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Global and Semi-global Regulated State Synchronization for Homogeneous Networks of Non-introspective Agents in Presence of Input Saturation– A Scale-free Protocol Design

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Abstract— **This paper studies global and semi-global regulated state synchronization of homogeneous networks of non-introspective agents in presence of input saturation based on additional information exchange where the reference trajectory is given by a so-called exosystem which is assumed to be globally reachable. Our protocol design methodology does not need any knowledge of the directed network topology and the spectrum of the associated Laplacian matrix. Moreover, the proposed protocol is scalable and achieves synchronization for any arbitrary number of agents.**

I. Introduction

The synchronization problem of multi-agent systems (MAS) has attracted substantial attention during the past decade, due to the wide potential for applications in several areas such as automotive vehicle control, satellites/robots formation, sensor networks, and so on. See for instance the books [16] and [34] or the survey paper [12].

State synchronization inherently requires homogeneous networks (i.e. agents which have identical dynamics). Therefore, in this paper we focus on homogeneous networks. State synchronization based on diffusive *full-state coupling* has been studied where the agent dynamics progress from single- and double-integrator dynamics (e.g. [13], [14], [15]) to more general dynamics (e.g. [19], [29], [32]). State synchronization based on diffusive *partial-state coupling* has also been considered, including static design ([9] and [10]), dynamic design ([6], [20], [21], [27], [30]), and the design with additional communication ([1] and [19]).

Meanwhile, it is worth to note that actuator saturation is pretty common and indeed is ubiquitous in engineering applications. Some researchers have tried to establish the (semi) global state and output synchronization results for MAS in the presence of input saturation. Global synchronization for neutrally stable agents has been studied by [11] (continuous-time) and [35] (discrete-time) for either undirected or detailed balanced graph. Then, global synchronization via static protocols for MAS with partial state coupling and linear general dynamics is developed in [8]. Reference [7] provides the design which can deal with networks that are not detailed balanced but intrinsically requires the agents to be single integrators. Similar scenarios also appear in [3] (finite-time consensus), and [37] (event-triggered control).

Semi-global leader-follower state synchronization has been studied in [24] and [25] in the case of full-state coupling. References [23], [26] and [31] provide the semi global result via partial state coupling but they all require extra communication and are introspective. Adaptive approach also is studied in [2] but the observer requires extra communication and is introspective. A low gain design is introduced in [22] for heterogeneous MAS with introspective agents and requires extra communication to track any trajectory from exosystem. The paper [36] considers non-introspective agents and requires extra communication for heterogeneous MAS, and [38] has similar design for discrete-time MAS. Then, [28] studied MAS with non-introspective agents and does not require extra communication, however it requires solution of a nonconvex optimization problem to find a dynamic protocol. Recently, [39] studid the semi-global state synchronization of homogeneous networks for both continuous/discrete-time MASs with non-introspective agents with both full-state and partial-state coupling in the presence of input saturation.

In this paper, we deal with global and semi-global regulated state synchronization problems for MAS in presence of input saturation by tracking the trajectory of an exosystem. We design dynamic protocols by using additional information exchange for MAS with non-introspective agents and for both networks with full- and partial-state coupling. The protocol design is scalable and does not need any information of communication network except connectivity. In other words, the proposed protocols work for any MAS with an arbitrary number of agents.

Notations and definitions

Given a matrix $A \in \mathbb{R}^{m\times n}$, A^{T} denotes the transpose of A and $\|A\|$ denotes the induced 2-norm of A. For a vector $x \in \mathbb{R}^q$, $\|x\|$ denotes the 2-norm of x and for a vector signal v, we denote the $\mathscr{L}_1$, $\mathscr{L}_2$, and $\mathscr{L}_\infty$ norm by $\|v\|_1$, $\|v\|_2$ and $\|v\|_\infty$ respectively. A square matrix A is said to be Hurwitz stable if all its eigenvalues are in the open left half complex plane. We denote by $\mathrm{diag}\{A_1,\dots,A_N\}$, a block-diagonal matrix with $A_1,\dots,A_N$ as its diagonal elements. $A \otimes B$ depicts the Kronecker product between A and B. I_n denotes the n-dimensional identity matrix and 0_n denotes

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$n \times n$ zero matrix; sometimes we drop the subscript if the dimension is clear from the context.

To describe the information flow among the agents we associate a *weighted graph* $\mathcal{G}$ to the communication network. The weighted graph $\mathcal{G}$ is defined by a triple $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, \ldots, N\}$ is a node set, $\mathcal{E}$ is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is the weighted adjacency matrix with non negative elements a_{ij}. Each pair in $\mathcal{E}$ is called an *edge*, where $a_{ij} > 0$ denotes an edge $(j,i) \in \mathcal{E}$ from node j to node i with weight a_{ij}. Moreover, $a_{ij} = 0$ if there is no edge from node j to node i. We assume there are no self-loops, i.e. we have $a_{ii} = 0$. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, \ldots, k-1$. A *directed tree* is a subgraph (subset of nodes and edges) in which every node has exactly one parent node except for one node, called the *root*, which has no parent node. The *root set* is the set of root nodes. A *directed spanning tree* is a subgraph which is a directed tree containing all the nodes of the original graph. If a directed spanning tree exists, the root has a directed path to every other node in the tree.

For a weighted graph $\mathcal{G}$, the matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^{N} a_{ik}, & i = j, \\ -a_{ij}, & i \neq j, \end{cases}$$

is called the *Laplacian matrix* associated with the graph $\mathcal{G}$. The Laplacian matrix L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $\mathbf{1}$ [4]. Moreover, if the graph contains a directed spanning tree, the Laplacian matrix L has a single eigenvalue at the origin and all other eigenvalues are located in the open right-half complex plane [16].

II. Problem formulation

Consider a MAS consisting of N identical dynamic agents with input saturation:

$$\begin{cases} \dot{x}_i = Ax_i + B\sigma(u_i), \\ y_i = Cx_i, \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}^n$, $y_i \in \mathbb{R}^q$ and $u_i \in \mathbb{R}^m$ are the state, output, and the input of agent $i = 1, \ldots, N$, respectively. Meanwhile,

$$\sigma(v) = \begin{pmatrix} \text{sat}(v_1) \\ \text{sat}(v_2) \\ \vdots \\ \text{sat}(v_m) \end{pmatrix} \quad \text{where} \quad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \in \mathbb{R}^m$$

with $\text{sat}(w)$ is the standard saturation function:

$$\text{sat}(w) = \text{sgn}(w) \min(1, |w|).$$

Assumption 1 *Assume agents are at most weakly unstable, namely, all eigenvalues of A are in the closed left half plane. Moreover, let (A, B, C) be stabilizable and detectable.*

The network provides agent i with the following information,

$$\zeta_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j), \tag{2}$$

where $a_{ij} \geqslant 0$ and $a_{ii} = 0$. This communication topology of the network can be described by a weighted graph $\mathcal{G}$ associated with (2), with the a_{ij} being the coefficients of the weighted adjacency matrix $\mathcal{A}$. In terms of the coefficients of the associated Laplacian matrix L, ζ_i can be rewritten as

$$\zeta_i = \sum_{j=1}^{N} \ell_{ij} y_j. \tag{3}$$

We refer to this as *partial-state coupling* since only part of the states are communicated over the network. When $C = I$, it means all states are communicated over the network and we call it *full-state coupling*. Then, the original agents are expressed as

$$\dot{x}_i = Ax_i + B\sigma(u_i) \tag{4}$$

and ζ_i is rewritten as

$$\zeta_i = \sum_{j=1}^{N} \ell_{ij} x_j.$$

Obviously, state synchronization is achieved if

$$\lim_{t \to \infty} (x_i - x_j) = 0. \quad \text{for all } i, j \in 1, \ldots, N \tag{5}$$

For homogeneous MAS such as in this paper, almost all papers considered state synchronization without imposing requirements on the synchronized trajectory. However, for heterogenous agents, it has been shown in [33], [5] that we basically need to consider regulated state synchronization where the objective of the agents is to ensure that their state asymptotically tracks a reference trajectory generated by a so-called exosystem. Although we consider homogeneous MAS, we will study regulated state synchronization in this paper.

The reference trajectory is generated by the following exosystem

$$\begin{aligned} \dot{x}_r &= Ax_r \\ y_r &= Cx_r. \end{aligned} \tag{6}$$

with $x_r \in \mathbb{R}^n$. Our objective is that the agents achieve regulated state synchronization, that is

$$\lim_{t \to \infty} (x_i - x_r) = 0, \tag{7}$$

for all $i \in \{1, \ldots, N\}$. Clearly, we need some level of communication between the exosystem and the agents. We assume that a nonempty subset $\mathscr{C}$ of the agents have access to their own output relative to the output of the exosystem. Specially, each agent i has access to the quantity

$$\psi_i = \iota_i(y_i - y_r), \qquad \iota_i = \begin{cases} 1, & i \in \mathscr{C}, \\ 0, & i \notin \mathscr{C}. \end{cases} \tag{8}$$

By combining this with (3), we have the following information exchange

$$\bar{\zeta}_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j) + \iota_i(y_i - y_r). \tag{9}$$

Meanwhile, for full-state coupling case (9) will change as

$$\bar{\zeta}_i = \sum_{j=1}^{N} a_{ij}(x_i - x_j) + \iota_i(x_i - x_r). \tag{10}$$

To guarantee that each agent can achieve the required regulation, we need to make sure that there exists a path to each node starting with node from the set $\mathscr{C}$. Therefore, we define the following set of graphs.

Definition 1 *Given a node set $\mathscr{C}$, we denote by $\mathbb{G}_{\mathscr{C}}^N$ the set of all graphs with N nodes containing the node set $\mathscr{C}$, such that every node of the network graph $\mathcal{G} \in \mathbb{G}_{\mathscr{C}}^N$ is a member of a directed tree which has its root contained in the node set $\mathscr{C}$.*

Remark 1 *Note that Definition 1 does not require necessarily the existence of directed spanning tree.*

In the following, we will refer to the node set $\mathscr{C}$ as root set in view of Definiton 1. For any graph $\mathbb{G}_{\mathscr{C}}^N$, with the Laplacian matrix L, we define the expanded Laplacian matrix as:

$$\tilde{L} = L + diag\{\iota_i\} = [\tilde{\ell}_{ij}]_{N\times N}$$

which is not a regular Laplacian matrix associated to the graph, since the sum of its rows need not be zero. We know that Definition 1, guarantees that all the eigenvalues of $\tilde{L}$, have positive real parts. In particular matrix $\tilde{L}$ is invertible.

In this paper, we also introduce an additional information exchange among protocols. In particular, each agent $i = 1,\dots,N$ has access to additional information, denoted by $\hat{\zeta}_i$, of the form

$$\hat{\zeta}_i = \sum_{j=1}^{N} a_{ij}(\xi_i - \xi_j) \tag{11}$$

where $\xi_j \in \mathbb{R}^n$ is a variable produced internally by agent j and to be defined in next sections.

First, we formulate the following problem for global regulated state synchronization of a MAS.

Problem 1 *Consider a MAS described by* (1) *and* (2) *and the associated exosystem* (6). *Let a set of nodes $\mathscr{C}$ be given which defines the set $\mathbb{G}_{\mathscr{C}}^N$. Let the associated network communication be given by* (9).

The ***scalable global regulated state synchronization problem with additional information exchange*** *of a MAS is to find, if possible, a dynamic protocol for each agent $i \in \{1,\dots,N\}$, using only knowledge of agent model, i.e. (A,B,C), of the form:*

$$\begin{cases} \dot{x}_{c,i} = f(x_{c,i}, \bar{\zeta}_i, \hat{\zeta}_i), \\ u_i \;= g(x_{c,i}), \end{cases} \tag{12}$$

where $\hat{\zeta}_i$ is defined in (11) *with $\xi_i = H_c x_{i,c}$, and $x_{c,i} \in \mathbb{R}^{n_c}$, such that regulated state synchronization* (7) *is achieved for any N and any graph $\mathcal{G} \in \mathbb{G}_{\mathscr{C}}^N$, and for all initial conditions of the agents $x_i(0) \in \mathbb{R}^n$, all initial conditions of the exosystem $x_r(0) \in \mathbb{R}^n$, and all initial conditions of the protocols $x_{c,i}(0) \in \mathbb{R}^{n_c}$.*

Next, we adopt semi-global framework to achieve regulated state synchronization by utilizing only *linear* protocols.

Problem 2 *Consider a MAS described by* (1) *and* (2) *and the associated exosystem* (6). *Let a set of nodes $\mathscr{C}$ be given which defines the set $\mathbb{G}_{\mathscr{C}}^N$ and let the associated network communication be given by* (9).

The ***scalable semi-global regulated state synchronization problem with additional information exchange*** *of a MAS is to find, if possible, a parametrized linear dynamic protocol with parameter $\varepsilon \in (0,1]$ for each agent $i \in \{1,\dots,N\}$, using only knowledge of agent model, i.e. (A,B,C), of the form:*

$$\begin{cases} \dot{x}_{c,i} = A_c^{\varepsilon} x_{c,i} + B_{c1}^{\varepsilon} \bar{\zeta}_i + B_{c2}^{\varepsilon} \hat{\zeta}_i, \\ u_i \;= F_c^{\varepsilon} x_{c,i}, \end{cases} \tag{13}$$

where $\hat{\zeta}_i$ is defined in (11) *with $\xi_i = E_c x_{i,c}$, and $x_{c,i} \in \mathbb{R}^{n_c}$, such that, for any given arbitrarily large compact sets $\mathbb{S}_a \in \mathbb{R}^n$, $\mathbb{S}_e \in \mathbb{R}^n$ and $\mathbb{S}_c \in \mathbb{R}^{n_c}$, and for any N and any graph $\mathcal{G} \in \mathbb{G}_{\mathscr{C}}^N$, there exists an ε^* such that, for all $\varepsilon \in (0,\varepsilon^*]$ regulated state synchronization* (7) *is achieved for all initial conditions of the agents in the set $\mathbb{S}_a$, all initial conditions of the exosystem in the set $\mathbb{S}_e$, and all initial conditions of the protocols in the set $\mathbb{S}_c$.*

Remark 2 *In the case of full-state coupling, matrix $C = I$ and we refer to Problems 2 and 1 with $\bar{\zeta}_i$ as* (10), *scalable semi-global and global regulated state synchronization problems for MAS with full-state coupling.*

III. Scalable global regulated state synchronization of MAS in presence of input saturation

In this section, we will consider the scalable global regulated state synchronization problem for a MAS with input saturation via scheduling (adaptive) design for both networks with full- and partial-state coupling.

A. Full-state coupling

In this case, the following nonlinear protocol is designed for each agent $i \in \{1,\dots,N\}$,

$$\begin{cases} \dot{\chi}_i = A\chi_i + Bu_i + \bar{\zeta}_i - \hat{\zeta}_i - \iota_i\chi_i \\ u_i \;= -B^{\mathrm{T}} P_{\varepsilon}(\chi_i)\chi_i, \end{cases} \tag{14}$$

where P_ρ is the unique solution of

$$A^{\mathrm{T}} P_\rho + P_\rho A - P_\rho B B^{\mathrm{T}} P_\rho + \rho P_\rho = 0. \tag{15}$$

and $\varepsilon(\chi_i)$ is defined as

$$\varepsilon(\chi_i) = \max\{\rho \in (0,1] : \chi_i^{\mathrm{T}} P_\rho \chi_i \operatorname{tr} B^{\mathrm{T}} P_\rho B \leqslant 1\}. \tag{16}$$

Note that [40] implies that P_ρ is increasing in ρ while $P_\rho \to 0$ as $\rho \to 0$. The agents communicate ξ_i which is chosen as $\xi_i = \chi_i$, therefore each agent has access to the following information:

$$\hat{\zeta}_i = \sum_{j=1}^{N} a_{ij}(\chi_i - \chi_j). \tag{17}$$

while $\bar{\zeta}_i$ is defined by (10).

Then, the synchronization result based on adaptation is stated in Theorem 1.

Theorem 1 *Consider a MAS described by* (4) *satisfying Assumption 1, and the associated exosystem* (6)*. Let a set of nodes $\mathscr{C}$ be given which defines the set $\mathbb{G}_{\mathscr{C}}^N$. Let the associated network communication be given by* (10)*.*

Then, the scalable global regulated state synchronization problem as stated in Problem 1 is solvable. In particular, the adaptive nonlinear dynamic protocol (14)*,* (15)*, and* (16) *solves the regulated state synchronization problem for any N and any graph $\mathcal{G} \in \mathbb{G}_{\mathscr{C}}^N$.*

Proof of Theorem 1: Firstly, let $\tilde{x}_i = x_i - x_r$, we have

$$\dot{\tilde{x}}_i = A\tilde{x}_i + B\sigma(u_i).$$

Let $e_i = \tilde{x}_i - \chi_i$. According to (16), it yields by construction that u_i does not get saturated, i.e., $\sigma(u_i) = u_i$, then, we can obtain

$$\begin{aligned} \dot{\tilde{x}}_i &= A\tilde{x}_i - BB^{\mathrm{T}} P_{\varepsilon(\tilde{x}_i - e_i)}(\tilde{x}_i - e_i), \\ \dot{e}_i &= Ae_i - \textstyle\sum_{j=1}^N \bar{\ell}_{ij} e_j. \end{aligned} \tag{18}$$

By defining

$$e = \begin{pmatrix} e_1 \\ \vdots \\ e_N \end{pmatrix}$$

we can obtain

$$\dot{e} = (I \otimes A - \bar{L} \otimes I)e \tag{19}$$

Since all eigenvalues of $\bar{L}$ are positive, we have

$$(T \otimes I)(I \otimes A - \bar{L} \otimes I)(T^{-1} \otimes I) = I \otimes A - \bar{J} \otimes I \tag{20}$$

for a non-singular transformation matrix T, where (20) is upper triangular Jordan form with $A - \lambda_i I$ for $i = 1, \cdots, N-1$ on the diagonal. Since all eigenvalues of A are in the closed left half plane, $A - \lambda_i I$ is stable. Therefore, all eigenvalues of $I \otimes A - \bar{L} \otimes I$ have negative real part. Therefore, we have that the dynamics for e_i are asymptotically stable.

We choose the following Lyapunov function:

$$V_i = (\tilde{x}_i - e_i)^{\mathrm{T}} P_{\varepsilon_\alpha}(\tilde{x}_i - e_i) \tag{21}$$

with $\varepsilon_\alpha = \varepsilon(\tilde{x}_i - e_i)$.

Assume V_i is non-increasing. Then we have

$$\frac{dV_i}{dt} \leqslant 0$$

On the other hand, if V_i is increasing then ε_α is non-increasing, which implies that P_{ε_α} is non-increasing. Meanwhile, we have

$$\begin{aligned} \frac{dV_i}{dt} \leqslant &(\tilde{x}_i - e_i)^{\mathrm{T}}[P_{\varepsilon_\alpha}(A - BB^{\mathrm{T}}P_{\varepsilon_\alpha}) + (A - BB^{\mathrm{T}}P_{\varepsilon_\alpha})^{\mathrm{T}} P_{\varepsilon_\alpha}](\tilde{x}_i - e_i) \\ &- e_i^{\mathrm{T}}[P_{\varepsilon_\alpha}(A - BB^{\mathrm{T}}P_{\varepsilon_\alpha}) + (A - BB^{\mathrm{T}}P_{\varepsilon_\alpha})^{\mathrm{T}} P_{\varepsilon_\alpha}]e_i \\ &+ 2e_i^{\mathrm{T}} P_{\varepsilon_\alpha} BB^{\mathrm{T}} P_{\varepsilon_\alpha}(\tilde{x}_i - e_i) - 2(\tilde{x}_i - e_i)^{\mathrm{T}} P_{\varepsilon_\alpha}\dot{e}_i \\ &+ (\tilde{x}_i - e_i)^{\mathrm{T}} \frac{dP_{\varepsilon_\alpha}}{dt}(\tilde{x}_i - e_i) \\ \leqslant &- \varepsilon V_i + \|e_i^{\mathrm{T}}[P_{\varepsilon_\alpha}(A - BB^{\mathrm{T}}P_{\varepsilon_\alpha}) + (A - BB^{\mathrm{T}}P_{\varepsilon_\alpha})^{\mathrm{T}} P_{\varepsilon_\alpha}]e_i\| \\ &+ 2\left\|P_{\varepsilon_\alpha}^{\frac{1}{2}} e_i\right\| \left\|P_{\varepsilon_\alpha}^{\frac{1}{2}} BB^{\mathrm{T}} P_{\varepsilon_\alpha}^{\frac{1}{2}}\right\| V_i^{\frac{1}{2}} + 2\left\|P_{\varepsilon_\alpha}^{\frac{1}{2}} \dot{e}_i\right\| V_i^{\frac{1}{2}} \end{aligned} \tag{22}$$

with P_{ε_α} satisfying (15).

Since e_i is the state of an asymptotically stable system, there exist z_1, z_2, z_3 such that

$$\|e_i^{\mathrm{T}}[P_{\varepsilon_\alpha}(A - BB^{\mathrm{T}}P_{\varepsilon_\alpha}) + (A - BB^{\mathrm{T}}P_{\varepsilon_\alpha})^{\mathrm{T}} P_{\varepsilon_\alpha}]e_i\|_1 \leqslant z_1$$
$$\left\|P_{\varepsilon_\alpha}^{\frac{1}{2}} e_i\right\|_1 \leqslant z_2, \left\|P_{\varepsilon_\alpha}^{\frac{1}{2}} \dot{e}_i\right\|_1 \leqslant z_3$$

Thus, we have

$$\frac{dV_i}{dt} \leqslant \beta_1(t) + \beta_2(t)V_i^{\frac{1}{2}} \leqslant (\beta_1(t) + \beta_2(t))(V_i + 1)^{\frac{1}{2}}$$

for suitable $\beta_1(t), \beta_2(t) \in \mathscr{L}_1$, and $\beta_1(t), \beta_2(t) \geq 0$, and

$$\dot{W}(t) = (\beta_1(t) + \beta_2(t))(W(t) + 1)^{\frac{1}{2}}$$

yields

$$W(t) = \left[\int_0^t (\beta_1(s) + \beta_2(s))ds + (W(0) + 1)^{\frac{1}{2}}\right]^2 - 1$$

Hence,

$$V_i(t) \leqslant \left(\|\beta_1\|_1 + \|\beta_2\|_1 + (V_i(0) + 1)^{\frac{1}{2}}\right)^2$$

Therefore, $V_i(t)$ is bounded which implies ε_α is bounded away from zero.

Remains to show that $V_i \to 0$. From [17, Lemma 6.1], we get

$$\left|(\tilde{x}_i - e_i)^{\mathrm{T}} \frac{dP_{\varepsilon_\alpha}}{dt}(\tilde{x}_i - e_i)\right| \leqslant k\frac{dV_i}{dt}$$

for some constant k.

Meanwhile, if V_i is non-increasing we can derive, similar to our early analysis (22), that

$$\begin{aligned} \frac{dV_i}{dt} &\leqslant - \varepsilon_\alpha V_i + \bar{\beta}(V_i + 1)^{\frac{1}{2}} + (\tilde{x}_i - e_i)^{\mathrm{T}} \frac{dP_{\varepsilon_\alpha}}{dt}(\tilde{x}_i - e_i) \\ &\leqslant - \varepsilon_\alpha V_i + \bar{\beta}(V_i + 1)^{\frac{1}{2}} - k\frac{dV_i}{dt} \end{aligned}$$

where $\bar{\beta} = \beta_1 + \beta_2$ and we get

$$\frac{dV_i}{dt} \leqslant -\frac{\varepsilon_\alpha}{1+k} V_i + \frac{\bar{\beta}}{1+k}(V_i + 1)^{\frac{1}{2}}.$$

But then we can prove that whether V_i is increasing or decreasing we always have

$$\frac{dV_i}{dt} \leqslant -\tilde{\alpha} V_i + \tilde{\beta}(V_i + 1)^{\frac{1}{2}}.$$

with $\tilde{\alpha}$ is a constant and lower bound of $\frac{\varepsilon_\alpha}{1+k}$ and $\tilde{\beta} = \frac{\bar{\beta}}{1+k} \in \mathscr{L}_1$. Clearly, that implies $V_i \to 0$ ■

B. Partial-state coupling

For partial-state coupling, we design the following adaptive nonlinear protocol for each agent $i \in \{1, \ldots, N\}$.

$$\begin{cases} \dot{\hat{x}}_i &= A\hat{x}_i + B\hat{\zeta}_{i2} + K(\bar{\zeta}_i - C\hat{x}_i) + \iota_i Bu_i \\ \dot{\chi}_i &= A\chi_i + Bu_i + \hat{x}_i - \hat{\zeta}_{i1} - \iota_i\chi_i \\ u_i &= -B^{\mathrm{T}} P_{\varepsilon(\chi_i)}\chi_i, \end{cases} \tag{23}$$

where K is a pre-design matrix such that $A - KC$ is Hurwitz stable. $P_{\varepsilon(\chi_i)}$ is the unique solution of (15) with $\rho = \varepsilon(\chi_i)$ where $\varepsilon(\chi_i)$ is defined as (16). In this protocol, the agents communicate $\xi_i = \begin{pmatrix} \xi_{i1}^{\mathrm{T}}, & \xi_{i2}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} \chi_i^{\mathrm{T}}, & u_i^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$, i.e. each agent has access to additional information $\hat{\zeta}_i = \begin{pmatrix} \hat{\zeta}_{i1}^{\mathrm{T}}, & \hat{\zeta}_{i2}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$, where:

$$\hat{\zeta}_{i1} = \sum_{j=1}^{N} a_{ij}(\chi_i - \chi_j), \tag{24}$$

and

$$\hat{\zeta}_{i2} = \sum_{j=1}^{N} a_{ij}(u_i - u_j). \tag{25}$$

while $\bar{\zeta}_i$ is defined via (9).

Then, we obtain the synchronization result based on adaptation as the following theorem.

Theorem 2 *Consider a MAS described by* (1) *satisfying Assumption 1, and the associated exosystem* (6)*. Let a set of nodes $\mathscr{C}$ be given which defines the set $\mathbb{G}_{\mathscr{C}}^N$. Let the associated network communication be given by* (9)*.*

Then, the scalable global regulated state synchronization problem as stated in Problem 1 is solvable. In particular, the adaptive nonlinear dynamic protocol (23)*,* (15)*, and* (16) *solves the scalable regulated state synchronization problem for any N and any graph $\mathcal{G} \in \mathbb{G}_{\mathscr{C}}^N$.*

Proof of Theorem 2: Similar to Theorem 1, let $\tilde{x}_i = x_i - x_r$. We also define

$$\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_N \end{pmatrix} \hat{x} = \begin{pmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_N \end{pmatrix} \chi = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_N \end{pmatrix}$$

According to (16), it yields by construction that u_i does not get saturated, i.e., $\sigma(u_i) = u_i$. By defining $e = \tilde{x} - \chi$ and $\bar{e} = (\bar{L} \otimes I)\tilde{x} - \hat{x}$, we can obtain

$$\begin{aligned} \dot{\tilde{x}}_i &= A\tilde{x}_i - BB^{\mathrm{T}}P_{\varepsilon(\tilde{x}_i - e_i)}(\tilde{x}_i - e_i) \\ \dot{\bar{e}} &= I \otimes (A - KC)\bar{e} \\ \dot{e} &= (I \otimes A - \bar{L} \otimes I)e + \bar{e} \end{aligned} \tag{26}$$

Since all eigenvalues of $A - KC$ and $I \otimes A - \bar{L} \otimes I$ have negative real part, we obtain dynamics of e and $\bar{e}$ are all asymptotically stable.

Then, we just need to prove the stability of

$$\dot{\tilde{x}}_i = A\tilde{x}_i - BB^{\mathrm{T}}P_{\varepsilon(\tilde{x}_i - e_i)}(\tilde{x}_i - e_i)$$

with e_i and $\dot{e}_i$ in $\mathscr{L}_1$. Then, similar to the proof of Theorem 1, the synchronization result can be obtained. ■

IV. Scalabale semi-global regulated state synchronization of MAS in presence of input saturation

In this section, we will consider the scalable semi-global regulated state synchronization problem for a MAS with input saturation for networks with full- and partial-state coupling.

A. Full-state coupling

We will design a parametrized linear dynamic protocol with parameter $\varepsilon \in (0, 1]$ for agent $i \in \{1, \ldots, N\}$ as follows.

$$\begin{cases} \dot{\chi}_i = A\chi_i + Bu_i + \bar{\zeta}_i - \hat{\zeta}_i - \iota_i\chi_i \\ u_i = -B^{\mathrm{T}}P_\varepsilon\chi_i, \end{cases} \tag{27}$$

where P_ε is the unique solution of the following ARE

$$A^{\mathrm{T}}P_\varepsilon + P_\varepsilon A - P_\varepsilon BB^{\mathrm{T}}P_\varepsilon + \varepsilon I = 0. \tag{28}$$

Note that [18] implies that (28), has a unique solution for any $\varepsilon > 0$ and $P_\varepsilon \to 0$. The protocol requires the additional information $\hat{\zeta}_i$ as (17), while $\bar{\zeta}_i$ is defined by (10).

Our formal result is stated in the following theorem.

Theorem 3 *Consider a MAS described by* (4) *satisfying Assumption 1, and the associated exosystem* (6)*. Let a set of nodes $\mathscr{C}$ be given which defines the set $\mathbb{G}_{\mathscr{C}}^N$. Let the associated network communication be given by* (10)*.*

Then, the scalable semi-global regulated state synchronization problem as stated in Problem 2 is solvable. In particular, for any given compact sets $\mathbb{S}_a \in \mathbb{R}^n$, $\mathbb{S}_e \in \mathbb{R}^n$ and $\mathbb{S}_c \in \mathbb{R}^n$, and for any N and any graph $\mathcal{G} \in \mathbb{G}_{\mathscr{C}}^N$, there exists $\varepsilon^* > 0$ such that, for any $\varepsilon \in (0, \varepsilon^*]$, the dynamic protocol (27) *and* (28) *solves the scalable regulated state synchronization problem.*

Proof: Due to the lack of space, the proof is provided in the extended version of the paper (See [**?**]). ■

B. Partial-state coupling

Now, we consider the case via partial-state coupling. We design a parametrized linear dynamic protocol with parameter $\varepsilon \in (0, 1]$ for agent $i \in \{1, \ldots, N\}$ as follows.

$$\begin{cases} \dot{\hat{x}}_i = A\hat{x}_i + B\hat{\zeta}_{i2} + K(\bar{\zeta}_i - C\hat{x}_i) + \iota_i Bu_i \\ \dot{\chi}_i = A\chi_i + Bu_i + \hat{x}_i - \hat{\zeta}_{i1} - \iota_i\chi_i \\ u_i = -B^{\mathrm{T}}P_\varepsilon\chi_i, \end{cases} \tag{29}$$

where K is a pre-design matrix such that $A - KC$ is Hurwitz stable and P_ε is the unique solutions of (28). Moreover, the agents communicate $\xi_i = \begin{pmatrix} \xi_{i1}^{\mathrm{T}}, & \xi_{i2}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} \chi_i^{\mathrm{T}}, & u_i^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$, i.e. each agent has access to additional information $\hat{\zeta}_i = \begin{pmatrix} \hat{\zeta}_{i2}^{\mathrm{T}}, & \hat{\zeta}_{i2}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$ as (24) and (25), while $\bar{\zeta}_i$ is defined via (9).

Then we have the following theorem for MAS via partial-state coupling.

Theorem 4 *Consider a MAS described by* (1) *satisfying Assumption 1, and the associated exosystem* (6)*. Let a set of nodes $\mathscr{C}$ be given which defines the set $\mathbb{G}_{\mathscr{C}}^N$. Let the associated network communication be given by* (9)*.*

Then, the scalable semi-global regulated state synchronization problem as stated in Problem 2 is solvable. In particular, for any given compact sets $\mathbb{S}_a \in \mathbb{R}^n$, $\mathbb{S}_e \in \mathbb{R}^n$, and $\mathbb{S}_c \in \mathbb{R}^{2n}$, and for any N and any graph $\mathcal{G} \in \mathbb{G}_{\mathscr{C}}^N$, there exists $\varepsilon^* > 0$ such that, for any $\varepsilon \in (0, \varepsilon^*]$ the dynamic protocol (29) *and* (28) *solves the scalable regulated state synchronization problem.*

Proof: Due to the lack of space, the proof is provided in the extended version of the paper (See [**?**]). ■

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