



# Cooperative sequencing games with position-dependent learning effect

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## ABSTRACT

This paper extends sequencing games as introduced by Curiel et al. (1989) to the setting with a position-dependent learning effect. We show that these games are balanced, and analyze the family of equal gains sharing (EGS) rules. In contrast to games without learning effect, we show that only a specific class of EGS rules leads to core allocations. This allocation rule is characterized axiomatically, and we also study its relationship with the  $\beta$ -rule, earlier introduced by Curiel et al. (1994).

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## 1. Introduction

Research on the intersection of cooperative game theory and scheduling was initiated by [7]. They introduced a class of cooperative games called sequencing games. They arise from the following single machine sequencing problem: There is one single machine set up to process a finite number of  $n$  jobs, together with an initial given order of these jobs. Each job has a processing time and is owned by one agent, and each agent is interested in minimizing its own job's completion time. Since the initial order is usually not the optimal one for minimizing the total costs of all players, any coalition of agents can in principle realize cost savings by changing their relative positions in the schedule. In this situation, one of the fundamental questions in cooperative sequencing games is how the globally optimal solution can be supported by redistributing the total cost savings among the agents so that no coalition has an incentive to deviate from it. These stable allocations are precisely the core of the underlying cooperative game.

[7] showed that such sequencing games are convex and therefore core allocations exist. They also introduced and characterized the equal gain splitting (EGS) rule, which can be interpreted as an algorithmic procedure to compute a core allocation. During the past few decades, different types of variations of sequencing situations and games have been presented in the literature, including ready times [11], due dates [2], multiple machines [3,12,17],

grouped jobs [4,10] multistage situations [5,6] and sequencing situations without an initial order [13].

A common assumption in all of the above works is that the processing times of jobs are constants and independent of their positions during the scheduling process. However, it is a common assumption that facilities such as workers, or computers augmented with artificial intelligence can improve their performance over time, by processing jobs. As a result, the processing time will be shorter if a job is scheduled in a later position. This phenomenon is well known in the literature as a “learning effect”, which was introduced e.g. by [1] who considers the actual processing time of a job as a decreasing power function of its position in the schedule.

The purpose of this note is to extend cooperative sequencing games to the situation where a position-dependent learning effect exists, meaning that the processing time of a job also depends on the number of jobs that are scheduled before it. We call these games LE (learning effect) sequencing games. The question that we ask is whether the results obtained by [7] for (ordinary) sequencing games will still hold in situations with a learning effect. In brief, the answer is yes, but with some non-trivial extensions and modifications of earlier ideas and proof techniques. Let us next sketch the major contributions of this note.

For sequencing games, [7] defined the worth of a coalition as the maximal cost savings it can obtain by admissible reordering the members of this coalition. However, a problem arises if we adopt this definition to LE sequencing games: the non-members who are placed later than the coalition in consideration will also benefit from the cooperation of this coalition, that is, their completion times will be reduced. We propose, in Section 3, how the worth of a coalition can be defined in LE sequencing games.

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How much of the “external” benefits of cooperative behavior should be allocated to the cooperative coalition, will be measured by a share function, introduced in Section 3. We show, maybe unsurprisingly, that the resulting LE sequencing games are balanced, hence have a nonempty core. In particular, LE sequencing game in which the cooperating coalition obtains the full benefits of cooperation, are even convex.

In Section 4, we then analyze how core allocations can be computed through adjacent exchanges of positions and EGS allocation rules. The main difficulty, in contrast to earlier works, lies in the fact that the cost savings contributed by an adjacent exchange does not only depend on the processing times of the two players involved, but also on the positions where these two players are in the schedule. This means that the outcomes could differ depending on the orders of neighbor switches. Unlike with [7], who showed that EGS rules always yield a core allocation no matter in which order the adjacent exchanges are realized, we show that *not* all LE-EGS rules lead to core elements for LE sequencing games. Interestingly, though, we identify one particular order of adjacent exchanges, and prove that it ensures that the corresponding LE-EGS rule yields a core element. This particular allocation rule is called the  $\Gamma$ -rule, and we provide its axiomatic characterization. We finally discuss the relationship between the  $\Gamma$ -rule and the  $\beta$ -rule that was earlier introduced by [9].

At the end of the paper, in Section 5, we discuss two possible extensions of LE sequencing games. Section 2 contains the notation and basic definitions.

## 2. Preliminaries

### 2.1. Cooperative game theory

A *cooperative game* is a pair  $(N, v)$ , where  $N$  is a nonempty, finite set and  $v : 2^N \rightarrow \mathbb{R}$  is a characteristic function satisfying  $v(\emptyset) = 0$ . An element of  $N$  (notation:  $i \in N$ ) and a subset  $S$  of  $N$  (notation:  $S \in 2^N$  with  $S \neq \emptyset$ ) are called a *player* and *coalition* respectively. The associated real number  $v(S)$  is called the *worth* of coalition  $S$ . The *core* of a cooperative game  $(N, v)$  is defined by

$$C(N, v) = \left\{ x \in \mathbb{R}^n \mid x(N) = v(N), x(S) \geq v(S) \text{ for all } S \subseteq N \right\},$$

where  $x(S) = \sum_{i \in S} x_i$ . The core is the set of efficient allocations of  $v(N)$  such that there is no coalition with an incentive to split off from the grand coalition. The core of a game can be empty and a game which has a nonempty core is called *balanced*. A game  $(N, v)$  is called *superadditive* if  $v(S) + v(T) \leq v(S \cup T)$  for any  $S, T \in 2^N$  with  $S \cap T = \emptyset$ . A game  $(N, v)$  is said to be *convex* if  $v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$  for every  $i \in N$  and for every  $S \subseteq T \subseteq N \setminus \{i\}$ .

### 2.2. Sequencing situations

There is a queue of  $n$  players, each of whom owns a single job which has to be processed on a machine. Player and job will be used interchangeably. The finite set of players is denoted by  $N$  where  $|N| = n$ . A processing order on the players is defined by a bijection  $\sigma : N \rightarrow \{1, \dots, n\}$ , where  $\sigma(i) = j$  means that player  $i$  is in position  $j$ . The initial, given order is denoted by  $\sigma_0$  and the set of all orders on  $N$  is denoted by  $\Pi_N$ . For every  $i \in N$ , player  $i$  has a processing time  $p_i$ .

We call a coalition  $S \subseteq N$  *connected* with respect to  $\sigma \in \Pi_N$  if for all  $i, j \in S$  and  $k \in N$  such that  $\sigma(i) < \sigma(k) < \sigma(j)$  it holds that  $k \in S$ . A  $\sigma$ -*component* of  $S$  is a maximally connected subset of  $S$  with respect to  $\sigma$ . Given a coalition  $S$ ,  $S/\sigma$  denotes the set of  $\sigma$ -components of  $S$ . Finally, the set of coalitions that are connected with respect to  $\sigma$  is denoted by  $con(\sigma)$ .

The set of predecessors of player  $j$  with respect to  $\sigma$  is defined by  $P(\sigma, i) = \{j \in N \mid \sigma(j) < \sigma(i)\}$  and the set of successors of player  $i$  with respect to  $\sigma$  is defined by  $F(\sigma, i) = \{j \in N \mid \sigma(j) > \sigma(i)\}$ . We also define  $\bar{P}(\sigma, i) = P(\sigma, i) \cup i$  and  $\bar{F}(\sigma, i) = F(\sigma, i) \cup i$ . Given  $\sigma_0$ , a processing order  $\sigma \in \Pi_N$  is called *admissible* for  $S$  if it satisfies the following condition:

$$P(\sigma_0, j) = P(\sigma, j) \text{ for all } j \in N \setminus S.$$

This means that only players inside  $\sigma_0$ -components of  $S$  are allowed to exchange their relative positions with each other, and in particular, all players outside  $S$  remain at the same positions as in  $\sigma_0$ . The set of all admissible orders for a coalition  $S$  is denoted by  $\Pi_S$ .

An *SPT* order (shortest processing time first) is the order in which the jobs are arranged according to non-decreasing processing times. Denote by  $\sigma_S$  the order which is attained from  $\sigma_0$  by reordering the members in each  $\sigma_0$ -component of  $S$  with respect to the SPT order, i.e., (i)  $\sigma_S(i) = \sigma_0(i)$  for every  $i \in N \setminus S$ , and (ii)  $\sigma_S(i) < \sigma_S(j)$  for every  $i, j \in T$  and every  $T \in S/\sigma_0$  such that  $p_i < p_j$ .

Let  $\sigma_0 \in \Pi_N$ . A cooperative game  $(N, v)$  is called  $\sigma_0$ -*component additive* if it satisfies the following three conditions:

- (i)  $v(i) = 0$  for all  $i \in N$ ,
- (ii)  $(N, v)$  is superadditive, and
- (iii)  $v(S) = \sum_{T \in S/\sigma_0} v(T)$  for all  $S \in 2^N$ .

[14] showed that  $\sigma_0$ -component additive games are always balanced.

## 3. Sequencing games with a learning effect

In a sequencing situation with learning effect, the machine has the ability to improve (by processing jobs). As a result, the later a job is scheduled in the sequence of jobs, the shorter its processing time. We assume that each player  $i \in N$  has a *nominal processing time*  $p_i$ . Given an order  $\sigma \in \Pi_N$ , the *actual processing time* of any job  $i$  decreases as a function of its position, and it equals

$$\sigma(i)^a p_i,$$

where  $a \leq 0$  is the so-called *learning index* [1]. In this paper, we assume that there is no idle time between jobs. Hence, the completion time of player  $i$  is

$$C(\sigma, i) = \sum_{\sigma(j) \leq \sigma(i)} \sigma(j)^a p_j,$$

which equals the cost of a job  $i$  under sequence  $\sigma$ . In other words, the cost of a job equals the time it spends in the system.

Define an *LE sequencing situation* by a 4-tuple  $(N, \sigma_0, p, a)$ , where  $N = \{1, \dots, n\}$  is the set of  $n$  players,  $\sigma_0 \in \Pi_N$  the initial order on the jobs,  $p = (p_i)_{i \in N} \in \mathbb{R}_+^n$  the vector representing the nominal processing times, and  $a$  the learning index.

For any order  $\sigma \in \Pi_N$ , the total costs of all players with respect to  $\sigma$  is  $\sum_{i \in N} C(\sigma, i)$ .  $\sigma$  is called *optimal* if the total costs of all players with respect to  $\sigma$  is minimized. [1] proved that if the jobs are arranged according to non-decreasing nominal processing times, i.e. in SPT order, the total costs of all players are minimal.

The following example shows that, in LE sequencing situations (with given  $\sigma_0$ ), even if  $\sigma \in \Pi_S$ , that is,  $\sigma$  is admissible for coalition  $S$ , player set  $S$  may have an “external” effect also on players outside  $S$ , because of the learning effect.

**Example 1.** Let  $N = \{1, 2, 3\}$ ,  $\sigma_0 = (1, 2, 3)$ ,  $p = \{3, 2, 1\}$ , and  $a = -1$ . Consider the coalition  $S = \{1, 2\}$ . If players 1 and 2 are willing to switch their positions, the total costs of coalition  $\{1, 2\}$  can be reduced by 1.5. At the same time, this switch decreases the completion times of  $\{1, 2\}$  by 0.5. Now, because there is no idle time, player 3 enjoys cost savings of 0.5, too.

Consequently, the question arises if and how much of these “external” cost savings should be attributed to coalition  $S$ ? We suggest to capture this issue by defining a *share function*  $\lambda : 2^N \rightarrow [0, 1]$ . This mapping represents a “tax rate” that is imposed by coalition  $S$ , on the members outside  $S$ . That said, we can fully define an *LE sequencing game*  $(N, v^\lambda)$  by defining the worth of a coalition  $S \subseteq N$  by

$$v^\lambda(S) = \max_{\sigma \in \Pi_S} \left\{ \sum_{i \in S} \Delta C(\sigma, i) + \lambda(S) \sum_{j \in N \setminus S} \Delta C(\sigma, j) \right\}, \quad (1)$$

where  $\Delta C(\sigma, i) = C(\sigma_0, i) - C(\sigma, i)$ . In particular, if  $\lambda(S) = 0$  for all  $S \subseteq N$ , the definition of the worth of coalitions in LE sequencing games concurs with those of earlier works. For notational simplicity, in the following we denote by  $(N, v)$  the LE sequencing games when  $\lambda(S) = 1$  for all  $S \subseteq N$ .

As [15] proved that the SPT rule also solves the makespan minimization problem with a (positional) learning effect, it follows that the order that achieves the maximal cost savings, and hence defines the worth of a coalition  $S$ , is exactly the SPT order. Recalling that  $\sigma_S$  is the order which is attained from  $\sigma_0$  by reordering the members in each  $\sigma_0$ -component of  $S$  with respect to the SPT order, the following is an easy exercise.

**Theorem 1.** *Let  $(N, \sigma_0, p, a)$  be an LE sequencing situation and  $(N, v^\lambda)$  any corresponding LE sequencing game. Then for any  $S \subseteq N$ ,*

$$v^\lambda(S) = \sum_{i \in S} \Delta C(\sigma_S, i) + \lambda(S) \sum_{j \in N \setminus S} \Delta C(\sigma_S, j).$$

It is also not hard to check that LE sequencing games  $(N, v)$  are  $\sigma_0$ -component additive, and thus they are balanced. Moreover, for any core element  $x \in C(N, v)$ , we have for any  $S \subseteq N$ ,  $\sum_{i \in S} x_i \geq v(S) \geq v^\lambda(S)$ . So we can conclude that LE sequencing game  $(N, v^\lambda)$  is balanced for any  $\lambda$ .

**Theorem 2.** *Let  $(N, \sigma_0, p, a)$  be an LE sequencing situation and  $(N, v^\lambda)$  the corresponding cooperative game. Then  $(N, v^\lambda)$  is balanced.*

[2] provided a simple expression for the coefficients in the unique linear decomposition of a  $\sigma_0$ -component additive game into unanimity games. By using this result, we can also give an expression for the value of these coefficients for an LE sequencing game  $(N, v)$  since it is  $\sigma_0$ -component additive. In fact, these coefficients can be proved to be nonnegative. Since unanimity games are convex, we therefore obtain that also  $(N, v)$  is convex. The proof of Theorem 3 is only technical but similar to that in [2], so we relegate it, together with the proofs of Theorems 1 and 2, to the supplement.

**Theorem 3.** *Let  $(N, \sigma_0, p, a)$  be an LE sequencing situation. Then the corresponding LE sequencing game  $(N, v)$  is convex.*

#### 4. The family of LE-EGS rules and the $\Gamma$ -rule

Motivated by the fact that core allocations for  $(N, v)$  are also core allocations for  $(N, v^\lambda)$  for any  $\lambda$ , we focus on LE sequencing games  $(N, v)$ . This is the setting where the worth of a coalition  $S$  includes all additional benefits that their cooperative behavior generates. In that sense, the constraints that define the core of  $(N, v)$  are tightest, and this version of the game could be considered as the “hardest”. One could also consider variations of the game  $(N, v^\lambda)$  where an additional player (a system designer) obtains the taxes implied by  $\lambda$ , but this would go beyond the scope of this note.

#### 4.1. The family of LE-EGS rules

Let  $(N, \sigma_0, p, a)$  be an LE sequencing situation. A pair  $(i, j)$  is *inverted* in  $S$  if  $i, j \in S$ ,  $\sigma_0(i) < \sigma_0(j)$ , and  $p_i > p_j$ . Given a connected coalition  $S \subseteq N$ , let us denote by  $I^S$  the number of all inverted pairs from  $S$  with respect to  $\sigma_0$ . An *adjacent exchange* is a mapping  $\tau : \Pi_N \rightarrow \Pi_N$ , and we denote the set of all adjacent exchanges by  $\Lambda_n$ . W.l.o.g. we can overload notation and assume that  $\tau$  exchanges the jobs on positions  $\tau$  and  $\tau + 1$  (i.e., we simply represent elements of  $\Lambda_n$  by the first position  $\tau$  of the adjacent exchange). To be precise, given  $\sigma \in \Pi_N$ , if  $\tau(\sigma) = \sigma'$ , we have  $\sigma^{-1}(\tau) = \sigma'^{-1}(\tau + 1)$ ,  $\sigma^{-1}(\tau + 1) = \sigma'^{-1}(\tau)$ , and  $\sigma^{-1}(k) = \sigma'^{-1}(k)$  for any  $k \neq \tau, \tau + 1$ .

The difference in total costs before and after an adjacent exchange  $\tau$  on order  $\sigma$  can be easily calculated. If we assume that  $\sigma(i) = \tau$  and  $\sigma(j) = \tau + 1$ , it will later be convenient to refer to this difference as  $g(\tau, \sigma, i, j)$ . It equals

$$g(\tau, \sigma, i, j) := (p_i - p_j) [(n - \tau + 1)(\tau)^a - (n - \tau)(\tau + 1)^a]. \quad (2)$$

Notice that the cost savings obtained by an adjacent exchange depends not only on the processing times of the two jobs  $i$  and  $j$ , but also on the position  $\tau$  of the adjacent exchange. The earlier the position, the larger the cost savings. For later use, it is convenient if we define

$$\theta(\tau) := [(n - \tau + 1)(\tau)^a - (n - \tau)(\tau + 1)^a]. \quad (3)$$

Any order  $\sigma$  can be obtained from any  $\sigma_0$  by successive adjacent exchanges. We define a *permutation process* as an ordered set of adjacent exchanges  $\rho = \{\tau_1, \dots, \tau_m\}$ ,  $\tau_1, \dots, \tau_m \in \Lambda_n$  such that  $\tau_1(\sigma_0) = \sigma_1$ ,  $\tau_2(\sigma_1) = \sigma_2$ , ...,  $\tau_m(\sigma_{m-1}) = \sigma$ . Let us write  $\rho(\sigma_0) = \sigma$  if permutation process  $\rho$  ends in  $\sigma$ . Moreover, recall that  $\sigma_S$  is the order attained from  $\sigma_0$  by reordering the members in each  $\sigma_0$ -component of  $S$  in SPT order, and  $I^S$  is the number of inverted pairs in  $S$  with respect to  $\sigma_0$ . Let us call a permutation process *feasible* if  $\rho(\sigma_0) = \sigma_S$  and  $|\rho| = I^S$ . That is,  $\rho$  successively exchanges inverted pairs of  $S$ . Finally, by  $\mathcal{P}(\sigma_0, \sigma_S)$  we denote the set of all feasible permutation processes for  $S$  and  $\sigma_0$ .

We now introduce the LE-EGS family, which is inspired by the equal gain splitting (EGS) rule defined and characterized in [7]. The idea of the EGS rule is to divide the cost savings obtained by an adjacent exchange equally between the two involved players. Here, we adopt this idea and define the *LE-EGS $^\rho$  rule* as follows. Let  $\rho = \{\tau_1, \dots, \tau_{I^N}\} \in \mathcal{P}(\sigma_0, \sigma_N)$  be a feasible permutation process so that  $\rho(\sigma_0) = \sigma_N$ ,  $\sigma_k = \tau_k(\sigma_{k-1})$  for any  $1 \leq k \leq I^N$ , and let  $i_k = \sigma_{k-1}^{-1}(\tau_k)$ ,  $j_k = \sigma_{k-1}^{-1}(\tau_k + 1)$  be the two players involved in the  $k$ th adjacent exchange. Then let the allocation to player  $i \in N$  be

$$LE\text{-EGS}_i^\rho(N, \sigma_0, p, a) = \frac{1}{2} \sum_{k=1}^{I^N} g(\tau_k, \sigma_{k-1}, i_k, j_k) \delta_i(i_k, j_k),$$

where  $\delta_i$  is simply an indicator variable to collect the payments made to player  $i$ ,

$$\delta_i(i_k, j_k) = \begin{cases} 1 & \text{if } i_k = i \text{ or } j_k = i \\ 0 & \text{otherwise.} \end{cases}$$

The family of all LE-EGS rules is defined by

$$LEF(N, \sigma_0, p, a) = \{LE\text{-EGS}^\rho(N, \sigma_0, p, a) \mid \rho \in \mathcal{P}(\sigma_0, \sigma_N)\}.$$

At this point, it is interesting to know whether any member of this family yields a core allocation. The following example shows that the answer is negative.

**Example 2.** Let  $(N, \sigma_0, p, a)$  be an LE sequencing situation, where  $N = \{1, 2, 3\}$ ,  $\sigma_0 = (1, 2, 3)$ ,  $p = \{3, 2, \frac{9}{2}\}$  and  $a = -1$ . Then it

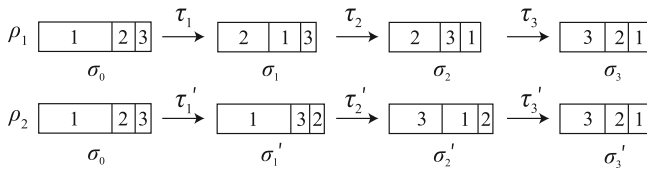


Fig. 1. The feasible permutation processes  $\rho_1$  and  $\rho_2$ .

Table 1

Worths of the LE sequencing game  $(N, v)$ .

$S$	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$v(S)$	0	0	0	2	0	$\frac{2}{15}$	$\frac{16}{5}$

can be observed that  $\sigma_N = (3, 2, 1)$ ,  $I^N = 3$  and there are two feasible permutation processes  $\rho_1 = \{\tau_1, \tau_2, \tau_3\}$ ,  $\rho_2 = \{\tau'_1, \tau'_2, \tau'_3\}$  which are illustrated in Fig. 1.

We can easily compute that  $LE\text{-EGS}^{\rho_1} = (\frac{7}{5}, \frac{6}{5}, \frac{3}{5})$  and  $LE\text{-EGS}^{\rho_2} = (\frac{23}{15}, \frac{2}{5}, \frac{19}{15})$ . The worths of the corresponding LE sequencing game  $(N, v)$  are displayed in Table 1. We can see that  $LE\text{-EGS}^{\rho_1} \in C(N, v)$ , but  $LE\text{-EGS}^{\rho_2} \notin C(N, v)$ .

#### 4.2. The $\Gamma$ -rule

We now introduce a particular feasible permutation process  $\rho \in \mathcal{P}(\sigma_0, \sigma_N)$  which actually yields a core allocation. It is defined by the following sequence of adjacent exchanges:

Step 1. Set  $k = 1$ .

Step 2. Find the earliest two neighboring jobs  $i_k, j_k$  that are inverted in order  $\sigma_{k-1}$ . If no such pair exists, then stop (and  $\sigma_{k-1} = \sigma_N$ ). Formally,

$$i_k := \arg \min \{ \sigma_{k-1}(\ell) \mid p_\ell > p_j, \sigma_{k-1}(j) = \sigma_{k-1}(\ell) + 1 \}.$$

Step 3. While  $(i_k, j_k)$  is an inverted pair in  $\sigma_{k-1}$ :

- Exchange  $i_k$  with  $j_k$  to obtain  $\sigma_k$  from  $\sigma_{k-1}$ , add the adjacent exchange  $\tau_k$  with  $\tau_k(\sigma_{k-1}) = \sigma_k$  to  $\rho$
- Let  $k = k + 1$
- Let  $i_k := i_{k-1}$ , let  $j_k := \sigma_{k-1}(i_k) + 1$  be the next job after  $i_k$  (break while if  $\sigma_{k-1}(i_k) = n$ )

Step 4. Goto Step 2

It is obvious that the so-computed permutation process  $\rho$  is feasible. For notational convenience, let us call the EGS rule that is based on this permutation process  $\rho$  the  $\Gamma$ -rule. We define for all  $i \in N$

$$\Gamma_i(N, \sigma_0, p, a) = LE\text{-EGS}_i^\rho(N, \sigma_0, p, a),$$

with  $\rho$  being the permutation process computed as above. In order to show that the  $\Gamma$ -rule of any LE sequencing situation yields a core element for the corresponding LE sequencing game, we derive the following lemmas.

The first lemma gives a new expression of  $v(S)$  for any  $S \in \text{con}(\sigma_0)$ , which is the sum of the gains of adjacent exchanges in a feasible permutation process for  $S$ .

**Lemma 1.** Let  $(N, \sigma_0, p, a)$  be an LE sequencing situation and  $(N, v)$  the corresponding LE sequencing game. For any  $S \in \text{con}(\sigma_0)$  and any feasible permutation process  $\rho \in \mathcal{P}(\sigma_0, \sigma_S)$ ,

$$v(S) = \sum_{k=1}^{I^S} g(\tau_k, \sigma_{k-1}, i_k, j_k).$$

**Proof.** From Theorem 1, we have that for any  $S \subseteq N$ ,

$$v(S) = \max_{\sigma \in \Pi_S} \sum_{i \in N} \Delta C(\sigma, i) = \sum_{i \in N} \Delta C(\sigma_S, i).$$

Since  $\rho = \{\tau_1, \tau_2, \dots, \tau_{I^S}\}$  is feasible, we have  $\sigma_{I^S} = \sigma_S$ . Hence,

$$\begin{aligned} \sum_{i \in N} \Delta C(\sigma_S, i) &= \sum_{i \in N} [C(\sigma_0, i) - C(\sigma_S, i)] \\ &= \sum_{i \in N} \sum_{k=1}^{I^S} (C(\sigma_{k-1}, i) - C(\sigma_k, i)) \\ &= \sum_{k=1}^{I^S} \sum_{i \in N} (C(\sigma_{k-1}, i) - C(\sigma_k, i)) \\ &= \sum_{k=1}^{I^S} g(\tau_k, \sigma_{k-1}, i_k, j_k). \quad \square \end{aligned}$$

The second lemma shows that in a feasible permutation process for  $N$ , the total gains of players in  $S$  are not less than  $v(S)$ .

**Lemma 2.** Let  $(N, \sigma_0, p, a)$  be an LE sequencing situation and  $(N, v)$  the corresponding LE sequencing game. If  $\rho \in \mathcal{P}(\sigma_0, \sigma_N)$  is a feasible permutation procedure computed by  $\Gamma$ , then for any  $S \in \text{con}(\sigma_0)$

$$\sum_{\substack{k=1, \dots, I^S \\ i_k, j_k \in S}} g(\tau_k, \sigma_{k-1}, i_k, j_k) \geq v(S).$$

**Proof.** Let  $\rho' = \{\tau'_1, \dots, \tau'_{I^S}\}$  be a feasible permutation process by adopting procedure  $\Gamma$  for coalition  $S$ . Let  $\tau'_k(\sigma'_{k-1}) = \sigma'_k$  for any  $1 \leq k \leq I^S$  and  $\sigma_0 = \sigma'_0$ . For every  $\tau'_k \in \rho'$ , the associated exchange players are denoted  $i'_k$  and  $j'_k$ . Then according to Lemma 1, we have

$$v(S) = \sum_{k=1}^{I^S} g(\tau'_k, \sigma'_{k-1}, i'_k, j'_k).$$

So, we are done if we can show that

$$\sum_{\substack{k=1, \dots, I^S \\ i_k, j_k \in S}} g(\tau_k, \sigma_{k-1}, i_k, j_k) \geq \sum_{k=1}^{I^S} g(\tau'_k, \sigma'_{k-1}, i'_k, j'_k). \quad (4)$$

Intuitively, this inequality should hold: First, every fixed inverted pair  $i, j \in S$  appears exactly once in each of the two sums. Second, since at the moment that we do an adjacent exchange of a fixed inverted pair  $i, j \in S$ , due to the procedure  $\Gamma$ , the position at which that exchange happens in  $\rho$  can only be earlier than in  $\rho'$ . Remembering (2), this yields the claim.

More formally, let  $i, j$  be a fixed inverted pair of jobs in  $S$ . Note that  $p_i > p_j$ . Let  $k$  be the iteration in  $\rho$  so that  $i_k = i, j_k = j$ , and let  $k'$  be the corresponding iteration in  $\rho'$  so that  $i'_k = i, j'_k = j$ . Define sets  $U', M', D'$  and  $U$  as follows

$$\begin{aligned} U' &= P(\sigma_0, i) \cap \{z \in S \mid p_z > p_i\}, \\ M' &= P(\sigma_0, j) \cap F(\sigma_0, i) \cap \{z \in S \mid p_z \geq p_i\}, \\ D' &= P(\sigma_0, j) \cap F(\sigma_0, i) \cap \{z \in S \mid p_z < p_i\}. \\ U &= P(\sigma_0, i) \cap \{z \in N \mid p_z > p_i\}, \end{aligned}$$

Then by definition of procedure  $\Gamma$ , in  $\sigma'_{k'-1}$ , all jobs in  $U' \cup M'$  must be in positions later than  $j$ . Moreover, all jobs in  $D'$  must be in positions before  $i$ . Hence, we conclude that

$$\sigma'_{k'-1}(i) = \sigma_0(i) + |D'| - |U'|.$$



Likewise, we have that

$$\sigma_{k-1}(i) = \sigma_0(i) + |D'| - |U|.$$

Since  $U' \subseteq U$ , it follows that  $\sigma_{k-1}(i) \leq \sigma'_{k-1}(i)$ . Recalling (2), we conclude

$$g(\tau_k, \sigma_{k-1}, i, j) \geq g(\tau'_k, \sigma'_{k-1}, i, j).$$

Inequality (4) now follows, since each inverted pair  $i, j \in S$  appears exactly once in both of the sums of (4).  $\square$

As promised, we now show that the  $\Gamma$ -rule yields a core element for any LE sequencing game  $(N, v)$ .

**Theorem 4.** *Let  $(N, \sigma_0, p, a)$  be an LE sequencing situation and  $(N, v)$  the corresponding LE sequencing game. Then the  $\Gamma$ -rule gives a core element of  $(N, v)$ .*

**Proof.** Let again  $\rho = \{\tau_1, \tau_2, \dots, \tau_{|N|}\}$  be the permutation process derived from  $\Gamma$ . Note that

$$\begin{aligned} \sum_{i \in N} \Gamma_i(N, \sigma_0, p, a) &= \sum_{i \in N} \frac{1}{2} \sum_{k=1}^{|N|} g(\tau_k, \sigma_{k-1}, i_k, j_k) \delta_i(i_k, j_k) \\ &= \sum_{k=1}^{|N|} g(\tau_k, \sigma_{k-1}, i_k, j_k) \\ &= v(N), \end{aligned}$$

where the third equality follows from Lemma 1. For any  $S \in \text{con}(\sigma_0)$ ,

$$\begin{aligned} \sum_{i \in S} \Gamma_i(N, \sigma_0, p, a) &= \sum_{i \in S} \frac{1}{2} \sum_{k=1}^{|N|} g(\tau_k, \sigma_{k-1}, i_k, j_k) \delta_i(i_k, j_k) \\ &\geq \sum_{i \in S} \frac{1}{2} \sum_{\substack{k=1 \dots |N| \\ i_k, j_k \in S}} g(\tau_k, \sigma_{k-1}, i_k, j_k) \delta_i(i_k, j_k) \\ &= \sum_{\substack{k=1 \dots |N| \\ i_k, j_k \in S}} g(\tau_k, \sigma_{k-1}, i_k, j_k) \\ &\geq v(S), \end{aligned}$$

the last inequality following from Lemma 2. From  $\sigma_0$ -component additivity, we finally conclude  $\Gamma(N, \sigma_0, p, a) \in C(N, v)$ .  $\square$

#### 4.3. A characterization of the $\Gamma$ -rule

Generally spoken, an allocation rule  $\phi$  assigns to each LE sequencing game an allocation  $x \in \mathbb{R}^n$ . We now aim at characterizing the  $\Gamma$ -rule in terms of (more or less) natural properties imposed on  $\phi$ . To that end, for given  $(N, \sigma_0, p, a)$ , denote the set of successors of a player's job  $i$  with smaller nominal processing time than that of player  $i$ , by  $D(\sigma_0, i)$ , i.e.,  $D(\sigma_0, i) := F(\sigma_0, i) \cap \{j \mid p_j < p_i\}$ . Player  $i$  is a *D-equivalence player* with respect to LE sequencing situations  $(N, \sigma_0, p, a)$ ,  $(N, \sigma_1, p, a)$  if  $F(\sigma_0, i) = F(\sigma_1, i)$ , and for any  $k, l \in D(\sigma_0, i)$ ,  $\sigma_0(k) < \sigma_0(l)$  implies  $\sigma_1(k) < \sigma_1(l)$ . We say that an inverted pair  $(i, j)$  is a *head pair* with respect to  $(N, \sigma_0, p, a)$  if  $\sigma_0(i) = \sigma_0(j) - 1$  and for any  $k \in P(\sigma_0, i)$ ,  $p_k \leq p_i$ . Consider the following properties.

- *D-equivalence:* Let  $(N, \sigma_0, p, a)$ ,  $(N, \sigma_1, p, a)$  be two LE sequencing situations. If some player  $i \in N$  is a *D-equivalence player* with respect to these two LE sequencing situations, then  $\phi_i(N, \sigma_0, p, a) = \phi_i(N, \sigma_1, p, a)$ .
- *Head switch property:* Let  $(N, \sigma_0, p, a)$ ,  $(N, \sigma_1, p, a)$  be two LE sequencing situations and  $i, j \in N$  be such that  $\sigma_0(i) = \sigma_0(j) - 1$ ,  $\sigma_1(i) = \sigma_0(j)$ ,  $\sigma_1(j) = \sigma_0(i)$  and  $\sigma_1(k) = \sigma_0(k)$

for any  $k \in N \setminus \{i, j\}$ . If  $(i, j)$  is a head pair with respect to  $(N, \sigma_0, p, a)$ , then

$$\begin{aligned} \phi_i(N, \sigma_1, p, a) - \phi_i(N, \sigma_0, p, a) \\ = \phi_j(N, \sigma_1, p, a) - \phi_j(N, \sigma_0, p, a). \end{aligned}$$

These two properties are the variations of *equivalence* and *switch* property introduced by [9] for characterizing the EGS rule. In the next theorem we show that the  $\Gamma$ -rule satisfies efficiency, the dummy property, *D*-equivalence and the head switch property. Since the proof of the characterization of the  $\Gamma$ -rule is along the same line as that of the corresponding result of [9] for the EGS rule, we relegate it to the supplement.

**Theorem 5.** *The  $\Gamma$ -rule is the unique allocation rule for LE sequencing situations that satisfies efficiency, the dummy property, *D*-equivalence and the head switch property.*

#### 4.4. Relationships between the $\Gamma$ -rule and the $\beta$ -rule

Let  $(N, v)$  be a cooperative game and  $\sigma_0$  an initial order of players. The  $\beta$ -rule is defined as follows [9]:

$$\beta_i(v) = \frac{1}{2} [v(\bar{P}(\sigma_0, i)) - v(P(\sigma_0, i)) + v(\bar{F}(\sigma_0, i)) - v(F(\sigma_0, i))],$$

for all  $i \in N$ . [9] showed that if  $(N, v)$  is  $\sigma_0$ -component additive, then the  $\beta$ -rule is in the core of  $(N, v)$ .

Since the LE sequencing game  $(N, v)$  is  $\sigma_0$ -component additive, we can deduce that  $\beta(v)$  must be in the core of  $(N, v)$ . Since the  $\Gamma$ -rule gives a core element of  $(N, v)$ , too, we next discuss the relationships between these two allocation rules. The following theorem provides two conditions for the coincidence of the  $\Gamma$ -rule and the  $\beta$ -rule.

**Theorem 6.** *Let  $(N, \sigma_0, p, a)$  be an LE sequencing situation and  $(N, v)$  the corresponding LE sequencing game.*

- (i) *When  $a = 0$ ,  $\Gamma(N, \sigma_0, p, a) = \beta(v)$ .*
- (ii) *When  $a < 0$ ,  $\Gamma(N, \sigma_0, p, a) = \beta(v)$  if and only if there are no three players  $i, j, k \in N$  satisfying  $\sigma_0(i) < \sigma_0(j) < \sigma_0(k)$  and  $p_i > p_j > p_k$ .*

**Proof.** (i) Obviously, the LE sequencing games coincide with the sequencing games defined by [7] and all LE-EGS rules coincide with the EGS rule when  $a = 0$ . Since [8] have shown that the EGS rule is equivalent to the  $\beta$ -rule in sequencing games, the result follows.

(ii) We first give another, equivalent formulation of the  $\Gamma$ -rule. Let  $(N, \sigma_0, p, a)$  be an LE sequencing situation. Define  $g_{ij}^r$  as the total cost savings generated from the adjacent exchange of players  $i, j$  with  $\sigma_0(i) = \sigma_0(j) - 1$  and  $\sigma_0(i) = r$ . For player  $i \in N$  fixed, let

$$U(\sigma_0, i) := P(\sigma_0, i) \cap \{k \mid p_k > p_i\} = \{u_1, \dots, u_{h_1}\}$$

with  $p_{u_1} \geq \dots \geq p_{u_{h_1}}$  and

$$D(\sigma_0, i) = F(\sigma_0, i) \cap \{j \mid p_j < p_i\} = \{d_1, \dots, d_{h_2}\}$$

with  $\sigma_0(d_1) < \dots < \sigma_0(d_{h_2})$ . We claim that, due to the definition of the  $\Gamma$ -rule,  $\Gamma_i$  can then be expressed in the following way:

$$\Gamma_i(N, \sigma_0, p, a) = \frac{1}{2} \left( \sum_{m=1}^{h_1} g_{u_m i}^{r-m} + \sum_{m=1}^{h_2} g_{i d_m}^{r-h_1+m-1} \right), \tag{5}$$

for all  $i \in N$ . To see why, recall how  $\Gamma$  is defined, and observe that for any  $1 \leq m \leq h_1$ , jobs  $u_m$  and  $i$  cannot become adjacent before  $u_1, \dots, u_{m-1}$  have moved to positions behind  $i$ . Hence,  $u_m$  is exactly in position  $(r - m)$  when the adjacent exchange of  $u_m$  and

$i$  takes place. This continues until  $u_{h_1}$  is on a position somewhere behind  $i$ . Consequently,  $i$  will be in positions  $r - h_1, r - h_1 + 1, \dots, r - h_1 + h_2 - 1$  when the adjacent exchanges with  $d_1, d_2, \dots, d_{h_2}$  take place. After  $i$  and  $d_{h_2}$  have exchanged their positions,  $i$  will remain at its position. Observe that the order in which jobs  $U(\sigma_0, i)$  exchange with  $i$ , only depends on the (nominal) sizes of  $u_1, \dots, u_{h_1}$ , while the order in which  $i$  exchanges with jobs  $D(\sigma_0, i)$ , only depends on the order of  $d_1, \dots, d_{h_2}$  in  $\sigma_0$ .

For any three orders  $\sigma_1, \sigma_2, \sigma_3 \in \Pi_N$ , any  $\rho_1 = \{\tau_1, \dots, \tau_m\} \in \mathcal{P}(\sigma_1, \sigma_2)$  and  $\rho_2 = \{\tau'_1, \dots, \tau'_k\} \in \mathcal{P}(\sigma_2, \sigma_3)$ , we define the connection of  $\rho_1$  and  $\rho_2$  by

$$\rho_1 \triangleright \rho_2 = \{\tau_1, \dots, \tau_m, \tau'_1, \dots, \tau'_k\}.$$

Clearly,  $\rho_1 \triangleright \rho_2(\sigma_1) = \sigma_3$ .

(if). Fix any  $i \in N$ , and assume  $\sigma_0(i) = r$ . Recall the above definitions of  $U(\sigma_0, i)$  and  $D(\sigma_0, i)$ . For convenience, we denote  $P(\sigma_0, i), \bar{P}(\sigma_0, i), F(\sigma_0, i)$ , and  $\bar{F}(\sigma_0, i)$  by  $P_i, \bar{P}_i, F_i$ , and  $\bar{F}_i$ .

Case 1: Suppose that  $U(\sigma_0, i) = \emptyset$ . Let  $\tau_m$  be the adjacent exchange for players  $i$  and  $d_m$  with  $i$  being at the  $(r + m - 1)$ -th position, where  $1 \leq m \leq h_2$ . Since there are no three players  $i, j, k \in N$  satisfying  $\sigma_0(i) < \sigma_0(j) < \sigma_0(k)$  and  $p_i > p_j > p_k$ , it follows that  $p_{d_1} \leq \dots \leq p_{d_{h_2}}$ . Then we must have for any  $\rho \in \mathcal{P}(\sigma_0, \sigma_{\bar{F}_i}), \rho \triangleright \{\tau_1, \dots, \tau_{h_2}\} \in \mathcal{P}(\sigma_0, \sigma_{\bar{F}_i})$ . From Lemma 1, it holds that

$$v(\bar{F}_i) - v(F_i) = \sum_{m=1}^{h_2} g(\tau_m, \sigma_m, i, d_m) = \sum_{m=1}^{h_2} g_{id_m}^{r+m-1}, \tag{6}$$

where  $\sigma_1 = \sigma_{\bar{F}_i}$ . Furthermore, since  $U(\sigma_0, i) = \emptyset$ , we have that

$$v(\bar{P}_i) - v(P_i) = 0. \tag{7}$$

It follows from (5)–(7) that

$$\begin{aligned} \beta_i(v) &= \frac{1}{2} [v(\bar{P}_i) - v(P_i) + v(\bar{F}_i) - v(F_i)] \\ &= \frac{1}{2} \sum_{m=1}^{h_2} g_{id_m}^{r+m-1} \\ &= \Gamma_i(N, \sigma_0, p, a). \end{aligned}$$

Case 2: Suppose that  $U(\sigma_0, i) \neq \emptyset$ . Then there is at least one player in  $P(\sigma_0, i)$  whose nominal processing time is larger than  $p_i$ . Since there are no three players  $i, j, k \in N$  satisfying  $\sigma_0(i) < \sigma_0(j) < \sigma_0(k)$  and  $p_i > p_j > p_k$ , we must have that  $D(\sigma_0, i) = \emptyset$ . Clearly, it follows that

$$v(\bar{F}_i) - v(F_i) = 0. \tag{8}$$

Let  $\tau_m$  be the adjacent exchange for players  $u_m$  and  $i$  with  $i$  being at the  $(r - m + 1)$ -th position, where  $1 \leq m \leq h_1$ . Then for any  $\rho \in \mathcal{P}(\sigma_0, \sigma_{\bar{P}_i})$ , we must have  $\rho \triangleright \{\tau_1, \dots, \tau_{h_1}\} \in \mathcal{P}(\sigma_0, \sigma_{\bar{P}_i})$ . From Lemma 1, it holds that

$$v(\bar{P}_i) - v(P_i) = \sum_{m=1}^{h_1} g(\tau_m, \sigma_m, u_m, i) = \sum_{m=1}^{h_1} g_{u_m i}^{r-m}, \tag{9}$$

where  $\sigma_1 = \sigma_{\bar{P}_i}$ . Then it follows from (5), (8) and (9) that

$$\begin{aligned} \beta_i(v) &= \frac{1}{2} [v(\bar{P}_i) - v(P_i) + v(\bar{F}_i) - v(F_i)] \\ &= \frac{1}{2} \sum_{m=1}^{h_1} g_{u_m i}^{r-m} \\ &= \Gamma_i(N, \sigma_0, p, a). \end{aligned}$$

(only if). We prove this part by contradiction. Suppose that  $\Gamma(N, \sigma_0, p, a) = \beta(v)$  and there are three players  $i, j, k \in N$  satisfying  $\sigma_0(i) < \sigma_0(j) < \sigma_0(k)$  and  $p_i > p_j > p_k$ . Define

$$i^* = \arg \min \{ \sigma_0(i) \mid \exists j, k \in N \text{ such that } \sigma_0(i) < \sigma_0(j) < \sigma_0(k) \text{ and } p_i > p_j > p_k \}.$$

It is easy to see that then,  $U(\sigma_0, i^*) = \emptyset$ . Let  $D(\sigma_0, i^*) = \{d_1, \dots, d_{h_2}\} = \{d'_1, \dots, d'_{h_2}\}$  be two orderings of  $D(\sigma_0, i^*)$  with  $\sigma_0(d_1) < \dots < \sigma_0(d_{h_2})$  and  $p_{d'_1} \leq \dots \leq p_{d'_{h_2}}$ . Assume that  $i^*$  is in position  $r^*$  in  $\sigma_0$ . We first show that

$$\sum_{m=1}^{h_2} g_{i^* d'_m}^{r^*+m-1} < \sum_{m=1}^{h_2} g_{i^* d_m}^{r^*+m-1}. \tag{10}$$

What (10) expresses is that the benefits of successive adjacent exchanges of  $i^*$  with jobs in  $D(\sigma_0, i^*)$  is larger, if the jobs in  $D(\sigma_0, i^*)$  are ordered by nominal processing times (SPT).

To prove (10), it suffices to show that the two successive adjacent exchanges  $(i^*, k)$  and  $(i^*, j)$  (in this order) are better than  $(i^*, j)$  and  $(i^*, k)$ , whenever  $p_j > p_k$ . Formally, this is expressed by

$$g_{i^* j}^r + g_{i^* k}^{r+1} < g_{i^* k}^r + g_{i^* j}^{r+1} \text{ for any } r^* \leq r \leq n - 1. \tag{11}$$

But this is true, as

$$\begin{aligned} g_{i^* k}^r - g_{i^* j}^r + g_{i^* j}^{r+1} - g_{i^* k}^{r+1} &= \theta(r)(p_{i^*} - p_k) - \theta(r)(p_{i^*} - p_j) \\ &\quad + \theta(r+1)(p_{i^*} - p_j) \\ &\quad - \theta(r+1)(p_{i^*} - p_k) \\ &= \theta(r)(p_j - p_k) + \theta(r+1)(p_k - p_j) \\ &= [\theta(r) - \theta(r+1)](p_j - p_k) \\ &> 0, \end{aligned}$$

where  $\theta(r)$  is defined in (3). Note that the inequality in (10) is strict, because of the choice of  $i^*$ , and since this implies the existence of  $j, k$  with  $\sigma_0(i^*) < \sigma_0(j) < \sigma_0(k)$  and  $p_{i^*} > p_j > p_k$ .

Therefore, (11) holds, and it follows that (10) holds. Let  $\tau_m$  be the adjacent exchange for players  $i^*$  and  $d'_m$  with  $i^*$  being at the  $(r + m - 1)$ -th position, where  $1 \leq m \leq h_2$ . Then for any  $\rho \in \mathcal{P}(\sigma_0, \sigma_{\bar{F}_{i^*}}), \rho \triangleright \{\tau_1, \dots, \tau_{h_2}\} \in \mathcal{P}(\sigma_0, \sigma_{\bar{F}_{i^*}})$ . Similar to (6), we have

$$v(\bar{F}_{i^*}) - v(F_{i^*}) = \sum_{m=1}^{h_2} g(\tau_m, \sigma_m, i^*, d'_m) = \sum_{m=1}^{h_2} g_{i^* d'_m}^{r^*+m-1},$$

where  $\sigma_1 = \sigma_{\bar{F}_{i^*}}$ . Since  $U(\sigma_0, i^*) = \emptyset$ , we have that

$$v(\bar{P}_{i^*}) - v(P_{i^*}) = 0.$$

Thus,

$$\begin{aligned} \beta_{i^*}(v) &= \frac{1}{2} [v(\bar{P}_{i^*}) - v(P_{i^*}) + v(\bar{F}_{i^*}) - v(F_{i^*})] \\ &= \frac{1}{2} \sum_{m=1}^{h_2} g_{i^* d'_m}^{r^*+m-1}. \end{aligned}$$

From (5), we have that

$$\Gamma_{i^*}(N, \sigma_0, p, a) = \frac{1}{2} \sum_{m=1}^{h_2} g_{i^* d'_m}^{r^*+m-1}.$$

Then it follows from (10) that  $\beta_{i^*}(v) > \Gamma_{i^*}(N, \sigma_0, p, a)$ , a contradiction.  $\square$

We can immediately conclude the following corollary from (9) in the proof of Theorem 6.

**Corollary 1.** For LE sequencing situation  $(N, \sigma_0, p, a)$ , let  $(N, v)$  be the corresponding LE sequencing game and  $i_n = \sigma_0^{-1}(n)$  be the last player in  $\sigma_0$ . Then  $\Gamma_{i_n}(N, \sigma_0, p, a) = \beta_{i_n}(v)$ .

**5. Final remarks**

One of the limitations of this paper is the restriction of the sequencing model to learning indices and cost functions which

are identical for all players. We briefly discuss extensions of the model to these more general cases.

First, assume different players have different learning indices (but identical cost functions). That is, any player  $i \in N$  will have her own learning index  $a_i$ . We can define an  $LE^\alpha$  sequencing situation by a 4-tuple  $(N, \sigma_0, p, \bar{a})$ , where  $\bar{a} = (a_i)_{i \in N} \in \mathbb{R}_{\leq 0}^n$ . Then the corresponding  $LE^\alpha$  sequencing game  $(N, u^\lambda)$  can be defined by letting, for all  $S \subseteq N$ ,

$$u^\lambda(S) = \max_{\sigma \in \Pi_S} \{c'_{\sigma_0}(S) - c'_\sigma(S) + \lambda(S) [c'_{\sigma_0}(N \setminus S) - c'_\sigma(N \setminus S)]\}.$$

where  $c'_\sigma(S) = \sum_{i \in S} \sum_{\sigma_0(j) \leq \sigma_0(i)} \sigma_0(i)^{a_i} p_i$ . [16] showed that the problem of finding the optimal order for the  $LE^\alpha$  sequencing situation can be solved in  $O(n^3)$  by formulating it as an assignment problem. By applying the same method, we can also prove that the optimal order for obtaining  $u^\lambda(S)$  can be found in polynomial time. Thus,  $u^\lambda(S)$  for any  $S$  is well-defined. Let  $(N, u)$  be the  $LE^\alpha$  sequencing game with the share function  $\lambda$  satisfying for  $S \subseteq N$ ,  $\lambda(S) = 1$ . One can easily verify that  $(N, u)$  is  $\sigma_0$ -component additive, and thus any corresponding  $LE^\alpha$  sequencing game is balanced and has a non-empty core. However, it is not clear how to extend the EGS rule to  $LE^\alpha$  sequencing situations, since there is no fixed principle for the optimal order.

Second, consider the possibility to allow different players to have different cost coefficients but identical learning indices. An  $LE^\alpha$  sequencing situation is then described by a 5-tuple  $(N, \sigma_0, \alpha, p, a)$ , where  $\alpha$  is a vector of cost coefficients, i.e.,  $\alpha = (\alpha_i)_{i \in N} \in \mathbb{R}_{\geq 0}^n$ . That is to say, the cost of player  $i \in N$  is now  $\alpha_i C(\sigma, i)$ . Analogous to the definition of LE sequencing games, we can define the corresponding  $LE^\alpha$  sequencing game  $(N, w^\lambda)$  by letting, for all  $S \subseteq N$ ,

$$w^\lambda(S) = \max_{\sigma \in \Pi_S} \{c^*_{\sigma_0}(S) - c^*_\sigma(S) + \lambda(S) [c^*_{\sigma_0}(N \setminus S) - c^*_\sigma(N \setminus S)]\},$$

$$\text{where } c^*_\sigma(S) = \sum_{i \in S} \alpha_i \sum_{\sigma_0(j) \leq \sigma_0(i)} \sigma_0(i)^{a_i} p_i.$$

However, finding an optimal order for minimizing the total costs of all players in an  $LE^\alpha$  sequencing situation may be NP-hard in general, in which case there is not much hope to find a closed form for working with  $w^\lambda(S)$ . One way to circumvent this problem is to restrict to  $LE^\alpha$  sequencing situations with agreeable weights, which says that for any  $i, j \in N$ ,  $p_i \leq p_j$  implies  $\alpha_i \geq \alpha_j$ . [18] showed that the total costs of all players in an  $LE^\alpha$  sequencing situation with agreeable weights can be minimized by the WSPT rule (weighted shortest processing time first, i.e., sequence jobs in non-decreasing order of  $p_i/\alpha_i$ ). In this case, one can work with closed form expressions for  $w^\lambda(S)$ , like in the present paper. However, the agreeability condition is quite strong. Further work is needed to get results for more general settings.

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## Appendix A. Supplementary data

Supplementary material related to this article can be found online at <https://doi.org/10.1016/j.orl.2020.04.013>.

## References

- [1] D. Biskup, Single-machine scheduling with learning considerations, *European J. Oper. Res.* 115 (1) (1999) 173–178.
- [2] P. Borm, G. Fiestras-Janeiro, H. Hamers, E. Sánchez, M. Voorneveld, On the convexity of games corresponding to sequencing situations with due dates, *European J. Oper. Res.* 136 (3) (2002) 616–634.
- [3] P. Calleja, P. Borm, H. Hamers, F. Klijn, M. Slikker, On a new class of parallel sequencing situations and related games, *Ann. Oper. Res.* 109 (1–4) (2002) 265–277.
- [4] B. Çiftçi, P. Borm, H. Hamers, M. Slikker, Batch sequencing and cooperation, *J. Sched.* 16 (4) (2013) 405–415.
- [5] I. Curiel, Multi-stage sequencing situations, *Internat. J. Game Theory* 39 (2010) 151–162.
- [6] I. Curiel, Compensation rules for multi-stage sequencing games, *Ann. Oper. Res.* 225 (1) (2015) 65–82.
- [7] I. Curiel, G. Pederzoli, S. Tijs, Sequencing games, *European J. Oper. Res.* 40 (3) (1989) 344–351.
- [8] I. Curiel, J. Potters, R. Prasad, S. Tijs, B. Veltman, Cooperation in one machine scheduling, *Z. Oper. Res.* 38 (2) (1993) 113–129.
- [9] I. Curiel, J. Potters, R. Prasad, S. Tijs, B. Veltman, Sequencing and cooperation, *Oper. Res.* 42 (3) (1994) 566–568.
- [10] S. Grundel, B. Çiftçi, P. Borm, H. Hamers, Family sequencing and cooperation, *European J. Oper. Res.* 226 (3) (2013) 414–424.
- [11] H. Hamers, P. Borm, S. Tijs, On games corresponding to sequencing situations with ready times, *Math. Program.* 69 (1–3) (1995) 471–483.
- [12] H. Hamers, F. Klijn, J. Suijs, On the balancedness of multiple machine sequencing games, *European J. Oper. Res.* 119 (3) (1999) 678–691.
- [13] F. Klijn, E. Sánchez, Sequencing games without initial order, *Math. Methods Oper. Res.* 63 (1) (2006) 53–62.
- [14] M. Le Breton, G. Owen, S. Weber, Strongly balanced cooperative games, *Internat. J. Game Theory* 20 (4) (1992) 419–427.
- [15] G. Mosheiov, Scheduling problems with a learning effect, *European J. Oper. Res.* 132 (3) (2001) 687–693.
- [16] G. Mosheiov, J.B. Sidney, Scheduling with general job-dependent learning curves, *European J. Oper. Res.* 147 (3) (2003) 665–670.
- [17] M. Slikker, Balancedness of sequencing games with multiple parallel machines, *Ann. Oper. Res.* 137 (1) (2005) 177–189.
- [18] C. Zhao, Q. Zhang, H. Tang, Machine scheduling problems with learning effects, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* 11 (5–6) (2004) 741–750.