearly and not exponentially with the number of control stops, thus resulting in problems with manageable sizes. It is also important to note that many of the generated sublines might not be deemed operational if the passenger demand is not sufficient to justify their use.

Assumptions on demand and passenger behavior
We assume that the bus stops of a minibus line are served only by the minibusses of this main line, e.g., there are no other bus or minibus lines covering (part of) the demand between these stops. Like in Furth and Wilson (*26*), we also assume that the deployed vehicles have sufficient capacity, so that we can neglect denied boarding related to overcrowding. Furthermore, we assume random passenger arrivals, as common in high-frequency services. Indeed, recent studies have shown that passengers do not coordinate their arrivals at stops with the arrival times of buses in high-frequency services, and thus their average waiting time is half the headway (see Welding, Hickman, Bartholdi III and Eisenstein, Cats (*27, 28, 29, 30*)).

Finally, we assume that passengers choose the next minibus that departs from their origin and brings them to their destination, irrespective of the subline. Ergo, the expected waiting time does not depend on the headways between buses of the same subline only, but on the headways between all *relevant* departures for the passengers, i.e., all departures from their origin that visit their destination.

Operations
To generalize the estimation of operational costs and passenger waiting times from the case of one line in Furth and Wilson (*26*) to the case of several sublines, we make the following assumptions on how the minibus system is going to be operated:

1. Each vehicle is exclusively assigned to one of the available sublines. That is, a vehicle is not allowed to operate in multiple sublines.

2. The minibusses operate according to a periodic schedule.

Note that Assumption 1 leads to a conservative estimate of the number of vehicles needed to operate the system. That is, the cost of operation in practice may be lower, if, in a subsequent step, a vehicle can be assigned to more than one line. Note also that Assumption 2 does not make a statement about synchronization of vehicles *within* a period, but only states that the schedule repeats every period. When operating several sublines we cannot expect that the departures relevant for a certain OD-pair will be perfectly synchronized with each other. In general, if f_{sy} denotes the number of relevant departures for OD-pair (s,y) per time period, the expected waiting time will lie somewhere between $\frac{T}{2f_{sy}}$ (if the relevant departures are perfectly synchronized) and $\frac{T}{2}$ (if all relevant departures take place at the same moment in time). We refer to f_{sy} as the *service frequency* of OD-pair (s,y). In our models, we use the value $\frac{T}{f_{sy}+1}$ to estimate the waiting time of OD-pair (s,y). This value is the expected waiting time under the assumption that vehicle departures are scheduled independently and randomly, assuming equal probability for each departure moment of a vehicle. In that sense, $\frac{T}{f_{sy}+1}$ constitutes a lower bound on the expected waiting time of a passenger with f_{sy} travel options within the hour. Once a set of sublines and their respective frequencies are known, these can be scheduled in a subsequent timetabling step, so that the actual expected waiting times will be lower than $\frac{T}{f_{sy}+1}$.

Proposed SFS mathematical programming model

Our proposed model extends the classical bus frequency settings model of Furth and Wilson (*26*) by borrowing the objectives and constraints of their model - while expanding the formulation in the case of sublines based on the discussion in the previous sections.

Before we proceed to the formulation of our model, we present its nomenclature:

NOMENCLATURE

Sets

$\mathscr{S}$	ordered set of stops of the minibus line in both directions, $\mathscr{S} = \langle 1,2,...,s,...\rangle$
$\mathscr{R}$	set of all potential lines $\mathscr{R} = \langle 1,2,...,r,...\rangle$, where line 1 is the original line that serves all stops $s \in S$ and $\langle 2,3,...r,...\rangle$ are the generated sublines. Note that $\lvert\mathscr{R}\rvert = \Omega + 1$
$\mathscr{O}$	set of OD-pairs with passenger demand. Note that if there is no passenger demand between stops $s \in \mathscr{S}$ and $\tilde{s} \in \mathscr{S}$, then $(s,\tilde{s}) \notin \mathscr{O}$
$\mathscr{F}$	discrete set of frequencies

Parameters

Ω	number of generated sublines, which is equivalent to the number of control stops
T	time interval of our demand-homogeneous planning period
B_{sy}	passengers willing to travel from stop s to y in our demand-homogeneous planning period, where $(s,y) \in \mathscr{O}$.
$\Delta_{r,sy}$	$\Delta_{r,sy} = 1$ if subline 1 serves the OD-pair $(s,y) \in \mathscr{O}$.
T_r	round-trip travel time of subline $r \in \mathscr{R}$
N	number of available minibusses
Θ	minimum allowed service frequency, $\Theta > 0$, to ensure a minimum level of service for any passenger traveling from stop s to stop y, where $(s,y) \in \mathscr{O}$
K	minimum number of minibusses that should be assigned to the original line, where $K \leq N$
W_1	the cost of operating an extra minibus
W_2	the cost of a marginal increase in the total vehicle running times
F	minimum frequency of a subline
M	a very large positive number

Variables

x_r	number of minibusses assigned to subline r
f_{sy}	the service frequency of OD-pair $(s,y) \in \mathscr{O}$.
f_r	the service frequency of subline $r \in \mathscr{R}$
a_r	binary variable, where $a_r = 1$ if subline r is deemed operational and 0 otherwise

Our SFS formulation contains 4 variables. Integer variable x_r specifies how many vehicles are assigned to each subline $r \in \mathscr{R}$. Note that according to Assumption 1, vehicles can only serve a single subline. Choosing $x_r = 0$ means that we do not operate subline r. Next, f_{sy} represents the realized service frequency for OD-pair $(s,y) \in \mathscr{O}$, which serves as input for the estimation of the average travel time. Then, f_r represents the selected service frequency for subline $r \in \mathscr{R}$. This subline frequeny needs to be integer since we assume a periodic timetable (Assumption 2). Finally, a_r is a binary variable that indicates whether subline $r \in \mathscr{R}$ is operational or not. Our SFS formulation is formally presented in the following mathematical model.

$$z(x,\pi) := \sum_{r\in\mathscr{R}} x_r \left(W_1 + W_2 T_r \left\lfloor \frac{T}{T_r} \right\rfloor \right) + \sum_{(s,y)\in\mathscr{O}} B_{sy} \frac{1}{f_{sy}+1} \tag{1}$$

$$f_r \leq \frac{x_r}{T_r} \qquad (\forall r \in \mathscr{R}) \tag{2}$$

$$f_{sy} \leq \sum_{r\in\mathscr{R}} \Delta_{r,sy} f_r \qquad (\forall (s,y) \in \mathscr{O}) \tag{3}$$

$$f_{sy} \geq \Theta \qquad (\forall (s,y) \in \mathscr{O}) \tag{4}$$

$$x_r \leq a_r M \qquad (\forall r \in \mathscr{R}) \tag{5}$$

$$x_r \geq a_r T_r F \qquad (\forall r \in \mathscr{R}) \tag{6}$$

$$\sum_{r\in\mathscr{R}} x_r \leq N \tag{7}$$

$$x_0 \geq K \tag{8}$$

$$x_r \in \mathbb{Z}_{\geq 0} \qquad (\forall r \in \mathscr{R}) \tag{9}$$

$$f_r \in \mathscr{F} \qquad (\forall r \in \mathscr{R}) \tag{10}$$

$$a_r \in \{0,1\} \qquad (\forall r \in \mathscr{R}) \tag{11}$$

The objective function (1) of our model strives to establish a trade-off between the reduction of (i) operational-related costs emerging from the use of additional minibusses and vehicle running times, and (ii) costs related to passenger waiting times estimated as discussed in 3.1.3 and multiplied by the passenger demand. It is subject to the following constraints: Constraint (2) describes that the round-trip travel time of each subline $r \in \mathscr{R}$, T_r, together with the number of assigned vehicles to this subline x_r, provides an upper bound on the subline frequency f_r, namely $f_r \leq \frac{x_r}{T_r}$. Constraint (3) sets the service frequency f_{sy} of each OD-pair $(s,y) \in \mathscr{O}$ to be no larger than the total frequency assigned to all sublines r that serve OD-pair (s,y). Note that the 0-1 parameter $\Delta_{r,sy}$ allows us to only consider the minibusses assigned to sublines $r \in \mathscr{R}$ that serve the particular OD-pair (s,y). Constraint (4) ensures each OD-pair (s,y) is served at least with minimum frequency Θ, thus guaranteeing a minimum level of service. Constraint (5) enforces that when subline $r \in \mathscr{R}$ is operational and therefore $x_r > 0$, a_r should be equal to one. Otherwise, $a_r = 0$. Constraint (6) states that every subline $r \in \mathscr{R}$ should have at least a minimum frequency of F to be deemed operational. Constraint (7) is the fleetsize constraint ensuring that no more vehicles are used than the available fleet N. Constraint (8) ensures that at least K minibusses will serve all stops $s \in \mathscr{S}$ by being assigned to the original line serving all stops, subline 0. Constraint (9) restricts the domain of x_r to positive integers, and constraint (10) restricts the domain of frequency f_r to that of feasible frequencies, thus allowing to require a minimum frequency if the subline is selected for operation, and constraint (11) defines variable a_r as binary.

Program (Q) is a MIP because variables a_r, x_r, f_{sy} receive discrete values. Our MIP is nonlinear because the objective function (1) contains a division by one of the variables. Despite its nonlinearity, program (Q) belongs to the category of mixed-integer convex programs (MICP) that can be solved to optimality for mid-sized problems Bonami et al. (*31*).

Reformulation to a MILP

Following the ideas presented in Claessens et al. (*32*) and van der Hurk et al. (*33*), we reformulate the SFS to a MILP. Let $u_{f,sy}$ be a binary decision variable, where $u_{f,sy}=1$ if the OD-pair $(s,y)\in\mathscr{O}$ is operated with frequency $f\in\mathscr{F}$, and 0 otherwise. Let also τ_f be a parameter indicating the passenger waiting time cost for a given frequency $f\in\mathscr{F}$. Note that the precomputation of the passenger waiting time cost as a function of the frequency would allow for arbitrary functions. In particular, we could use $\tau_f=\frac{1}{f+1}$ as proposed earlier.

Then, constraint (4) can be rewritten as:

$$\sum_{f\in\mathscr{F}} f u_{f,sy} \geq \Theta \qquad \big(\forall (s,y)\in\mathscr{O}\big) \tag{12}$$

$$\sum_{f\in\mathscr{F}} u_{f,sy} = 1 \qquad \big(\forall (s,y)\in\mathscr{O}\big) \tag{13}$$

Note that constraint (13) ensures that each OD-pair $(s,y)\in\mathscr{O}$ is served only by one frequency $f\in\mathscr{F}$. In addition, constraint (12) ensures that the operating frequency $f\in\mathscr{F}$ of the OD-pair $(s,y)\in\mathscr{O}$ is at least equal to the *minimum allowed frequency*, Θ.

Reckon that the frequency $f\in\mathscr{F}$ of all minibusses serving a stop pair $(s,y)\in\mathscr{O}$ cannot exceed the number of trips per hour that can be performed by the minibusses, x_r, assigned to each subline $r\in\mathscr{R}$ (constraints (2),(3),(10)). Constraints (2),(3),(10) are reformulated as:

$$\begin{aligned} f\cdot u_{f,sy} &\leq \sum_{r\in\mathscr{R}} \Delta_{r,sy} f_r && \big(\forall f\in\mathscr{F}, \forall (s,y)\in\mathscr{O}\big)\\ f_r &\leq \frac{x_r}{T_r} && \big(\forall r\in\mathscr{R}\big)\\ f_r &\in \mathscr{F} && \big(\forall r\in\mathscr{R}\big) \end{aligned} \tag{14}$$

Note that if $u_{f,sy}=0$ for some frequency $f\in\mathscr{F}$ the inequality constrain (3) holds because the OD-pair is not served by frequency f. Furthermore, $u_{f,sy}=1$ can only hold for a frequency f that does not exceed the maximum number of trips per hour that can be performed when fully utilizing the minibusses x_r assigned to sublines $r\in\mathscr{R}$, that is, if $f\leq\sum_{r\in\mathscr{R}}\Delta_{r,sy}f_r$.

Finally, reckon that the objective function was stated as:

$$z(x,f) := \underbrace{W_1\sum_{r\in\mathscr{R}} x_r}_{\text{cost of operating the minibusses}} + \underbrace{W_2\sum_{r\in\mathscr{R}} x_r T_r \left\lfloor \frac{T}{T_r}\right\rfloor}_{\text{vehicle running times cost}} + \underbrace{\sum_{(s,y)\in\mathscr{O}} B_{sy}\frac{1}{f_{sy}+1}}_{\text{passenger waiting times cost}}$$

where the third term is nonlinear.

Given that we now pre-calculate the waiting time costs $t_f=\frac{1}{f+1}$ for each frequency $f\in\mathscr{F}$, the third term of the objective function can be replaced by the linear term:

$$\sum_{(s,y)\in\mathscr{O}} B_{sy}\sum_{f\in\mathscr{F}} t_f u_{f,sy}$$

Consequently, for any frequency $f\in\mathscr{F}$, we have $t_f u_{f,sy}=\frac{1}{f_{sy}+1}$ if we operate the OD-pair $(s,y)\in\mathscr{O}$ with that frequency, and $t_f u_{f,sy}=0$ otherwise. Our objective function is reformulated as:

$$\tilde{z}(x,u) := \sum_{r\in\mathscr{R}} x_r \left(W_1 + W_2 T_r \left\lfloor \frac{T}{T_r} \right\rfloor \right) + \sum_{(s,y)\in\mathscr{O}} B_{sy} \sum_{f\in\mathscr{F}} t_f u_{f,sy} \tag{15}$$

With this reformulation, our mixed-integer linear program (MILP) is:

$$\begin{aligned}
(\tilde{Q}) \quad & \min_{x,u} \quad \tilde{z}(x,u) \\
& \text{s.t.} \quad (x,a,u) \text{ satisfy Eqs. } (5)-(8),(12)-(15) \\
& \qquad a_r \in \{0,1\} && (\forall r \in \mathscr{R}) \\
& \qquad x_r \in \mathbb{Z}_{\geq 0} && (\forall r \in \mathscr{R}) \\
& \qquad u_{f,sy} \in \{0,1\} && (\forall f \in \mathscr{F}, \forall (s,y) \in \mathscr{O})
\end{aligned} \tag{16}$$

The resulting MILP guarantees global optimality and results in significant computational improvements over (Q) because of its linear nature.

ASSIGNING MINIBUSSES UNDER TRAVEL TIME AND PASSENGER DEMAND UNCERTAINTY

Autonomous minibusses operate in dedicated lanes and exhibit stable inter-station travel times. Nonetheless, the passenger demand might vary significantly in space and time introducing uncertainties when determining the number of vehicles assigned to sublines. In the remainder of this section we treat the passenger demand B as an uncertain parameter, and present two approaches to determine optimal subline frequencies under demand uncertainty where we let $\tilde{z}(x,u,b)$ denote the value of the objective function in dependence of the variables x,u and the uncertain demand B.

Stochastic optimization model: Minimizing the expected value of our objective function

One frequently-used approach to cope with parameter uncertainty is to search for a solution that optimizes the *expected* value of the objective function. In general, this requires knowledge of the *probability distributions* governing the uncertain parameters (in our case: the demand distribution). For our model, however, knowledge on the *expected* demand per OD-pair is sufficient to compute the solution minimizing the expectation of the objective function: due to the linearity of the expected value operator, and due to the fact that the uncertain demand variables only appear in the objective function, we have

$$\begin{aligned}
\mathbb{E}_B[\tilde{z}(x,u,B)] :=& \mathbb{E}\left[\sum_{r\in\mathscr{R}} x_r \left(W_1 + W_2 T_r \left\lfloor \frac{T}{T_r} \right\rfloor \right) + \sum_{(s,y)\in\mathscr{O}} B_{sy} \sum_{f\in\mathscr{F}} t_f u_{f,sy} \right] \\
=& \sum_{r\in\mathscr{R}} x_r \left(W_1 + W_2 T_r \left\lfloor \frac{T}{T_r} \right\rfloor \right) + \sum_{(s,y)\in\mathscr{O}} \mathbb{E}[B_{sy}] \sum_{f\in\mathscr{F}} t_f u_{f,sy}.
\end{aligned} \tag{17}$$

In our experiments, we estimate $\mathbb{E}[B_{sy}]$ by the average observed demand $\bar{B}_{sy}$ for OD-pair (s,y) to compute the number of vehicles and the frequency assignment that minimizes the expected value of our objective function.

Robust optimization model

In robust optimization, the travel demand between two stops B_{sy} is considered as an uncertain *environmental variable* that varies for reasons outside of our control and is the *adversary* of our system. Unlike in stochastic optimization, in robust optimization we do not need to know the probability distribution of the random variables, but only the range (e.g., $B_{sy} \in [B_{sy}^{min}, B_{sy}^{max}]$) in which it can fall. Note that the main difference between robust and stochastic optimization is that in robust optimization we optimize the outcome in the worst case, while in our stochastic optimization approach we optimize the outcome in the average case. As it is very unlikely that all values B_{sy} for $(s,y) \in \mathscr{O}$ take their worst case value simultaneously, we assume that the overall number of passengers is bounded by a number $\mathscr{Y}$. That is, we consider the uncertainty set $\mathscr{U} := \{B : B_{sy} \in [B_{sy}^{min}, B_{sy}^{max}], \sum_{(s,y)\in\mathscr{O}} B_{sy} \leq \mathscr{Y}\}$ for an adequately chosen $\mathscr{Y}$.

Robust optimization seeks to find the number of vehicles assigned to each subline r, such that our solution performs well at worst-case passenger demand scenarios. This robust optimization objective that explicitly considers the passenger demand uncertainty is formulated as a min(i)max problem:

$$
\begin{aligned}
(P) \quad & \min_{x,u} \max_{B \in \mathscr{U}} \quad \tilde{z}(x,u,B) & & \\
\text{s.t.} \quad & (x,a,u) \text{ satisfy Eqs. } (5)-(8),(12)-(15) & & \\
& a_r \in \{0,1\} & & (\forall r \in \mathscr{R}) \\
& x_r \in \mathbb{Z}_{\geq 0} & & (\forall r \in \mathscr{R}) \\
& u_{f,sy} \in \{0,1\} & & (\forall f \in \mathscr{F}, \forall (s,y) \in \mathscr{O})
\end{aligned}
\tag{18}
$$

Note that for any given passenger demand 'noise' $B^0 = \{B_{sy}^0\}$, we can find the optimal minibus assignment to sublines by solving the previously described minimization program $\tilde{Q}$ using B^0 as passenger demand input. In some problems, the worst values of $B = \{B_{sy}\}$ are easy to guess based on prior problem knowledge and the minimax problem is reduced to a classical minimization one. In our case though, the worst-case values of the environmental variables B depend on the settings of the design variables x in a way that is not intuitively obvious.

To solve our minimax problem, one can employ evolutionary algorithms Cramer et al., Lung and Dumitrescu (*34, 35*). However, they do not guarantee convergence and do not exploit the convexity of our objective function because they treat it as a black box. One prominent strategy that we select for our study is the minimax approximation strategy that relaxes the original problem by introducing and updating a small discrete set of points in the continuous space of the environmental variables Shimizu and Aiyoshi (*36*). For a discussion with respect to the optimality conditions of the minimax problem, we refer to Shimizu and Aiyoshi, Polak, Ye and Zhu (*36, 37, 38*).

The minimax problem P searches for the minibus assignment x, u that minimizes the worst-case performance $\max_B \tilde{z}(x,u,B)$. With the minimax approximation strategy this problem is relaxed by performing the maximization over a finite set $\mathscr{G}_e$ instead of all possible $B \in \mathscr{X}_e = ([B_{s,y}^{min}, B_{s,y}^{max}])_{s,y}$.

For any *discretization* $\mathscr{G}_e \subset \mathscr{X}_e$, we introduce the following robust optimization problem that replaces problem P

$$
\begin{aligned}
\tilde{P}(\mathscr{G}_e) : \min_{x,u} \max_{B} \quad & \tilde{z}(x,u,B) \\
\text{s.t.} \quad & (x,a,u) \text{ satisfy Eqs. } (5)-(8),(12)-(15) \\
& a_r \in \{0,1\} && \big(\forall r \in \mathscr{R}\big) \\
& x_r \in \mathbb{Z}_{\geq 0} && \big(\forall r \in \mathscr{R}\big) \\
& u_{f,sy} \in \{0,1\} && \big(\forall f \in \mathscr{F}, \forall (s,y) \in \mathscr{O}\big) \\
& B \in \mathscr{G}_e
\end{aligned}
\tag{19}
$$

Given $\mathscr{G}_e$, program $\tilde{P}(\mathscr{G}_e)$ has favorable mathematical properties compared to P. To solve this numerically, Marzat et al. (*39*) proposed to start with a set $\mathscr{G}_e$ of just one randomly chosen point $B^0 \in \mathscr{X}_e$. Then, $x^0 \triangleq \{\text{Solves } \tilde{P}(\mathscr{G}_e) \text{ for } \mathscr{G}_e = \{B^0\}\}$ is the best solution in the domain of our design variables. Given x^0, the next step searches for $B^1 \in \mathscr{X}_e$ that disturbs the overall performance as much as possible. To this end, we solve:

$$
T(x^0,u^0): \qquad \max_{B} \tilde{z}(x^0,u^0) \quad \text{s.t.} \quad
\begin{array}{l}
B_{s,y}^{min} \leq B_{s,y} \leq B_{s,y}^{max} \\
\sum_{(s,y)\in\mathscr{O}} B_{sy} \leq \mathscr{Y} \\
\text{Eq.}(15)
\end{array}
\tag{20}
$$

If the maximum possible demand disturbance B^1 does not worsen the performance too much, that is, $\tilde{z}(x^0,u^0,B^1) - \tilde{z}(x^0,u^0,B^0) < \varepsilon$ for some threshold $\varepsilon \in \mathbb{R}_{\geq 0}$, then x^0 is an *acceptable solution approximation* of the minimax problem $\tilde{P}$ and the search terminates. If not, the point B^1 is added to the set $\mathscr{G}_e$ and the procedure is repeated (see alg.1) Shimizu and Aiyoshi (*36*).

Algorithm 1 Minimax approximation via relaxation of the environmental variables

0: Set $\varepsilon \in \mathbb{R}_{\geq 0}$;

1: Choose randomly $B^0 \in \mathscr{U}$ and set $\mathscr{G}_e \leftarrow (B^0)$. Then, set $k = 0$.

2: Solve $\tilde{P}(\mathscr{G}_e)$ and obtain x^k;

3: Solve $T(x^k,u^k)$ and obtain (B^{k+1});

4: If $\tilde{z}(x^k,u^k,B^{k+1}) - \tilde{z}(x^k,u^k,B^k) < \varepsilon$, STOP. Else, extend $\mathscr{G}_e \leftarrow \mathscr{G}_e \cup \{B^{k+1}\}$, $k \leftarrow k+1$ and go to Step 2.

Note that when Algorithm 1 stops with a solution (x^k,u^k) and a scenario B^{k+1}, we have that

$$
\tilde{z}(x^k,u^k,B^{k+1}) < \min_{x,u} \max_{B\in\mathscr{G}} \tilde{z}(x,k,B) + \varepsilon \tag{21}
$$

because for an optimal solution (x^*, u^*) to (P) we have that

$$\max_{B\in\mathscr{U}} \tilde{z}(x^*,u^*,B) \geq \max_{B\in\mathscr{G}_e^{k-1}} \tilde{z}(x^*,u^*,B) \geq \max_{B\in\mathscr{G}_e^{k-1}} \tilde{z}(x^k,u^k,B) = \tilde{z}(x^k,u^k,B^k)$$
$$> \tilde{z}(x^k,u^k,B^{k+1}) - \varepsilon = \max_{B\in\mathscr{U}} \tilde{z}(x^k,u^k,B) - \varepsilon. \quad (22)$$

NUMERICAL EXPERIMENTS

Simulation setup

We create our simulation environment based on data from a pilot of an autonomous electric minibus project in Frankfurt, Germany. Autonomous minibusses operate in many European cities (e.g., Luxembourg, Lyon, Paris, Berlin) in the form of pilot projects that serve typically up to 4 bus stops covering a distance from 0.5km to 2km. The autonomous minibus line in Frankfurt is operated by 6-seat buses that drive completely independently at a maximum speed of 15 km per hour. The vehicles have a 700-meter long test track at their disposal, thus avoiding mixed traffic environments and maintaining stable inter-station travel times. Services are operated from 1pm to 7pm on a daily basis. The bus line is bi-directional, and we consider two control stops resulting in three potential sublines when considering the originally planned line as one of them (see Fig.2). In our simulation setup, our originally planned bi-directional line serves $\mathscr{S} = \langle 1,2,...,8\rangle$ stops. The minimum number of minibusses that need to be assigned to the original line that serves all stops is $K = 2$ and the total number of available minibusses is $N = 6$.

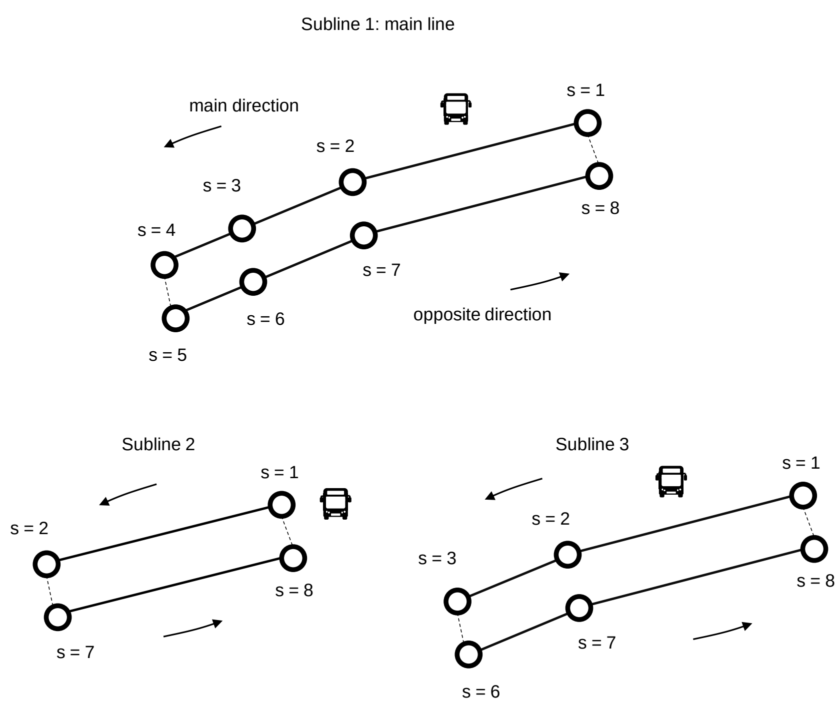

FIGURE 2 : Topology of the autonomous electric minibus line in Frankfurt, Germany

From Fig.2, the set of all lines where we can assign vehicles is $\mathscr{R} = \langle 1,2,3\rangle$. The time period of our planning is $T = 6$ h because the service operates daily from 1pm until 7pm. A subline is deemed operational if it has a frequency of at least $F = 1$ minibus per hour. The minimum allowed frequency to ensure a minimum level of service between any OD-pair $(s,y) \in \mathscr{O}$ is $\Theta = 2$ trips/h. Reckon that the minimum allowed frequency requirement does not hold for OD-pairs with no passenger demand, e.g., $(s,y) \notin \mathscr{O}$. The average round-trip travel times of the lines in set $\mathscr{R}$ are $(T_1,T_2,T_3) = (12,6,10)$ expressed in minutes. The cost of operating an extra minibus is set to $W_1 = 3$, and the cost of a marginal increase in the total running times $W_2 = 1.5$.

The number of passengers willing to travel between any OD-pair s,y can vary significantly from day to day. For this purpose, we present the passenger demand patterns among OD-pairs in Fig.3. In this figure, the lowest observed passenger demand, B_{sy}^{min}, the median, B_{sy}^{median}, and the highest observed passenger demand, B_{sy}^{max}, are reported for each OD-pair (s,y). Reckon that pairs $(s,y) \notin \mathscr{O}$ if there is no passenger demand associated with them.

The varying passenger demand presented in Fig.3 is used as input in our stochastic and robust optimization models (sections 5.2-5.3).

s	y	B_{sy}^{min}	B_{sy}^{median}	B_{sy}^{max}
1	2	62	208	304
1	3	73	319	421
1	4	2	7	36
2	3	51	250	397
2	4	2	6	31
3	4	0	1	17
5	6	0	2	16
5	7	1	9	52
5	8	0	2	14
6	7	41	140	203
6	8	12	67	182
7	8	31	110	279

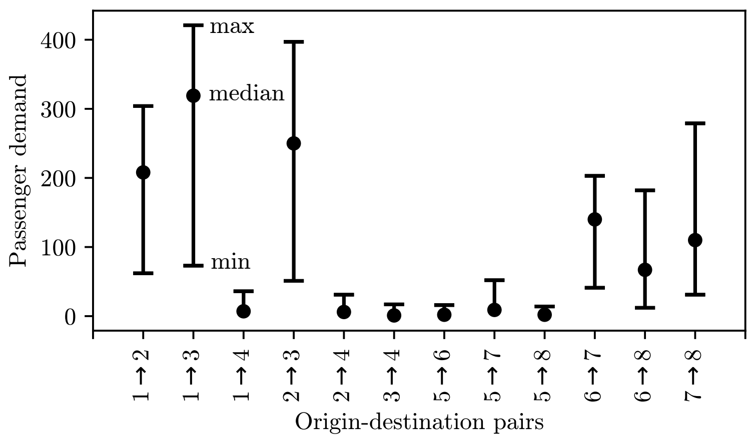

FIGURE 3 : Total number of passengers willing to travel from origin stop s to destination stop y in our homogeneous planning period $T = 6$ h.

Before proceeding to the stochastic and robust solutions, we calculate the solution of the baseline model that does not consider *sublines* and *passenger demand variations*. That is, the considered demand in the baseline model is the B_{sy}^{median} in Fig.3. Solving this baseline model returns solution:

$$(x_1,x_2,x_3) = (5,0,0)$$

with a frequency of 20 trips per hour for the original line. Note that $x_2,x_3 = 0$ since the baseline 'no sublines' model assigns vehicles only to the originally planned line.

Stochastic solution

To compute the stochastic solution, we first fit normal probability distributions to the passenger demand data expressed in Fig.3 with a mean value B_{sy}^{median}, $\forall(s,y) \in \mathscr{O}$ and standard deviation σ_{sy}, $\forall(s,y) \in \mathscr{O}$ derived from a distribution fitting process. Then, we apply the SAA method by generating $i = 1,2,...,1000$ Monte Carlo simulation scenarios via sampling from the respective normal distributions:

$$B^{i}_{sy} = \mathcal{N}(B^{median}_{sy}, \sigma^2_{sy}), \;\; \forall (s,y) \in \mathcal{O}, \;\; \forall i \in \{1,2,...,1000\} \tag{23}$$

The sampled B^{i}_{sy}, $i = 1,2,...,1000$ values are used to solve the stochastic optimization problem $(\hat{P})$ in the optimization solver LINDO 10.0 resulting in the following vehicle assignment:

$$(x_1, x_2, x_3) = (2, 0, 2)$$

Note that this stochastic optimization solution assigns four vehicles in total (two to the originally planned line and two to the 3rd subline presented in Fig.2). This differs from the baseline 'no sublines' solution that assigned five vehicles to the originally planned line.

Robust solution

To compute a robust solution that performs well at worst-case demand scenarios, we use the lower and upper boundary values $(B^{min}_{sy}, B^{max}_{sy})$ from Fig.3 to solve the min(i)max problem $\tilde{P}(\mathcal{G}_e)$ expressed in Eq.(19). To find a robust design, we apply Alg.1 with $\varepsilon = 0.05$. We initialize our set $\mathcal{G}_e$ by selecting a random noise $B^0_{sy} \in \mathcal{X}_e$ and setting $B^0_{sy} \leftarrow \mathcal{G}_e$. We set $\mathcal{Y} = 1.3 \sum_{(s,y)\in\mathcal{O}} B^{median}_{sy}$ as an upper bound of the overall passenger demand because the highest observed daily demand in the actual service is approximately 30% higher than the daily median. We initially let the random choice B^0_{sy} be equal to B^{median}_{sy}, $\forall (s,y) \in \mathcal{O}$.

The solution of $\tilde{P}(\mathcal{G}_e)$ can be easily obtained by solving the minimization program $\tilde{Q}$ for B^0_{sy}. That is, $x^0 \triangleq \{\text{Solves } \tilde{Q} \text{ for } B_{sy} \leftarrow B^0_{sy}\}$. The resulting solution in LINDO 10.0 is:

$$x^0 = (3, 0, 1)$$

with an objective function score $\tilde{z}(x^0, u^0, B^0) = 101.53$.

To obtain the worst-case passenger demand noise B^1_{sy} for x^0, u^0, we solve the maximization problem $T(x^0, u^0)$. This yields

$$B^1 = (304, 421, 2, 386, 2, 1, 1, 1, 1, 203, 18, 115) \text{ passengers per respective OD-pair}$$

with $\tilde{z}(x^0, u^0, B^1) = 533.51$. Given that $\tilde{z}(x^0, u^0, B^1) - \tilde{z}(x^0, u^0, B^0) \nleq \varepsilon$, we add B^1 to $\mathcal{G}_e$ and solve the updated $\tilde{P}(\mathcal{G}_e)$. The updated $\tilde{P}(\mathcal{G}_e)$ is solved by solving $\tilde{P}(\mathcal{G}_e)$ for all $B \in \mathcal{G}_e$ and return x^k that minimizes the worst-case performance for the environmental variables in $\mathcal{G}_e$. For B^1, the solution of $\tilde{Q}(B^1)$ is:

$$x^1 = (2, 0, 3)$$

and the performance of designs x^0, x^1 for the environmental variables yields the solution of $\tilde{P}(\mathcal{G}_e)$ with the lowest worst-case performance in $\mathcal{G}_e$: $x^* = (2, 0, 3)$. The corresponding performance of solution $x^* = (2, 0, 3)$ is 111.66.

After solving $T(x^1, u^1)$ we receive the same worst-case passenger demand noise:

$$B^2 = B^1 = (304, 421, 2, 386, 2, 1, 1, 1, 1, 203, 18, 115)$$

and the algorithm is terminated with solution

$$(x_1, x_2, x_3) = (2, 0, 3)$$

that performs best in the worst-case scenario of passenger demand noise.

Performance evaluation of the proposed solutions

To evaluate the performance of the proposed solutions of (a) the baseline ‘no sublines’, (b) the stochastic, and (c) the robust optimization models, we sample passenger demand data from Fig.3 generating a dataset of 1000 demand values for each OD-pair $(s,y) \in \mathscr{O}$. This sampled data generates 1000 daily scenarios and at each one of those scenarios we apply the solutions of the three models (baseline ‘no sublines’, stochastic, and robust). The purpose of this evaluation is to investigate which solution yields the best on-average performance and is more resilient to demand changes from day to day.

To implement the aforementioned solutions, we need to assign different numbers of vehicles to the three sublines resulting in deviating operational costs. Fig.6 presents the average daily running times when implementing the solutions of the no sublines, stochastic, and robust models indicating that the stochastic model yields a 20% improvement in terms of running times. This improvement was partly expected because the stochastic solution $(x_1,x_2,x_3) = (2,0,2)$ requires one less vehicle than the ‘no sublines’ $(x_1,x_2,x_3) = (5,0,0)$ and the ‘robust’ solution $(x_1,x_2,x_3) = (2,0,3)$. Additionally, the ‘robust’ solution results in less running times compared to the ‘no sublines’ solution because it assigns 3 vehicles to the third subline that does not serve stops 4 and 5.

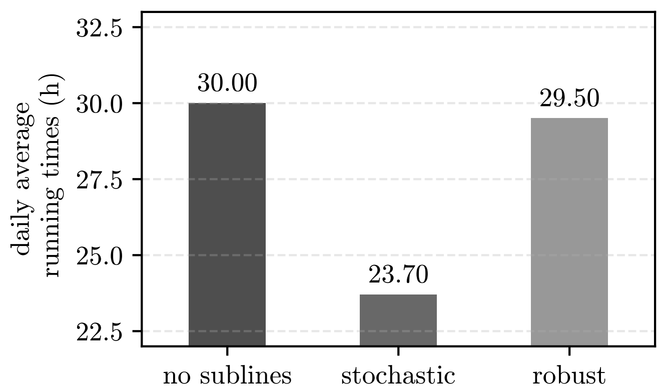

FIGURE 4 : Average vehicle running times on a daily basis when applying the ‘no sublines’ solution, the stochastic solution, and the robust solution

At each one of the 1000 days sampled from Fig.3 we apply the no sublines, the stochastic, and the robust solutions and we evaluate the performance of the respective objective functions at the end of the day for that passenger demand scenario. The performance of the stochastic against the no sublines solution is presented in the left sub-figure of Fig.5 and the performance of the robust against the no sublines solution is presented in the right sub-figure of the same graph. In Fig.5 one can observe whether the objective function improves or deteriorates when substituting the no sublines solution by the stochastic or the robust solutions that consider sublines.

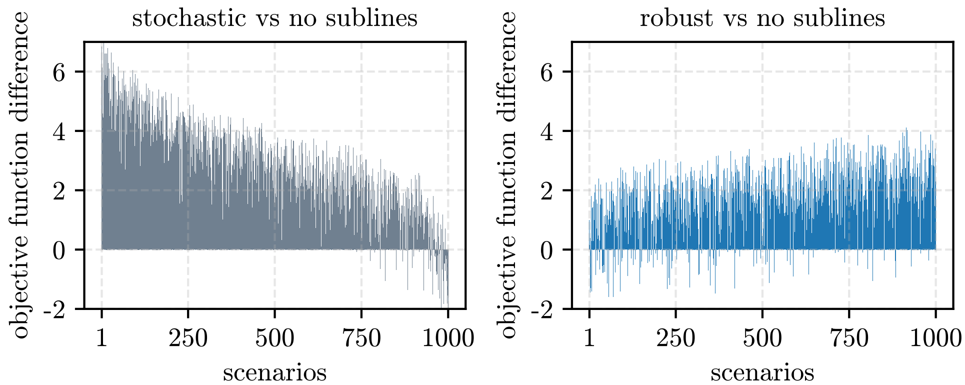

FIGURE 5 : Objective function performance improvement (or deterioration) of the stochastic and the robust solutions compared to the baseline 'no sublines' solution in each one of the 1000-day passenger demand scenarios

From Fig.5 one can note that the stochastic solution has clearly a better performance compared to the no sublines solution until day 750 since it improves the performance of the objective function by 0 to 7 points. After that, the no sublines solution performs better in several days. At this point, we should note that the days are ordered according to the performance of the no sublines solution presenting the days where the performance of the no sublines solution is deteriorating in an ascending order. Continuing with the interpretation of the results, interestingly, the robust solution has a more consistent patterns. However, it does not perform considerably better in any of the cases, and this can be probably explained by the fact that this solution tries to perform best at worst-case scenarios without trying to overperform in every scenario. The results from Fig.5 are summarized in the boxplot of Fig.6 which indicates the median performance, the interquartile ranges, the minimum and maximum performances according to the boxplot convention of Tukey, and the outlier performances for some days where the performances of our solutions lied outside of the minimum-maximum performance range.

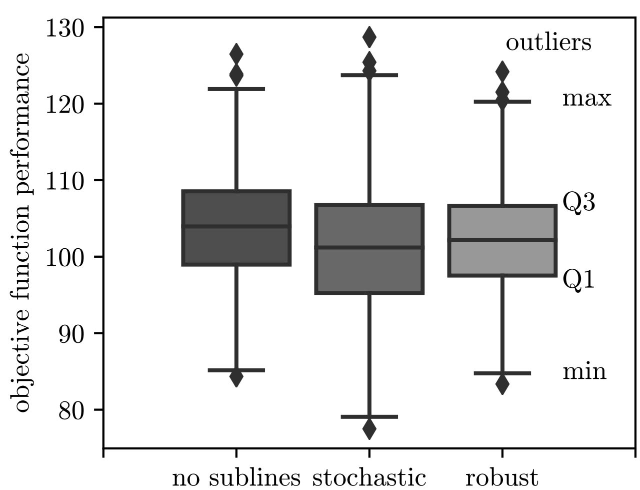

FIGURE 6 : Boxplot of the performances of the 'no sublines', 'stochastic', and 'robust' solutions in the 1000 demand scenarios

The quantitative values from Fig.6 demonstrate that:

- the stochastic solution has the best performance on most of the 1000 days resulting in an improved median value;
- the robust solution has the most stable performance with an interquartile range of only 9.1 indicating that its performance is very stable across different passenger demand scenarios;
- even though the stochastic solution has the best median performance, its interquartile range is very high and fluctuates significantly between good and bad performing days.

All in all, both the robust and the stochastic solutions have advantages and disadvantages. The robust solution appears to be the selection of choice for more conservative transport operators who want a minimum level of service under different demand fluctuations. In reverse, the stochastic solution is the selection of choice for transport operators who are willing to have the biggest benefits without being sensitive to the severe performance deterioration observed at particular days.

CONCLUDING REMARKS

In this work, we introduced a novel frequency setting model that assigns autonomous buses to sublines which serve the entire originally planned line or parts of it. This model, originally formulated as a MINLP, is reformulated as a MILP that can be easily solved to global optimality. Based on that model, we explicitly consider the uncertainty of passenger demand in the optimization process by formulating a stochastic and a robust optimization model, respectively. Notably, the stochastic model is based on the sample average approximation method and maintains an MILP formulation.

The performance of the stochastic and robust optimization models that assign vehicles to sublines to improve the trade-off between operational costs and passenger waiting time costs was compared against the baseline model that does not consider sublines. When evaluating the performance of those models in the autonomous minibus line operating in Frankfurt, we developed scenarios with different demand patterns to study the optimal distribution of vehicles among sublines under various demand conditions. When the demand levels among different OD-pairs differ significantly, the proposed stochastic and robust models return improved solutions. Namely, the vehicle assignment proposed by the stochastic model yields the best on-average performance with a 2.6% improvement, whereas the robust solution yields the most stable design with an interquartile range of 9.1.

In this work, the generated sublines serve segments of the originally planned lines and are a product of short-turning. In future research, this can be expanded by considering interlining lines where the same vehicle can be used by more than one line as an additional option. Finally, experiments can be expanded beyond autonomous minibusses. This will be of interest since regular bus services serve significantly more stops and have higher vehicle capacity which might lead to higher uncertainty in terms of passenger demand and vehicle loads.

REFERENCES

[1] Fabulos, *Robot buses as part of urban public transportation*, 2020.

[2] Muezner, P., *First Self-Driving Bus Service ioki Operating in Germany*, 2018.

[3] Duss, A.-C., *Contern to host Luxembourg's first driverless bus*, 2018.

[4] Stein, J. and N. Goebel, *Berlin tests driverless buses*, 2019.

[5] Modijefsky, M., *Lyon launches new autonomous public transport service*, 2019.

[6] Ceder, A., *Public transit planning and operation: Modeling, practice and behavior*. CRC press, 2016.

[7] Yu, B., Z. Yang, and J. Yao, Genetic algorithm for bus frequency optimization. *Journal of Transportation Engineering*, Vol. 136, No. 6, 2010, pp. 576–583.

[8] Szeto, W. Y. and Y. Wu, A simultaneous bus route design and frequency setting problem for Tin Shui Wai, Hong Kong. *European Journal of Operational Research*, Vol. 209, No. 2, 2011, pp. 141–155.

[9] Gkiotsalitis, K. and O. Cats, Reliable frequency determination: Incorporating information on service uncertainty when setting dispatching headways. *Transportation Research Part C: Emerging Technologies*, Vol. 88, 2018, pp. 187–207.

[10] Ceder, A. A., Public-transport vehicle scheduling with multi vehicle type. *Transportation Research Part C: Emerging Technologies*, Vol. 19, No. 3, 2011, pp. 485–497.

[11] Hassold, S. and A. A. Ceder, Public transport vehicle scheduling featuring multiple vehicle types. *Transportation Research Part B: Methodological*, Vol. 67, 2014, pp. 129–143.

[12] Ceder, A., Public transport scheduling. *Handbook of Transport Systems and Traffic Control*, Vol. 3, 2001, pp. 539–558.

[13] Gkiotsalitis, K. and F. Alesiani, Robust timetable optimization for bus lines subject to resource and regulatory constraints. *Transportation Research Part E: Logistics and Transportation Review*, Vol. 128, 2019, pp. 30–51.

[14] Ibarra-Rojas, O., F. Delgado, R. Giesen, and J. Muñoz, Planning, operation, and control of bus transport systems: A literature review. *Transportation Research Part B: Methodological*, Vol. 77, 2015, pp. 38–75.

[15] Gkiotsalitis, K. and O. Cats, Public transport planning adaption under the COVID-19 pandemic crisis: literature review of research needs and directions. *Transport Reviews*, 2020, pp. 1–19.

[16] Gkiotsalitis, K. and O. Cats, At-stop control measures in public transport: Literature review and research agenda. *Transportation Research Part E: Logistics and Transportation Review*, Vol. 145, 2021, p. 102176.

[17] Ceder, A., Bus frequency determination using passenger count data. *Transportation Research Part A: General*, Vol. 18, No. 5-6, 1984, pp. 439–453.

[18] dell'Olio, L., A. Ibeas, and F. Ruisánchez, Optimizing bus-size and headway in transit networks. *Transportation*, Vol. 39, No. 2, 2012, pp. 449–464.

[19] Cipriani, E., S. Gori, and M. Petrelli, Transit network design: A procedure and an application to a large urban area. *Transportation Research Part C: Emerging Technologies*, Vol. 20, No. 1, 2012, pp. 3–14.

[20] Cats, O. and S. Glück, Frequency and vehicle capacity determination using a dynamic transit assignment model. *Transportation Research Record*, Vol. 2673, No. 3, 2019, pp. 574–585.

[21] Delle Site, P. and F. Filippi, Service optimization for bus corridors with short-turn strategies and variable vehicle size. *Transportation Research Part A: Policy and Practice*, Vol. 32, No. 1, 1998, pp. 19–38.

[22] Cortés, C. E., S. Jara-Díaz, and A. Tirachini, Integrating short turning and deadheading in the optimization of transit services. *Transportation Research Part A: Policy and Practice*, Vol. 45, No. 5, 2011, pp. 419–434.

[23] Verbas, İ. Ö. and H. S. Mahmassani, Exploring trade-offs in frequency allocation in a transit network

using bus route patterns: Methodology and application to large-scale urban systems. *Transportation Research Part B: Methodological*, Vol. 81, 2015, pp. 577–595.

[24] Verbas, İ. Ö. and H. S. Mahmassani, Integrated frequency allocation and user assignment in multimodal transit networks: Methodology and application to large-scale urban systems. *Transportation Research Record*, Vol. 2498, No. 1, 2015, pp. 37–45.

[25] Gkiotsalitis, K., Z. Wu, and O. Cats, A cost-minimization model for bus fleet allocation featuring the tactical generation of short-turning and interlining options. *Transportation Research Part C: Emerging Technologies*, Vol. 98, 2019, pp. 14–36.

[26] Furth, P. G. and N. H. Wilson, Setting frequencies on bus routes: Theory and practice. *Transportation Research Record*, Vol. 818, No. 1981, 1981, pp. 1–7.

[27] Welding, P., The instability of a close-interval service. *Journal of the operational research society*, Vol. 8, No. 3, 1957, pp. 133–142.

[28] Hickman, M. D., An analytic stochastic model for the transit vehicle holding problem. *Transportation Science*, Vol. 35, No. 3, 2001, pp. 215–237.

[29] Bartholdi III, J. J. and D. D. Eisenstein, A self-coördinating bus route to resist bus bunching. *Transportation Research Part B: Methodological*, Vol. 46, No. 4, 2012, pp. 481–491.

[30] Cats, O., Regularity-driven bus operation: Principles, implementation and business models. *Transport Policy*, Vol. 36, 2014, pp. 223–230.

[31] Bonami, P., M. Kilinç, and J. Linderoth, Algorithms and software for convex mixed integer nonlinear programs. In *Mixed integer nonlinear programming*, Springer, 2012, pp. 1–39.

[32] Claessens, M., N. M. van Dijk, and P. J. Zwaneveld, Cost optimal allocation of rail passenger lines. *European Journal of Operational Research*, Vol. 110, No. 3, 1998, pp. 474–489.

[33] van der Hurk, E., H. N. Koutsopoulos, N. Wilson, L. G. Kroon, and G. Maróti, Shuttle planning for link closures in urban public transport networks. *Transportation Science*, Vol. 50, No. 3, 2016, pp. 947–965.

[34] Cramer, A. M., S. D. Sudhoff, and E. L. Zivi, Evolutionary algorithms for minimax problems in robust design. *IEEE Transactions on Evolutionary Computation*, Vol. 13, No. 2, 2009, pp. 444–453.

[35] Lung, R. I. and D. Dumitrescu, A new evolutionary approach to minimax problems. In *Evolutionary Computation (CEC), 2011 IEEE Congress on*, IEEE, 2011, pp. 1902–1905.

[36] Shimizu, K. and E. Aiyoshi, Necessary conditions for min-max problems and algorithms by a relaxation procedure. *IEEE Transactions on Automatic Control*, Vol. 25, No. 1, 1980, pp. 62–66.

[37] Polak, E., *Basics of Minimax Algorithms*, Springer US, Boston, MA, pp. 343–369, 1989.

[38] Ye, J. and D. Zhu, Optimality conditions for bilevel programming problems. *Optimization*, Vol. 33, No. 1, 1995, pp. 9–27.

[39] Marzat, J., E. Walter, and H. Piet-Lahanier, A new expected-improvement algorithm for continuous minimax optimization. *Journal of Global Optimization*, Vol. 64, No. 4, 2016, pp. 785–802.