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(Eds.)

Modeling and Control of Complex Physical Systems

The Port-Hamiltonian Approach

Springer

This book is dedicated to all the people that worked in the GEOPLEX project and to all the students and researchers that joined the GEOPLEX public events: their enthusiasm deeply motivated us to prepare this book

Foreword

The story behind this book begins during the International Conference on Robotics and Automation 2001 which took place in Seoul, Korea in May of 2001. During the farewell reception I had a nice talk with Henrik Christensen, founder and scientific director of EURON, the European Robotic Network. Henrik mentioned that during some discussion with the European commission it came out that there was interest for good fundamental projects for the last call of FP5 related to the KA 4, Action Line IV2.1 on Advance Control Systems. At the end of August, after discussions with Arjan van der Schaft also at the University of Twente, it was decided to work on a proposal related to the great potentials of port-Hamiltonian systems initially introduced by Arjan van der Schaft and Bernhard Maschke. The deadline for the Submission was October 17th 2001 and it was at that stage not even clear who would be involved in the consortium. Around the end of September the real writing began and the consortium was formed by the University of Twente (NL) as a coordinator under my responsibility, Control Lab Products (NL) who with the 20-sim modeling and simulation program should have provided the tools implementing the new ideas in the project, Université Claude Bernard Lyon 1 (F) under the leadership of Bernhard Maschke, Universitat Politècnica de Catalunya (SP) under the leadership of Enric Fossas Colet, Supelec (F) under the leadership of Romeo Ortega, Johannes Kepler Universitat Linz (A) under the leadership of Kurt Schlacher, Katholieke Universiteit Leuven (B) under the leadership of Ir. Herman Bruyninckx, l'Università degli Studi di Bologna (I) under the leadership of Claudio Melchiorri and finally the CNRS (F) with Francoise Couenne: a great consortium was born.

During an incredible active period and difficult moments in which I did not believe we were going to make it, many sleepless nights and hard work brought us to a successful submission before the deadline. The project name was *"Geometric Network Modeling and Control of Complex Physical Systems"* with Acronyms GEOPLEX proposed by Herman Bruyninckx on an email dated 7th September 2001.

During the project preparation we decided to have as a deliverable a book which would collect some of the major results of the project. This volume is the final result of this effort. Many people have contributed to this volume and tough decisions have

been made in what to include and what to leave out in order to have a volume with didactic value and with a reasonably homogeneous notation.

The official "book deliverable" was a draft and a lot of work still needed to be done in order to get it to a useful publishable volume. The initial editorial efforts where done by Vincent Duindam, Herman Bruyninckx and myself during many meetings and discussions about the structure, the homogeneity and the various possibilities. After the end of the project, due to many obligations, the book has been in stand-by for a while until I kindly asked the help and support of Alessandro Macchelli (Alex) from the University of Bologna who has been also extremely scientifically (and not only) active during the project. Thanks to Alex this major group effort has finally become a real book and I can speak for the all consortium that we are really proud of this result.

The four years of the project beside having been scientifically productive and brought us to many new results, have created a wonderful interpersonal synergy which is still bringing fruits to this beautiful field. The GEOPLEX journey has been a great one and even if not all of you have enjoyed the great atmosphere and scientific discussions, I hope you will enjoy the result of this successful project and wonderful theory on port-Hamiltonian systems.

Enschede (NL), March 2009 *Stefano Stramigioli*

Preface

This preface gives a "bird's eye" view on the *paradigm* of port-Hamiltonian systems [137] for modelling and control of complex dynamical systems, which will be explained in detail in the rest of this book. The mentioned complexity comes, in the first place, from the *scale* of the systems, which is too large to be captured and controlled reliably by the traditional "block-diagram" approaches. This preface explains why this paradigm has a large potential to be successful in tackling some of the big challenges in modern control theory and engineering. Three of the paradigm's major features are:

i) its scalability to very large interconnected multi-physics systems;
ii) its ability for incorporating non-linearities while retaining underlying conservation laws;
iii) its integration of the treatment of both finite-dimensional and infinite-dimensional components.

But also for more traditional control problems, the port-Hamiltonian systems paradigm provides a solid foundation, which suggests new ways to look at control problems and offers powerful tools for analysis and control.

The port-Hamiltonian systems paradigm has, over the last decade, succeeded in matching the "old" framework of port-based network modeling of multi-domain physical systems with the "new" framework of geometric dynamical systems and control theory. It provides a very systematic approach to modelling, analysis and control, via

i) the separation of the network interconnection structure of the system from the constitutive relations of its components;
ii) the emphasis on *power flow* and the ensuing distinction between different kind of variables;
iii) the analysis of the system through the properties of its interconnection structure and the component constitutive relations;
iv) the achievement of *control by interconnection*, by means of *stabilization by Casimir generation and energy shaping* , *energy routing control* (transferring energy between components in the system), and *port and impedance control*.

Some familiarity with 'geometry', in particular 'coordinate-free' thinking and the identification of physically different types of variables with different mathematical objects, is the price to pay for a complete understanding of the paradigm: while port-Hamiltonian systems may at first sight make things "unnecessarily complicated" for the most simple systems, it can reach much further than traditional paradigms, with not much more than the same set of concepts that are used for these simple systems. This book's major ambition is to convince its readers that

i) the extra mathematical complexity introduced in port-Hamiltonian systems is the *necessary minimum* to represent the essential inherent properties of large-scale interconnected physical systems;
ii) understanding these mathematical concepts drastically reduces the human effort to master the intellectual "curse of dimensionality" created by tackling complex dynamical systems by only the traditional "block-diagram" control approaches.

Complex dynamical systems

Modern control engineering is continuously challenged to provide modelling, analysis and control for ever more *complex* systems:

- Complexity in the sense that the modern consumer expects to see more 'intelligent', better performing, and yet more miniaturized and/or lighter products. Most of the current sensing and actuation components do not scale well towards these meso- and milli-scales; at least, much less than the computational components. Inevitably, this evolution requires the development of sensor/actuator components that are integrated into the mechanical structure of the products. This will most certainly lead to components that cannot be modelled (and produced !) anymore as traditional finite-dimensional, i.e. "lumped parameter", systems.
- Complexity in the sense that production, logistics and service facilities evolve towards more distributed systems, with more decentrally controlled degrees of freedom, more interactions between various controlled subsystems, less possibilities to define, let alone measure, all relevant variables in the system, etc. Continental-scale power grids are a nice example: electricity is often produced further and further away from the final consumer; new energy sources (e.g., wind, biomass, energy recuperation) require more flexibility and bring higher load and source irregularities in the grid control system. Experience has shown that the traditional "optimized" control of the power grids has quite some problems with robustness against sudden transient effects, such as line breakage.

The traditional control approaches have, up to now, to a very large extent been focusing on "human-scale" systems, where one single control engineer can comprehend the whole system, one centralized controller can do the whole job, and one "Simulink" block diagram suffices to model the whole system dynamics to the required level of detail and to optimize its control to the required level of performance.

This approach, however, seldom scales well in the above-mentioned evolution towards more complex systems. Some of the major scaling problems are:

- Block diagram modeling of physical systems lacks compositionality: whenever interconnecting a physical system to additional components the block diagram modeling usually needs to be redone completely.
- The presence of fundamental physical properties such as conservation laws and energy balance is not reflected in the block diagram structure, and easily gets out of sight when the scale of the system grows.
- *Block diagram causality:* the large majority of control engineers is only familiar with the "Simulink"-like block diagram approach. Most of them don't even realize that, for modelling and computational simplicity, this approach *imposes* one specific physical and computational causality (i.e., a fixed choice of what is *input* and what is *output*) onto the system model and onto its controller, while the real physical system does not have these causal constraints.
- *Non-linearities:* the more nonlinear components appear in the system, the more difficult it gets to provide stable, efficient and optimized controllers with the traditional linear state space control theory that most engineers are trained in.
- *Integration of finite-dimensional and infinite-dimensional components:* no modelling and control paradigm has yet achieved the breakthrough in this domain.
- *Network size:* when the number of components in the system under control grows, the number of state variables also grows, as well as the communication delays between actuators, plant and sensors, the transmission line effects, etc. This means that a traditional centralized controller will not work anymore.
- *Robustness:* the traditional state space control paradigm has a big focus on optimized control, and the algorithms for designing robust controllers are still (implicitly) targeted at centralized systems. However, all the above-mentioned complexities drastically reduce the robustness of any optimal controller, if it has to work on a real-world complex system.

The port-Hamiltonian systems paradigm

Port-Hamiltonian systems represent a control *paradigm*, in the sense that they provide a set of models, thought patterns, techniques, practices, beliefs, systematic procedures, terminology, notations, symbols, implicit assumptions and contexts, values, performance criteria, ..., shared by a community of scientists, engineers and users in their modelling, analysis, design and control of complex dynamical systems. Being a coherent paradigm by itself does not mean that port-Hamiltonian systems have nothing in common with other control paradigms. The name "port-Hamiltonian" systems, for example, refers to the two major components of the paradigm, which exist for quite some time already:

- *Port:* the modelling approach is *port-based*, more in particular it builds upon the successful multi-domain Bond Graph way of composing complex systems by means of *power*-preserving interconnections.
- *Hamiltonian:* the mathematical framework extends the geometric Hamiltonian formulation of mechanics, by emphasizing the geometry of the state space and the Hamiltonian function (total stored *energy*) as basic concepts for modelling multi-physics systems.

Because of its roots in port-based modeling he port-Hamiltonian systems paradigm extends the geometric Hamiltonian formulation of physics by generalizing the geometry of the classical *phase space* to the geometry of the state space of energy variables which is determined by the (power-conserving) *interconnection structure*. Furthermore, it allows for the incorporation of *energy-dissipating*[1] components, and the presence of open ports modelling interaction with an (unknown) environment or accessible for controller interaction. Port-Hamiltonian systems theory relies on a rather limited amount of concepts from *geometry*. Geometry, in particular geometric linear algebra in the linear case and differential geometry in the nonlinear case, has proven to be a very appropriate mathematical formalism whenever one wants to separate generic, coordinate-free and (hence) intrinsic descriptions of systems from the details and particularities of specific representations, and if one wants to capture the physical characteristics of the variables involved in the mathematical description. Port-Hamiltonian systems shares this sympathy for geometry with, among others, the *geometric nonlinear control theory* paradigm [93, 156], and geometric mechanics [24, 40].

The *systematic procedure* for the modelling and control of complex dynamical systems, as it is beginning to materialize in the port-Hamiltonian systems theory, is as follows:

i) Model the system as energy storing and energy dissipating components, connected via ports to power conserving transmissions and conversions.
ii) Separate the network structure from the constitutive relations of the components.
iii) (Optionally) reduce the order of the system model while respecting the invariant *structure* of the system dynamics.
iv) Identify the *Casimir functions* (conservation laws) in the system, in order to use them in the design of the controller.
v) Control the system by *interconnecting* it to *energy shaping* and/or *damping injection* components, and by adding energy routing controllers.

The first focus of a port-Hamiltonian controller is to achieve a *feasible, stable and robust* control, instead of being driven by performance optimization from the start. As motivated above, this focus is already difficult enough when controlling increasingly complex dynamical systems.

[1] Of course, from a thermodynamical perspective energy is not dissipated but converted from, say, the mechanical or electrical domain to, e.g., the thermal domain. It would therefore be more appropriate to speak of 'free energy' that is dissipated.

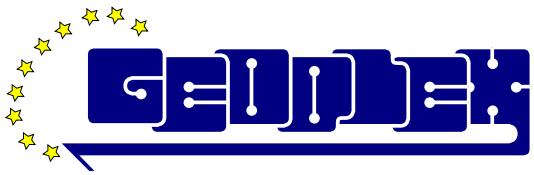


Fig. P.1 The logo of the GEOPLEX project.

The GEOPLEX project

The last few years, significant progress has been made in the port-Hamiltonian systems paradigm, to a large extent thanks to the concerted efforts of the GEOPLEX project [1], whose logo is displayed in Fig. P.1. Some of the major evolutions are: the port-Hamiltonian approach has excellent results in the systematic separation (conceptually, as well as in the mathematical representation) of the *interconnection structure* and the *dynamical properties of interconnected system*; finite-dimensional systems and infinite-dimensional systems can be described with unified concepts and mathematical representations, and one is nearing the above-mentioned breakthrough towards unified *control* of both domains; the *energy shaping* and *damping injection*, as well as the *energy routing*, control approaches begin to mature and show their advantages for the construction of safe and predictable controllers for complex systems.

GEOPLEX does not only have theoretical developments, but also real-world experimental verifications of the port-Hamiltonian systems approach: walking robots, teleoperated and haptic devices, piezo-controlled beams and plates, chemical engineering of reaction processes, electrical grids with sources and flywheels, etc..

Port-based modelling

Port-based modelling as in the Bond Graph formalism [165] models a physical system as the *interconnection* of (possibly a large amount) of components from a rather small set of dynamic elements: energy storage, energy dissipation, energy transportation or energy conversion. Each element interacts with the system via a *port*, that consists of a couple of "dual" *effort* and *flow* quantities, whose product gives the power flow in and out of the component. For example, force and velocity for a mechanical system, or current and voltage for an electric network. The network allows (loss-less) power exchange between all components and describes the power flows within the system and between the system and the environment. Some advantages of the Bond Graph approach are:

- It focuses on *energy* (in all its instantiations) as the fundamental physical concept to appropriately model the real world.

- It is *multi-domain*: the same concepts and mathematical representations are used for mechanical, electrical, hydraulic, pneumatic, thermo-dynamical, ..., components.
- It is *multi-scale*: components can be decomposed hierarchically in smaller interconnected components.
- Its models are *acausal*: each component contains only "constitutive relationships" which describe the dynamic relations between the port variables, without imposing which ones are inputs and which ones are output.

The Bond Graph approach has proven to be very successful in the modelling of complex systems, at least in the lumped parameter domain.

Differential geometry for systematic structuring

The above-mentioned system model networks, derived from port-based network modelling, can be mathematically described and analysed by means of the concept of a *Dirac structure* [54, 59, 183], at least for "lumped parameter" (finite-dimensional) systems. A Dirac structure can be regarded as a generalization of the well-known Kirchhoff laws of electrical circuit theory. It separates the (power conserving) *network topology* ("interconnection") from the (power storing or dissipating) dynamics of the components; both together provide a complete model of the dynamical system under study. The Dirac structure allows to bring *all* possible system models, however complex, into the same mathematical form, strongly facilitating a highly systematic treatment.

This systematic treatment has been extended to infinite-dimensional systems, for which the Dirac structure is generalized into the *Stokes-Dirac structure*, by incorporating Stokes theorem applied to the underlying conservation laws. Again, the same concepts are being reused.

Control by interconnection

The port-Hamiltonian systems paradigm uses the system's *interconnection structure* and its *Hamiltonian* (i.e., its total energy) as the primary vehicles for modelling and control. If one wants to steer the system to one of its stable equilibrium states, it is easy to do so: the Hamiltonian of the system assumes its minimum at this state, so, by introducing dissipation in the controller ("damping injection"), the energy in the system decreases until the minimum of energy, or, equivalently, the desired equilibrium configuration is reached.

However, the natural equilibrium states of the system seldom correspond to the desired system state. So, "energy shaping" is necessary, i.e., one has to add a control component to the system network, in such a way that the desired state in one way or another corresponds to a stable equilibrium of the new system. In summary,

the port-Hamiltonian approach to set-point control is "control by interconnection": the controller components that one adds to the plant network are the same kind of components as the ones that make up the model of the plant itself. In doing so, the first emphasis is on getting the controlled system robustly stable, possibly giving up on "optimal performance" of some of the subsystems.

How is this energy-shaping achieved? Clearly, the Hamiltonian of the control components may be chosen arbitrarily, but do not directly influence the shape of the Hamiltonian with respect to the state variables of the original system. One way to achieve energy-shaping is to add controller components in such a way that conserved quantities (*Casimirs*) are enforced involving the state variables of the original system and the controller states. Interestingly enough, these conserved quantities are determined by the Dirac structure of the interconnected system (original system *plus* controller system), thus leading to the problem of how to 'shape' or manipulate this Dirac structure by the interconnection with a controller Dirac structure. The total closed-loop energy function is then shaped by combining the Hamiltonian of the original system with the Casimirs and suitably chosen Hamiltonians for the controller systems. This procedure is systematic, but not explicit (that is, it still requires insight from the control designer), and it is not guaranteed to be applicable in all situations. Another (but very much related !) approach to stabilization, which in principle offers more options, is the Interconnection-Damping-Assignment methodology, where the energy function, the interconnection structure, as well as the damping structure, are directly modified by state feedback

Software support

The GEOPLEX project consists of some academic research institutions, plus one company: Controllab Products, which develops and markets the 20-sim [51] simulation and control software. 20-sim has Bond Graph modelling tools under the hood, but can also provide more user-friendly (because domain-specific) iconic diagrams; it has extensive algorithmic support for causality determination, for solving the differential equations governing a particular model (including controller), and for transforming the control design into code for embedded control computers. In addition, the software is being extended with some of the differential-geometric concepts and tools that are developed by the academic GEOPLEX partners.

Who should read this book?

The target public of this book consists of "traditional" control engineers, confronted with complex, multi-domain control problems, and graduate students in *Systems and Control*. This book can extend their knowledge and understanding of advanced modelling, analysis and control methods using the port-Hamiltonian systems paradigm,

because port-Hamiltonian systems bring the systems' *inherent structure* to the surface *explicitly*. This is an advantage because every additional structure that the engineer knows about its systems helps *to improve their analysis and control*.

- *Better insight* into complex systems, via the explicit distinction between the *inherent, coordinate-free physical properties* of the systems, and the *artificial, non-physical "properties"* that often show up in the specific *coordinate-based* algorithms that are used to implement analysis and control;
- The *acausal* description of components is very appropriate for the modelling, analysis and control of *open systems*, i.e., systems that can be interconnected to other open systems;
- The port-Hamiltonian approach can interconnect finite-dimensional and infinite-dimensional systems;
- *Scalability*: port-Hamiltonian structure is preserved by interconnection of multiple components modeled as port-Hamiltonian systems, also when components come from different domains;
- *"Re-use"* of the same theory for finite- and infinite-dimensional systems;
- More *modular, structured, and re-usable software framework*, leading to more user-friendly and more reliable modelling, simulation and design software tools;
- More structure means more "constraints" that make the "solution search space" smaller, hence *leading to potentially more efficient and more precise algorithms*.

Outline of the book

This book aims at presenting a unified framework for modelling, analysis, simulation and control of *complex* dynamical systems based on the port-Hamiltonian formalism. Background and concepts of a port-based approach to integrated modelling and simulation of physical systems and their controllers are illustrated in **Chapter 1**. These important notions are the conceptual motivation from a physical point of view of what is elaborated mathematically and applied to particular cases in the remaining chapters. In fact, in **Chapter 2**, it is shown how the representation of a lumped-parameter physical system as a bond graph naturally leads to a dynamical system endowed with a geometric structure, called port-Hamiltonian system. The notion of Dirac structure is here introduced as the key mathematical concept to unify the description of complex interactions in physical systems. Moreover, it is shown how the port-Hamiltonian structure is related, among others, to the classical Hamiltonian structure of physical systems. Furthermore, different representations of port-Hamiltonian systems are discussed, as well as the ways to navigate between them, and tools for analysis are introduced.

Port and port-Hamiltonian concepts are the basis of the detailed examples of modelling in several domains illustrated in **Chapter 3**. Here, it is shown how port-Hamiltonian systems can be fruitfully used for the structured modelling of electromechanical systems, robotic mechanisms and chemical systems. As far as the chemical domain is concerned, expressions of the models representing momentum,

heat and mass transfer as well as chemical reactions within homogeneous fluids are reported in the port-based formalism. Furthermore, some insights are also given concerning the constitutive equations and models allowing to calculate transport and thermodynamic properties.

These last concepts serve as a starting point for the generalization of the port-Hamiltonian description of lumped parameter systems towards the distributed parameter ones. This is accomplished in **Chapter 4** by extending the definition of Dirac structure. In fact, it is shown how the Dirac structure and the port-Hamiltonian formulation arise from the description of distributed parameter systems as systems of conservation laws. In case of systems of two conservation laws, which describe two physical domains in reversible interaction, they may be formulated as port-Hamiltonian systems defined on a canonical interconnection structure, called canonical Stokes-Dirac structure. Several examples of physical systems are provided in order to illustrate the power of the proposed approach.

In the remaining chapters, it is shown how the port-Hamiltonian formulation offers powerful methods for control of complex multi-physics systems. In **Chapter 5**, a number of approaches to exploit the model structure of port-Hamiltonian systems for control purposes is illustrated. Formulating physical systems as port-Hamiltonian systems naturally leads to the consideration of *impedance* control problems, where the *behavior* of the system at the interaction port is sought to be shaped by the addition of a controller system. As an application of this strategy of *control by interconnection* within the port-Hamiltonian setting, the problem of stabilization of a desired equilibrium by shaping the Hamiltonian into a Lyapunov function for this equilibrium is considered. The mathematical formalism of port-Hamiltonian systems provides various useful techniques, ranging from Casimir functions, Lyapunov function generation, shaping of the Dirac structure by composition, and the possibility to combine finite-dimensional and infinite-dimensional systems.

In this respect, the control problem of distributed parameter port-Hamiltonian systems is discussed in **Chapter 6**. This chapter aims at extending some of the well-established control techniques developed for finite dimensional port-Hamiltonian systems illustrated in Chapter 5 to the infinite dimensional case. First result concerns the control by damping injection, which is applied to the boundary and distributed control of the Timoshenko beam. Then, the control by interconnection and energy shaping via Casimir generation is also discussed, giving particular emphasis to the stabilization of *mixed finite and infinite dimensional* port Hamiltonian system and to the dynamical control of a Timoshenko beam.

March 2009 *Herman Bruyninckx*, *Arjan J. van der Schaft*

Acknowledgements

It is difficult to write acknowledgments of a book because undoubtedly there will be people which will not be specifically mentioned which directly or indirectly have contributed to the realization of this work.

I would like to acknowledge first the two persons which can be considered the inventors of the line of research presented in this volume which are Arjan van der Schaft and Bernhard Maschke who beside being two incredibly inspiring scientists, are wonderful people to work with and to have informal discussions or a nice dinner at the end of a working day.

Furthermore, the people which I would like to specifically mention for the birth and support of the project are Henrik Christensen who suggested to work on such a proposal, as also mentioned in the foreword, Alkis Konstantellos, the EC project officer who has always supported the project with great enthusiasm and Alberto Isidori and Andreas Kugi, the reviewers of the project which have always given useful feedback during the formal project reviews and elsewhere.

On the editorial side I would like to stress the incredible work that especially Vincent Duindam and Alessandro Macchelli have done without whom we would still be waiting for this volume.

But of course, the people which are really the architects of this book are the authors reported in each chapter but not only, because the result of each chapter is a consequence of many discussions which took place among all members of the consortium; thank you GEOPLEX consortium! The project has been a wonderful journey and this book is the result of a unique scientific collaboration.

Enschede (NL), March 2009 *Stefano Stramigioli*

Contents

Chapter 1
Port-Based Modeling of Dynamic Systems

P. C. Breedveld

Abstract Many engineering activities, in particular mechatronic design, require that a multi-domain or 'multi-physics' system and its control system be designed as an integrated system. This chapter discusses the background and concepts of a port-based approach to integrated modeling and simulation of physical systems and their controllers, with parameters that are directly related to the real-world system, thus improving insight and direct feedback on modeling decisions. It serves as the conceptual motivation from a physical point of view that is elaborated mathematically and applied to particular cases in the remaining chapters.

1.1 Introduction

1.1.1 Modeling of dynamic systems

If modeling, design and simulation of (controlled) systems are to be discussed, some initial remarks at the meta-level are required. It should be clear and it probably will be, due to the way it is phrased next, that no global methodology exists that deals with each problem that might emerge. In other words, no theory or model can be constructed independently of some problem context. Nevertheless, in practice, not only well-established theories are treated as some form of absolute 'truth', but also (sub-)models of physical components are often considered as constructs that can be independently manipulated, for instance in a so-called model library. Without some reference to a problem context, such a library would be useless, unless there is an implicit agreement about some generic problem context, such that some generic sub-models can be stored for re-use. However, such a foundation is rather weak, as implicit agreements tend to diverge, especially in case of real world problems.

Herein, we will focus on the generic problem context of the dynamic behavior of systems that primarily belong to the domain of the (control) engineer, but also of the physicist, the biologist, the physiologist, etc. These systems can be roughly

characterized as systems that can be considered to consist for a large part of subsystems for which it is relevant to their dynamic behavior that they obey the basic principles of macroscopic physics, like the conservation principles for fundamental physical quantities like 'charge', 'momentum', 'matter', 'energy' etc. as well as the positive-entropy-production principle, in other words: the 'laws' of thermodynamics. The other part is considered to consist of sub-models for which the energy bookkeeping is generally not considered relevant for the dynamic behavior. Such parts are generally addressed as the signal processing part ('controller') that is commonly for a large part realized in digital form for the common reasons of flexibility, reproducibility, robustness (error correction), maintainability, etc., even though analogue solutions are in some cases much less costly in terms of material and energy consumption.

This chapter focuses on the description of the part for which energy bookkeeping is relevant for the dynamic behavior, while keeping a more than open eye for the connection to the signal part, either in digital or in analogue form. It is argued that so-called 'port-based modeling' is ideally suited for the description of the energetic part of a multi-domain system, sometimes also called multi-physics system. This means that the approach by definition deals with multidisciplinary systems like those encountered in mechatronics for example.

Port-based physical system modeling aims at providing insight, not only in the behavior of systems that an engineer working on multidisciplinary problems wishes to design, build, troubleshoot or modify, but also in the behavior of the environment of that system. A key aspect of the physical world around us is that 'nature knows no domains'. In other words, all boundaries between disciplines are man-made, but highly influence the way humans interact with their environment. A key point each modeler should be aware of is that any property of a model that is a result of one of his own choices, should not affect the results of the model. Examples of modeler's choices are: relevance of time and space scales, references, system boundaries, domain boundaries, coordinates and metric. If a variation in one of these choices leads to completely different conclusions about the problem for which the model is constructed, the model obviously does not serve its purpose as it tells us more about the modeler than about the problem to be solved. Again, when the issue is phrased in this manner, it is hard to disagree, but practice shows that this modeling principle is often violated.

1.1.2 History of physical systems modeling of engineering systems

Several attempts to unified or systematic approaches of modeling have been launched in the past. In the upcoming era of the large-scale application of the steam engine over 150 years ago, the optimization of this multi-domain device (thermal, pneumatic, mechanical translation, mechanical rotation, mechanical controls, etc.) created the need for the first attempt to a systems approach. This need for such a 'mecha-thermics' approach was then named thermodynamics. Although many will

not recognize a modern treatment of thermodynamics as the first systems theory, it certainly was aimed originally in trying to describe the behavior of such a system independently of the involved domains. However, it required a paradigm shift or 'scientific revolution' in the sense of Kuhn (see [106]), due to the fact that the concept of entropy had to be introduced for reasons of consistency, i.e. to be able to properly 'glue' these domains together by means of the concept of a conserved quantity called energy. The rather abstract nature of the concept of entropy, and to some extent of the concept of conserved energy too, as energy was up till than considered a quantity that could be dissipated, has caused that students have considered thermodynamics a 'difficult' subject ever since, resulting in only a relatively limited number of engineers and scientists actively using the thermodynamic approach in modeling of system behavior and system design.

Despite the fact that the first evidence of the use of feedback dates back to 200-100 BC when water clocks required the water level in a reservoir to be kept constant, followed by Cornelis Drebbel's thermostat and James Watt's fly-ball governor, it was not before the late nineteen twenties that feedback was realized by means of electric signals (Harold Stephen Black's 1927 famous patent that he wrote on a copy of the New York Times). At first, electronic feedback was used internally, to reduce distortion in electric amplifiers, but later, especially during World War II, this concept was used in radar control and missile guidance. One might say that the multi-domain approach to feedback was transferred to a signal approach in which the external power supply did not need to be part of the behavioral analysis. However, a more important paradigm shift was still to come, viz. the idea that the use of feedback allowed the construction of components, viz. operational amplifiers, with which basic mathematical operations could be mimicked, leading to analogue computers. This gave a new meaning to the terminology 'analogue simulation' that until then was conceived as mimicking behavior by means of analogue circuits or mechanisms.

Just after World War II, due to the rapidly increasing demand for electric power, the USA was in great need for power plants, in particular hydro-power plants, which should be able to deal with large and sometimes rapid fluctuations in the power grid. Obviously, the success of control theory (cybernetics) during World War II inspired many to apply control theory to the dynamic problems involved in electric power production. One such a civil engineer by the name of Henry M. Paynter[1] (1923-2002, professor at MIT & UT Austin, Fig. 1.1) tried to use the early analogue computers that he had invented together with James Philbrick, to simulate the dynamics of the power plants to be built. He used the at that time common description of block diagrams that display the computational structure of the differential and algebraic equations being used, as these mathematical operations were to be mapped directly on the basic components of the analogue computer. However, for reasons that will become clear in the course of this treatise (viz. related to the concept of so-called 'computational causality') he ran into model formulation problems. At the beginning of the fifties he realized himself that the concept of a 'port'

[1] See also `http://www.me.utexas.edu/~lotario/paynter/hmp/index.html`.

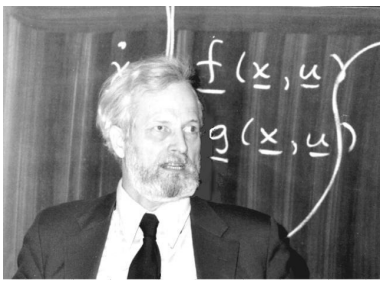


Fig. 1.1 Prof. Henry M. Paynter (Enschede, 1984).

introduced in electrical circuit theory a few years earlier by Wheeler & Dettinger (see [219]), should be extended to arbitrary power ports that can be applied domain-independently. Power ports include mechanical ports, hydraulic ports, thermal ports, electric ports, etc., i.e. everything Paynter needed for the description of the dynamic behavior of hydro-power plants.

In the following decade, after moving to the MIT mechanical engineering department, he designed a notation based on the efficient representation of the relation between two ports by just one line that he called a 'bond'. This so-called 'bond graph' notation was completed when he finally introduced the concept of the junction in a lecture in 1959 [165]. Junctions not only make a bond graph a powerful tool, but they are rather abstract concepts that require a similar paradigm shift as the one mentioned for thermodynamics. Once this shift is made, it often induces over-enthusiasm and over-expectations that not only lead to disappointment, but also unnecessarily scare off experienced engineers and scientists who have learned to accept the limitations of modeling.

As a result, just like thermodynamics, bond graphs never became widely popular, although they spread over the whole world and are still alive and continue to grow after almost fifty years. By contrast, signal processing as well as analogue and later digital computing are less constrained by physical reality. This allows mimicking virtually everything, from physically correct or incorrect models to arbitrary mathematical relations that describe imaginary systems. In the previous decade, this even led to concepts like a 'cyber world', etc., even though the level of physical modeling in most virtual environments is rather low, as demonstrated by the unnatural features of much virtual behavior, even though there has been quite some progress recently due to more awareness for the importance of the underlying physics.

Nevertheless, the introduction of rapid and flexible machinery for production, assembly, manipulation (incl. surgery), etc., that has truly taken off in the nineties, raised the need for a systems approach again. In these application areas physical constraints continue to limit imagination. The deliberate dynamic behavior as well as the suppression of undesired dynamics of such devices heavily leans on the application of digital electronics (microcomputers) and software, but a domain-independent description of the parts in which power plays a role is crucial to make a designer aware of the fact that a considerable part of these systems is constrained by the limits of the physical world. This mix of mechanics, or rather physical sys-

tem engineering in general, at the one hand and digital electronics, software and control at the other hand has been named ‘mechatronics’, even though ‘digiphysics’ would have been a more appropriate label, as this probably would have prevented the common misinterpretation that mechatronics is basically a modern form of motion synchronization where the control by means of mechanisms (e.g. by cam wheels) is replaced by electronic motion synchronization (e.g. by digitally controlled electric motors). Nevertheless, we will continue to use the label ‘mechatronics’, even though ‘behavior’ has a wider meaning than ‘motion’.

1.1.2.1 History of bond graphs

When Henry Paynter introduced the junctions in April 1959, he concluded a period of about a decade in which most of the underlying concepts were formed and put together into a conceptual framework and corresponding notation. In the sixties the bond graph notation, e.g. the half arrow to represent positive orientation and insightful node labeling, was further elaborated by his students, in particular Dean C. Karnopp, later professor at UC Davis (California, US), and Ronald C. Rosenberg, later professor at Michigan State University (Michigan, US) who also designed the first computer tool (ENPORT) that supported simulation of bond graph models. In the early seventies Jan J. van Dixhoorn, professor at the University of Twente, Netherlands, and Jean U. Thoma, professor at the University of Waterloo (Ontario, Canada). and independent consultant in Zug, Switzerland, were the first to introduce bond graphs in Europe.

These pioneers in the field and their students have been spreading these ideas worldwide. Jan van Dixhoorn realized that an early prototype of the block-diagram-based software TUTSIM could be used to input simple causal bond graphs, which, about a decade later, resulted in a PC-based tool. This work laid the basis for the development of a port-based computer tool at the University of Twente (‘20-sim’ or ‘Twente-sim’). He also initiated research in modeling more complex physical systems, in particular thermo-fluid systems and spatial mechanisms.

In the last two decades, bond graphs either have been a topic of research or are being used in research at many universities worldwide and are part of (engineering) curricula at a steadily growing number of universities. In the last decade, industrial use has become increasingly important.

1.1.3 Tools needed for the integrated design of controlled physical systems

Obviously, a smooth connection is needed between the information-theoretical descriptions of the behavior of digital systems and physical systems theory. Since their introduction bond graphs have allowed the combined use of power ports and signal ports, both in- and output, and a corresponding mix with block diagrams. As

block diagrams can successfully represent all digital operations that are similar to mathematical operations, the common bond graph/block diagram representation is applicable. This graphical view supports a hierarchical organization of a model, supporting reuse of its parts.

However, many systems that are studied by engineers, in particular control and mechatronic engineers, differ from the engineering systems that were previously studied in the sense that the spatial description of complex geometrical configurations often plays an important role in the dynamic behavior, thus including the control aspects of these systems. While dynamic influences of such mechanisms could be suppressed by simply limiting the operating speed previously, the increasing need for higher speeds and lighter mechanisms has made this solution obsolete. This shows the need for a consistent aggregation of – at the one hand – the description of the configuration of a mechanism and – at the other hand – the displacements in a system that in some way are related to the storage of potential or elastic energy. Later this will be addressed as 'the dual role of the displacement variable'. The geometric interpretation of physical structure that will be one of the main topics of this book and can be seen as a generalization by abstraction of the description of configuration space into the resulting approach in terms of manifolds, therefore also carries a danger: a too high level of abstraction, despite its power in analysis, may blur the above insight as the distinction between the two roles of the displacement variable seems lost.

Another aspect of these systems is that only few realistic models can be solved analytically, emphasizing the important role of a numerical approximation of the solution (simulation). The aggregation of numerical properties in the representation of dynamic systems allows that a proper trade-off is made between numerical and conceptual complexity of a model, however, without confusing the two, a common pitfall. The approach discussed herein offers a basis for making such a trade-off, resulting in both a higher modeling efficiency and a higher numerical simulation efficiency.

In a mechatronics approach, where a controlled system is designed as a whole, it is advantageous that model structure and parameters are directly related to physical structure in order to have a direct connection between design or modeling decisions and physical parameters [49]. In addition, it is desired that (sub-)models be reusable, despite the danger of non-matching problem contexts addressed earlier. Common simulation software based on block-diagram or equation input does not sufficiently support these features. The port-based approach towards modeling of physical systems allows the construction of easily extendible models. As a result it optimally supports reuse of intermediate results within the context of one modeling or design project. Potential reuse in other projects depends on the quality of the documentation, particularly of the modeling assumptions. In addition, full integration of user in- and output of configuration information in a modeling and simulation tool, i.e. CAD-like input and 3D visualization of simulation results as one of the views on model results, is required to support the insight in the influence of the kinematic structure on the dynamic behavior. Currently, research is even focused on extending this approach from multi-body systems to systems that contain parts

that are until now analyzed separately by means of Finite Element Models (FEM), which is limited to a vibration analysis in terms of eigen modes.

1.1.4 Object-oriented modeling

The port-based approach may be considered a kind of object-oriented approach to modeling: each object is determined by constitutive relations at the one hand and its interface, being the power and signal ports to and from the outside world, at the other hand. Other realizations of an object may contain different or more detailed descriptions, but as long as the interface (number and type of ports) is identical, they can be exchanged in a straightforward manner. This allows top-down modeling as well as bottom-up modeling. Straightforward interconnection of (empty) sub-models supports the actual decision process of modeling, not just model input and manipulation. Empty sub-model types may be filled with specific descriptions with various degrees of complexity – these sub-models are said to be polymorphic [214] – to support evolutionary and iterative modeling and design approaches [218]. Additionally, sub-models may be constructed from other sub-models resulting in hierarchical structures. Without trying to become part of the everlasting argument between object-oriented and process-oriented approaches, it should be noted that the 'objects' in this particular case represent behaviors or 'processes' (storage, transformation, etc.) with respect to energy, such that the label 'process-oriented approach' is applicable too. In this context it is interesting to note that when Paynter introduced the elements of the bond graph notation, he was heavily inspired by the American philosopher Charles Sanders Peirce, who proposed to 'dualize' graphical representation: where commonly the nodes of a graph represent 'objects' and its edges 'operations', Peirce proposed to represent 'objects' by edges and 'operations' by nodes: we will see that the nodes of a bond graph indeed represent basic behaviors with respect to energy, while its edges, the bonds, represent the relevant conjugate power variables.

1.1.5 Design phases of engineering systems

Modeling, simulation and identification is often done for already existing systems. The design of a controller has to be done for an already realized and given 'process' (incremental design). By contrast, in case of a full design the system does not yet exist, which not only means that there is a large initial uncertainty, but also that there is much more freedom to modify the design, not just the controller, but the complete 'process', including the mechanical construction. However, the next discussion will show that a port-based approach is also crucial in modern design.

In a design process the following, iterative phases can be distinguished:

Phase 1: A conceptual design is made of the system that has to be constructed, taking into account the tasks that have to be performed and identifying and modeling the major components and their dominant dynamic behaviors, as well as the already existing parts of the system or its environment that cannot be modified. The part of the model that refers to the latter parts can be validated already.

Phase 2: Controller concepts can be evaluated on the basis of this simple model. This requires that the model is available in an appropriate form, e.g. as a transfer function or a state space description. If this phase provides the insight that modification of some dominant behavior would be quite beneficial, revisiting Phase 1 can lead to the desired improvement. Another reason to return to Phase 1 may be an alternative choice of system boundary or the boundary between physical process ('plant') and controller.

Phase 3: When the controller evaluation is successful, the different components in the system can be selected and a more detailed model can be made. The controller designed in Phase 2 can be evaluated with the more detailed model and controller and component selection can be changed. If the effects of the detailed model prove to distort the originally foreseen performance, revisiting either Phase 2 or even Phase 1 with the newly obtained insights can lead to improved performance.

Phase 4: When Phase 3 has been successfully completed the controller can be realized electronically or downloaded into a dedicated microprocessor (embedded system). This hardware controller can be tested with a hardware-in-the-loop simulation that mimics the physical system ('plant') still to be built. It is to be preferred that the translation from the controller tested in simulations is automatically transferred to e.g. C-code, without manual coding; not only because of efficiency reasons, but also to prevent coding errors. If this phase results in new insights given the non-modeled effects of the implementation of the controller, the previous phases may be revisited, depending on the nature of the encountered problem.

Phase 5: Finally the physical system itself or a prototype can be built. As this is usually the most cost intensive part of the process, this should be done in such a way that those physical parameters that proved to be most critical in the previous phases are open for easy modification as much as possible, such that final tuning can lead to an optimal result. During the construction, new awareness of not yet modeled, but relevant physical behavior, may result in changes to the model of the physical systems, such that the impact on the final performance can be immediately estimated and, if necessary, compensated by physical means or by signal processing means.

Given the key role of structured, multi domain system modeling in the above process, special attention is given to domain-independent modeling of physical systems and the role of multiple views.

1.1.6 Multiple views in the design and modeling process

Engineering deals with the integrated design of a tangible ('mechanical') system and its embedded control system. In practice, this 'mechanical system' has a rather wide scope. It may also contain hydraulic, pneumatic and even thermal parts that influence its dynamic characteristics. This definition implies that it is important that the system be designed as a whole as much as possible. This requires a systems approach to the design problem. Because in mechatronics the scope is limited to controlled mechanical systems, it will be possible to come up with more or less standard solutions. An important aspect of mechatronic systems is that the synergy realized by a clever combination of a physical system and its embedded control system leads to superior solutions and performances that could not be obtained by solutions in one domain. Because the embedded control system is often realized in software, the final system will be flexible with respect to the ability to be adjusted for varying tasks.

The interdisciplinary field of design thus requires tools that enable the simultaneous design of the different parts of the system. The most important disciplines playing a role are mechanical engineering, electrical engineering and software engineering, all based on a solid knowledge of physics and mathematics. One of the ideas behind mechatronic design is that functionality can be achieved either by solutions in the (physical) mechanical domain, or by information processing in electronics or software. This implies that models should be closely related to the physical components in the system. It also requires software tools that support such an approach. In an early stage of the design process simple models are required to make some major conceptual design decisions. In a later stage (parts of the) models can be more detailed to investigate certain phenomena in depth. The relation to physical parameters like inertia, compliance and friction is important in all stages of the design. Because specialists from various disciplines are involved in mechatronic design, it is advantageous if each specialist is able to see the performance of the system in a representation that is common in his or her own domain. Accordingly, it should be possible to see the performance of the mechatronic system in multiple views. Typical views that are important in this respect are: ideal physical models (IPMs) represented by 'iconic diagrams', bond graphs, block diagrams, Bode plots, Nyquist plots, state space description, time domain, animation, C-code of the controller.

This has been formalized as the so-called multiple view approach that is particularly well supported by window-based computer tools: a number of graphical representations like iconic diagrams, which are domain-dependent, linear graphs, which are more or less domain-independent, but limited to the existence of analogue electric circuits [192] block diagrams, which represent the computational structure, bond graphs, which are domain-independent, etc. as well as equations, which represent the mathematical form in different shapes (transfer functions, state space equations in matrix form, etc.) can serve as model representations in different windows. The tool in which all examples of this chapter are treated, 20-sim, has been designed on the basis of such a multiple view approach. Possible views in 20-sim are: equations, including matrix-vector form, block diagrams, (multi-)bond

graphs, transfer functions, state-space representations (system, in- and output matrices), time responses, phase planes, functional relationships, step responses, bode plots, pole-zero plots, Nyquist diagrams, Nichols charts and 3D-animation. Where possible, automatic transformation is provided and results are linked.

The port-based approach has been taken as the underlying structure of 20-sim, formulated in the internal language SIDOPS [38], which makes it an ideal tool for demonstration of the port-based and multiple view approaches. A more detailed introduction to ports, bonds and the bond graph representation is given later. This will give the reader sufficient insight in order to exercise it with the aid of a port-based modeling and simulation software like 20-sim. This tool allows high level input of models in the form of iconic diagrams, equations, block diagrams or bond graphs and supports efficient symbolic and numerical analysis as well as simulation and visualization. Elements and sub-models of various physical domains (e.g. mechanical or electrical) or domain-independent ones can be easily selected from a library and combined into a model of a physical system that can be controlled by block-diagram-based (digital) controllers[2]. For more advanced issues, the interested reader is referred to the references. However, first we return to the meta-level of modeling in order to address some important issues that should be clear upfront.

1.2 Modeling philosophy

1.2.1 'Every model is wrong'

This paradoxical statement seeks to emphasize that any 'model' that perfectly represents all aspects of an original system, assuming it can exist, would not be a *model*, but an exact copy of that system (*identity*). When modeling, by contrast, one looks for simple but relevant *analogies*, not for complex identities. As a consequence, a model is much simpler than reality. This is its power and its weakness at the same time. The weakness is that its validity is constrained to the problem context it was constructed for, whereas its strength is the gain of insight that may be obtained in the key behaviors that play a role in this particular context. In other words: 'no model has absolute validity'. The resulting advice is that one should always keep the limitations of a model in mind and always try to make them explicit first. Especially in an early phase of a modeling or design process, such a focus may result in interesting insights.

[2] A demonstration copy of 20-sim that allows the reader to get familiar with the ideas presented in this contribution can be downloaded from the Internet, `http://www.20sim.com`.

1.2.2 'A model depends on its problem context'

Models should be *competent* to support the solution of a specific problem. This also means that any type of archiving of a model or sub-model should always include information about the corresponding problem context. Without this context, the model has no meaning in principle. Training of specialists and experts is often related to what is sometimes called a 'culture' and that they are said to speak a 'jargon'. This culture and jargon reflect the existence of a particular (global) problem context, even though this context is not explicitly described when models are made. For electrical circuit designers, this problem context consists of the behavior of electric charges and in particular of the voltages and currents related to this behavior, in a specific part of the space-time scales. This behavior is such that electromagnetic radiation plays no dominant role. Mechanical systems mostly belong to another part of the space-time scale, although there may be considerable overlap, in particular in precision engineering.

These cultures and jargons easily lead to implicit assumptions too. The assumptions, in turn, may lead to model extrapolations that have no validity in the specific problem context at hand due to the danger of ignoring these earlier assumptions. These extrapolations often start from well-known classroom problems with analytical solutions like the model of a pendulum [35]. In other words: 'implicit assumptions and model extrapolations should be avoided'. The resulting advice is that one should focus at the model's competence to represent the behavior of interest, not at its 'truth content' and that it is better to start from scratch as much as possible instead of trying to extrapolate standard (textbook) models without complete understanding of the underlying assumptions.

1.2.3 Physical components versus conceptual elements

At all times it should be clear that (physical) components, i.e. identifiable, tangible system parts that can be physically disconnected and form a so-called physical, often visible, structure, are to be clearly distinguished from (conceptual) elements, i.e. abstract entities that represent some basic behavior, even though these conceptual elements are sometimes given the same name as the physical component. For example, a resistor may be an electrical component with two connection wires and some color code (cf. Fig. 1.2a), while the same name is used for the conceptual element (commonly represented by Fig. 1.2b) that represents the dominant behavior of the component with the same name, but also that of a piece of copper wire through which a relatively large current flows or even the leakage in the component 'electrical capacitor'. Note that this model of the dominant behavior requires that the problem context is such that the component 'resistor' is part of a current loop in a network in which the behavior of the voltages and currents plays a role. By contrast, other realistic problem contexts exist in which the dominant behavior of the *component* 'resistor' is not represented by the *element* 'resistor', but by the

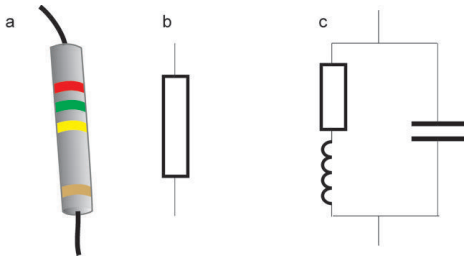


Fig. 1.2 Component electrical resistor (a) with two different conceptual models (b & c).

element 'mass' or a combination of mechanical conceptual elements like 'mass', 'spring' and 'damper'. For example, when this component is to be rapidly manipulated in assembly processes, i.e. before it becomes part of an active electric circuit, the element 'mass' could be a competent model.

Often, not only the dominant behavior of a component has to be described, but also some other properties that are often called 'parasitic', because they generate a conceptual structure and destroy the one-to-one mapping between components and elements that seems to simplify modeling and design in a quite misleading way (cf. Fig. 1.2c). Those areas of engineering in which materials can be manipulated to an extent that all other behaviors than the dominant one are sufficiently suppressed to achieve the desired functionality (like electrical engineering), have been the first to apply network style dynamic models successfully. In our daily life we have learned to make quick intuitive decisions about dominant behaviors ('survival of the fittest'). This type of learning stimulates implicit and intuitive decisions that may fail in more complex engineering situations (counter-intuitive solutions).

Implicit assumptions are commonly not only made about the problem context, but also about the reference, the orientation, the coordinates, and the metric and about 'negligible' phenomena. Famous classroom examples may have an impact on the understanding of real behavior for generations, especially due to the textbook copying culture that is the result of what may be called a 'quest for truth' motivation, ignoring model competence. A notorious example is the false explanation that the lift of an aircraft wing is solely due to the air speed differences and resulting dynamic pressure differences in the boundary layer generated by the wing profile. This explanation has survived many textbooks, even though the simple observation that airplanes with such wing profiles can fly upside down falsifies this explanation in an extremely simple and evident way.

Another example is a model of which the behavior changes after a change of coordinates: as coordinates are a modeler's choice, they cannot have any impact on the behavior of the described system. Not keeping an open eye for these aspects of modeling may lead to exercises that are documented in the scientific literature in which controllers are designed to deal with model behaviors that are due to imperfections of the model and that are not observed at all in the real system or rather the actual problem context.

Summarizing the crucial issues in the process of modeling of dynamic behavior:

- Determination of the purpose of the model in a specific *problem context* in order to be able to judge whether a model is *competent* for a particular problem context. In other words: no generic, 'true' (sub)model exists by definition in the sense that (sub)models are not exact copies of the (sub)systems to be modeled, but they may be competent to support the solution of a *particular problem* related to the actual system. This problem may be related to the past (trouble shooting), to the future (conceptual design) and to the present (model-based, real-time control, including the control of user interfaces in simulators).
- Identification of *dominant* and relevant *behaviors* and decomposition into *elementary behaviors*.
- Generation of a *conceptual structure* that combines these elementary behaviors into a computable dynamic model of the relevant system behavior(s).

The discussion of these meta-level issues in modeling has paved the way for the introduction of the concept of a 'port' in the next section.

1.3 Ports in dynamical systems models

1.3.1 Bilateral bonds versus unilateral signals

The concept of a *port* is generated by the fact that sub-models in a model have to *inter*act with each other by definition and accordingly need some form of *conceptual interface*. In physical systems, such an interaction is always (assumed to be) coupled to an exchange of energy, i.e. a power. In domain-independent terminology, such a relation is called a (power) *bond* accordingly. This bond represents a bilateral relation and connects two (power) ports of the elements or sub-models that are interacting (Fig. 1.3). Note that the energy conservation principle does not require the energy to traverse the intermediate space in a flow-like manner: if energy is generated at one place at the same rate that energy is annihilated in other place, the energy conservation principle would still hold. However, in addition we will at all times apply Heaviside's (macroscopic) principle, which states that this cannot be the case and energy has to traverse the intermediate space, even when this space is a conceptual space, thus giving meaning to the concept of a bond.

In the signal domain, the power of a signal relation is assumed to be negligible compared to the powers that do play a role, such that a signal relation may be considered a 'unilateral' relation. Note that *ideal* operational amplifiers have an infinite input impedance and a zero output impedance in order to suppress the back-effect and to be purely unilateral, but can only be approximated by adding external power. The bilateral nature of the power relations (as opposed to unilateral signal relations) suggests the presence of two variables that have some relation to the power represented by the bond. These so-called power-conjugate variables can be defined in different ways, but they are commonly related to the power P by means of a product operation and in the domain-independent case named effort e and flow f:

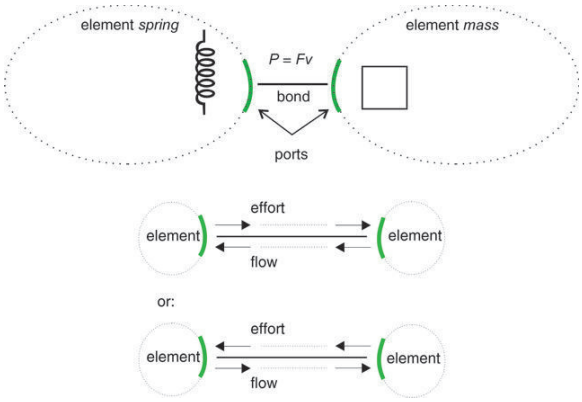


Fig. 1.3 Bond connecting two ports: bilateral signal flows.

$$P = e \times f.$$

Domain-dependent examples are force and velocity in the mechanical domain, voltage and current in the electrical domain, pressure and volume flow in the hydraulic domain, etc. In principle, the flow variable can be seen as the rate of change of some state, or 'equilibrium-*establishing* variable', whereas the effort variable can be seen as the equilibrium-*determining* variable. In thermodynamics the latter are called intensive states that do not depend on the extent of the system as opposed to the extensive states that are proportional to the extent and not necessarily equal in equilibrium. The common approach to port-based modeling distinguishes, similar to modeling electrical networks and simple mechanical systems, two *dual* types of storage: capacitive or C-*type* storage and inertial or I-*type* storage (examples of C's: electrical capacitor, spring, etc.; examples of I's: coil, mass, etc.). This disables the use of the distinction between flow and effort as rate of change of state and equilibrium-determining variable, respectively, for variable identification during modeling. In other words: the common approach unnecessarily symmetrizes the roles of effort and flow in the models. The so-called Generalized Bond Graph or GBG approach introduced in 1979 [29] and further developed between 1979 and 1984 [32] circumvents this problem by using one type of storage and splitting the corresponding domains into two that are explicitly connected by a so-called *symplectic gyrator* (SGY, to be elaborated later), thus leaving the discussion about the force-voltage versus force-current analogy a non-issue [30, 32]. Nevertheless, there is a clear didactic preference to introduce the common approach using the force-voltage analogy [91]. As this part of the book is to lay the physical foundations for a rigorous mathematical treatment of system models that have been named 'port-Hamiltonian systems' (cf. Chapter 2), where the Dirac structure is the abstraction of a conceptual structure, it is relevant to note at this point that the GBG approach not only leaves the symplectic gyrator explicit such that the distinction between flows as elements of a linear space and the efforts as the elements of a dual linear space (see Sect. B.1.1 in Appendix B) is also maintained in the mathematical picture, but also, even more importantly, because the coupling between the domains that can

be represented by a symplectic gyrator, is a special case from the point of view of physics. To be more specific: in the mechanical case the coupling between the kinetic and the potential domain only simplifies into a unit gyrator (or symplectic gyrator) when working in an inertial frame, i.e. a non-accelerating frame, where the coupling can be described by Newton's second law; in the electromagnetic case the coupling between the electric and the magnetic domain is described by Maxwell's equations, which only simplify into a symplectic gyrator under the quasi-stationary assumption, which reflects the assumption that an electric circuit is not radiating, or rather that its radiation may be neglected [30]. As a result, the GBG approach provides more insight during modeling than the conventional approach. However, in case of the simple examples that are inevitable during an introduction, the difference between the two approaches lies merely in the fact that the GBG represents the SGY using only the C-type of storage port, while the conventional approach can be considered to have eliminated this SGY by partial dualization, thus introducing the dual type (I-type) of storage port. This is why the conventional approach will often be used, while referring to the GBG approach as much as possible.

However, before we can discuss this in more detail, some more general issues need to be addressed first.

1.3.2 Dynamic conjugation versus power conjugation

The two signals of the bilateral signal flow representing a physical interaction are dynamically conjugated in the sense that one variable represents the rate of change of the characteristic physical property, like electric charge, amount of moles, momentum, while the other variable represents the equilibrium-determining variable. This is called dynamic conjugation. As long as no other domains are of interest, the concept of energy is not particularly relevant, such that these variables do not need to be related to a power, like the effort and flow discussed earlier, as the domain is based on the conserved quantity that characterizes it. Examples are: temperature and heat flow (product is not a power, heat is not a proper state if other domains are involved), molar flow and concentration or mole fraction (product is not a power), etc. The power-conjugated variables effort and flow are a subset of these dynamically conjugated variables, due to the additional constraint that their product represents a power.

This illustrates that the concept of a domain-independent conserved quantity, the energy, is crucial for the consistent interconnection of physical phenomena in different domains. The discussion of basic behaviors in Sect. 1.6 is based on this and thus requires either the consistent use of power-conjugated variables or carefully defined domain transitions that are power continuous and energy conserving.

1.3.3 Bond graph notation

Bond graphs are *labeled di-graphs*: its *edges* are the bonds and represent the bilateral signal flow of the power-conjugate variables effort and flow. The common convention for the position of the symbols for the effort and flow variables in a bond graph with respect to their bond is that efforts are written above or to the left of a bond and flows below or to the right. As this is ambiguous when the bond has a 'north-west inclination' (considering the top of the paper to be 'north') the symbol for the bond orientation is also used to indicate the position of the flow and is supposed to be in line with the common convention. This edge orientation of the di-graph is represented by a little stroke that forms a *half-arrow* with the line representing the edge. This is the typical appearance of a bond (cf. the bond graph in Fig. 1.14). The meaning of the half arrow will be discussed in more detail later in Sect. 1.6.1. The *labeled nodes* of the bond graph are (multiport) elements that can be distinguished on the basis of their behavior with respect to energy, power, and the conserved quantities typical for a domain. An important property is the power continuity of a node: a power continuous node satisfies an instantaneous power balance at all times, i.e. the net power into the node is zero at all times. Obviously, non-degenerate one-ports cannot be power continuous. The node labels are represented by mnemonic codes that refer to the basic behavior.

1.3.3.1 Node types in a bond graph

There are *nine* (*eight* in the GBG!) basic node types that can be categorized in *five* groups of basic physical behaviors:

1. Storage ('first law', *energy conservation*), node labels: C, I (I not in GBG)
2. Supply and demand (*boundary conditions*), labels: Se, Sf
3. Reversible transformation (configuration constraints, *inter-domain* connections), labels: TF, GY
4. Distribution (topological constraints, *intra-domain* connections), labels: 0, 1
5. Irreversible transformation ('second law', *positive entropy production*, dissipation of *free* energy), label: R(S)

The storage elements can store energy reversibly and are consequently not power continuous. The *sources* supply power to the system (from the environment) or drain power from the system (to the environment) and are also not power continuous with respect to the system accordingly. In fact, *sources* can be considered storage elements that are infinitely large with respect to the storage processes of interest, such that, if the energy would be tracked, it would still satisfy the energy conservation principle. The energy conservation principle prohibits other forms of power discontinuity, i.e. all other elements should be power continuous in principle. The elementary transducers are power continuous two-ports, while the junctions are power continuous multi-ports, i.e. with two or more ports. The junctions are not parametrized. In standard bond graphs, energy is replaced by its Legendre transform with respect

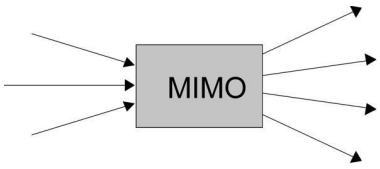


Fig. 1.4a conventional MIMO.

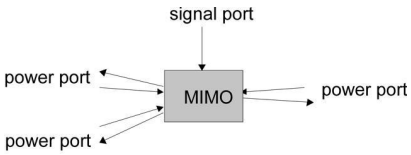


Fig. 1.4b MIMO system with bilateral power ports and modulating signal ports.

to entropy, the *free energy*, which is not conserved and allows the thermal port of the irreversible transducer to be omitted, thus changing the RS in a one-port resistor R, which is a power discontinuous one-port. Common introductions to bond graphs generally do not mention this and introduce the R as just another one-port, without pointing out that its use implies the implicit assumption that the temperature of the system can be considered constant with respect to the dynamics of interest. It would be too disturbing to always point this out, but during the modeling process it is good to be aware of this underlying assumption. Each of the above behaviors is discussed in more detail in Sect. 1.6.

1.4 Computational causality

In pure mathematical terms, one can state that a subsystem with a number of (power) ports, called multiport, is a multiple-input-multiple-output or MIMO system model, of which the set of inputs and the set of outputs is not chosen a priori. The relation between the input and output variables, the so-called constitutive relation, determines the nature of this multiport.

If the number of input variables is not equal to the number of output variables, this means that there has to be at least one unilateral signal port as opposed to a bilateral power port, as the latter is by definition characterized by one input and one output. If this signal port is an input signal, the multiport is called ‘modulated’. Modulation does not affect the power balance, in other words: no energy can be exchanged via a signal port as its conjugate variable and thus the associated power are zero by definition. Situations can exists too in which the model is modulated although the number of inputs equals the number of outputs, because for each modulating signal there is also some output signal.

Although ports and bonds illustrate that two bilateral signals are involved in a relation, no a priori choice about the direction of the corresponding signals needs to be made. This is an important distinction with a conventional MIMO system (Figures 1.4a and 1.4b). A particular choice of this computational direction or causality is needed before a set of computable relations can be found or some particular analysis can be performed. Often, such a ‘causality assignment’ leads to computational forms that are not obvious and would have led to modeling problems in

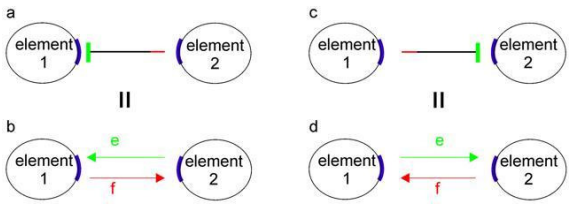


Fig. 1.5 Causal stroke showing computational direction of effort signal.

conventional approaches, in particular when domain boundaries are crossed (cf. the remarks about Paynter's motivation in the introduction, Sect. 1.1.2). As a result, bond causality, in particular its algorithmic assignment, does not only support the solution of computational and analytical issues, it also gives the modeler immediate feedback about the physical meaning of his modeling decisions and the trade-off he has to make between conceptual and computational complexity (cf. Sect. 1.7). This means that computational causality is not merely computational, but in fact expresses the fact that time differentiation cannot be realized by physical means, while time integration is naturally related to the process of storage (Riemann *sum*). The adjective 'computational' is used merely to make a distinction with what is commonly called the 'causality' of signals, meaning that the effect of some input signal in some output cannot precede the input signal, in other words a system model is causal if it obeys the principle of the 'arrow of time' (time-reflection asymmetry).

If information about its (computational) causality is represented on a bond in a bond graph by means of a so-called 'causal stroke' (cf. Fig. 1.5), the bond graph simultaneously represents physical and computational structure [33, 98]. From the latter point of view, a bond graph can be seen as a condensed block diagram. However, although any causal bond graph can be converted into a block diagram, the reverse does not hold, as physical structure is lost in the first transformation.

The causal stroke is attached to that end of the bond where the effort signal comes out, i.e. where it enters the connected port. This automatically means that the so-called open end of the bond represents the computational direction of the flow signal (cf. Fig. 1.5). The actual use and impact of the representation of computational causality in the bond graph will become clear after the introduction of the basic elements that each have particular causal port properties.

1.5 System versus environment: system boundary

The distinction between system and environment is determined by the role of these parts: the environment can influence the system, but not dynamically interact with it. In signal terminology: the environment may 'influence' the system via the system's inputs and 'observe' the system via its outputs, but the inputs cannot depend on these outputs at the time scale of interest. In case of normal use, a car battery for example, may be considered the environment of a dashboard signal light, as the discharge caused by this small bulb will not affect the voltage of the battery in a

considerable way. In other words, the car battery in this problem context (regular car use) can be modeled by a voltage source. However, in a context of a car being idle for three months (other time scale!) the car battery has to be made part of the system and dominantly interacts with the resistance of the bulb like a discharging capacitor. The resulting RC-model is competent in this problem context to predict the time-constant of the discharge process. In severe winter conditions the thermal port of this capacitor will have to be made part of this system model too, etc.

Note that, after a particular choice of the separation between unbounded environment and bounded system, the influence of the environment on the system may be conceptually concentrated in this finite system boundary by means of so-called sources and sinks, also called boundary conditions or constraints, depending on the domain background. They are part of the ideal conceptual elements to be discussed in the next section.

From an energy point of view, the following viewpoint emerges. Since the universe is assumed to obey the first and second law of thermodynamics, viz. energy conservation and positive entropy production, only being interested in a relatively small part of the universe (system) that may still interact with the rest (its environment) not only means that exchange with the environment of energy and all conserved quantities can take place, but also that the entropy of the system may decrease. However, it is additionally assumed that this decrease can only be due to a net flow out of the system, not by local annihilation. In other words: it is assumed that *the positive entropy production principle also holds locally*. This justifies the use of the concept of an irreversible transducer (RS).

1.6 Elementary behaviors and basic concepts

This section discusses in some more detail the conceptual elementary behaviors that can be distinguished in the common description of the behavior of physical systems, in particular from a port-based point of view. Before the individual elements can be discussed, first the notation for the positive orientation in the form of the so-called half-arrow needs to be elaborated.

1.6.1 Positive orientation and the half-arrow

Each bond represents a connection between two ports. However, with one loose end it can be used to visualize the port it is connected to. The three variables involved, viz. effort, flow and power, may have different signs with respect to this port. In order to be able to indicate this, a *half arrow*, as opposed to the full arrow that is commonly used to graphically represent the direction of a signal, is attached to the bond, expressing the *positive orientation* of these variables, similar to the plus and minus signs and the arrow that are used for an electric two-pole to represent the

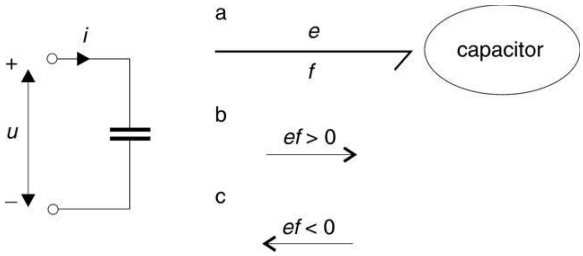


Fig. 1.6 *Positive orientation* represented by a half arrow (a), *direction* depends on sign (b,c).

Fig. 1.7 Half-arrow does not influence causality.

positive orientation of the voltage and the current respectively (Fig. 1.6). The half-arrow does *not* indicate the direction of the flow or of the power: the direction is opposite in case the corresponding variable has a negative value.

Just like the causal stroke, the half-arrow is an additional label to the bond, but they do not affect each other (Fig. 1.7): the causal stroke merely fixates the direction of the individual signal flows in the bilateral signal flow pair, whereas the half-arrow merely represents positive orientation with respect to the connected ports.

1.6.2 Constitutive relations of elements

Each port of a bond graph node requires one *constitutive relation*, while the node type determines the shape of this relation in the sense that it constrains the possible forms of these constitutive relations.

Often, relatively small variations around the origin or some operating point can be linearly approximated, resulting in just one parameter per port, e.g. capacitance, resistance, etc. These constitutive parameters always consist of a combination of geometric parameters and material parameters. If a configuration is made time-variant, a consequence can be that a geometric parameter becomes an *energy state* and requires an additional power port of a storage element (e.g. condenser microphone, coil with moving core, etc.) or a signal port of the other elements, resulting in *state-modulation*.

However, as most physical variables have some upper limit, *saturation*, and thus non-linearity, will occur in all constitutive relations of parametrized ports. Examples are: the speed of light that shows that the parameter 'mass' cannot remain constant at all times, breakdown voltage of a capacitor, force at which a spring breaks, magnetic saturation, etc. It depends on the problem context whether or not such a nonlinear range should be included in the model.

Each constitutive relation of a port may contain quantities of a different nature:

- conjugate power variables or port variables effort and flow (e, f);
- energy states (q, p);

- dynamic parameters that characterize the constitutive relation (capacitance, resistance, mass, spring constant, etc.); these parameters commonly depend on both material and geometric parameters;
- material parameters (parameters that are characteristic for a specific material like specific densities of all kinds);
- geometric or configuration parameters (constant spatial quantities like length, surface, content, distance, etc.).

Making this distinction is of great help during modeling of dynamic behavior. Take for example a simple ohmic resistor. Its constitutive relation relates the port-variables u (voltage) and i (current) by means of a constant dynamic parameter, the resistance R, resulting in the constitutive relation commonly addressed as Ohm's law: $u = Ri$. In turn the resistance depends on the specific resistance ρ of the material (e.g. carbon) and its configuration, e.g. its length l and the cross-section area A: $R = \frac{\rho l}{A}$. The variables in a dynamic model (port variables as well as state variables) play a quite different role than the parameters, even though problem contexts may exist in which a resistance is to be computed from a given voltage and a given current. However, such a problem context cannot be considered to require a *dynamic* model. This is different when the configuration is not constant, such that configuration parameters become (configuration state) variables. An example is a potentiometer, where the length of the carbon path in the circuit can be changed. However, this does not give the resistance R the same nature as the port variables u and i: the resistor keeps the same port variables and just becomes position-modulated due to the position dependence of R.

Another example is a spring that may be described by Hooke's law: $F = Kx$. Many are inclined to categorize the variables and parameters in a similar manner as those of the resistor: force F and displacement x as dynamic variables and the spring constant K as the dynamic parameter that is a function of a material parameter (Young modulus) and the configuration parameters (depending on the shape of the spring). However, the product of the force F and the displacement x does not equal the power of a spring, which is the product of the force F and the velocity v, i.e. the rate of change of the displacement x. This confirms that the ideal spring is a quite different element than the ideal resistor with respect to its energetic properties: it stores elastic energy and is characterized by an energy state that is the time integral of the flow. The detailed discussion of the basic elements will start with this key element (concept) of dynamic system models: storage.

1.6.3 Storage

The most elementary behavior that needs to be present in a system in order to be dynamic is 'storage'. In mathematical terms, one can describe this behavior by the integration of the rate of change of some *conserved quantity*, viz. the stored quantity or *state*, and by the relation of this state with the equilibrium determining variable, the so-called *constitutive relation*, also called *port characteristic*.

The storage ports are somewhat exceptional as the relation between the conjugate variables effort and flow contains two stages: the first stage is always *integration with respect to time* into an *energy state*. This operation can, if necessary, be inverted into a differentiation with respect to time although this means that physically relevant information about the *initial condition*, i.e. the initial content of the storage element, cannot be given a place in the model (cf. the later discussion of causal port properties in Sect. 1.7). The second part is an unambiguous functional relation between the (extensive) *energy state* (*q*- or *p*-type) and the conjugate power variable (intensive state). The latter relation is not a priori constrained, except for the constraint that if a node contains more than one storage port, i.e. if it is a multiport storage element, it should satisfy the *Maxwell reciprocity conditions* in order to satisfy the energy conservation principle. However, qualitative properties of a storage (multi)port, like *intrinsic stability*, may lead to additional constraints such as positive-definiteness and positive diagonal elements of the Jacobian of the relation (cf. Sect. 1.9.3).

The storage ports can also be classified as '*history ports*', a terminology that reflects the required presence of an integration with respect to time, while all other ports belong to the class of '*non-history ports*', which means that their constitutive relations only relate the variables at the current time instant, in other words the constitutive relations are algebraic, although it can still be the case that states modulate these elements. This state modulation particularly occurs in mechanism models in which the geometric constraints can be represented by position-modulated transformers and their multiport generalizations. The importance of choosing variables that lead to insightful representations of complex mechanisms that can be easily manipulated should not be underestimated.

At the signal level, other forms of history operations can exist, like flip-flops, sample and hold, pure integration, etc. This distinction is helpful when preparing a numerical simulation as it indicates that variables of previous numerical steps need to be stored for future use. The presence of history ports is required to obtain dynamic behavior.

If measurement of the relation between intensive and extensive states results in a loop in the port characteristic (hysteresis), the port that is observed cannot be simply represented by just one storage port, but contains at least one other storage port through which power is exchanged. If this port is connected to a dissipative port, the cycle will have to be clockwise due to the positive entropy production principle (cf. Sect. 1.9.3 on multi-ports).

In the common classification of domains, many domains are characterized by two types of states, viz. the generalized displacement and the generalized momentum, following the common approach in the mechanical domain (Table 1.1). It has been noted before that a different classification of domains, which for instance separates the mechanical domain into a kinetic domain and a potential or elastic domain, can easily resolve the paradoxical situation that results from the common choice [32].

Application of the common classification leads to two types of storage elements:

- the C-type storage element in which the flow is integrated into a generalized displacement and related to the conjugate effort;

Table 1.1 Domains with corresponding flow, effort, generalized displacement, and generalized momentum.

	f flow	e effort	$q = \int f\,\mathrm{d}t$ generalized displacement	$p = \int e\,\mathrm{d}t$ generalized momentum
electro-magnetic	i current	u voltage	$q = \int i\,\mathrm{d}t$ charge	$\lambda = \int u\,\mathrm{d}t$ magnetic flux linkage
mechanical translation	v velocity	F force	$x = \int v\,\mathrm{d}t$ displacement	$p = \int F\,\mathrm{d}t$ momentum
mechanical rotation	ω angular velocity	T torque	$\theta = \int \omega\,\mathrm{d}t$ angular displacement	$b = \int T\,\mathrm{d}t$ angular momentum
hydraulic pneumatic	ϕ volume flow	p pressure	$V = \int \phi\,\mathrm{d}t$ volume	$\Gamma = \int p\,\mathrm{d}t$ momentum of a flow tube
thermal	T temperature	f_S entropy flow	$S = \int f_S\,\mathrm{d}t$ entropy	
chemical	μ chemical potential	f_N molar flow	$N = \int f_N\,\mathrm{d}t$ number of moles	

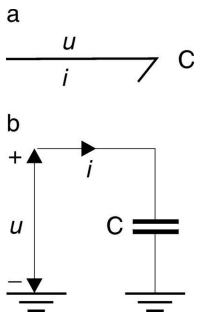


Fig. 1.8 Bond graph representation (a) of an electrical capacitor (b).

- the I-type storage element in which the effort is integrated into a generalized momentum and related to the conjugate flow.

Both are dual in the sense that they can be transformed into each other by interchanging the roles of the conjugate variables effort and flow. Simple examples of C-type storage elements are:

- ideal spring (mechanical domain, GBG: elastic or potential domain);
- ideal capacitor (electric domain, Fig. 1.8);
- ideal reservoir (hydraulic/pneumatic domain, GBG: potential domain);
- ideal heat capacitor (thermal domain).

The explicit use of the adjective 'ideal' tries to emphasize the difference between elements and components although the naming is usually based on the component that dominantly displays a particular elementary behavior.

Simple examples of I-type storage elements are:

- ideal mass (mechanical domain, GBG: C of the kinetic domain);

Table 1.2 Thermodynamic framework of domains and variables.

	f flow, equilibrium establishing	e effort, equilibrium determining, intensive state	$q = \int f \mathrm{d}t$ generalized state, extensive state
electric	i current	u voltage	$q = \int i dt$ charge
magnetic	u voltage	i current	$\lambda = \int u \mathrm{d}t$ magnetic flux linkage
elastic / potential translation	v velocity	F force	$x = \int v \mathrm{d}t$ displacement
kinetic translation	F force	v velocity	$p = \int F \mathrm{d}t$ momentum
elastic / potential rotation	ω angular velocity	T torque	$\theta = \int \omega \mathrm{d}t$ angular displacement
kinetic rotation	ω angular velocity	T torque	$b = \int T \mathrm{d}t$ angular momentum
elastic hydraulic	ϕ volume flow	p pressure	$V = \int \phi \mathrm{d}t$ volume
kinetic hydraulic	p pressure	ϕ volume flow	$\Gamma = \int p \mathrm{d}t$ momentum of a flow tube
thermal	T temperature	f_S entropy flow	$S = \int f_S \mathrm{d}t$ entropy
chemical	μ chemical potential	f_N molar flow	$N = \int f_N \mathrm{d}t$ number of moles

- ideal inductor (electric domain, GBG: C of the magnetic domain);
- ideal fluid inertia (hydraulic/pneumatic domain, GBG: C of the kinetic domain).

This common choice of domains leads to two disadvantages:

1. the asymmetry between effort and flow is destroyed as both can be considered a rate of change;
2. no insight can be obtained from the fact that the concept of storage can take two different forms.

This is why the GBG framework (Table 1.2) should get ample attention too.

Storage elements can be used in a domain-independent way due to the built-in representation of the energy-conservation principle. Not only the stored quantity, e.g. charge, matter, momentum, flux linkage, etc. is stored, but also the energy related to this storage. In case that more than one quantity is stored (multi-port storage) the principle of energy conservation supports the description of the potential *power transfer* from one domain into the other by means of *cycle processes*. Almost all engineering transduction processes can be related to this concept and usefully analyzed with the tools that thermodynamics provides, even when the model contains no thermal port. For instance, the insight that a set of two coupled coils, i.e. the component 'transformer', does not transform direct current is easily explained this way [36].

1.6.4 Irreversible transformation

Next to the first law of thermodynamics, i.e. energy conservation, the second law of thermodynamics, i.e. positive entropy production, has to be satisfied. However, the entropy production is assumed to take place only in the two-port irreversible transformers that are usually addressed as one-port 'dissipators' or 'resistors' due to the fact that the thermal port can be omitted if the temperature is assumed to be homogeneous and constant at the time scale of interest. This implicit assumption is often not explicitly mentioned, which may lead to modeling inconsistencies, as these one-ports are clearly power discontinuous: the energy of the system is in fact be replaced in this case by its *Legendre transform* (see Sect. B.2 in Appendix B) with respect to the entropy, i.e. the so-called *free energy*, which can be dissipated. This reduces the irreversible, power continuous two-port transducer into a virtually *power discontinuous*, i.e. '*free energy dissipating*' *one*-port that is commonly called *dissipator*, *resistor* or *damper*.

As the rest of the system has to satisfy the second principle too, all entropy production is assumed zero there, which results in entropy continuity for all elements except for the storage elements where *reversible* storage of entropy is allowed. 'Reversible storage' is a tautology, as irreversibilities would violate the basic concept of storage, but is used here to make the distinction with the irreversible production. The common acronym for an irreversible transducer is RS, derived from the common acronym in the isothermal case, R, to which an S for source is added to represent the entropy production.

Simple examples of irreversible transforming (resistive) elements are:

- ideal electric resistor;
- ideal friction;
- ideal fluid resistor;
- ideal heat resistance.

Due to the second principle of thermodynamics (positive entropy production), the relation between the conjugate variables at the R-port can be linear or nonlinear as long as the relation remains in the 1st and 3rd quadrant. However, the relation at the S-port (always in the thermal domain) is intrinsically nonlinear, due to the absolute zero-point of temperature (linear two-ports can be proven to be reversible).

In other words: the constraint on an R-port is that the functional relation should satisfy the positive entropy production principle. For the common orientation definitions (i.e. one-ports *except sources* positive *towards* the port; two-ports one inward, other port outward) this means that this function cannot be in the second or fourth quadrant and thus has to intersect with the origin.

There is no demand of linearity of the R-port characteristic, such that a *diode* belongs to the class of electrical R-ports, even though it does not have an ohmic (i.e. linear) resistance. Similarly, a check valve belongs to the class of hydraulic R-ports. Nonlinear friction in a mechanical contact with Coulomb and static friction and the Stribeck effect can still be described by a nonlinear R-port, although its implementation requires special attention from a port-based perspective.

The *irreversible transducer* does not change type when dualized. In principle, it is a power-continuous, port-asymmetric two-port just like transformers and gyrators, which will appear an uncommon conclusion at first sight. The notion of a port-asymmetric multiport will be clarified further when port-symmetric multi-ports are discussed.

1.6.5 Reversible transformation

Irreversible transformation more or less suggests the 'possibility' of, or rather, the need for, the ideal concept of a reversible transducer. Reversible transducers cannot store or produce entropy, as these properties are already concentrated in the storage and RS elements, and hence, they have to be power continuous. Their most elementary form is the two-port. It can be formally proven that, independent of the domain, only two types of port-asymmetric, i.e. with non-exchangeable ports, power-continuous two-ports can exist, at the one hand the so-called transformer (acronym: TF) that relates the efforts of both ports and also the flows of both ports ('non-mixing'), and at the other hand the so-called gyrator (acronym: GY) that relates the flow of one port with the effort of the other vice versa ('mixing'). The constitutive relations of two-ports are all *multiplicative* in form: the multiplication factor (transformation or gyration ratio) can be constant (regular TF and GY) or depend on an arbitrary time-dependent variable, the so-called modulating signal (acronyms: MTF and MGY) and, in some cases, on the port variables, in which case modulation changes into non-linearity. An example of the latter situation is the dominant behavior of a centrifugal pump or turbine: a nonlinear GY (often incorrectly written as a 'port-modulated' MGY) with a hydraulic port (p,ϕ) and a rotation port (T,ω) with ratio $(a\omega + b\phi)$, i.e.

$$p = (a\omega + b\phi)\omega = a\omega^2 + b\phi\omega$$
$$T = (a\omega + b\phi)\phi = a\omega\phi + b\phi^2$$

where a and b depend on the geometry and the fluid properties.

In fact, it is possible to 'see' the power-continuous, port-asymmetric RS as a port-modulated gyrator (MGY) or transformer (MTF), depending the causality of the R-port (the S-port has a fixed effort-in causality), modulated by the input of the S-port (the absolute temperature) and the output of the R-port. However, not only does this representation hide the property of irreversibility, it also results in a set of incomputable relations if directly applied for simulation, so it is better to continue to use it as a separate concept with its own acronym.

Simple examples of reversible transforming elements are:

- ideal (or perfect) electric transformer;
- ideal lever;
- ideal gear box;
- ideal piston-cylinder combination;

- ideal positive displacement pump.

Simple examples of reversible gyrating elements are:

- ideal centrifugal pump;
- ideal turbine;
- ideal electric motor.

An ideal, continuously variable transmission is a simple example of a reversible, modulated transforming element, while an ideal turbine with adjustable blades is a simple example of a reversible, position-modulated gyrating element, albeit nonlinear (cf. earlier remark).

In port-based models of planar and spatial mechanisms, specific types of (configuration) state-modulated (multiport) transformers play a crucial role, which exposes the dual role of the displacement variable.

The *reversible transformations* appear in dual form just like most of the other node types: the *non-mixing*, *reciprocal* transformer or *TF-type* transducer, and the mixing, anti-reciprocal gyrator or *GY-type* transducer.

1.6.6 Supply & demand (sources & sinks / boundary conditions)

As already announced, the supply and demand from and to the environment can be concentrated in the (conceptual!) system boundary and represented by sources or sinks. As sinks can be considered negative sources, only ideal sources are used as ideal elements. Given that a port has two kinds of variables, effort and flow, two kinds of sources may exist, Sources of effort and Sources of flow (acronyms: Se and Sf). These two, dual sources correspond to the two, dual types of *boundary conditions* (called Dirichlet and Neumann conditions in the context of partial differential equations).

Generally speaking, all storage elements that are large compared to the dynamics of interest (note that this cannot be considered independently of the resistance of its connection to the rest of the system) may be approximated by infinitely large storage elements that are identical to sources. An infinitely large C-type storage element becomes an Se, an infinitely large I-type storage element becomes an Sf. However, feedback control may turn a port into a source too, cf. a stabilized voltage source. As the voltage may be adapted or modulated, these kinds of sources are called modulated sources (MSe, MSf).

Simple examples are of (modulated) effort sources are:

- ideal (controlled) voltage source;
- ideal (controlled) pressure source, etc.

Simple examples are of (modulated) flow sources are:

- ideal (controlled) current source;
- ideal (controlled) velocity source, etc.

A source is degenerate in the sense that its constitutive 'relation' merely states that there should be *no relation* between its conjugate variables: the only constraint is that the imposed variable is independent of the conjugate variable. So-called 'non-ideal sources' violate this constraint, but can always be considered a combination of an ideal source with one of the other node types (usually a resistor that represents the so-called internal resistance). However, non-ideal sources influence the dynamic characteristics of a system model while ideal sources do not.

1.6.7 Distribution

The *topological constraints* also appear in dual form: the so-called *0-junction* and *1-junction*. The fact that these topological constraints are represented by *nodes* of the graph are the most powerful feature of the bond graph representation, but at the same time the most uncommon and potentially confusing aspect. The constitutive relations (one per port) of the first type of junction require all efforts to be identical and the flows to sum up zero with the choice of sign related to their positive orientation, similar to a Kirchhoff current law. The 0-junction not only represents a generalized, i.e. domain independent, Kirchhoff Current Law (KCL), but also the identity of the conjugate efforts, such that it can be considered to represent a common effort. Paynter called this junction a zero-junction, due to the similarity between the symbol for zero and the shape of a node in an electric circuit that satisfies the KCL.

The constitutive relations of the second type of junction, simply called 1-junction, are dual: all flows should be identical and the efforts sum to zero with the choice of sign related to their positive orientation (effort balance), similar to a Kirchhoff voltage law. Similar to its dual node, the 0-junction, a 1-junction not only represents a generalized, i.e. domain independent, Kirchhoff Voltage Law (KVL), but also the identity of the conjugate flows, such that it can be considered to represent a common flow.

The common approach to model mechanical constraints at the *position* level is related to the dual nature of the *position* variable, both *energy state* and *configuration state*. Merely from an energy point of view the mechanical constraints lie at the velocity (mechanical flow) level and should be treated as such. However, the description of the variable configuration requires a formulation at the *position* level, commonly resulting in *position modulation* of the mechanical junction structure.

However, the (topological) structure may not be constant. In that case, the junction may depend on a *logical state* that, if it were, switches it 'on' and 'off'. This 'switched junction' is represented by adding the letter X to the junction symbol, i.e. X0 and X1, and is modulated by a Boolean signal. In the 'off'-state, all connected ports have zero power.

Another way to justify the need for the concept of a junction is that, in order to be able to distribute power between subsystems in an arbitrary way, distributing elements with three or more ports are required. By assigning all energy storage to the storage elements, all entropy production to the irreversible transducers

('dissipators') and all exchange with the environment to the sources, only the property of *power continuity* remains. Furthermore, the requirement that ports should be connectable at will, requires that an interchange of ports of these distributing or interconnecting elements has no influence. This is the so-called property of *port symmetry*. It is important to note that it can be formally proven that only the requirements of power continuity and port symmetry result in two solutions, i.e. two types of multi-ports (i.e. interconnection elements with two or more ports) with constitutive relations that turn out to be linear and non-parametrized, the so-called junctions. No assumption about domain or form of the constitutive relations is required. This supports the above conclusion about the mechanical constraints. Port-*symmetric*, power continuous two-ports are junctions too, which explains why the assumption of port-asymmetry was required when discussing the TF and GY.

As mentioned before, really manipulating the concept of the junction in a way that supports the modeling process, i.e. without using other modeling techniques and translation first, requires some skill as the true understanding of the junctions requires the paradigm shift mentioned earlier. Nevertheless, the results are powerful, as will be demonstrated after the discussion of the causal port properties.

1.6.8 Summary of elements

Summarizing, we repeat that the following *nine* (*GBG: eight*) *basic node-types* are distinguished:

- 4 (GBG: 3) one-ports: C, I (not in GBG), (M)Se, (M)Sf;
- 2 two-ports: (M)TF, (M)GY;
- 2 n-ports with $n > 1$: 0, 1;
- 1 one- or two-port: (M)R(S).

The basic one-ports are *power discontinuous*, the basic two-ports are *power-continuous* and *port-asymmetric* and the basic multi-ports are *power continuous* and *port-symmetric*.

The power-continuous elements, with the exception of the RS, form together the so-called Generalized Junction Structure. If the anti-reciprocal part of the GJS (the gyrators) are split from this JS, a reciprocal, so-called Weighted Junction Structure (WJS) remains, in which the transformation ratios are the weighting factors. If these are taken from the JS too, a so-called Simple Junction Structure (SJS) remains, which consists of junctions and bonds.

1.6.9 Modulation and bond activation

It was already shortly mentioned that the letter M in the node symbol of some of the parametrized nodes stands for '*modulated*', expressing that the constitutive equation

can depend on an external signal (*modulation*) without changing the nature of the node or affecting the power balance. Storage elements are parametrized, but not modulated in principle, as this would violate the basic concept of storage. However, when it is obvious that either the power or the dynamic interaction related to one of the ports of a multiport version of the storage element can be neglected at all times with respect to the other port(s), *modulation* can be used (e.g. a variable capacitor in a receiver circuit).

Modulation usually requires '*bond activation*', i.e. the bond, a bi-lateral relation, reduces to a uni-lateral relation, the signal, due to the fact that the other conjugate variable can be neglected in the particular context. The terminology refers to the fact that an *active* element, e.g. an operational amplifier, is required to obtain this situation. However, decomposition of nonlinear elements can also lead to junction structures containing internally modulated elements that are modulated by 'true signals', in the sense there is no conjugate variable by definition. This means that *internal modulation* that is related to decomposition cannot be considered bond activation.

Internal modulation can be useful in principle, but should be used as a modeling instrument with great care, as it can be used, in particular in case of internal modulation by one of the port variables of the modulated node, to construct one 'elementary' behavior out of another one. For example, a voltage source directly or indirectly modulated by its own conjugate flow behaves like a resistor, etc. In other words: *internally modulated* sources not only violate the basic definition of a source, they can also be used to construct virtually 'anything'. Nevertheless, if used with sufficient care, they can enhance insight in specific cases, such that a 'veto' on their use would be inappropriate.

1.7 Causal port properties

Each of the nine (GBG: 8) basic port-types (C, I, R(S), TF, GY, Se, Sf, 0, 1) introduced above has its own causal port properties, that can be categorized as follows: fixed causality, preferred causality, arbitrary causality and causal constraints. The graphical representation of causality by means of the causal stroke has been introduced already (cf. Fig. 1.5).

1.7.1 Fixed causality

It needs no explanation that a source of effort (Se) always has an effort as output signal, in other words, the causal stroke is attached to the end of the bond that is connected to the rest of the system (Figures 1.9 and 1.10a). Mutatis mutandis the causal stroke of a flow source (Sf) is connected at the end of the bond connected to the source (Fig. 1.10b). These causalities are called 'fixed causalities' accordingly. Apart from these fundamentally fixed causalities, all ports of elements that may

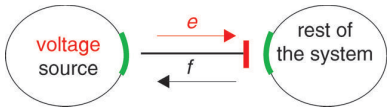


Fig. 1.9 Fixed effort-out causality of an effort (voltage) source.

Fig. 1.10 Fixed causality of sources.

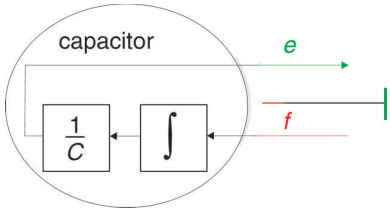


Fig. 1.11 Preferred integral causality of a capacitor.

become nonlinear and non-invertible, i.e. all but the junctions, may become fixed due to the fact that the constitutive relation may only take one form.

1.7.2 Preferred causality

A less strict causal port property is that one of the two possibilities is, for some reason, preferred over the other. Commonly, this kind of property is assigned to storage ports, as the two forms of the constitutive relation of a storage port require either differentiation with respect to time or integration with respect to time (Fig. 1.11). On the basis of numerical arguments, the integral form is preferred, due to the fact that numerical differentiation amplifies numerical noise, but there are more fundamental arguments too. A first indication is found in the fact that the integral form allows the use of an initial condition, while the differential form does not. An initial state or content of some storage element is a physically relevant property that clearly illustrates the statement that integration ‘exists’ in nature, whereas differentiation does not. Although one should be careful with the concept ‘existence’ when discussing modeling, this statement seeks to emphasize that differentiation with respect to time requires information about future states in principle, whereas integration with respect to time does not. The discussion of causal analysis will make clear that violation of a preferred causality gives important feedback to the modeler about his modeling decisions. Some forms of analysis require that the differential form is preferred, but this requirement is never used in order to prepare the constitutive relations for numerical simulation.

1.7.3 Arbitrary causality

The expected next possibility in the sequence is that the causality of a port is neither fixed nor preferred, thus arbitrary. Examples of arbitrary port causality are linear, thus invertible, resistive ports. For example, the a-causal form of the constitutive relation of an ohmic resistor is $u - Ri = 0$, the effort-out causal form is $u = Ri$, while the flow-out causal form is $i = u/R$ (cf. Fig. 1.16).

1.7.4 Causal constraints

Causal constraints only exist for basic multi-ports, i.e. elements with two or more ports, like the transducers (TF, GY) and the junctions (0, 1). If the constitutive relation of the two-port transducers is linear (the junctions are intrinsically linear), the first port to which causality is assigned is arbitrary, but the causality of the second port is immediately fixed. For instance, the two-port transformer always has one port with effort-out causality and one with flow-out causality. By contrast, the causalities of the ports of a two-port gyrator always have the same type of causality. In graphical terms: a TF has only one causal stroke directed to it, while a GY has either both causal strokes directed to it or none.

The fundamental feature of the junctions that either all efforts are common (0-junction) or all flows are common (1-junction) shows that only one port of a 0-junction can have 'effort-in causality' i.e. flow-out causality, viz. the result of the flow-balance. By contrast, only one port of a 1-junction can have 'flow-in causality' i.e. effort-out causality, viz. the result of the effort-balance. In graphical terms: only one causal stroke can be directed towards a 0-junction, while only one open end can be directed towards a 1-junction.

1.7.5 Causal paths

A bond path between two ports of the type C, I, R, Se, or Sf via the (G)JS containing 0, 1, TF, and GY is called a *causal path* if the sequence of the causal strokes is such that they have one direction, with the exception of a path through a GY where this causal stroke direction is always altered. A causal path is equivalent with a signal loop in a block diagram or signal flow graph, except for the case that a source port (Se or Sf) is part of the path (cf. Fig. 1.12).

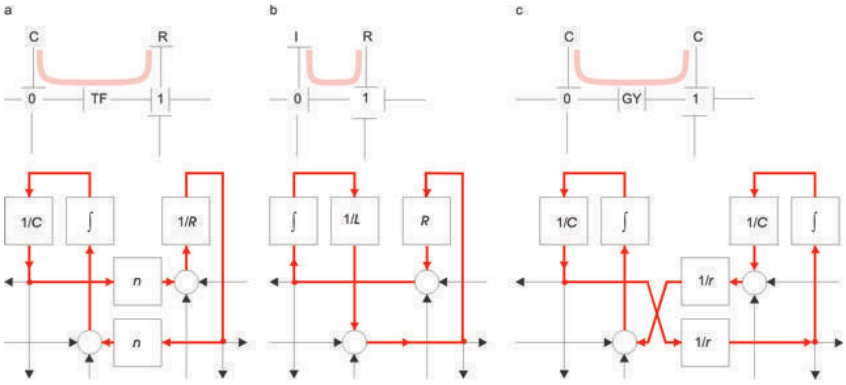


Fig. 1.12 Causal paths and block diagram expansion to signal loops.

Fig. 1.13 Propagation of a fixed causality via a 1-junction.

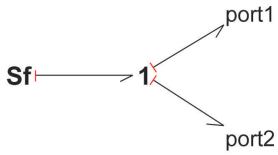


Fig. 1.14 Dependent inertia via the causal constraints of 1-junctions and transformer.

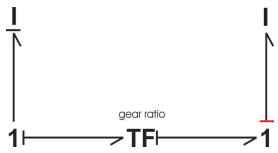
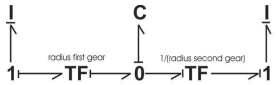


Fig. 1.15 Independent inertia's by adding the elasticity of the transmission (e.g. belt drive).



1.8 Causal analysis: feedback on modeling decisions

1.8.1 Sequential Causality Assignment Procedure (SCAP)

Causal analysis, also called causality assignment or causal augmentation, is the algorithmic process of putting the causal strokes at the bonds on the basis of the causal port properties induced by the nature of the constitutive relations. Not only the final result, but also the assignment process provides immediate feedback on modeling decisions.

All sorts of causality assignment algorithms can be applied for different purposes. The common purpose is to write the model equations in a form suitable for simulation, i.e. maximizing the number of storage ports with integral causality. The most common algorithm is the so-called *Sequential Causality Assignment Procedure* (SCAP) [99]. It is not perfect, in the sense that it fails in some rare cases, but it

not only generates a set of computable equations, but also gives feedback on modeling decisions. This is the reason for not discussing a more robust, but also more complex method that can be used for computer implementation [64, 65, 86]. Herein, this distinction will not be made for the sake of clarity, as it is not relevant for most simple models. A short description of the steps of the SCAP is:

Step 1a: If present, assign a fixed causal source port and propagate this causality along the nodes with causal constraints until propagation terminates due to multiple possibilities. For instance, if a flow source is connected to a 1-junction, the source-port immediately gets flow-out causality, which in turn means that the corresponding port at the 1-junction gets flow-in causality, which means that all other ports of the 1-junction get flow-out causality, etc. (Fig. 1.13). Repeat this step until all source ports are augmented with a fixed causality. If propagation leads to conflicts with other fixed causalities, the model is ill-posed, e.g. two voltage sources in parallel or two force sources trying to impose the same force (mechanically 'in series'). If propagation leads to conflicts with preferred causalities, the model contains differentiations of the inputs (input-dependent 'states'). However, also those storage ports that obtain integral causality as a result of propagation of the fixed causality of one or more source-ports do not result in independent states: only their initial conditions can be freely chosen, the rest of their behavior is fully dictated by the source port(s), such that they do not contribute to the characteristic dynamic behavior of the model. If all bond ports are causal at this point, the model does not have its own dynamics, but is completely determined by its inputs.

Step 1b: If present, assign a fixed causal port that is not a source port and propagate this causality along the nodes with causal constraints until propagation terminates due to multiple possibilities. Repeat this step until all ports of this type are augmented with a fixed causality. If propagation leads to conflicts with other fixed causalities or with preferred causalities, the *causal path* (signal loop) that causes the conflict should be analyzed symbolically as to obtain a solution. This propagation should not lead to non-preferred causalities as this would lead to misleading conclusions about the order of the model, unless the fixed causal port is a storage port itself. In that case, the non-preferred causality is similar to the dependency that can occur during the next step.

Step 2: If present, assign a preferred causal port and propagate this causality along the nodes with causal constraints until propagation terminates due to multiple possibilities. Repeat this step until all ports with preferred causality are assigned. If propagation leads to conflicts with other preferred causalities the model contains dependent states (no independent initial condition). Fig. 1.14 shows the bond graph of two rigidly linked inertia's, e.g. the motor inertia and the load inertia in a servo system model, including a transmission (TF), but without any compliance. This shows the modeler that he has chosen a model in which two storage ports depend on each other and form a signal loop (causal path) with an integration that is compensated by a differentiation, i.e. a net algebraic loop. The computational problem may be solved either by the application of implicit numerical integration, by changing the model (the sequence of putting the causal

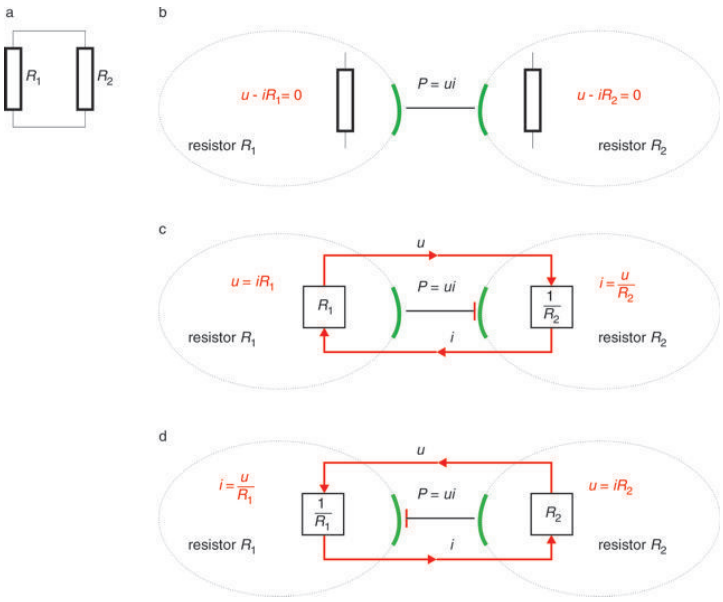


Fig. 1.16 Arbitrary causality of two resistors causing an algebraic loop.

strokes hints the modeler where a model change should be made, e.g. adding the compliance of the transmission between the two rigid bodies, cf. Fig. 1.15), or by symbolic manipulation (either manually or automatically) of the model. A technique to deal with this problem by adding some advanced control schemes to the model is under investigation. This also changes the model, but not in a way that can be physically interpreted [103].

Step 3: If not all ports are causal after the above steps, there are at least two ports with arbitrary causality, resulting in a number of possibilities that all will lead to *causal paths* between ports of elements that are described by algebraic constitutive relations thus causing algebraic signal loops (Fig. 1.16). Choose the causality of these ports not only in such a way that the number of algebraic loops is minimized, but also in such a way that the loop gains of these algebraic loops are smaller than one as much as possible. This step is to be repeated until all bond have their causality assigned. In a similar manner as in case of differential causality, the assignment procedure itself hints the modeler how to change the model in order to prevent this kind of loop.

The causality assignment procedure is completely algorithmic. More advanced variations on this algorithm exists and are implemented that can handle all possible situations [86]. As a result, it can be used without using the notation itself, e.g. by replacing the bond graph by the more common iconic diagram representation or the linear graph notation. However, this largely reduces the amount of feedback that can be given to the modeler about his modeling decisions, and the effect of model modifications becomes less obvious. Nevertheless, if one is merely interested in

converting a simple iconic diagram into code ready for simulation, this is a powerful option.

A causal bond graph can always be straightforwardly expanded into a *block diagram* or signal flow graph. The experienced user will be able to obtain the same information from a causal bond graph as from a block diagram, viz. the computational structure, while the bond graph already represents the physical structure in a domain-independent way. This demonstrates one of the main advantages of the bond graph representation: it can be seen immediately how changes in the physical structure affect the computational structure and thus the dynamic characteristics vice versa. This is particularly helpful during conceptual design, troubleshooting and solving problems related to numerical simulation.

At the other hand, not any block diagram or signal flow graph can be converted into a causal bond graph as they generally do not contain conjugate port variables. However, an attempt to convert a block diagram that represents the computational structure of a model of a physical system into a bond graph can be a quite insightful experience, as it may explicate earlier choices about the nature of the physical ports as well as eliminations of physically relevant variables.

The earlier mentioned trade-off between conceptual and computational complexity is illustrated by the simple example of a rigid constraint between two rigid bodies (Fig. 1.14). Conceptual simplicity leads to a causal problem (a so-called dependent inertia with differential causality) – the example already showed that a loop emerges containing an integration and a differentiation, i.e. a 'net' algebraic loop – and consequently to numerical complexity (DAE). A DAE is a mixed set of differential and algebraic equations that cannot be solved straightforwardly by means of explicit numerical integration (e.g. with the common Runge-Kutta 4^{th}-order method). However, the way in which the causal problem emerges in the model during causal analysis clearly suggests how the model can be modified in order to prevent the causal problem. In this example, the rigid constraint can be replaced by an elastic element, i.e. a finite rigidity. Although this gives the model some more conceptual complexity, the numerical (structural) complexity is reduced, due to the fact that the resulting equations are a set of ordinary differential equations (ODE) that can be solved by explicit numerical integration schemes [10] [3].

The model still needs a rather stiff constraint and thus introduces dynamics at a time scale that is not of interest. This means not only that both options to formulate the model can be a solution depending on the problem context, the available tools, etc., but also that a third solution can be obtained, viz. a symbolic transformation of the model as to eliminate the dependent inertia. In other words: two rigidly connected rigid bodies may be considered as one rigid body. This possibility is directly induced by the causal analysis of the bond graph model.

[3] See also the course slides at `http://www.npac.syr.edu/users/gcf/CPS615NI95/`.

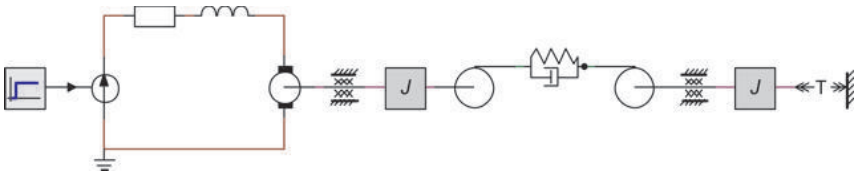


Fig. 1.17 Iconic representation of a servo system with belt drive (graphical 20-sim input).

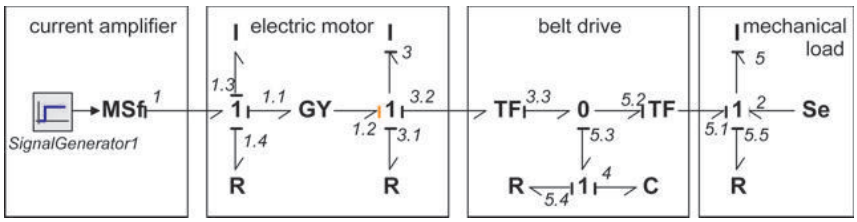


Fig. 1.18 Simple, linear bond graph model of the servo system in Fig. 1.17.

1.8.2 Example of causal analysis

Fig. 1.17 shows an iconic diagram representation of the servo-system containing a belt drive. The bond graph in Fig. 1.18 represents this simple linear model. It is graphical input to 20-sim. This software puts the causal strokes automatically, and immediately while drawing the graph. The order in which the strokes are put can be indicated by sequence numbers, where *i.j* represents the *j*th propagation of putting stroke *i*.

The fixed causalities are (M)Sf (1) and Se (2), where only Sf propagates via the 1-junction and imposes causality to the electrical I and R and the electrical port of the GY, thus eliminating the electrical time constant that would have been present in the model if the electrical source would have been a voltage source. The propagation stops at the next 1-junction, after the mechanical port gets its causality via the constraint of the GY (1.2). The preferred causalities are the remaining storage elements, i.e. the inertia of the rotor (I 3), the compliance of the belt (C 4) and the inertia of the mechanical load (I 5). The motor inductance (I 1.3) plays no dynamic role as its current is imposed and its voltage (that is computed by differentiation) does not affect the current amplifier (Sf 1), like the motor voltage (GY 1.1) and the ohmic voltage drop in the circuit (R 1.4). Propagation of the motor inertia (I 3) reaches as far as the 0-junction representing the force in the belt, and propagation of the inertia of the load completes the causality of this graph. Following causal strokes through the graph (causal path) identifies the existence of signal loops.

1.9 Hierarchical modeling

1.9.1 Word bond graphs

Organizing and representing larger system models in a hierarchy can increase efficiency and overview of the modeling process. An example is the so-called word bond graph, in which the nodes represent physical components. They can also represent phenomena that may require sub-models that contain more than one basic element. Word bond graphs are represented by words or text enclosed by ellipses or circles. These words describe the basic behavior or purpose of a sub-model (as for example in Fig. A.16 in Appendix A). This notation can also be used to support the first modeling phase, in which the relevant physical components in a system are identified, without further specification than their dominant behavior. It can be decided later whether other elementary behaviors are also required to obtain a competent model of this physical component.

1.9.2 Multi-bonds

In many cases, multiple bonds connect the nodes of a (word) bond graph. Similar to the notation of multiple signals as 'double-lined arrows', it can be useful to represent multiple bonds by 'double-lined half-arrows' that are called *multi-bonds*. The *dimension* of a multi-bond, i.e. the number of constituent bonds, can optionally be written between the two lines of a multi-bond. Multi-bonds have initially been introduced as vector bonds. As multi-bonds are frequently used to represent the coordinates of vectors in planar and spatial mechanisms, while it merely represents a column matrix and *not* a vector in space, this terminology appeared to be highly confusing for a graphical representation and has been abandoned in the early eighties. Apart from the advantages of efficiency and overview, one major disadvantage of a multi-bond is that it is not suited to properly represent the causality, except for the situation that the causalities of all constituent bonds are identical. In order to cope with this problem and in order to be able to combine multi-bond representations with single bond representations, the concept of the *direct sum* was introduced, represented by a vertical line perpendicular to the connected bonds. Formally speaking, it can be considered a special multiport transformer that is characterized by a unit matrix of which the order of the rows can be changed as to represent a change of order of the participating bonds. In that case, this matrix has to be provided in order to characterize the direct sum, otherwise a unit matrix (no change of order) is assumed.

1.9.3 Multiport generalizations

A word bond graph adds only one level to the model hierarchy in principle, although multiple layers are possible if physical sub-components can be identified. However, complex models also require different layers at the conceptual level. This requires multiport generalizations of the nodes.

1.9.3.1 Sources

By definition, a multiport requires that its ports are interrelated. This means that the sources cannot be combined into a multiport due to the nature of a source, i.e. no dependency on the conjugate variables, but source *arrays* can be used (cf. the last part of this section).

1.9.3.2 Multiport storage elements

The storage elements can be generalized into a multiport in which the number of ports is equal to the number of *energy states*. The energy functions of these states can be used to generate the constitutive relations of this multiport, similar to the Gibbs relation in thermodynamics or to the Hamiltonian description of mechanical systems. It is obvious that this makes this notation and approach an ideal instrument to establish a link between these two huge scientific areas. The constitutive relation has to satisfy the *Maxwell reciprocity condition*[4] in order to satisfy the energy conservation principle. This condition is also called Maxwell symmetry as it requires the symmetry of the Jacobian of the constitutive relations.

However, a multiport storage element adds the potential of a new behavior that is not represented by one of the basic elements, viz. *reversible transformation* by cycle processes as opposed to the instantaneous reversible transformation represented by a transformer or a gyrator. From a conceptual design point of view, it is worthwhile to note that, in principle, instantaneous power *transduction* between domains does not 'exist' (e.g. *passive* DC transformers cannot be realized, often rotating parts or cycling 'working fluids' are required to construct continuous power transducers), but can only be approximated by relatively fast cycles or cycles in which the storage can be neglected (e.g. intermittent elastic storage in the touching gears of a gear box).

Another important observation with respect to multiport storage elements is that the integral causality of the ports corresponds to a generating function that is equal to the energy. If an integral causality of a port is changed into a differential one, this corresponds to replacing the (extensive) *energy state* by its (intensive) conjugate variable (partial derivative of the energy with respect to the conjugate state). This, in turn, corresponds mathematically to a *Legendre* transform of a function of multiple

[4] In Hamiltonian mechanics this is expressed as the energy being a so-called closed two-form.

variables (cf. Sect. B.2 in Appendix B). *Legendre transforms* are not only often used in thermodynamics, when the conjugate variable of a state (intensive state, effort) can be assumed constant (e.g. *free energy* in case of constant temperature, *enthalpy* in case of constant pressure, *Gibbs free energy* in case of constant temperature and pressure), but also in mechanics, where the dual nature of the *position* variable, i.e. *energy state* and *configuration state*, has led to a preference for the *position* and its derivative, the *velocity*, instead of the true extensive *energy states*: *position* and *momentum*. As a consequence, the *Hamiltonian* (kinetic energy T + potential energy V) is often Legendre transformed into the *Lagrangian* (kinetic co-energy T^* - potential energy V), although this generally does not lead to equations that are optimally suited for numerical simulation. This wide field of research is still under study, but many important results have been obtained that appear not yet generally known.

A final observation to be mentioned is that, in the linear case, a multiport storage element can always be decomposed into some one-port storage elements and a power continuous junction structure called Generalized Junction Structure (GJS) or Dirac structure in mathematical terminology (cf. Sect. 2.1). If the number of independent parameters required to characterize the multiport or n-port, viz. $n + (n^2 - n)/2 = n(n+1)/2$, is equal to the number of parameters in the decomposition, the decomposition is called a *canonical decomposition*. Decompositions depend on the causality of the ports. Reversibly, a bond graph that only contains storage elements and a non-modulated (except state modulation by the states of the participating storage elements) junction structure with open ports can be composed into one multiport storage element. There are only two types of canonical decompositions of a linear two-port storage element: three linear storage elements connected by a Simple Junction Structure (SJS) (direct or immediate canonical decomposition) or two linear storage elements connected by a GJS with only one linear transducer (congruence canonical decomposition, see [31]).

Example 1.1 (Solenoid with configuration dependent inductance). Many different configurations exist in which a solenoid has an inductance that is configuration dependent, like a relay, a magnetic bearing or suspension system, a solenoid with moving core like a linear motor, or an LVDT (linear variable displacement transmitter), etc. They all have in common that the linear constitutive relation of the electric (magnetic) port of a coil, viz. $\lambda = Li$, where λ is the flux linkage[5], L the self-inductance, and i the current, can be written: $\lambda = L(x)i$, where x is some displacement that represents a change of configuration (position of the core, changing air gap in the magnetic circuit, etc.).

Since a storage element cannot be simply modulated by such a configuration variable, it means that a second port emerges, which, due to the mechanical nature of the configuration variable, is a mechanical port and this displacement also starts playing a role in the stored magnetic energy

[5] Its rate of change is the voltage of the electrical port: $u = \frac{\mathrm{d}\lambda}{\mathrm{d}t} = L\frac{\mathrm{d}i}{\mathrm{d}t}$

$$E = \frac{\lambda^2}{2L(x)} = E(\lambda, x).$$

Obviously, the constitutive relations of the ports can be found by taking the partial derivatives of this energy with respect to the energy states:

$$i(\lambda, x) = \frac{\partial E}{\partial \lambda} = \frac{\partial}{\partial \lambda} \frac{\lambda^2}{2L(x)} = \frac{\lambda}{L(x)}$$

the already known relation for the magnetic port and

$$F = \frac{\partial E}{\partial x} = \frac{\partial}{\partial L} \left(\frac{\lambda^2}{2L(x)} \right) \frac{\mathrm{d}L(x)}{\mathrm{d}x} = F(\lambda, x)$$

or

$$F(i, x) = -\frac{i^2}{2} \frac{\mathrm{d}L(x)}{\mathrm{d}x}$$

The latter, more common relation can also be found by taking the co-energy E^* (negative Legendre transform of the magnetic energy) with respect to λ:

$$\begin{aligned} E^*(i, x) &= i\lambda - E(\lambda, x) = i\lambda - \frac{\lambda^2}{2L(x)} = L(x)i^2 - \frac{L(x)i^2}{2} = \frac{L(x)i^2}{2} \\ \lambda(i, x) &= \frac{\partial E^*}{\partial i} = \frac{\partial}{\partial i} \frac{L(x)i^2}{2} = L(x)i \\ F(i, x) &= -\frac{\partial E^*}{\partial x} = -\frac{\partial}{\partial x} \frac{L(x)i^2}{2} = -\frac{i^2}{2} \frac{\mathrm{d}L(x)}{\mathrm{d}x} \end{aligned}$$

Note that in this case of a linear magnetic port, confusing energy and co-energy would lead to a change of sign of the force. In case of a nonlinear magnetic port, confusion between energy and co-energy may lead to even more serious differences, thus showing that the distinction between energy and co-energy should not be neglected, as, unfortunately, is often the case.

Without further specifying $L(x)$ it is still possible to perform some generic analysis, e.g. by finding the Jacobian of the constitutive relations:

$$\frac{\partial(i, F)}{\partial(\lambda, x)} = \begin{bmatrix} \frac{1}{L(x)} & -\frac{\lambda}{L^2(x)} \frac{\mathrm{d}L(x)}{\mathrm{d}x} \\ -\frac{\lambda}{L^2(x)} \frac{\mathrm{d}L(x)}{\mathrm{d}x} & \left(\frac{\lambda^2}{L^3(x)} \left(\frac{\mathrm{d}L(x)}{\mathrm{d}x} \right)^2 - \frac{\lambda^2}{2L^2(x)} \frac{\mathrm{d}^2L(x)}{\mathrm{d}x^2} \right) \end{bmatrix}$$

Requiring that:

$$
\begin{aligned}
L(x) &> 0 \\
\left(\frac{\mathrm{d}L(x)}{\mathrm{d}x}\right)^2 - \frac{L(x)}{2}\frac{\mathrm{d}^2L(x)}{\mathrm{d}t^2} &> 0 \\
\det\frac{\partial(i,F)}{\partial(\lambda,x)} &= \left(\frac{\lambda^2}{L^4(x)}\left(\frac{\mathrm{d}L(x)}{\mathrm{d}x}\right)^2 - \frac{\lambda^2}{2L^3(x)}\frac{\mathrm{d}^2L(x)}{\mathrm{d}x^2}\right) \\
&\quad - \frac{\lambda^2}{L^4(x)}\left(\frac{\mathrm{d}L(x)}{\mathrm{d}x}\right)^2 \\
&= -\frac{\lambda^2}{2L^3(x)}\frac{\mathrm{d}^2L(x)}{\mathrm{d}x^2} > 0
\end{aligned}
$$

means that $\frac{\mathrm{d}^2L(x)}{\mathrm{d}x^2} < 0$, such that $\left(\frac{\mathrm{d}L(x)}{\mathrm{d}x}\right)^2 - \frac{L(x)}{2}\frac{\mathrm{d}^2L(x)}{\mathrm{d}x^2} > 0$ is also satisfied. On the other hand,

$$
\frac{\partial(\lambda,F)}{\partial(i,x)} = \begin{bmatrix} L(x) & i\frac{\mathrm{d}L(x)}{\mathrm{d}x} \\ -i\frac{\mathrm{d}L(x)}{\mathrm{d}x} & -\frac{i^2}{2}\frac{\mathrm{d}^2L(x)}{\mathrm{d}x^2} \end{bmatrix}
$$

and

$$
\begin{aligned}
L(x) &> 0 \\
\frac{\mathrm{d}^2L(x)}{\mathrm{d}x^2} &< 0 \\
\det\frac{\partial(\lambda,F)}{\partial(i,x)} &= -\frac{i^2}{2}\frac{\mathrm{d}^2L(x)}{\mathrm{d}x^2}L(x) + i^2\left(\frac{\mathrm{d}L(x)}{\mathrm{d}x}\right)^2 > 0
\end{aligned}
$$

with the latter equation is always true if the two earlier conditions are satisfied.

As intrinsic stability conditions require the diagonal elements of the Jacobian as well as its determinant to be positive, this translates in both cases in the condition that the self-inductance should always be positive and that the second derivative of $L(x)$ should be negative, independent of the way in which the magnetic port is driven (i.e. by imposing a current or a flux linkage). The examples mentioned before can be split into two groups: those in which $L(x)$ is a bell-shaped function and those where $L(x)$ is hyperbolic. In the first case the middle part of the bell, between the flex points, describes an intrinsically stable area. For instance, if a coil with moving core is given a constant current, the core will react as if it is attached to a regular spring as long as it is brought not to far from its equilibrium point, otherwise it will fly out. By contrast, a magnetic circuit with a variable air gap always has a second derivative of $L(x)$ that is positive, which means that the air gap will always tend to collapse when the magnetic circuit is activated, unless a mechanical spring is added, like in a relay, or a virtual spring is created by means of a proportional feedback between air gap (position of the moving part) and current with sufficient gain to make the spring constant-like element in the Jacobian positive (magnetic levitation in bearings and suspension systems).

The shape of $L(x)$ in magnetic levitation is often incorrectly chosen as $C_1 + \frac{C_2}{x}$, where it should be

$$L(x) = C_1 + \frac{C_2}{C_3 + x} = L_{\min} + \frac{C_3(L_{\max} - L_{\mix})}{C_3 + x}$$

with $C_i > 0$, thus showing that the self-inductance remains finite when the air gap closes ($x = 0$). As

$$\frac{\mathrm{d}^2 L}{\mathrm{d}x^2} = +\frac{2C_2}{(C_3 + x)^3} > 0,$$

this two-port storage element is intrinsically unstable for all values of $x \geq 0$ and if $i \neq 0$.

The common derivation of the self-inductance of a solenoid while neglecting fringing ('infinitely long solenoid') and while assuming that the field lines run through a cross-section area A over a distance l through iron with permeability $\mu_r \gg 1$ and over a distance x through air with $\mu_r = 1$, leads to

$$L(x) = \left(\frac{x}{n^2 \mu_0 A} + \frac{l}{n^2 \mu_o \mu_r A} \right)^{-1} = \left(\frac{x\mu_r + l}{n^2 \mu_0 \mu_r A} \right)^{-1} = \frac{n^2 \mu_0 \mu_r A}{x\mu_r + l}$$

where n is the number of windings. In other words:

$$C_1 = L_{\min} \qquad C_2 = \frac{l}{\mu_r}(L_{\max} - L_{\min}) \qquad C_3 = \frac{l}{\mu_r}$$

where $L_{\max} = L(0) = \frac{n^2 \mu_0 \mu_r A}{l}$. For larger values of x, the non-fringing assumption does not hold, which leads to the addition of $C_1(L_{\min})$ to the above expression. However, given that $\mu_r \gg 1$, there are values of x for which l can be neglected, but if the air gap x approaches 0, $L(0)$ will definitely have a finite value as l cannot be neglected in that situation. In problem contexts where x is assumed to vary only around some operating point for which $x\mu_r \gg l$, the approximation where $C_3 = 0$ can be valid, but this constraint seems to be easily overlooked. In Example 2.2 in Chapter 2, even a linear relation is chosen, inspired by the required control setting.

1.9.3.3 Multiport resistors

The resistive port of an *irreversible transducer* can also be generalized in multiport form. The *Jacobian* of its relations has to be symmetric, as only this symmetric part contributes to the entropy production. A potential non-symmetric Jacobian can always be separated into a symmetric part that can be represented by a resistive port and an antisymmetric part that can be represented by the multiport generalization of a gyrator, which is a power continuous junction structure element. This issue has been a source of conflict in thermodynamics: in the thirties Onsager introduced firstly his Onsager symmetry for the relation between generalized forces (efforts) and generalized fluxes (flows), i.e. for what is now called a multiport re-

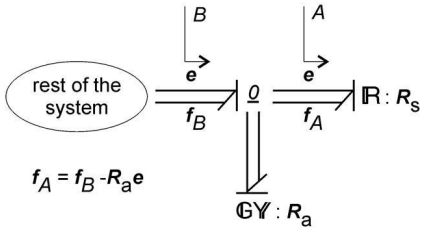


Fig. 1.19 Symmetrizing a dissipative multiport ($R_s + R_a$) into R_s by changing the 'port of view' (B→A).

sistor; next Casimir, inspired by a discussion with Tellegen, who introduced the gyrator in electrical engineering in the late forties, extended this by showing that in some cases there is an antisymmetric contribution. Finally, Truesdell showed that a transformation of the conjugate variables can always symmetrize these relations, which corresponds to changing the 'port of view' in a bond graph (Fig. 1.19). There exist (canonical) decompositions of multiport R(S) elements similar to those of the storage elements but the constraint on linearity is much less severe.

1.9.3.4 Multiport transducers

The elementary two-port elements, TF and GY, can be generalized in a straightforward manner by changing the scalar conjugate variables in their relations into column matrices. The scalar transduction ratio then becomes a transduction matrix. In case of a multiport transformer, the matrix itself describes the flow-relation and its transpose the effort relation, as can be derived from power continuity.

In case of the gyrator there is simply one relation between efforts and flows that is characterized by the gyration matrix. This makes clear that the format of the constitutive relation of a gyrator is similar to that of a resistive port, even though the gyrator belongs to the (generalized) junction structure. It can thus be seen as the antisymmetric counterpart of a symmetric R-port too (cf. Fig. 1.19).

The causality constraints of the multiport transformer are related to the (partial) invertibility of the transformation matrix. If the dimension of the inward multi-bond is not equal to the dimension of the outward multi-bond, the matrix is not square and singular as a result. This means that the causality of the multi-bonds cannot be inverted. However, it may still be possible to invert the constitutive matrix partially. This requires a mixed causality of the multi-bonds and accordingly the use of the direct sum.

1.9.3.5 Multiport components

The port relations of arbitrary multi-ports can be used as a starting point for decomposing them into basic elements. The nature of the variables plays an important role: constitutive relations of true power ports should be formulated in terms of efforts and flows or their time integrals (*energy states*). If the latter case occurs, this indicates

that the port should at least contain one storage element. Depending on the shape of this relation, other elements may be needed to represent the corresponding behavior. For instance, the presence of a cycle demonstrates the presence of either another coupled storage port, or a form of hysteresis caused by a resistive phenomenon.

1.9.3.6 Arrays

In the multi-bond notation it is sometimes helpful to be able to collect a number of the same symbols, even if they are not directly related. For instance, a collection of 1-port I-type elements representing the storage of momentum of a body in three independent coordinate directions has no power relation (at least, not in the *inertial frame*), but should conceptually be connected. The concept of an array of bonds or elements, represented by underlining the corresponding symbol is used. Nesting (arrays of arrays, etc.) is possible, but only advised as long as it enhances insight. For instance, three n-dimensional multi-bonds connected to an (n-dimensional) junction array (each bond connects only to the junction in the array matching its index) has a different meaning than three n-dimensional multi-bonds connected to a single junction (all bonds connected to one and the same junction), even though the difference in notation is just the underlining of the junction symbol. The first is often encountered in models of planar and spatial mechanisms, while the second is encountered in models of chemical reactions for example.

1.10 Example of the use of the port concept

Only the actual use of the port-concept can fully clarify its importance. Therefore, a simple, but meaningful case study is discussed to illustrate it. A component that may be used in engineering systems, viz. a control valve, but in which the control is not realized by (digital) electronic signal processing, but physically, i.e. as an energetic process, is taken as an example. This choice is made in order to focus on the multidisciplinary modeling part on the basis of power ports.

1.10.1 Problem context

Under some operating conditions of a low-vacuum control valve (cf. Fig. 1.20a) spontaneous, self-sustained oscillations occur [34]. Given the purpose of the valve, viz. to maintain a constant ‘low’ vacuum in particular in medical applications, this behavior is clearly undesired. In order to solve this problem, insight is to be obtained in the source(s) of this behavior and the design parameters of the system that should be modified in order to prevent it. Some simple oscilloscope measurements of these oscillations, mainly showing shape and frequency, are available to the mod-

eler as well as a construction drawing of the valve with data on geometry and used materials.

1.10.2 Functional description of the valve

The intended basic operation of this control valve is that an orifice can be opened and closed by a valve body that is connected to a diaphragm loaded by a coil spring. Changing the position of the other end of this spring with a screw knob can set its pretension. The diaphragm is part of the wall of the valve chamber that is at one end pneumatically connected to the 'supply' pressure (a relatively high-level under-pressure or 'vacuum') via the valve opening, and at the other end via an orifice and a hose to the 'mouth piece' to suck superfluous body fluids away in a device as used by dentists and surgeons. Given some desired low-level under-pressure or 'low-vacuum', the pressure difference over the diaphragm will cause the valve opening to get smaller if the actual pressure gets too low compared to the desired pressure. Due to the increasing flow resistance of the variable orifice, the pressure difference with the supply pressure ('high' vacuum) will increase again vice versa.

1.10.3 Analysis

If this common functional explanation is translated into a block diagram, it becomes clear that the resulting model is not dynamic at all (Fig. 1.20b) as all relations are algebraic. If oscillations occur, it is tempting to identify a damped second-order system consisting of the valve body, the spring and the mechanical damping that is always present. As such a model is not competent to explain sustained oscillations, it seems natural to argue that the airflow is likely to drive these oscillations. The next step that seems obvious is to conclude that the common chaotic behavior of flow phenomena (turbulence) that is hard to model deterministically is likely to form the onset of the oscillations, such that no attempt is made to create a competent dynamic model and the problem is approached in an ad hoc way by changing the geometry of the valve by trial and error. However, if one approaches this problem from a port-based point of view, the analysis will make a distinction between power relations and modulation and leads to another result, not only of the analysis, but also of the identification of the actual physics that play a role in such a valve.

In a regular valve, a screw modulates the position of the body of the valve. The fluid acts with a force on this body, trying to move it out of the valve seat. The reason that the fluid cannot displace the valve body, while the human hand can do this, is the presence of the transforming action of the screw/spindle. This amplifies the static friction of the screw seen from the translating port of the screw/spindle. As this static friction is only overcome during a hand turning the valve and the dynamics of this process are at a completely different time scale than the flow phenomena

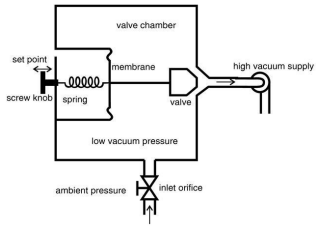


Fig. 1.20a Sketch of the low-vacuum control valve.

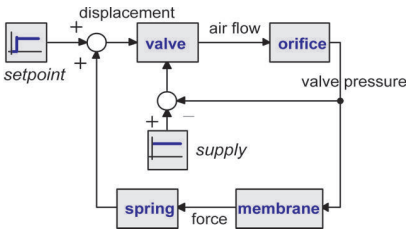


Fig. 1.20b Stationary, qualitative model resulting from functional description.

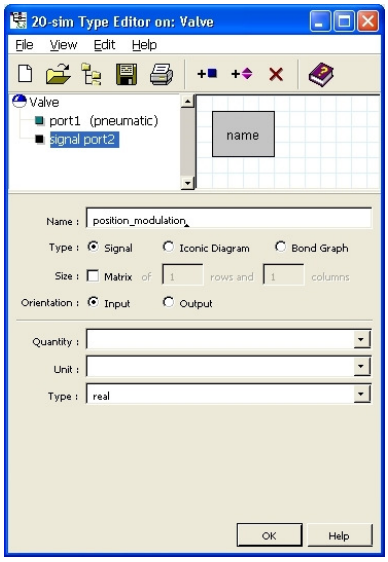


Fig. 1.21a Definition of ports in 20-sim.

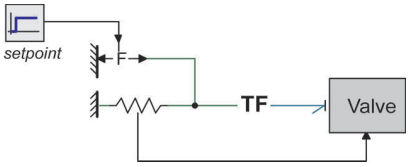


Fig. 1.21b Port definition and model construction in 20-sim.

in the valve, a change in position of the valve body is commonly modeled as a modulation of the flow resistance of the valve. Hence, a position-modulated resistor can describe the dominant behavior of an arbitrary valve. Fig. 1.21a shows how the ports and port properties of such a valve can be defined in 20-sim, without having to define the exact constitutive relations yet.

Feedback can be introduced by a diaphragm (membrane) that transforms the difference in pressure at its sides into a force that can cause a displacement. By connecting the body of the valve to the membrane, such that an increasing pressure difference will close the valve and a decreasing pressure difference will open it, it will thus have a counteraction in both cases, i.e. a negative feedback. The relation between force and displacement is characterized by the stiffness of the diaphragm. It needs to be increased in order to attenuate the position changes of the valve body. This is achieved by connecting a spring. By connecting the other end of the spring to

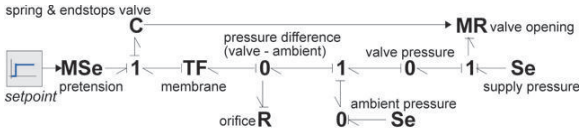


Fig. 1.22 Addition of boundary conditions, flow resistor and structural details (pressures).

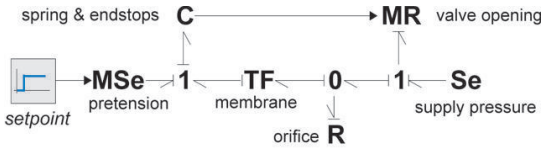


Fig. 1.23 Simplification by choosing ambient pressure as reference pressure.

the screw, the screw can be used to change the set-point for the pressure difference by changing its pretension. The screw serves as a combination of a (Coulomb) friction and a transformer that amplifies its effect similar to the regular valve described above. The model of the complete valve has to be at least extended by an ideal transformer (TF) to represent the dominant behavior of the diaphragm, an ideal spring to represent the elasticity of the spring and the diaphragm and a modulated force source to introduce the pretension of the set-point. Fig. 1.21b shows this with a mixed use of bond graph (TF, valve), block diagram (modulation and signal generator) and iconic diagram elements (spring, force source and fixed world).

The source of the pressure difference described earlier has not been accurately defined. One might conclude that the pressure difference between some supply pressure and the ambient pressure is meant, as these are the two evidently present pressures. However, this would cause the output pressure to fluctuate with the supply pressure, which is commonly not desired. Furthermore, the output pressure is required to cause some fluid exchange with the environment, i.e. some flow connection to the environment. As a consequence one is usually interested in setting the pressure difference between the output pressure and the supply pressure. This means that the valve needs to contain a more or less closed volume, the so-called valve chamber, in which the output pressure is allowed to be different from both the supply pressure and the ambient pressure. Some opening needs to connect this chamber to the environment in order to allow the desired flow. The dominant behavior of this restriction is that of an ideal (fluid) resistor, whether a hose is attached to the orifice or not. Parasitic behavior as fluid inertia (in case of a long hose) may be added later when fine-tuning the model. Summarizing, the following ideal elements are required in the model: a position-modulated resistor, a transformer, a spring and a resistor (Fig. 1.22). As the spring is the only dynamic element (containing an integration with respect to time) in this model, oscillatory solutions are not likely.

The labeled nodes in the bond graph merely represent the elementary behaviors, while their exact constitutive relations have not been determined yet. Some of them will be nonlinear though. If the ambient pressure is chosen as the reference pressure (zero-point), all pressures will obtain negative values in a low-vacuum control valve, but the bond graph is simplified into the one in Fig. 1.23.

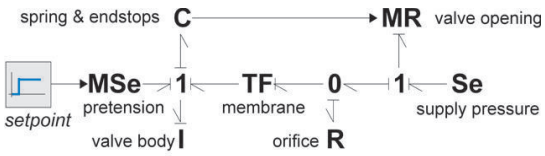


Fig. 1.24 Addition of the valve body mass.

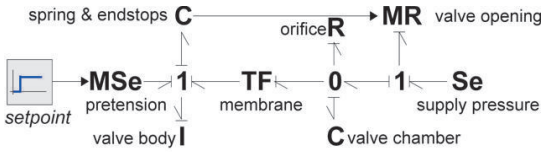


Fig. 1.25 Addition of the compressibility of the air in the valve chamber (C).

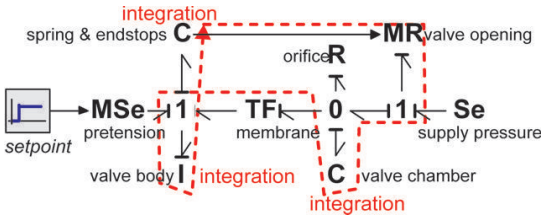


Fig. 1.26 Third-order loop (three integrations) via a causal path and the modulation signal.

The flows through the resistors are mainly dictated by the pressures imposed by sources, except for the contribution to the valve chamber pressure by the spring. It can be concluded that a linearization around an operating point leads to a first-order model characterized by a time constant. At this point, one might be inclined to bring the possibility of oscillatory behavior into the model by adding an ideal mass to represent the dominant behavior of the valve body. Together with the ideal spring, it forms a (damped) second-order system that has the potential of oscillatory solutions. However, such oscillations are not self-exciting and not self-sustained, unless the system would contain negative damping which would violate the laws of physics. Note the change of position of some of the causal strokes and the causal path from the R to the valve that indicates an algebraic loop (Fig. 1.24).

The causality assignment process hints the modeler to put a C-type storage element at the 0-junction representing the pressure in the valve chamber in order to prevent this algebraic loop. This element represents the compressibility of the air in the valve chamber (Fig. 1.25) and will appear crucial in obtaining a model that is competent to represent self-sustained oscillations.

Fig. 1.26 shows that this model contains a third-order loop via the position modulation of the valve and a causal path. It can be interpreted as follows: the position that modulates the valve is (inversely) proportional to the flow through the valve. The capacitance of the valve chamber relates the displaced volume (first integration!) of this flow to the pressure in the chamber. Via the diaphragm, this pressure acts with a force on the valve body. The resulting change of its momentum (second integration!) results in a change of its velocity. Finally this velocity causes its displacement (third integration!) and thus results in the position that modulates the valve resistor (closure of the loop). Under certain conditions, this third-order loop

may have unstable solutions that are bounded by the non-linearities of the model, like the valve body hitting the valve seat (end stops that can be added to the model easily, but discussion is beyond the scope of this contribution). The causality of the ports is derived automatically by 20-sim while drawing this graph and automatically results in a computable set of equations for simulation. The same procedure is used in case of iconic diagrams and other representations that contain the concept of a port, although in those cases the feedback to the modeler that a third-order loop is present cannot be obtained immediately.

At this point, this example should have illustrated that modeling should be focused on the relevant elementary behaviors present in a system, not merely on a (one-to-one) translation of the functional relations as the designer of the valve intended them, because this would never lead to taking into account the compressibility of the air in the valve chamber. The key elements in this model to represent the observed behavior are: the nonlinear, position-modulated resistor, the valve body, the diaphragm, and the capacitance of the valve chamber to create the third-order loop, but also the spring with its adjustable pretension, the fluid resistor at the inlet, the supply pressure and the valve body hitting the valve seat. The number of elementary one- and multi-ports is relatively small.

After identification of the proper parameter values from the provided measurement data, first simulation runs showed indeed self-starting and self-sustained oscillations with a shape that coincided with the shapes observed on the oscilloscope. The frequency of these first results was only 10% off the observed frequency. Fine-tuning of the model allowed these frequencies to be matched. However, the actual problem was already solved before the parameter identification phase, because the process of setting up the model structure already indicated the crucial role of the valve chamber that was confirmed by an experienced senior craftsman at the work floor where these valves were produced and assembled. He then remembered that long ago the role of this valve chamber had been identified by trial and error. A result that had been forgotten over the years and didn't play a role in the design of the new valve that was causing the oscillation problems.

After this example, it should be clear that a bond graph without modulating signals can never result in three integrations in a loop. A causal path can only exist between at most two storage elements, such that the number of integrations in the corresponding signal loop is at most two. Hence, the modulating signal of the valve that contains a third integration is also one of the crucial elements to create a model that is competent to represent the instabilities.

The possibility of the oscillations that can result from the third-order loop is inherent to this particular type of design. None of the parts can be omitted or changed as to break the third-order loop. For this reason, every designer of such valves should have the insights discussed above in order to be able to choose the dimensions of the valve such that it never displays undesired behavior in or near the range of operation. This insight is more related to model structure than to particular simulation results, although simulation results can help to identify the influence of the valve chamber size on the modes of operation. This example demonstrates that a port-

based approach provides this insight quite easily, although the use of this approach should be supported by sufficient knowledge of engineering physics.

It is worth mentioning that similar types of valves are not only used as low-vacuum control valves, but also as fuel-injection valves, pressure reduction valves, etc.

1.11 Conclusion

This chapter has shown how port-based modeling of physical systems can help to make proper decisions during the design of engineering systems and to create more insight in the physics of the object to be modeled, in particular in a control setting. This approach enables to easily move between finding solutions in the controller domain and in the physical structure itself, which is the key aspect of a mechatronics approach. Software tools that cover the different domains support this process.

Emphasis was on the background of physical modeling in general. In particular, the paradigm shift to the port-based approach via the introduction of the concepts of a port and a junction were discussed. An example demonstrated that one of the major achievements is that a notation allowing a multiple-view approach provides insight into the nature and background of the observed behavior. A bond graph representation gives the user who has gained some expertise in this graphical language, feedback about his modeling decisions via the representation of computational causality by the causal stroke. As the approach focuses on insight, it is also particularly suited for education [36]. All sorts of generalizations exist, but are beyond the scope of this contribution. The interested reader is referred to the extensive literature on bond graphs and port-based modeling.

1.12 Future Trends

The following general future trends in bond graphs and port-based modeling can be distinguished:

- continuous improvement and extension of computer support for bond graph representation, analysis and generation of numerical simulation models;
- mathematical formalization (port-Hamiltonian systems) of all aspects of the approach thus establishing a relation with other model views and analysis techniques (see also Chapter 2);
- extension of port-interfaces to other sub-model descriptions like wave-scattering variables (cf. Sect. 2.8) and finite elements;
- true integration of model parts that need modal analysis with a ‘lumped approach’;
- heuristic tools that support the port-based modeling decision process as well as the settings of the numerical analysis;

- support of knowledge management in order to store and use relevant information about the problem context, model performance, etc.;
- use of the port-based approach for *co-simulation*.

Chapter 2
Port-Hamiltonian Systems

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Abstract In this chapter, we will show how the representation of a lumped-parameter physical system as a bond graph naturally leads to a *dynamical system* endowed with a *geometric structure*, called a *port-Hamiltonian system*. The dynamics are determined by the storage elements in the bond graph (cf. Sect. 1.6.3), as well as the resistive elements (cf. Sect. 1.6.4), while the geometric structure arises from the generalized junction structure of the bond graph. The formalization of this geometric structure as a *Dirac structure* is introduced as the key mathematical concept to unify the description of complex interactions in physical systems. It will also allow to extend the definition of a finite-dimensional port-Hamiltonian systems as given in this chapter to the infinite-dimensional case in Chapter 4, thus dealing with distributed-parameter physical systems. We will show how this port-Hamiltonian formulation offers powerful methods for the analysis of complex multi-physics systems, also paving the way for the results on control of port-Hamiltonian systems in Chapter 5 and in Chapter 6. Furthermore, we describe how the port-Hamiltonian structure relates to the classical Hamiltonian structure of physical systems as being prominent in e.g. classical mechanics, as well as to the Brayton-Moser description of RLC-circuits.

2.1 From junction structures to Dirac structures

In the preceding chapter, we have seen how port-based network modeling of lumped-parameter physical systems leads to a representation of the physical system by *generalized bond graphs*. Generalized bond graphs consist of energy-storing elements, resistive elements and power-continuous elements like transformers, gyrators, 0- and 1-junctions. These elements are linked by *bonds*, each carrying a pair of flow and effort variables, whose product equals the *power* through the bond. In order to fix the direction of power flow, a half arrow is attached to each bond, indicating the positive direction of power flow. Thus, a generalized bond graph is an oriented graph with its nodes being decorated by one of the elements indicated above, and ev-

ery edge (called 'bond') labeled by two scalar conjugate variables $f \in \mathbb{R}$ (flow) and $e \in \mathbb{R}$ (effort). Furthermore, the elements at every node only involve the flow and effort variables associated with the bonds that are incident on that node. An important extension of this definition of a bond graph is obtained by allowing for *multi-bonds* (cf. Sect. 1.9.2) which are labeled by flow vectors $f \in \mathbb{R}^k$ and dual effort vectors $e \in (\mathbb{R}^k)^*$ (cf. Sect. B.1.1 in Appendix B). Still a further extension (see Sect. 3.2), which comes in naturally for 3-D mechanical systems, is to consider flows f which take value in the Lie algebra $se(3)$ ('twists') and efforts e which take value in the dual Lie algebra $se^*(3)$ ('wrenches').

The key concept in the formulation of port-based network models of physical systems as port-Hamiltonian systems is the geometric notion of a *Dirac structure*. Loosely speaking, a Dirac structure is a subspace of the space of flows f and efforts e such that for every pair (f,e) in the Dirac structure the power $e \times f$ is equal to zero, and, furthermore, the subspace has maximal dimension with respect to this property. This means that it is not possible to extend the subspace to a larger subspace that still has this power-conserving property.

2.1.1 From 0- and 1-junctions to Dirac structures

Before mathematically formalizing the notion of a Dirac structure, we will start with showing how the basic bond graph elements of 0-junctions and 1-junctions as encountered in the previous chapter share these properties of power-conservation and maximal dimension.

Let us start with the simple 0-junction relating two pairs of flows and efforts (f_1,e_1) and (f_2,e_2) by

$$e_1 = e_2 \qquad\qquad f_1 + f_2 = 0 \tag{2.1}$$

Clearly, the 0-junction is power-conserving, that is,

$$e_1 f_1 + e_2 f_2 = 0 \tag{2.2}$$

But there is more: the 0-junction is described by two independent equations involving 4 variables, and thus represents a 2-dimensional subspace of the 4-dimensional space of total vectors (f_1,f_2,e_1,e_2) of flow and effort variables. Furthermore, it can be seen (this will be shown later on in full generality) that we cannot leave out one of the equations in (2.1) while still retaining the power-conservation property (2.2), that is, the dimension 2 is the maximal achievable dimension with respect to the power-conservation property.

The same situation occurs for the simple 1-junction described by the relations

$$f_1 = f_2 \qquad\qquad e_1 + e_2 = 0 \tag{2.3}$$

Fig. 2.1 Equivalence of bonds.

As for the 0-junction, the 1-junction describes a two-dimensional subspace of the four-dimensional space of total vectors (f_1, f_2, e_1, e_2) of flow and effort variables, which is satisfying the power-conservation property (2.2).

Higher-dimensional 0- and 1-junctions share the same properties. Recall that the 0-junction linking k bonds with pairs of flow and effort variables $(f_1, e_1), \cdots, (f_k, e_k)$ is given by the equations

$$e_1 = e_2 = \cdots = e_k \qquad\qquad f_1 + f_2 + \cdots + f_k = 0 \tag{2.4}$$

In this case we have k independent equations involving $2k$ flow and effort variables, and thus the 0-junction specifies a k-dimensional subspace of the space $\mathbb{R}^{2k}$ of total vectors $(f_1, f_2, \cdots, f_k, e_1, e_2, \cdots, e_k)$. Furthermore, all vectors in this subspace satisfy the power-conservation property

$$e_1 f_1 + e_2 f_2 + \cdots + e_k f_k = 0, \tag{2.5}$$

while k is the *maximal* dimension of a subspace with this property.

Similarly, the 1-junction linking k pairs of flow and effort variables $(f_1, e_1), \cdots, (f_k, e_k)$ is given by the k independent equations

$$f_1 = f_2 = \cdots = f_k \qquad\qquad e_1 + e_2 + \cdots + e_k = 0 \tag{2.6}$$

specifying a k-dimensional subspace of the space $\mathbb{R}^{2k}$ of vectors in the form $(f_1, f_2, \cdots, f_k, e_1, e_2, \cdots, e_k)$, satisfying the power-conservation property (2.5).

Remark 2.1. The case of 0- or 1-junctions where the incident k bonds do not all have the same orientation can be handled similarly. For example, in the case of a 0-junction the last equation of (2.4) changes into

$$\varepsilon_1 f_1 + \varepsilon_2 f_2 + \cdots + \varepsilon_k f_k = 0$$

where ε_i is 1 or -1 depending on the fact that the half-arrow of the i-th bond incident on the 0-junction is incoming or outgoing. Consequently, the power-conservation property (2.5) changes into

$$\varepsilon_1 e_1 f_1 + \varepsilon_2 e_2 f_2 + \cdots + \varepsilon_k e_k f_k = 0.$$

For conceptual and notational simplicity we will throughout only consider bonds with *incoming* half-arrows on every incident node. In this respect we note that any bond between two nodes with arbitrary half-arrow direction can be always represented by two bonds linked by a zero-junction in such a way that the half-arrows are incoming for both nodes, see Fig. 2.1. (Although this may change the signs of the flow and effort variables.)

2.1.2 Dirac structures

0- and 1-junctions are prime examples of the general concept of a (constant) Dirac structure, which is defined as follows. We start with an abstract finite-dimensional linear space of *flows* $\mathscr{F}$ (for simplicity one can think of $\mathscr{F} = \mathbb{R}^k$). The elements of $\mathscr{F}$ will be denoted by $f \in \mathscr{F}$, and are called *flow vectors*. The space of *efforts* is given by the *dual* linear space $\mathscr{E} := \mathscr{F}^*$, and its elements are denoted by $e \in \mathscr{E}$. In the case of $\mathscr{F} = \mathbb{R}^k$ the space of efforts is $\mathscr{E} = (\mathbb{R}^k)^*$, and as the elements $f \in \mathbb{R}^k$ are commonly written as *column* vectors the elements $e \in (\mathbb{R}^k)^*$ are appropriately represented as *row* vectors. Then the *total space* of flow and effort variables is $\mathscr{F} \times \mathscr{F}^*$, and will be called the space of *port variables*. On the total space of port variables, the *power* is defined by

$$P = \langle e \mid f \rangle \qquad (f,e) \in \mathscr{F} \times \mathscr{F}^*, \tag{2.7}$$

where $\langle e \mid f \rangle$ denotes the dual product, that is, the linear functional $e \in \mathscr{F}^*$ acting on $f \in \mathscr{F}$. If f is written as a column vector and e as a row vector, then the power is simply the product $\langle e \mid f \rangle = ef$. However, for simplicity, we will throughout write the effort e also as a column vector, in which case

$$\langle e \mid f \rangle = e^{\mathrm{T}} f$$

Definition 2.1. A *Dirac structure* on $\mathscr{F} \times \mathscr{F}^*$ is a subspace $\mathscr{D} \subset \mathscr{F} \times \mathscr{F}^*$ such that

i) $\langle e \mid f \rangle = 0$, for all $(f,e) \in \mathscr{D}$,
ii) $\dim \mathscr{D} = \dim \mathscr{F}$.

Property *i)* corresponds to *power-conservation*, and expresses the fact that the total power entering (or leaving) a Dirac structure is zero. It can be shown that the *maximal dimension* of any subspace $\mathscr{D} \subset \mathscr{F} \times \mathscr{F}^*$ satisfying property *i)* is equal to $\dim \mathscr{F}$. Instead of proving this directly, we will give an equivalent definition of a Dirac structure from which this claim immediately follows. Furthermore, this equivalent definition of a Dirac structure has the advantage that it generalizes to the case of an *infinite-dimensional* linear space $\mathscr{F}$, leading to the definition of an infinite-dimensional Dirac structure. This will be instrumental in the definition of a *distributed-parameter* port-Hamiltonian system later on in Chapter 4.

In order to give this equivalent characterization of a Dirac structure, we look more closely at the geometric structure of the total space of flow and effort variables $\mathscr{F} \times \mathscr{F}^*$. In fact, related to the definition of power, there exists a canonically defined *bi-linear form* $\ll \cdot, \cdot \gg$ on the space $\mathscr{F} \times \mathscr{F}^*$, defined as

$$\ll (f^a, e^a), (f^b, e^b) \gg := \left\langle e^a \mid f^b \right\rangle + \left\langle e^b \mid f^a \right\rangle \tag{2.8}$$

with $(f^a, e^a), (f^b, e^b) \in \mathscr{F} \times \mathscr{F}^*$. Note that this bi-linear form is *indefinite*, that is, $\ll (f,e), (f,e) \gg$ may be positive or negative, but it is *non-degenerate*, that is,

$\ll (f^a,e^a),(f^b,e^b) \gg = 0$ for all (f^b,e^b) implies that $(f^a,e^a)=0$. We can then give the following fundamental definition, [54, 67].

Proposition 2.1. *A (constant) Dirac structure on $\mathscr{F}\times\mathscr{F}^*$ is a subspace $\mathscr{D}\subset\mathscr{F}\times\mathscr{F}^*$ such that*

$$\mathscr{D}=\mathscr{D}^\perp, \tag{2.9}$$

where $\perp$ denotes the orthogonal complement with respect to the bi-linear form $\ll\cdot,\cdot\gg$.

Proof. Let $\mathscr{D}$ satisfy (2.9). Then for every $(f,e)\in\mathscr{D}$

$$0=\ll (f,e),(f,e)\gg = \langle e\mid f\rangle + \langle e\mid f\rangle = 2\langle e\mid f\rangle$$

By non-degeneracy of $\ll\cdot,\cdot\gg$

$$\dim\mathscr{D}^\perp = \dim(\mathscr{F}\times\mathscr{F}^*) - \dim\mathscr{D} = 2\dim\mathscr{F} - \dim\mathscr{D}$$

and hence property (2.9) implies $\dim\mathscr{D}=\dim\mathscr{F}$. Conversely, let $\mathscr{D}$ be a Dirac structure and thus satisfying properties *i)* and *ii)* of Definition 2.1. Let $(f^a,e^a),(f^b,e^b)$ be any vectors contained in $\mathscr{D}$. Then by linearity also $(f^a+f^b,e^a+e^b)\in\mathscr{D}$. Hence by property *i)*

$$\begin{aligned} 0 &= \left\langle e^a+e^b \mid f^a+f^b\right\rangle \\ &= \left\langle e^a\mid f^b\right\rangle + \left\langle e^b\mid f^a\right\rangle + \langle e^a\mid f^a\rangle + \left\langle e^b\mid f^b\right\rangle \\ &= \left\langle e^a\mid f^b\right\rangle + \left\langle e^b\mid f^a\right\rangle = \ll (f^a,e^a),(f^b,e^b)\gg \end{aligned} \tag{2.10}$$

since by another application of property *i)*, $\langle e^a\mid f^a\rangle = \left\langle e^b\mid f^b\right\rangle = 0$. This implies that $\mathscr{D}\subset\mathscr{D}^\perp$. Furthermore, by property *ii)* and $\dim\mathscr{D}^\perp = 2\dim\mathscr{F}-\dim\mathscr{D}$ it follows that

$$\dim\mathscr{D}=\dim\mathscr{D}^\perp$$

yielding $\mathscr{D}=\mathscr{D}^\perp$.

Remark 2.2. Note that we have actually shown that property *i)* implies $\mathscr{D}\subset\mathscr{D}^\perp$. Together with the fact that $\dim\mathscr{D}^\perp = 2\dim\mathscr{F}-\dim\mathscr{D}$ this implies that any subspace $\mathscr{D}$ satisfying property *i)* has the property that $\dim\mathscr{D}\leq\dim\mathscr{F}$. Thus, as claimed before, a Dirac structure is a linear subspace *of maximal dimension* satisfying property *i)*.

Remark 2.3. The property $\mathscr{D}=\mathscr{D}^\perp$ can be regarded as a generalization of Tellegen's theorem in circuit theory, since it describes a constraint between two *different* realizations of the port variables, in contrast to property *i)*.

Remark 2.4. In the infinite-dimensional case (cf. Chapter 4), the property $\mathscr{D}=\mathscr{D}^\perp$ will be taken as the *definition* of an infinite-dimensional Dirac structure.

From a mathematical point of view, there are a number of direct examples of Dirac structures $\mathscr{D} \subset \mathscr{F} \times \mathscr{F}^*$. We leave the proofs as an exercise to the reader.

1. Let $J : \mathscr{F}^* \to \mathscr{F}$ be a skew-symmetric linear mapping, that is, $J = -J^*$, where $J^* : \mathscr{F}^* \to (\mathscr{F})^{**} = \mathscr{F}$ is the adjoint mapping. Then
$$\text{graph } J := \Big\{ (f,e) \in \mathscr{F} \times \mathscr{F}^* \mid f = Je \Big\}$$
is a Dirac structure.
2. Let $\omega : \mathscr{F} \to \mathscr{F}^*$ be a skew-symmetric linear mapping, then
$$\text{graph } \omega := \Big\{ (f,e) \in \mathscr{F} \times \mathscr{F}^* \mid e = \omega f \Big\}$$
is a Dirac structure.
3. Let $\mathscr{G} \subset \mathscr{F}$ be any subspace. Define
$$\mathscr{G}^{\text{orth}} = \Big\{ e \in \mathscr{F}^* \mid \langle e \mid f \rangle = 0 \text{ for all } f \in \mathscr{G} \Big\}$$
Then $\mathscr{G} \times \mathscr{G}^{\text{orth}} \subset \mathscr{F} \times \mathscr{F}^*$ is a Dirac structure.

2.1.3 Examples of Dirac structures

In this subsection we will discuss a number of physical examples of Dirac structures.

2.1.3.1 Transformers, gyrators, and ideal constraints

We have seen above that the bond graph elements of 0- and 1-junctions are key examples of Dirac structures. Also *transformers*, *gyrators* and *ideal constraints* are seen to be examples of Dirac structures. Indeed, recall the definition of a transformer. A transformer is a 2-port linking two bonds with flow and effort variables (f_1, e_1) and (f_2, e_2) by

$$f_2 = \alpha f_1 \qquad\qquad e_1 = -\alpha e_2 \tag{2.11}$$

with α being a constant, called the *transformer ratio*. The subspace defined by (2.11) is easily checked to be a Dirac structure. Also the multi-dimensional version of (2.11)

$$f^b = T f^a \qquad\qquad e^a = -e^b T \tag{2.12}$$

with (f^a, e^a) and (f^b, e^b) being pairs of column vectors of flow variables and *row* vectors of effort variables of the same dimension, and T being a matrix of appropriate dimensions, is immediately seen to define a Dirac structure.

Similarly, recall that a gyrator is given by the relations

$$f_1 = \beta e_2 \qquad\qquad \beta e_1 = -f_2, \tag{2.13}$$

which again is defining a Dirac structure. The resulting unit gyrator for $\beta = 1$ is called the *symplectic gyrator*. The multi-dimensional version is given as the Dirac structure defined by

$$f^a = Ge^b \qquad\qquad f^b = -G^{\mathrm{T}}e^a, \tag{2.14}$$

where now, again for simplicity of notation, e^a and e^b denote column vectors, and G is a matrix of appropriate dimensions.

Finally, ideal effort and flow constraints are trivial examples of Dirac structures. Let (f,e) denote a (multi-dimensional) pair of flows and efforts. Then the effort constraint

$$\mathscr{D} := \Big\{(f,e) \;\mid\; e = 0\Big\}$$

is defining a Dirac structure $\mathscr{D}$, and the same holds for the ideal flow constraint

$$\mathscr{D} := \Big\{(f,e) \;\mid\; f = 0\Big\}$$

2.1.3.2 Kirchhoff's laws as Dirac structures

Consider an electrical circuit with n edges where the current through the i-th edge is denoted by I_i and the voltage over the i-th edge is V_i. Collect the currents in a single column vector I (of dimension n) and the voltages in an n-dimensional column vector V. The following consequence of Kirchhoff's current and voltage laws is well-known. Let Kirchhoff's *current* laws be written in matrix form as

$$\mathscr{A}I = 0 \tag{2.15}$$

for some matrix $\mathscr{A}$ (with n columns). Then Kirchhoff's *voltage* laws can be written in the following form. All allowed vectors of voltages V in the circuit are given as

$$V = \mathscr{A}^{\mathrm{T}}\lambda \tag{2.16}$$

for any vector λ of appropriate dimension. It is immediately seen that the total space of currents and voltages allowed by Kirchhoff's current and voltage laws

$$\mathscr{D} := \Big\{(I,V) \mid \mathscr{A}I = 0,\; V = \mathscr{A}^{\mathrm{T}}\lambda\Big\} \tag{2.17}$$

defines a Dirac structure. Consequently

$$(V^a)^{\mathrm{T}}I^b + (V^b)^{\mathrm{T}}I^a = 0$$

for all pairs $(I^a, V^a), (I^b, V^b) \in \mathscr{D}$. In particular, by taking V^a, I^b equal to zero, we obtain

$$(V^b)^{\mathrm{T}} I^a = 0$$

for all I^a satisfying (2.15) and all V^b satisfying (2.16). This nothing else than *Tellegen's theorem*.

2.1.3.3 Kinematic pairs

The equations describing a kinematic pair (e.g. a revolute or prismatic joint) in a three-dimensional mechanical system are, from the Dirac structure point of view, of the same type as Kirchhoff's current and voltage laws.

Indeed, the constraint forces F generated in a (frictionless and infinitely stiff) kinematic pair produce no power on the velocities V allowed by the pair:

$$\mathscr{A} V = 0 \qquad\qquad F = \mathscr{A}^{\mathrm{T}} \lambda \tag{2.18}$$

where the columns of $\mathscr{A}^{\mathrm{T}}$ form a basis for the space of allowed reaction forces, and λ is a vector of scalar reaction force coordinates.

2.1.3.4 The principle of virtual work

The principle of virtual work can be formulated as

$$\sum_{i=1}^{n} F_i \delta q_i = 0 \tag{2.19}$$

where $F = (F_1, \cdots, F_n)$ is the vector of impressed forces, and $\delta q = (\delta q_1, \cdots, \delta q_n)$ denotes the vector of *virtual* displacements that are compatible with the kinematic constraints of the system. The expression $\sum_{i=1}^{n} F_i \delta q_i$ equals the infinitesimal work (or *power*) due to the impressed forces and the infinitesimal displacement. If the kinematic constraints of the system are given as $\mathscr{A} \delta q = 0$ then it follows that the impressed forces should be given as $F = \mathscr{A}^{\mathrm{T}} \lambda$, as in the previous subsection.

Originally, the principle of virtual work is formulated as an *equilibrium* condition: it expresses that a system with configuration coordinates $q = (q_1, q_2, \cdots, q_n)$, which is subject to forces $F(q)$, is at equilibrium $\bar{q}$ if the virtual work $\sum_{i=1}^{n} F_i(\bar{q}) \delta q_i$ corresponding to any admissible virtual displacement δq from $\bar{q}$ is equal to zero. In D'Alembert's principle this was extended by adding the *inertial forces* $\dot{p}$ to the impressed forces.

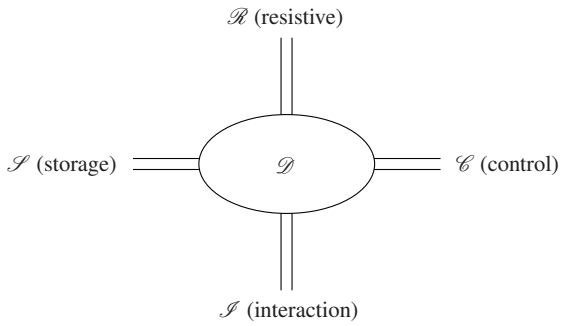

Fig. 2.2 Port-Hamiltonian system.

2.2 Port-Hamiltonian systems

Crucial property of the concept of Dirac structure is that the standard interconnection of Dirac structures is again a Dirac structure. This fact, formally to be proved in Sect. 2.5, has the following important consequence. Recall from Chapter 1 that any bond graph representation of a physical system can be summarized as follows. The bond graph consists of *energy-storing elements*, *resistive elements*, *power-conserving elements* such as *transformers, gyrators, and ideal constraints*, and *0- and 1-junctions* linked by *bonds*. Furthermore, we may take together all the power-conserving elements with the 0- and 1-junctions in order to obtain the *generalized junction structure* (cf. Sect. 1.6.8). Since all the components of the generalized junction structure are Dirac structures, the generalized junction structure, being the interconnection of Dirac structures, is also a Dirac structure. Hence the bond graph can be compactly represented as an energy-storing multi-port containing all the energy-storing elements and a resistive multi-port containing all resistive elements, *linked by a Dirac structure.* This is the starting point for the formulation of a bond graph as a port-Hamiltonian system.

2.2.1 Geometric definition of a port-Hamiltonian system

In general, a port-Hamiltonian system can be represented as in Fig. 2.2. Central in the definition of a port-Hamiltonian system is the notion of a *Dirac structure*, depicted in Fig. 2.2 by $\mathscr{D}$. Basic property of a Dirac structure is *power conservation*: the Dirac structure links the various port variables in such a way that the total power associated with the port-variables is zero.

The port variables entering the Dirac structure have been split in Fig. 2.2 in different parts. First, there are two *internal* ports. One, denoted by $\mathscr{S}$, corresponds to energy-storage and the other one, denoted by $\mathscr{R}$, corresponds to internal energy-dissipation (resistive elements). Second, two *external* ports are distinguished. The external port denoted by $\mathscr{C}$ is the port that is accessible for controller action. Also the presence of *sources* may be included in this port. Finally, the external port de-

noted by $\mathscr{I}$ is the interaction port, defining the interaction of the system with (the rest of) its environment.

2.2.1.1 Energy storage port

The port variables associated with the internal storage port will be denoted by (f_S, e_S). They are interconnected to the energy storage of the system which is defined by a finite-dimensional state space manifold $\mathscr{X}$ with coordinates x, together with a Hamiltonian function $H : \mathscr{X} \to \mathbb{R}$ denoting the energy. The flow variables of the energy storage are given by the *rate* $\dot{x}$ of the energy variables x. Furthermore, the effort variables of the energy storage are given by the *co-energy* variables $\frac{\partial H}{\partial x}(x)$, resulting in the energy balance[1]

$$\frac{\mathrm{d}}{\mathrm{d}t}H = \left\langle \frac{\partial H}{\partial x}(x) \mid \dot{x} \right\rangle = \frac{\partial^{\mathrm{T}} H}{\partial x}(x)\dot{x} \tag{2.20}$$

The interconnection of the energy storing elements to the storage port of the Dirac structure is accomplished by setting

$$f_S = -\dot{x} \qquad\qquad e_S = \frac{\partial H}{\partial x}(x)$$

Note that this corresponds to an ordinary 0-junction. Hence the energy balance (2.20) can be also written as

$$\frac{\mathrm{d}}{\mathrm{d}t}H = \frac{\partial^{\mathrm{T}} H}{\partial x}(x)\dot{x} = -e_S^{\mathrm{T}} f_S \tag{2.21}$$

2.2.1.2 Resistive port

The second internal port corresponds to internal energy dissipation (due to friction, resistance, etc.), and its port variables are denoted by (f_R, e_R). These port variables are terminated on a static resistive relation $\mathscr{R}$. In general, a static resistive relation will be of the form

$$R(f_R, e_R) = 0, \tag{2.22}$$

with the property that for all (f_R, e_R) satisfying (2.22)

$$\langle e_R \mid f_R \rangle \leq 0 \tag{2.23}$$

A typical example of such a nonlinear resistive relation will be given in Example 2.15. In many cases we may restrict ourselves to *linear* resistive relations. (Note that some types of non-linearity already can be captured in the description of the

[1] Throughout we adopt the convention that $\frac{\partial H}{\partial x}(x)$ denotes the *column* vector of partial derivatives of H.

resistive port of the Dirac structure.) This means that the resistive port variables (f_R, e_R) satisfy linear relations of the form

$$R_f f_R + R_e e_R = 0 \tag{2.24}$$

The inequality (2.23) corresponds to the square matrices R_f and R_e satisfying the following properties of symmetry and semi-positive definiteness

$$R_f R_e^{\mathrm{T}} = R_e R_f^{\mathrm{T}} \geq 0, \tag{2.25}$$

together with the dimensionality condition

$$\operatorname{rank}\left[R_f \,|\, R_e\right] = \dim f_R \tag{2.26}$$

Indeed, by the dimensionality condition (2.26) and the symmetry (2.25) we can equivalently rewrite the kernel representation (2.24) of $\mathscr{R}$ into an image representation

$$f_R = R_e^{\mathrm{T}} \lambda \qquad\qquad e_R = -R_f^{\mathrm{T}} \lambda \tag{2.27}$$

That is, any pair (f_R, e_R) satisfying (2.24) can be written into the form (2.27) for a certain λ, and conversely any (f_R, e_R) for which there exists λ such that (2.27) holds is satisfying (2.24). Hence by (2.25) for all f_R, e_R satisfying the resistive relation

$$e_R^{\mathrm{T}} f_R = -\left(R_f^{\mathrm{T}} \lambda\right)^{\mathrm{T}} R_e^{\mathrm{T}} \lambda = -\lambda^{\mathrm{T}} R_f R_e^{\mathrm{T}} \lambda \leq 0 \tag{2.28}$$

Without the presence of additional external ports, the Dirac structure of the port-Hamiltonian system satisfies the power-balance

$$e_S^{\mathrm{T}} f_S + e_R^{\mathrm{T}} f_R = 0 \tag{2.29}$$

which leads by substitution of the equations (2.21) and (2.28) to

$$\frac{\mathrm{d}}{\mathrm{d}t} H = -e_S^{\mathrm{T}} f_S = e_R^{\mathrm{T}} f_R \leq 0 \tag{2.30}$$

An important special case of resistive relations between f_R and e_R occurs when the resistive relations can be expressed as an *input-output* mapping

$$f_R = -F(e_R) \tag{2.31}$$

where the resistive characteristic $F : \mathbb{R}^{m_r} \to \mathbb{R}^{m_r}$ satisfies

$$e_R^{\mathrm{T}} F(e_R) \geq 0, \quad e_R \in \mathbb{R}^{m_r} \tag{2.32}$$

In many cases, F will be derivable from a so-called *Rayleigh dissipation function* $R : \mathbb{R}^{m_r} \to \mathbb{R}$, in the sense that

$$F(e_R) = \frac{\partial R}{\partial e_R}(e_R).$$

For *linear* resistive elements, (2.31) specializes to

$$f_R = -\tilde{R} e_R \tag{2.33}$$

for some positive semi-definite symmetric matrix $\tilde{R} = \tilde{R}^{\mathrm{T}} \geq 0$.

2.2.1.3 External ports

Now, let us consider in more detail the *external* ports to the system. We shall distinguish between two types of external ports. One is the *control port* $\mathscr{C}$, with port variables (f_C, e_C), which are the port variables which are accessible for controller action. The other type of external port is the *interaction port* $\mathscr{I}$, which denotes the interaction of the port-Hamiltonian system with its environment. The port variables corresponding to the interaction port are denoted by (f_I, e_I). Taking both the external ports into account the power-balance (2.29) extends to

$$e_S^{\mathrm{T}} f_S + e_R^{\mathrm{T}} f_R + e_C^{\mathrm{T}} f_C + e_I^{\mathrm{T}} f_I = 0, \tag{2.34}$$

whereby (2.30) extends to

$$\frac{\mathrm{d}}{\mathrm{d}t} H = e_R^{\mathrm{T}} f_R + e_C^{\mathrm{T}} f_C + e_I^{\mathrm{T}} f_I \tag{2.35}$$

2.2.1.4 Port-Hamiltonian dynamics

The port-Hamiltonian system with state space $\mathscr{X}$, Hamiltonian H corresponding to the energy storage port $\mathscr{S}$, resistive port $\mathscr{R}$, control port $\mathscr{C}$, interconnection port $\mathscr{I}$, and total Dirac structure $\mathscr{D}$ will be succinctly denoted by $\Sigma = (\mathscr{X}, H, \mathscr{R}, \mathscr{C}, \mathscr{I}, \mathscr{D})$. The dynamics of the port-Hamiltonian system is specified by considering the constraints on the various port variables imposed by the Dirac structure, that is,

$$(f_S, e_S, f_R, e_R, f_C, e_C, f_I, e_I) \in \mathscr{D},$$

and to substitute in these relations the equalities $f_S = -\dot{x}$ and $e_S = \frac{\partial H}{\partial x}(x)$. This leads to the implicitly defined dynamics

$$\left(-\dot{x}(t), \frac{\partial H}{\partial x}(x(t)), f_R(t), e_R(t), f_C, (t), e_C(t), f_I(t), e_I(t)\right) \in \mathscr{D} \tag{2.36}$$

with $f_R(t), e_R(t)$ satisfying for all t the resistive relation

$$R(f_R(t), e_R(t)) = 0 \tag{2.37}$$

In many cases of practical interest, the equations (2.36) will constrain the allowed states x, depending on the values of the external control and interaction port variables (f_C, e_C) and (f_I, e_I). Thus in a coordinate representation (as will be treated in detail in the next section), port-Hamiltonian systems generally will consist of a mixed set of *differential* and *algebraic* equations (DAEs).

Example 2.1 (General LC-circuits). Consider an LC-circuit with arbitrary network topology. Kirchhoff's current and voltage laws take the general form

$$\begin{aligned} A_L^{\mathrm{T}} I_L + A_C^{\mathrm{T}} I_C + A_P^{\mathrm{T}} I_P &= 0 \\ V_L &= A_L \lambda \\ V_C &= A_C \lambda \\ V_P &= A_P \lambda \end{aligned} \tag{2.38}$$

for certain matrices A_L, A_C and A_S. Here I_L, I_C and I_P denote the currents, respectively through the inductors, capacitors and external ports. Likewise, V_L, V_C and V_P denote the voltages over the inductors, capacitors and external ports. (The matrices A_L, A_C and A_S are in fact of a special nature, consisting of 0's, 1's, and -1's, corresponding to the circuit graph.) Kirchhoff's current and voltage laws define a Dirac structure $\mathscr{D}$ between the flows and efforts

$$\begin{aligned} f &= (I_C, V_L, I_P) = (-\dot{Q}, -\dot{\phi}, I_P) \\ e &= (V_C, I_L, V_P) = \left(\frac{\partial H}{\partial Q}, \frac{\partial H}{\partial \phi}, V_P \right) \end{aligned}$$

with Hamiltonian $H(Q,\phi)$ the total energy. Indeed, it easily follows that for all $I_C, V_L, I_P, V_C, I_L, V_P$ satisfying (2.38) $I_C^T V_C + I_L^T V_L + I_P^T V_P = 0$, while $\dim \mathscr{D} = \dim I_C + \dim V_L + \dim I_P$. This leads to the port-Hamiltonian system in implicit form

$$\begin{aligned} -\dot{\phi} &= A_L \lambda \\ \frac{\partial H}{\partial Q} &= A_C \lambda \\ V_P &= A_P \lambda \\ 0 &= A_L^{\mathrm{T}} \frac{\partial H}{\partial \phi} - A_C^{\mathrm{T}} \dot{Q} + A_P^{\mathrm{T}} I_P \end{aligned}$$

with state vector $x = (Q, \phi)$. Clearly, in general these implicit equations are not easily amenable to analysis. However, more convenient coordinate representations can be obtained using the theory exposed in Sect. 2.4.

Example 2.2 (Electro-mechanical system). Consider the dynamics of an iron ball in the magnetic field of a controlled inductor, shown in Fig. 2.3. The port-Hamiltonian description of this system (with q the height of the ball, p the vertical momentum, and φ the magnetic flux of the inductor) is given as

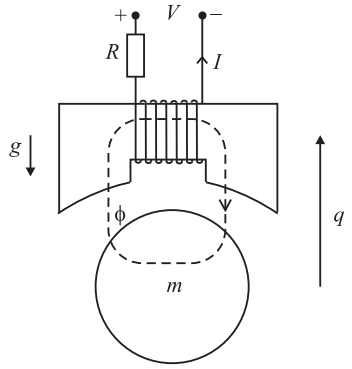

Fig. 2.3 Magnetically levitated ball.

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{R} \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial \varphi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} V, \qquad I = \frac{\partial H}{\partial \varphi} \tag{2.39}$$

This is a typical example of a system where the *coupling* between two different physical domains (mechanical and magnetic) takes place via the Hamiltonian

$$H(q,p,\varphi) = mgq + \frac{p^2}{2m} + \frac{\varphi^2}{2k_1(1-\frac{q}{k_2})}$$

where the last term depends both on a magnetic variable (in this case φ) and a mechanical variable (in this case the height q).

2.2.2 Modulated Dirac structures and port-Hamiltonian systems on manifolds

For many systems, especially those with 3-D mechanical components, the Dirac structure is actually *modulated* by the *energy* or by *configuration* variables, as described in Sect. 1.6.9. Furthermore, the state space $\mathscr{X}$ is a *manifold* (Sect. B.1.2) and the flows $f_S = -\dot{x}$ corresponding to energy-storage are elements of the tangent space $T_x\mathscr{X}$ at the state $x \in \mathscr{X}$, while the efforts e_S are elements of the co-tangent space $T_x^*\mathscr{X}$. The modulation of the Dirac structure is usually intimately related to the underlying geometry of the system.

Example 2.3 (Spinning rigid body). Consider a rigid body spinning around its center of mass in the absence of gravity. The energy variables are the three components of the body angular momentum p along the three principal axes: $p = (p_x, p_y, p_z)$, and

the energy is the kinetic energy

$$H(p) = \frac{1}{2}\left(\frac{p_x^2}{I_x} + \frac{p_y^2}{I_y} + \frac{p_z^2}{I_z}\right),$$

where I_x, I_y, I_z are the principal moments of inertia. Euler's equations describing the dynamics are

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}}_{J(p)} \begin{bmatrix} \frac{\partial H}{\partial p_x} \\ \frac{\partial H}{\partial p_y} \\ \frac{\partial H}{\partial p_z} \end{bmatrix} \tag{2.40}$$

The Dirac structure is given as the graph of the skew-symmetric matrix $J(p)$, i.e., modulated by the non-constant energy variables p.

Modulated Dirac structures often arise as a result of *ideal constraints* imposed on the generalized velocities of the mechanical system by its environment, called *kinematic constraints*. In many cases, these constraints will be configuration dependent, causing a Dirac structure modulated by the configuration variables.

Consider a mechanical system with n degrees of freedom, locally described by n configuration variables $q = (q_1, \ldots, q_n)$. Expressing the kinetic energy as $\frac{1}{2}\dot{q}^{\mathrm{T}} M(q)\dot{q}$, with $M(q) > 0$ being the generalized mass matrix, we define in the usual way the Lagrangian function $L(q, \dot{q})$ as the *difference* of kinetic energy and potential energy $P(q)$, i.e.

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^{\mathrm{T}} M(q)\dot{q} - P(q) \tag{2.41}$$

Suppose now that there are constraints on the generalized velocities $\dot{q}$, described as

$$A^{\mathrm{T}}(q)\dot{q} = 0 \tag{2.42}$$

with $A(q)$ an $n \times k$ matrix of rank k everywhere (that is, there are k independent kinematic constraints). Classically, the constraints (2.42) are called *holonomic* if it is possible to find new configuration coordinates $\overline{q} = (\overline{q}_1, \ldots, \overline{q}_n)$ such that the constraints are equivalently expressed as

$$\dot{\overline{q}}_{n-k+1} = \dot{\overline{q}}_{n-k+2} = \cdots = \dot{\overline{q}}_n = 0 \tag{2.43}$$

in which case one may eliminate the configuration variables $\overline{q}_{n-k+1}, \ldots, \overline{q}_n$, since the kinematic constraints (2.43) are equivalent to the *geometric* constraints

$$\overline{q}_{n-k+1} = c_{n-k+1}, \ldots, \overline{q}_n = c_n \tag{2.44}$$

for certain constants $c_{n-k+1}, \ldots, c_n$ determined by the initial conditions. Then the system reduces to an *unconstrained* system in the $(n-k)$ remaining configuration coordinates $(\overline{q}_1, \ldots, \overline{q}_{n-k})$. If it is *not* possible to find coordinates $\overline{q}$ such that (2.43)

holds (that is, if we are not able to *integrate* the kinematic constraints as above), then the constraints are called *non-holonomic*.

The equations of motion for the mechanical system with Lagrangian $L(q,\dot{q})$ and constraints (2.42) are given by the Euler-Lagrange equations [154]

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} &= A(q)\lambda + B(q)u \qquad\qquad \lambda \in \mathbb{R}^k,\ u \in \mathbb{R}^m \\ A^{\mathrm{T}}(q)\dot{q} &= 0 \end{aligned} \tag{2.45}$$

where $B(q)u$ are the external forces (controls) applied to the system, for some $n \times m$ matrix $B(q)$, while $A(q)\lambda$ are the *constraint forces*. The Lagrange multipliers $\lambda(t)$ are uniquely determined by the requirement that the constraints $A^{\mathrm{T}}(q(t))\dot{q}(t) = 0$ have to be satisfied for all t.

Defining the generalized momenta

$$p = \frac{\partial L}{\partial \dot{q}} = M(q)\dot{q}, \tag{2.46}$$

the constrained Euler-Lagrange equations (2.45) transform into *constrained Hamiltonian equations*

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + A(q)\lambda + B(q)u \\ y &= B^{\mathrm{T}}(q)\frac{\partial H}{\partial p}(q,p) \\ 0 &= A^{\mathrm{T}}(q)\frac{\partial H}{\partial p}(q,p) \end{aligned} \tag{2.47}$$

with $H(q,p) = \frac{1}{2}p^{\mathrm{T}}M^{-1}(q)p + P(q)$ the total energy. The constrained Hamiltonian equations (2.47) define a port-Hamiltonian system, with respect to the modulated Dirac structure

$$\begin{aligned} \mathscr{D} = \Bigg\{ (f_S, e_S, f_C, e_C) \mid\ & 0 = A^{\mathrm{T}}(q)e_S,\ e_C = B^{\mathrm{T}}(q)e_S, \\ & -f_S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} e_S + \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} f_c,\ \lambda \in \mathbb{R}^k \Bigg\} \end{aligned} \tag{2.48}$$

Example 2.4 (Rolling euro). Let x, y be the Cartesian coordinates of the point of contact of the coin with the plane. Furthermore, φ denotes the heading angle, and θ the angle of Queen Beatrix' head[2]. With all constants set to unity, the constrained Lagrangian equations of motion are

[2] On the Dutch version of the Euro.

$$\begin{aligned}
\ddot{x} &= \lambda_1 \\
\ddot{y} &= \lambda_2 \\
\ddot{\theta} &= -\lambda_1 \cos\varphi - \lambda_2 \sin\varphi + u_1 \\
\ddot{\varphi} &= u_2
\end{aligned} \tag{2.49}$$

with u_1 the control torque about the rolling axis, and u_2 the control torque about the vertical axis. The total energy is $H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}p_\theta^2 + \frac{1}{2}p_\varphi^2$. The rolling constraints are $\dot{x} = \dot{\theta}\cos\varphi$ and $\dot{y} = \dot{\theta}\sin\varphi$, i.e. rolling without slipping, which can be written in the form (2.42) by defining

$$A^{\mathrm{T}}(x,y,\theta,\phi) = \begin{bmatrix} 1 & 0 & -\cos\phi & 0 \\ 0 & 1 & -\sin\phi & 0 \end{bmatrix}$$

This motivates to extend the definition of a *constant* Dirac structure $\mathscr{D} \subset \mathscr{F} \times \mathscr{F}^*$ (with $\mathscr{F}$ a linear space) as given before in Proposition 2.1 to *Dirac structures on manifolds*. Simply put, a Dirac structure on a manifold $\mathscr{X}$ is point-wise (that is, for every $x \in \mathscr{X}$) a constant Dirac structure $\mathscr{D}(x) \subset T_x\mathscr{X} \times T_x^*\mathscr{X}$.

Definition 2.2. Let $\mathscr{X}$ be a manifold. A Dirac structure $\mathscr{D}$ on $\mathscr{X}$ is a vector subbundle of the Whitney sum[3] $T\mathscr{X} \oplus T^*\mathscr{X}$ such that

$$\mathscr{D}(x) \subset T_x\mathscr{X} \times T_x^*\mathscr{X}$$

is for every $x \in \mathscr{X}$ a constant Dirac structure as before.

If, next to the energy storage port, there are additional ports (such as resistive, control or interaction ports) with port variables $f \in \mathscr{F}$ and $e \in \mathscr{F}^*$, then a modulated Dirac structure is point-wise specified by a constant Dirac structure

$$\mathscr{D}(x) \subset T_x\mathscr{X} \times T_x^*\mathscr{X} \times \mathscr{F} \times \mathscr{F}^* \tag{2.50}$$

2.2.3 Input-state-output port-Hamiltonian systems

An important special case of port-Hamiltonian systems as defined above is the class of *input-state-output port-Hamiltonian systems*, where there are no algebraic constraints on the state space variables, and the flow and effort variables of the resistive, control and interaction port are split into conjugated input-output pairs

Input-state-output port-Hamiltonian systems are defined as dynamical systems of the following form

[3] The Whitney sum of two vector bundles with the same base space is defined as the vector bundle whose fiber above each element of this common base space is the product of the fibers of each individual vector bundle.

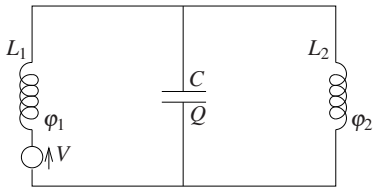

Fig. 2.4 Controlled LC-circuit.

$$\Sigma : \begin{cases} \dot{x} = \big[J(x) - R(x)\big] \dfrac{\partial H}{\partial x}(x) + g(x)u + k(x)d \\ y = g^{\mathrm{T}}(x) \dfrac{\partial H}{\partial x}(x) \\ z = k^{\mathrm{T}}(x) \dfrac{\partial H}{\partial x}(x) \end{cases} \qquad x \in \mathscr{X} \tag{2.51}$$

where (u,y) are the input-output pairs corresponding to the control port $\mathscr{C}$, while (d,z) denote the input-output pairs of the interaction port $\mathscr{I}$. Note that $y^{\mathrm{T}}u$ and $z^{\mathrm{T}}d$ equal the power corresponding to the control, respectively, interaction port. Here the matrix $J(x)$ is skew-symmetric, that is $J(x) = -J^{\mathrm{T}}(x)$. The matrix $R(x) = R^{\mathrm{T}}(x) \geq 0$ specifies the resistive structure. From a resistive port point of view, it is given as $R(x) = g_R^{\mathrm{T}}(x)\tilde{R}g_R(x)$ for some linear resistive relation $f_R = -\tilde{R}e_R$ with $\tilde{R} = \tilde{R}^{\mathrm{T}} \geq 0$ and g_R representing the input matrix corresponding to the resistive port.

The underlying Dirac structure of the system is then given by the graph of the skew-symmetric linear map

$$\begin{bmatrix} -J(x) & -g_R(x) & -g(x) & -k(x) \\ g_R^{\mathrm{T}}(x) & 0 & 0 & 0 \\ g^{\mathrm{T}}(x) & 0 & 0 & 0 \\ k^{\mathrm{T}}(x) & 0 & 0 & 0 \end{bmatrix} \tag{2.52}$$

In general, the Dirac structure defined as the graph of the mapping (2.52) is a *modulated Dirac structure* since the matrices J, g_R, g and k may all depend on the energy variables x.

Example 2.5 (LC-circuit with independent storage elements). Consider a controlled LC-circuit (see Fig. 2.4) consisting of two inductors with magnetic energies $H_1(\varphi_1)$ and $H_2(\varphi_2)$ (φ_1 and φ_2 being the magnetic flux linkages), and a capacitor with electric energy $H_3(Q)$ (Q being the charge). If the elements are linear, then

$$H_1(\varphi_1) = \frac{1}{2L_1}\varphi_1^2 \qquad H_2(\varphi_2) = \frac{1}{2L_2}\varphi_2^2 \qquad H_3(Q) = \frac{1}{2C}Q^2$$

Furthermore, let $V = u$ denote a voltage source. Using Kirchhoff's laws, one immediately arrives at the input-state-output port-Hamiltonian system

$$\begin{bmatrix} \dot{Q} \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{J} \begin{bmatrix} \frac{\partial H}{\partial Q} \\ \frac{\partial H}{\partial \varphi_1} \\ \frac{\partial H}{\partial \varphi_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \frac{\partial H}{\partial \varphi_1} \qquad (= \text{current through first inductor})$$

with $H(Q,\varphi_1,\varphi_2) := H_1(\varphi_1) + H_2(\varphi_2) + H_3(Q)$ the total energy. Clearly the matrix J is skew-symmetric. In [145] it has been shown that, in this way, every LC-circuit with independent storage elements can be modelled as an input-state-output port-Hamiltonian system (with respect to a constant Dirac structure).

2.2.4 Input-state-output port-Hamiltonian systems with direct feed-through

Input-state-output port-Hamiltonian systems with *feed-through terms*, i.e., direct input-to-output coupling, are given as (for simplicity we do not take the interaction port into account) [74, 180]

$$\begin{cases} \dot{x} = [J(x) - R(x)] \dfrac{\partial H}{\partial x}(x) + [g(x) - P(x)]u \\ y = [g(x) + P(x)]^{\mathrm{T}} \dfrac{\partial H}{\partial x}(x) + [M(x) + S(x)]u \end{cases} \tag{2.53}$$

with the matrices P, R and S satisfying

$$Z = \begin{bmatrix} R(x) & P(x) \\ P^{\mathrm{T}}(x) & S(x) \end{bmatrix} \geq 0 \tag{2.54}$$

Compared with the skew-symmetric map (2.52) we see that in this case we have

$$\begin{bmatrix} f_x \\ y \end{bmatrix} = \begin{bmatrix} -J(x) & -g(x) \\ g^{\mathrm{T}}(x) & M \end{bmatrix} \begin{bmatrix} e_x \\ u \end{bmatrix} + \begin{bmatrix} R(x) & P(x) \\ P^{\mathrm{T}}(x) & S(x) \end{bmatrix} \begin{bmatrix} e_x \\ u \end{bmatrix} \tag{2.55}$$

It follows that

$$e_x^{\mathrm{T}} f_x + u^{\mathrm{T}} y = \begin{bmatrix} e_x^{\mathrm{T}} & u^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} R(x) & P(x) \\ P^{\mathrm{T}}(x) & S(x) \end{bmatrix} \begin{bmatrix} e_x \\ u \end{bmatrix} \geq 0$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t} H(x) = -e_x^{\mathrm{T}} f_x = u^{\mathrm{T}} y - \begin{bmatrix} e_x^{\mathrm{T}} & u^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} R(x) & P(x) \\ P^{\mathrm{T}}(x) & S(x) \end{bmatrix} \begin{bmatrix} e_x \\ u \end{bmatrix} \leq u^{\mathrm{T}} y$$

thus recovering the basic energy balance for port-Hamiltonian systems.

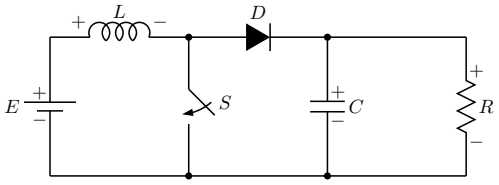

Fig. 2.5 Boost circuit with clamping diode

Port-Hamiltonian input-state-output systems with feed-through terms readily show up in the modeling of power converters [74], as well as in friction models (see e.g. [104] for a port-Hamiltonian description of the dynamic *LuGre* friction model).

2.2.5 Port-Hamiltonian systems with variable topology

In a number of cases, it is useful to model fast transitions in physical systems as *instantaneous switches*. Examples include the description of switching elements, like diodes and thyristors in electrical circuits, and impacts in mechanical systems. Within the port-Hamiltonian description, one obtains in all these cases an (idealized) model where the Dirac structure depends on the position of the switches, but, on the other hand, the Hamiltonian H and the resistive elements are *independent* of the position of the switches.

In both examples below, we obtain a *switching* port-Hamiltonian system, specified by a Dirac structure $\mathscr{D}_s$ depending on the switch position $s \in \{0,1\}^n$ (here n denotes the number of independent switches), a Hamiltonian $H : \mathscr{X} \to \mathbb{R}$, and a resistive structure $\mathscr{R}$. Furthermore, every switching may be internally induced (like in the case of a diode in an electrical circuit or an impact in a mechanical system) or externally triggered (like an active switch in a circuit or mechanical system).

Example 2.6 (Boost converter). Consider the power converter in Fig. 2.5. The circuit consists of an inductor L with magnetic flux linkage ϕ_L, a capacitor C with electric charge q_C and a resistance load R, together with a diode D and an ideal switch S, with switch positions $s = 1$ (switch closed) and $s = 0$ (switch open). The diode is modeled as an ideal diode with voltage-current characteristic $v_D i_D = 0$, with $v_D \leq 0$ and $i_D \geq 0$. The circuit is used to obtain a voltage at the resistance load (the output voltage) that is higher than the voltage E of the input source (a *step-up* converter).

Taking as continuous state (energy) variables the electric charge q_C and the magnetic flux linkage ϕ_L, and as stored energy the quadratic function $\frac{1}{2C}q_C^2 + \frac{1}{2L}\phi_L^2$, we obtain the following port-Hamiltonian model of the circuit:

$$\begin{bmatrix} \dot{q}_C \\ \dot{\phi}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{R} & 1-s \\ s-1 & 0 \end{bmatrix} \begin{bmatrix} \frac{q_C}{C} \\ \frac{\phi_L}{L} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} E + \begin{bmatrix} s \cdot i_D \\ (s-1) \cdot v_D \end{bmatrix}$$

$$I = \frac{\phi_L}{L}$$

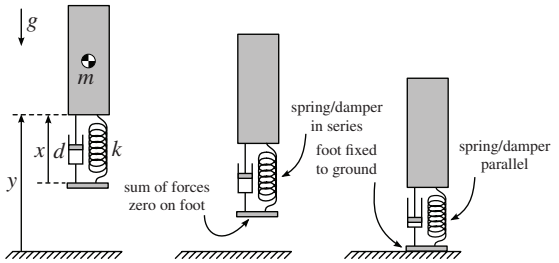

Fig. 2.6 Model of a bouncing pogo-stick: definition of the variables (left), situation without ground contact (middle), and situation with ground contact (right).

Here $s \in \{0,1\}$ denotes the switch, E and I are the voltage, respectively current, of the input source, and i_D and v_D are respectively the current through and the voltage across the ideal diode.

Example 2.7 (Bouncing pogo-stick). Consider the example of the vertically bouncing pogo-stick in Fig. 2.6: it consists of a mass m and a mass-less foot, interconnected by a linear spring (stiffness k and rest-length x_0) and a linear damper d. The mass can move vertically under the influence of gravity g until the foot touches the ground. The states of the system are taken as x (length of the spring), y (height of the bottom of the mass), and p (momentum of the mass, defined as $p = m\dot{y}$). Furthermore, the contact situation is described by a variable s with values $s = 0$ (no contact) and $s = 1$ (contact). The total energy (Hamiltonian) of the system equals

$$H(x,y,p) = \frac{1}{2}k(x-x_0)^2 + mg(y+y_0) + \frac{1}{2m}p^2 \tag{2.56}$$

where y_0 is the distance from the bottom of the mass to its center of mass.

When the foot is not in contact with the ground (middle figure), the total force on the foot is zero (since it is mass-less), which implies that the spring and damper forces must be equal but opposite. When the foot is in contact with the ground (right figure), the variables x and y remain equal, and hence also $\dot{x} = \dot{y}$. For $s = 0$ (no contact) the system can be described by the port-Hamiltonian system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \\ p \end{bmatrix} = \begin{bmatrix} -\frac{1}{d} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} k(x-x_0) \\ mg \\ \frac{p}{m} \end{bmatrix} \tag{2.57}$$

i.e. two independent systems (spring plus damper, and mass plus gravity), while for $s = 1$, the port-Hamiltonian description of the system is given as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \\ p \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & -d \end{bmatrix} \begin{bmatrix} k(x-x_0) \\ mg \\ \frac{p}{m} \end{bmatrix} \tag{2.58}$$

In this last case the resistive force $-d\dot{x}$ is added to the spring force and the gravitational force exerted on the mass, while for $s = 0$ the resistive force is equal to the spring force.

The two situations can be taken together into one port-Hamiltonian system with variable Dirac structure as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \\ p \end{bmatrix} = \begin{bmatrix} \frac{s-1}{d} & 0 & s \\ 0 & 0 & 1 \\ -s & -1 & -s\cdot d \end{bmatrix} \begin{bmatrix} k(x-x_0) \\ mg \\ \frac{p}{m} \end{bmatrix} \tag{2.59}$$

In addition, the conditions for switching of the contact are functions of the states, namely as follows: contact is switched from off to on when $y-x$ crosses zero in the negative direction, and contact is switched from on to off when the velocity $\dot{y}-\dot{x}$ of the foot is positive in the no-contact situation, i.e. when $\frac{p}{m}+\frac{k}{d}(x-x_0)>0$.

2.3 Relationships with classical Hamiltonian and Euler-Lagrange equations

Historically, the Hamiltonian approach has its roots in analytical mechanics and starts from the principle of least action, via the Euler-Lagrange equations and the Legendre transformation, towards the Hamiltonian equations of motion. On the other hand, the network approach stems from electrical engineering, and constitutes a cornerstone of systems theory. While much of the *analysis* of physical systems has been performed within the Lagrangian and Hamiltonian framework, the network modelling point of view is prevailing in *modelling* and *simulation* of (complex) physical systems. The framework of port-Hamiltonian systems *combines* both points of view, by associating with the interconnection structure (generalized junction structure in bond graph terminology) of the network model a geometric structure given by a *Dirac structure*. This is in contrast with the classical Hamiltonian equations of motion where the geometric structure is basically determined by the geometry of the phase space given as the cotangent bundle of the configuration manifold.

In the first subsection we briefly describe the classical framework of Lagrangian and Hamiltonian differential equations as originating from analytical mechanics, and indicate how it naturally extends to port-Hamiltonian systems. Conversely, in the second subsection we discuss how, starting from the port-Hamiltonian description, Legendre transformations may be useful in the description and analysis of the system.

2.3.1 From Euler-Lagrange equations to port-Hamiltonian systems

The standard *Euler-Lagrange equations* are given as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) - \frac{\partial L}{\partial q}(q,\dot{q}) = \tau, \tag{2.60}$$

where $q = (q_1, \ldots, q_k)^{\mathrm{T}}$ are generalized configuration coordinates for the system with k degrees of freedom, the Lagrangian L equals the *difference* $K - P$ between kinetic co-energy $K(q,\dot{q})$ and potential energy $P(q)$, and $\tau = (\tau_1, \ldots, \tau_k)^{\mathrm{T}}$ is the vector of generalized forces acting on the system. Furthermore, $\frac{\partial L}{\partial \dot{q}}$ denotes the column-vector of partial derivatives of $L(q,\dot{q})$ with respect to the generalized velocities $\dot{q}_1, \ldots, \dot{q}_k$, and similarly for $\frac{\partial L}{\partial q}$. In standard mechanical systems the kinetic co-energy K is of the form

$$K(q,\dot{q}) = \frac{1}{2}\dot{q}^{\mathrm{T}} M(q)\dot{q} \tag{2.61}$$

where the $k \times k$ inertia (generalized mass) matrix $M(q)$ is symmetric and positive definite for all q. In this case the vector of generalized *momenta* $p = (p_1, \ldots, p_k)^{\mathrm{T}}$, defined for any Lagrangian L as $p = \frac{\partial L}{\partial \dot{q}}$, is simply given by

$$p = M(q)\dot{q}, \tag{2.62}$$

and by defining the state vector $(q_1, \ldots, q_k, p_1, \ldots, p_k)^{\mathrm{T}}$ the k second-order equations (2.60) transform into $2k$ first-order equations

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) \qquad \left(= M^{-1}(q)p\right) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + \tau \end{aligned} \tag{2.63}$$

where

$$H(q,p) = \frac{1}{2}p^{\mathrm{T}} M^{-1}(q)p + P(q) \quad \left(= \frac{1}{2}\dot{q}^{\mathrm{T}} M(q)\dot{q} + P(q)\right) \tag{2.64}$$

is the total energy of the system[4]. The equations (2.63) are called the *Hamiltonian equations* of motion, and H is called the *Hamiltonian*. The state space of (2.64) with local coordinates (q,p) is usually called the *phase space*.

The following *energy balance* immediately follows from (2.63):

$$\frac{\mathrm{d}}{\mathrm{d}t}H = \frac{\partial^{\mathrm{T}} H}{\partial q}(q,p)\dot{q} + \frac{\partial^{\mathrm{T}} H}{\partial p}(q,p)\dot{p} = \frac{\partial^{\mathrm{T}} H}{\partial p}(q,p)\tau \qquad \left(= \dot{q}^{\mathrm{T}}\tau\right), \tag{2.65}$$

expressing that the increase in energy of the system is equal to the supplied work (*conservation of energy*).

If the Hamiltonian $H(q,p)$ is assumed to be the sum of a positive kinetic energy and a potential energy which is *bounded from below*, that is

$$H(q,p) = \frac{1}{2}p^{\mathrm{T}} M^{-1}(q)p + P(q)$$

[4] Note that, because of the fact that the kinetic energy is a *quadratic* function of the momenta p, it equals the kinetic co-energy $K(q,\dot{q})$, cf. Sect. B.2.

and

$$M(q) = M^{\mathrm{T}}(q) > 0, \qquad \exists C > -\infty \quad \text{such that } P(q) \geq C$$

then it follows that the system (2.63) with inputs $u := \tau$ and outputs $y := \dot{q}$ is a *passive* (in fact, *loss-less*) state space system with storage function $H(q,p) - C \geq 0$[5]. Since energy is defined up to a constant, we may as well take as potential energy the function $P(q) - C \geq 0$, in which case the total energy $H(q,p)$ becomes non-negative and thus itself is a storage function.

System (2.63) is an example of a Hamiltonian system with collocated inputs and outputs, which more generally is given in the following form

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + B(q)u \\ y &= B^{\mathrm{T}}(q)\frac{\partial H}{\partial p}(q,p) \end{aligned} \tag{2.66}$$

with $u, y \in \mathbb{R}^m$. Here $B(q)$ is the input force matrix, with $B(q)u$ denoting the generalized forces resulting from the control inputs u. In case $m < k$ we speak of an *under-actuated* system. If $m = k$ and the matrix $B(q)$ is invertible for all q, then the system is fully actuated.

By definition of the output $y = B^{\mathrm{T}}(q)\dot{q}$, we again obtain the energy balance

$$\frac{\mathrm{d}H}{\mathrm{d}t}(q(t),p(t)) = u^{\mathrm{T}}(t)y(t) \tag{2.67}$$

For a system-theoretic treatment of the Hamiltonian systems (2.66), especially if the output y can be written as the time-derivative of a vector of generalized configuration coordinates, we refer to e.g. [39, 55, 156, 177, 178].

A major generalization of the class of Hamiltonian systems (2.66) consists in considering systems which are described in local coordinates as

$$\begin{cases} \dot{x} = J(x)\dfrac{\partial H}{\partial x}(x) + g(x)u \\ y = g^{\mathrm{T}}(x)\dfrac{\partial H}{\partial x}(x) \end{cases} \tag{2.68}$$

with $x \in \mathscr{X}$ and $u, y \in \mathbb{R}^m$. Here $J(x)$ is an $n \times n$ matrix with entries depending smoothly on x, which is assumed to be *skew-symmetric*

$$J(x) = -J^{\mathrm{T}}(x), \tag{2.69}$$

[5] 'Loss-less' is a strong form of 'passive'; in the latter case, (2.65) need only be satisfied with the equality sign '=' replaced by the inequality sign '$\leq$'.

and $x = (x_1, \ldots, x_n)$ are local coordinates for an n-dimensional state space manifold $\mathscr{X}$. Because of (2.69), we easily recover the energy-balance $\frac{\mathrm{d}H}{\mathrm{d}t}(x(t)) = u^{\mathrm{T}}(t)y(t)$, showing that (2.68) is loss-less if $H \geq 0$.

The system (2.68) with J satisfying (2.69) is an input-state-output port-Hamiltonian system with Dirac structure determined by $J(x)$ and $g(x)$ and Hamiltonian H. Note that (2.66) (and hence (2.63)) is a particular case of (2.68) with $x = (q, p)$, and

$$J(x) = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix} \qquad\qquad g(q,p) = \begin{bmatrix} 0 \\ B(q) \end{bmatrix}.$$

If the matrix J satisfies the *integrability* conditions, see (2.219)

$$\sum_{l=1}^{n} \left[J_{lj}(x)\frac{\partial J_{ik}}{\partial x_l}(x) + J_{li}(x)\frac{\partial J_{kj}}{\partial x_l}(x) + J_{lk}(x)\frac{\partial J_{ji}}{\partial x_l}(x) \right] = 0, \tag{2.70}$$

$i, j, k = 1, \ldots, n$, then we may find canonical coordinates in which the equations (2.68) take the form

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p,s) + g_q(q,p,s)u \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p,s) + g_p(q,p,s)u \\ \dot{s} &= g_s(q,p,s)u \\ y &= g_q^{\mathrm{T}}(q,p,s)\frac{\partial H}{\partial q}(q,p,s) + g_p^{\mathrm{T}}(q,p,s)\frac{\partial H}{\partial p}(q,p,s) + \\ &\qquad + g_s^{\mathrm{T}}(q,p,s)\frac{\partial H}{\partial s}(q,p,s) \end{aligned} \tag{2.71}$$

Apart from the appearance of the variables s, these equations are very close to the standard Hamiltonian form (2.66). In particular, if $g_s = 0$, then the variables s are merely an additional set of *constant* parameters.

2.3.2 Port-Hamiltonian systems and Legendre transformations

First we will go into more detail about the Legendre transformation as already encountered before, and discussed in more detail in Sect. B.2. A slightly different interpretation is given below. Consider a real-valued function $F(u_1, u_2, \cdots, u_n)$ of n variables $u_1, u_2, \cdots, u_n$. Consider a second set of variables $v_1, v_2, \cdots, v_n$, and link both sets of variables to each other by setting

$$v_i = \frac{\partial F}{\partial u_i}, \qquad i = 1, 2, \cdots, n \tag{2.72}$$

Now define the function $\bar{F}$ as

$$\bar{F}(u_1,\cdots,u_n,v_1,\cdots,v_n) := v_1u_1+\cdots v_nu_n - F(u_1,\cdots,u_n) \tag{2.73}$$

In order to simplify notation we will henceforth write $u=(u_1,u_2,\cdots,u_n), v=(v_1,v_2,\cdots,v_n)$ and $\bar{F}(u,v)=v^{\mathrm{T}}u-F(u)$. It is immediately checked that the partial derivatives of $\bar{F}(u,v)$ with respect to the variables $u_1,\cdots,u_n$

$$\frac{\partial \bar{F}}{\partial u_i}(u,v), \qquad i=1,\cdots,n$$

are all zero whenever these partial derivatives are evaluated at points $\left(u, v=\frac{\partial F}{\partial u}\right)$. Indeed

$$\frac{\partial \bar{F}}{\partial u_i} = v_i - \frac{\partial F}{\partial u_i} \tag{2.74}$$

Furthermore, if the mapping from u to $v=\frac{\partial F}{\partial u}$ is *invertible* then we can express u as a function $u(v)$ of v, and we may define the function

$$F^*(v) := \bar{F}(u(v),v) = v^{\mathrm{T}}u(v) - F(u(v)) \tag{2.75}$$

The function $F^*(v)$ is called the *Legendre transform* of $F(u)$.

Remark 2.5. A sufficient condition for local invertibility of the mapping from u to $v=\frac{\partial F}{\partial u}$ is that the *Hessian* matrix $\frac{\partial^2 F}{\partial u^2}$ is invertible.

Differentiating $F^*(v)$ with respect to v yields

$$\frac{\partial F^*}{\partial v_i} = \sum_{j=1}^{n} \frac{\partial \bar{F}}{\partial u_j}\frac{\partial u_j(v)}{\partial v_i} + \frac{\partial \bar{F}}{\partial v_i} \tag{2.76}$$

where all expressions at the right-hand side are evaluated at $\left(u, v=\frac{\partial F}{\partial u}\right)$. Since by (2.74), the first term at the right-hand side is zero, it follows that

$$\frac{\partial F^*}{\partial v_i}(v) = \frac{\partial \bar{F}}{\partial v_i}(u(v),v)) = u_i \tag{2.77}$$

Thus we have obtained a completely *symmetric* relation between the variables u and v: if v is linked to u via (2.72) in an invertible manner, then conversely u is obtained from v via (2.77).

This construction is immediately extended to functions $F(u,w)$ depending on additional variables $w=(w_1,\cdots,w_m)$. Linking in the same way the variables $v=(v_1,\cdots,v_n)$ to $u=(u_1,\cdots,u_n)$ via

$$v_i = \frac{\partial F}{\partial u_i}(u,w), \qquad i=1,2,\cdots,n, \tag{2.78}$$

and defining the function $\bar{F}(u,v,w)=v^{\mathrm{T}}u-F(u,w)$ one obtains the Legendre transform of $F(u,w)$ with respect to u as the function $F^*(v,w)$ defined as

$$F^*(v,w) := v^{\mathrm{T}}u(v,w) - F(u(v,w),w) \tag{2.79}$$

under the assumption that the map $u \mapsto v$ defined by (2.78) is invertible and can be solved as $u = u(v,w)$. It follows as before that this inverse mapping is determined by

$$\frac{\partial F^*}{\partial v_i}(v,w) = \frac{\partial \bar{F}}{\partial v_i}(u(v,w),v,w) = u_i \tag{2.80}$$

Furthermore, with regard to the partial derivatives with respect to the additional variables w, we obtain the following relationship between $F(u,w)$ and $F^*(v,w)$:

$$\frac{\partial F}{\partial w_i}(u,w) = -\frac{\partial F^*}{\partial w_i}(v(u),w), \qquad i = 1,\cdots,n \tag{2.81}$$

A classical application of the Legendre transformation is the following. Consider a classical Hamiltonian system

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \end{cases} \qquad i = 1,\cdots,n \tag{2.82}$$

and suppose that the mapping

$$v_i = \frac{\partial H}{\partial p_i}, \qquad i = 1,\cdots,n$$

from $p = (p_1,\cdots,p_n)$ to $v = (v_1,\cdots,v_n)$ is invertible, leading to the Legendre transform of $H(q,p)$ with respect to p denoted as $H^*(q,v)$. It follows that

$$\begin{aligned} p_i &= \frac{\partial H^*}{\partial v_i} \\ \frac{\partial H}{\partial q_i} &= -\frac{\partial H^*}{\partial q_i} \end{aligned} \qquad i = 1,\cdots,n \tag{2.83}$$

and thus the Hamiltonian system (2.82) transforms into

$$\begin{cases} \dot{q}_i = v_i \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial H^*}{\partial v_i} = \dfrac{\partial H^*}{\partial q_i} \end{cases} \qquad i = 1,\cdots,n \tag{2.84}$$

Denoting $L(q,v) := H^*(q,v)$ (the *Lagrangian* function) the equations (2.84) can be taken together into the classical Euler-Lagrange equations

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v_i}(q,v) - \frac{\partial L}{\partial q_i}(q,v)\right)\Bigg|_{v=\dot{q}} = 0, \qquad i = 1,\cdots,n$$

2.3.2.1 From port-Hamiltonian systems to the Brayton-Moser equations

Consider a port-Hamiltonian system without dissipation and external ports, which can be represented by the set of differential equations

$$\dot{x} = J(x)\frac{\partial H}{\partial x}(x) \tag{2.85}$$

with $J(x)$ being skew-symmetric. Suppose that the mapping from the energy variables x to the co-energy variables $e := \frac{\partial H}{\partial x}(x)$ is invertible, so that the inverse transformation from the co-energy variables to the energy variables is given by

$$x = \frac{\partial H^*}{\partial e}(e) \tag{2.86}$$

with H^* the Legendre transformation of H given as

$$H^*(e) = e^{\mathrm{T}}x - H(x) \tag{2.87}$$

Then the dynamics of the port-Hamiltonian system (2.85) can be equally well expressed in the co-energy variables e. Indeed, the time-evolution of e may be obtained from (2.85) by substituting $x(t) = \frac{\partial H^*}{\partial e}(e(t))$ into the differential equation (2.85), leading to

$$\frac{\partial^2 H^*}{\partial e^2}(e)\dot{e} = J(x)e, \tag{2.88}$$

where one may finally substitute $x = \frac{\partial H^*}{\partial e}$ in order to obtain a differential equation solely in the co-energy variables e.

What can we say about the particular structure of the port-Hamiltonian system expressed in the co-energy variables e? *Assume* that we may find coordinates $x = (x_q, x_p)$, with $\dim x_q = k$ and $\dim x_p = n-k$, such that in these coordinates the matrix $J(x)$ takes the form

$$J(x) = \begin{bmatrix} 0 & -B(x) \\ B^{\mathrm{T}}(x) & 0 \end{bmatrix} \tag{2.89}$$

with $B(x)$ a $k \times (n-k)$ matrix, and moreover the Hamiltonian H splits into a function of x_q and x_p, that is, H can be written as

$$H(x_q, x_p) = H_q(x_q) + H_p(x_p) \tag{2.90}$$

for certain functions H_q and H_p. Write, accordingly to the splitting $x = (x_q, x_p)$ of the energy variables, the co-energy variables as $e = (e_q, e_p)$ with

$$e_q = \frac{\partial H_q}{\partial x_q} \qquad\qquad e_p = \frac{\partial H_p}{\partial x_p}$$

It follows that the Legendre transform $H^*(e)$ of $H(x)$ splits as $H^*(e) = H_q^*(e_q) + H_p^*(e_p)$. In such coordinates (2.88) takes the form

$$\begin{bmatrix} \frac{\partial^2 H_q^*}{\partial e_q^2} & 0 \\ 0 & \frac{\partial^2 H_p^*}{\partial e_p^2} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}e_q}{\mathrm{d}t} \\ \frac{\mathrm{d}e_p}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} 0 & -B(x) \\ B^{\mathrm{T}}(x) & 0 \end{bmatrix} \begin{bmatrix} e_q \\ e_p \end{bmatrix} \tag{2.91}$$

Defining the function

$$P(e_q, e_p, x) := e_q^{\mathrm{T}} B(x) e_p \tag{2.92}$$

it follows (after multiplication of the last $n-k$ equations in (2.91) by a factor -1) that (2.91) can be alternatively written as

$$\begin{bmatrix} \frac{\partial^2 H_q^*}{\partial e_q^2} & 0 \\ 0 & -\frac{\partial^2 H_p^*}{\partial e_p^2} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}e_q}{\mathrm{d}t} \\ \frac{\mathrm{d}e_p}{\mathrm{d}t} \end{bmatrix} = - \begin{bmatrix} \frac{\partial P}{\partial e_q} \\ \frac{\partial P}{\partial e_p} \end{bmatrix} \tag{2.93}$$

This is the type of equations that were obtained for RLC-circuits in [27, 28], and which are commonly called the *Brayton-Moser equations* (cf. Example 2.8 below). These equations can be interpreted as *gradient equations* with respect to the *mixed potential function P* and the indefinite inner product or pseudo-*Riemannian metric* defined by the symmetric matrix

$$\begin{bmatrix} \frac{\partial^2 H_q^*}{\partial e_q^2} & 0 \\ 0 & -\frac{\partial^2 H_p^*}{\partial e_p^2} \end{bmatrix} \tag{2.94}$$

Note however that if the matrix B and therefore the function P *non-trivially depends* on x then (2.93) is *not* valid if we substitute $x = \frac{\partial H^*}{\partial e}(e)$ in the definition of $P(e_q, e_p, x)$ *before* taking the partial derivatives of P with respect to e_q and e_p.

Remark 2.6. If the Hamiltonian H does *not* split as $H(x_p, x_q) = H_p(x_p) + H_q(x_q)$ then we obtain instead of (2.93) the more general type of equations

$$\begin{bmatrix} \frac{\partial^2 H^*}{\partial e_q^2} & \frac{\partial^2 H^*}{\partial e_q \partial e_p} \\ -\frac{\partial^2 H^*}{\partial e_p \partial e_q} & -\frac{\partial^2 H^*}{\partial e_p^2} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}e_q}{\mathrm{d}t} \\ \frac{\mathrm{d}e_p}{\mathrm{d}t} \end{bmatrix} = - \begin{bmatrix} \frac{\partial P}{\partial e_q} \\ \frac{\partial P}{\partial e_p} \end{bmatrix} \tag{2.95}$$

Hence in this case the left-most matrix appearing in (2.95) is not symmetric anymore, and hence does not define a (pseudo-) Riemannian metric.

By (2.85) and (2.89), the mixed potential function P can be rewritten as

$$P(e_q, e_p, x) = e_q^{\mathrm{T}} B(x) e_p = -e_q^{\mathrm{T}} \dot{x}_q = \dot{x}_p^{\mathrm{T}} e_p \tag{2.96}$$

Thus the function P denotes minus the power associated to the q-part of the system, which is (since the total power is zero) also equal to the power of the p-part of the system. Hence the mixed potential function P captures the interconnection structure (or Dirac structure) of the port-Hamiltonian system, while the pseudo-Riemannian metric (2.94) is determined by its Hamiltonian.

Example 2.8. Consider an LC-circuit with independent capacitors C and independent inductors L. Denoting the voltages and currents of the capacitors by v_C and i_C and the voltages and currents of the inductors by v_L and i_L, Kirchhoff's current and voltage laws can be written as

$$\begin{bmatrix} i_C \\ v_L \end{bmatrix} = \begin{bmatrix} 0 & -B \\ B^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} \tag{2.97}$$

for some *constant* matrix B (consisting of elements $0, 1$, and -1). Furthermore, the Hamiltonian (total energy) splits as the sum of an electrical and magnetic energy

$$H(Q_C, \varphi_L) = H_C(Q_C) + H_L(\varphi_L)$$

Thus, the above assumptions for deriving the Brayton-Moser equations are automatically satisfied, leading to the standard Brayton-Moser equations (in the case of no energy dissipation)

$$\begin{bmatrix} \frac{\partial^2 H_C^*}{\partial v_C^2} & 0 \\ 0 & -\frac{\partial^2 H_L^*}{\partial i_L^2} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}v_C}{\mathrm{d}t} \\ \frac{\mathrm{d}i_L}{\mathrm{d}t} \end{bmatrix} = - \begin{bmatrix} \frac{\partial P}{\partial v_C} \\ \frac{\partial P}{\partial i_L} \end{bmatrix} \tag{2.98}$$

with $P(v_C, i_L) = v_C^{\mathrm{T}} B i_L = -v_C^{\mathrm{T}} i_C = v_L^{\mathrm{T}} i_L$.

Example 2.9 (Example 2.14, continued). Consider the example of the rolling euro. In (2.148), the explicit dynamical equations of the system on the constrained state space has been obtained. Denoting $x_q = (x, y, \theta, \varphi)$ and $x_p = (p_1, p_2)$ it is clear that the above assumptions for the derivation of the Brayton-Moser equations are satisfied, with

$$B = \begin{bmatrix} 0 & \cos\varphi \\ 0 & \sin\varphi \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.99}$$

and

$$H_q(x_q) = 0 \qquad\qquad H_p(x_p) = \frac{1}{2} p_1^2 + \frac{1}{4} p_2^2 \tag{2.100}$$

However, in this case the mapping $e = \frac{\partial H}{\partial x}$ is not invertible since the potential energy $H_q = 0$, and thus we cannot derive Brayton-Moser equations for the rolling euro. On the other hand, let us modify the example by adding a potential energy H_q in such a way that the mapping

$$e_x = \frac{\partial H_q}{\partial x} \qquad e_y = \frac{\partial H_q}{\partial y} \qquad e_\theta = \frac{\partial H_q}{\partial \theta} \qquad e_\varphi = \frac{\partial H_q}{\partial \varphi} \tag{2.101}$$

is an invertible mapping from (x,y,θ,φ) to $(e_x,e_y,e_\theta,e_\varphi)$, allowing for the definition of the Legendre transform $H_q^*(e_x,e_y,e_\theta,e_\varphi)$[6]. Then the Brayton-Moser equations are given by (2.93) with $B(\varphi)$ as above, and $H_p^*(e_{p_1},e_{p_2}) = \frac{1}{2}e_{p_1}^2 + e_{p_2}^2$.

The Brayton-Moser equations can be extended to the case of energy-dissipation (and perhaps is most useful in this context). The idea is to modify the mixed potential function P (covering until now the power-conserving interconnection structure) with Rayleigh functions corresponding to the resistive elements. Let us consider a port-Hamiltonian system *with* dissipation (but still without external ports):

$$\begin{cases} \dot{x} = J(x)\dfrac{\partial H}{\partial x}(x) + g_R(x)f_R \\ e_R = g_R^{\mathrm{T}}(x)\dfrac{\partial H}{\partial x}(x) \end{cases} \tag{2.102}$$

where the resistive relation is specified by an (effort-controlled) Rayleigh dissipation function $R(e_R)$, that is,

$$f_R = -\frac{\partial R}{\partial e_R}(e_R) \tag{2.103}$$

Suppose as before that we may find coordinates $x = (x_q,x_p)$, $\dim x_q = k$, $\dim x_p = n-k$ such that, in these coordinates, the matrix $J(x)$ takes the form

$$J(x) = \begin{bmatrix} 0 & -B(x) \\ B^{\mathrm{T}}(x) & 0 \end{bmatrix} \tag{2.104}$$

with $B(x)$ a $k \times (n-k)$ matrix, and moreover, that the Hamiltonian H splits as $H(x_q,x_p) = H_q(x_q) + H_p(x_p)$. Furthermore, assume for simplicity that, in these coordinates, $g_R(x)$ has the form

$$g_R(x_q,x_p) = \begin{bmatrix} 0 \\ -I \end{bmatrix} \tag{2.105}$$

This implies that $e_R = e_p$. Then (2.91) extends to

$$\begin{bmatrix} \frac{\partial^2 H_q^*}{\partial e_q^2} & 0 \\ 0 & \frac{\partial^2 H_p^*}{\partial e_p^2} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}e_q}{\mathrm{d}t} \\ \frac{\mathrm{d}e_p}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} 0 & -B(x) \\ B^{\mathrm{T}}(x) & 0 \end{bmatrix} \begin{bmatrix} e_q \\ e_p \end{bmatrix} - \begin{bmatrix} 0 \\ -I \end{bmatrix} \frac{\partial R}{\partial e_p}(e_p) \tag{2.106}$$

Defining now the *modified mixed potential function*

$$P(e_q,e_p,x) = e_q^{\mathrm{T}} B(x) e_p + R(e_p) \tag{2.107}$$

then the system (2.106) still takes the Brayton-Moser form (2.88) with respect to this modified mixed potential function. Note that this new mixed potential function

[6] For example, one may think of a rolling disc on a curved surface with spring-like elements corresponding to the motion in the θ and φ directions.

captures the interconnection structure as well as the specification of the resistive port relation (by means of a Rayleigh dissipation function).

2.4 Representations of Dirac structures and port-Hamiltonian systems

In the preceding section, we have provided the geometric definition of a port-Hamiltonian system containing three main ingredients. First, the energy storage which is represented by a state space manifold $\mathscr{X}$ specifying the space of energy variables together with a Hamiltonian $H : \mathscr{X} \to \mathbb{R}$ defining the energy. Secondly, there are the static resistive elements, and thirdly there is the Dirac structure linking all the flows and efforts associated to the energy storage, resistive elements, and the external ports (e.g. control and interaction ports) in a power-conserving manner. This, together with the general formulation (Definition 2.2) of a Dirac structure, leads to a completely *coordinate-free* definition of a port-Hamiltonian system, because of three reasons: *(a)* we do not start with coordinates for the state space manifold $\mathscr{X}$, *(b)* we define the Dirac structure as a *subspace* instead of a set of equations, *(c)* the resistive relations are defined as a subspace constraining the port variables (f_R, e_R).

This geometric, coordinate-free, point of view has a number of advantages. It allows one to reason about port-Hamiltonian systems without the need to choose specific representations. For example, in Sect. 2.6 we will see that a number of properties of the port-Hamiltonian system, such as passivity, existence of conserved quantities and algebraic constraints, can be analyzed without the need for choosing coordinates and equations. On the other hand, for many purposes, e.g. simulation, the need for a representation in coordinates of the port-Hamiltonian system is indispensable. Then the emphasis shifts to finding the most convenient coordinate representation for the purpose at hand. The examples of the previous section have already been presented in this way. In this section, we will briefly discuss a number of possible representations of port-Hamiltonian systems. It will turn out that the key issue is the representation of the Dirac structure.

2.4.1 Representations of Dirac structures

Dirac structures admit different *representations*. Here we list the most important ones, with proofs provided only for the first two cases. Further information can be found in [25, 54, 59, 87, 179].

Kernel and Image representation

Every Dirac structure $\mathscr{D} \subset \mathscr{F} \times \mathscr{F}^*$ can be represented in *kernel representation* as

$$\mathscr{D} = \left\{ (f,e) \in \mathscr{F} \times \mathscr{F}^* \mid Ff + Ee = 0 \right\} \tag{2.108}$$

for linear maps $F : \mathscr{F} \to \mathscr{V}$ and $E : \mathscr{F}^* \to \mathscr{V}$ satisfying

$$\begin{aligned} &(i) \quad EF^* + FE^* = 0, \\ &(ii) \quad \operatorname{rank}(F+E) = \dim \mathscr{F}, \end{aligned} \tag{2.109}$$

where $\mathscr{V}$ is a linear space with the same dimension as $\mathscr{F}$, and where $F^* : \mathscr{V}^* \to \mathscr{F}^*$ and $E^* : \mathscr{V}^* \to \mathscr{F}^{**} = \mathscr{F}$ are the adjoint maps of F and E, respectively.

It follows from (2.109) that $\mathscr{D}$ can be also written in *image representation* as

$$\mathscr{D} = \left\{ (f,e) \in \mathscr{F} \times \mathscr{F}^* \mid f = E^*\lambda,\ e = F^*\lambda,\ \lambda \in \mathscr{V}^* \right\} \tag{2.110}$$

Sometimes it will be useful to relax this choice of the linear mappings F and E by allowing $\mathscr{V}$ to be a linear space of dimension greater than the dimension of $\mathscr{F}$. In this case we shall speak of *relaxed* kernel and image representations.

Matrix kernel and image representations are obtained by choosing linear coordinates for $\mathscr{F}$, $\mathscr{F}^*$ and $\mathscr{V}$. Indeed, take any basis $f_1, \cdots, f_n$ for $\mathscr{F}$ and the *dual basis* $e_1 = f_1^*, \cdots, e_n = f_n^*$ for $\mathscr{F}^*$, where $\dim \mathscr{F} = n$. Furthermore, take any set of linear coordinates for $\mathscr{V}$. Then the linear maps F and E are represented by $n \times n$ matrices F and E satisfying

$$\begin{aligned} &(i) \quad EF^{\mathrm{T}} + FE^{\mathrm{T}} = 0, \\ &(ii) \quad \operatorname{rank} \begin{bmatrix} F & E \end{bmatrix} = \dim \mathscr{F}. \end{aligned} \tag{2.111}$$

In the case of a relaxed kernel and image representation F and E will be $n' \times n$ matrices with $n' > n$.

A (constructive) proof for the existence of matrix kernel and image representations can be given as follows. Consider a Dirac structure $\mathscr{D} \subset \mathscr{F} \times \mathscr{F}^*$ where we have chosen linear coordinates for $\mathscr{F}$, $\mathscr{F}^*$ and $\mathscr{V}$. In particular, choose any basis $f_1, \cdots, f_n$ for $\mathscr{F}$ and the *dual basis* $e_1 = f_1^*, \cdots, e_n = f_n^*$ for $\mathscr{F}^*$, where $\dim \mathscr{F} = n$. Since $\mathscr{D}$ is a subspace of $\mathscr{F} \times \mathscr{F}^*$ it follows that there exist square $n \times n$ matrices F and E such that

$$\mathscr{D} = \operatorname{Im} \begin{bmatrix} E^{\mathrm{T}} \\ F^{\mathrm{T}} \end{bmatrix}$$

where $\operatorname{rank} \begin{bmatrix} F & E \end{bmatrix} = \dim \mathscr{F}$. Thus any element $(f,e) \in \mathscr{D}$ can be written as

$$f = E^{\mathrm{T}}\lambda \qquad\qquad e = F^{\mathrm{T}}\lambda$$

for some $\lambda \in \mathbb{R}^n$. Since $e^{\mathrm{T}} f = 0$ for every $(f,e) \in \mathscr{D}$ this implies that

$$\lambda^{\mathrm{T}} F E^{\mathrm{T}} \lambda = 0$$

for every λ, or equivalently, $EF^{\mathrm{T}} + FE^{\mathrm{T}} = 0$. Conversely, any subspace $\mathscr{D}$ given by (2.111) is a Dirac structure, since it satisfies $e^{\mathrm{T}} f = 0$ for every $(f,e) \in \mathscr{D}$ and its dimension is equal to n.

Constrained input-output representation

Every Dirac structure $\mathscr{D} \subset \mathscr{F} \times \mathscr{F}^*$ can be represented as

$$\mathscr{D} = \Big\{ (f,e) \in \mathscr{F} \times \mathscr{F}^* \mid f = Je + G\lambda,\ G^{\mathrm{T}} e = 0 \Big\}, \tag{2.112}$$

for a skew-symmetric mapping $J : \mathscr{F} \to \mathscr{F}^*$ and a linear mapping G such that $\operatorname{Im} G = \big\{ f \mid (f,0) \in \mathscr{D} \big\}$. Furthermore, $\ker J = \big\{ e \mid (0,e) \in \mathscr{D} \big\}$.

The proof that (2.112) defines a Dirac structure is straightforward. Indeed, for any (f,e) given as in (2.112) we have

$$e^{\mathrm{T}} f = e^{\mathrm{T}}(Je + G\lambda) = e^{\mathrm{T}} Je + e^{\mathrm{T}} G\lambda = 0$$

by skew-symmetry of J and $G^{\mathrm{T}} e = 0$. Furthermore, let rank $G = r \leq n$. If $r = 0$ (or equivalently $G = 0$) then the dimension of $\mathscr{D}$ is clearly n since in that case it is the graph of the mapping J. For $r \neq 0$ the freedom in e will be reduced by dimension r, while at the other hand the freedom in f will be increased by dimension r (because of the term $G\lambda$).

Conversely, let $\mathscr{D} \subset \mathscr{F} \times \mathscr{F}^*$ be a Dirac structure. Define the subspace

$$\mathscr{F}^*_{\mathscr{D}} = \Big\{ e \in \mathscr{F}^* \ \mid\ \exists f \text{ s.t. } (f,e) \in \mathscr{D} \Big\}$$

Define a matrix G such that $\mathscr{F}^*_{\mathscr{D}} = \ker G^{\mathrm{T}}$. Now choose any subspace $\bar{\mathscr{F}}^*_{\mathscr{D}}$ which is complementary to $\mathscr{F}^*_{\mathscr{D}}$, that is,

$$\mathscr{F}^* = \mathscr{F}^*_{\mathscr{D}} \oplus \bar{\mathscr{F}}^*_{\mathscr{D}}$$

Define the following subspaces of $\mathscr{F}$:

$$\bar{\mathscr{F}}_{\mathscr{D}} := (\mathscr{F}^*_{\mathscr{D}})^{\perp} \qquad\qquad \mathscr{F}_{\mathscr{D}} := (\bar{\mathscr{F}}^*_{\mathscr{D}})^{\perp}$$

Then, $\mathscr{F} = \mathscr{F}_{\mathscr{D}} \oplus \bar{\mathscr{F}}_{\mathscr{D}}$. Define now $J : \mathscr{F}^* \to \mathscr{F}$ as follows:

1. Define J to be zero on $\bar{\mathscr{F}}^*_{\mathscr{D}}$.
2. For every $e \in \mathscr{F}^*_{\mathscr{D}}$ there exists $f \in \mathscr{F}_{\mathscr{D}}$ with $(f,e) \in \mathscr{D}$. Define $Je = f$. Do this for a basis of $\mathscr{F}^*_{\mathscr{D}}$ and extend J to a linear map from $\mathscr{F}^*_{\mathscr{D}}$ to $\mathscr{F}_{\mathscr{D}}$.

Since $(f,e) \in \mathscr{D}$ satisfies $e^{\mathrm{T}} f = 0$ it follows that $e^{\mathrm{T}} Je = 0$ for all $e \in \mathscr{F}^*$, and thus J is skew-symmetric.

Hybrid input-output representation

Let $\mathscr{D}$ be given in matrix kernel representation by square matrices E and F as in 1. Suppose rank $F = m (\leq n)$. Select m independent columns of F, and group them into a matrix F_1. Write (possibly after permutations) $F = \begin{bmatrix} F_1 \mid F_2 \end{bmatrix}$ and correspondingly $E = \begin{bmatrix} E_1 \mid E_2 \end{bmatrix}$

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \qquad\qquad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Then it can be shown [25] that the matrix $\begin{bmatrix} F_1 \mid E_2 \end{bmatrix}$ is invertible, and

$$\mathscr{D} = \left\{ \left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \right) \;\middle|\; \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = J \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \right\} \tag{2.113}$$

with $J := -\begin{bmatrix} F_1 \mid E_2 \end{bmatrix}^{-1} \begin{bmatrix} F_2 \mid E_1 \end{bmatrix}$ skew-symmetric.

It follows that any Dirac structure can be written as the graph of a skew-symmetric map. The vectors e_1, f_2 can be regarded as *input* vectors, while the complementary vectors f_1, e_2 can be seen as *output* vectors. (This is very much like the multi-port description of a passive linear circuit, where it is known that although it is *not* always possible to describe the port as an admittance or as an impedance, it *is* possible to describe it as a hybrid admittance/impedance transfer matrix, for a suitable selection of input voltages and currents and complementary output currents and voltages [19].)

Canonical coordinate representation

There exists a basis for $\mathscr{F}$ and dual basis for $\mathscr{F}^*$, such that, in these bases, the vector (f,e), when partitioned as $(f_q, f_p, f_r, f_s, e_q, e_p, e_r, e_s)$, is in $\mathscr{D}$ if and only if

$$\begin{aligned} f_q &= -e_p \\ f_p &= e_q \\ f_r &= 0 \\ e_s &= 0 \end{aligned} \tag{2.114}$$

For a proof we refer to [54]. The representation of a Dirac structure by canonical coordinates is very close to the classical Hamiltonian equations of motion, see Sect. 2.4.4.

Remark 2.7. A special type of kernel representation occurs if not only $EF^* + FE^* = 0$ but in fact $FE^* = 0$ (while still $\operatorname{rank}(F+E) = \dim \mathscr{F}$). In this case it follows that $\operatorname{Im} E^* \subset \ker F$. However, it follows from the kernel/image representation of any Dirac structure that $\ker F \subset \operatorname{Im} E^*$, and thus $\operatorname{Im} E^* = \ker F$ Hence in this case the Dirac structure is the product of the subspace $\ker F \subset \mathscr{F}$ and the subspace

$\ker F^{\mathrm{orth}} = \ker E \subset \mathscr{F}^*$, with $\cdot^{\mathrm{orth}}$ denoting the orthogonal complement with respect to the duality product between $\mathscr{F}$ and $\mathscr{F}^*$. We have already encountered this special type of Dirac structure in the case of Kirchhoff's current and voltage laws (Sect. 2.1.3.2) and in the case of kinematic pairs (Sect. 2.1.3.3), while mathematically it has been exemplified at the end of Sect. 2.1.1.

In [25, 59, 59, 185] it is shown how one may convert any of the above representations into any other. An easy transformation that will be used frequently in the sequel is the transformation of the *constrained input-output* representation into the *kernel* representation. Consider the Dirac structure $\mathscr{D}$ given in constrained input-output representation by (2.112). Construct a linear mapping $G^\perp$ of maximal rank satisfying

$$G^\perp G = 0$$

Then, pre-multiplying the first equation of (2.112) by $G^\perp$, one eliminates the Lagrange multipliers λ and obtains

$$\mathscr{D} = \Big\{ (f,e) \in \mathscr{F} \times \mathscr{F}^* \mid G^\perp f = G^\perp J e,\ G^{\mathrm{T}} e = 0 \Big\}, \tag{2.115}$$

which is easily seen to lead to a kernel representation. Indeed,

$$F = \begin{bmatrix} -G^\perp \\ 0 \end{bmatrix} \qquad\qquad E = \begin{bmatrix} G^\perp J \\ G^{\mathrm{T}} \end{bmatrix}$$

defines a kernel representation.

2.4.2 Representations of port-Hamiltonian systems

Coordinate representations of the port-Hamiltonian system (2.36) are obtained by choosing a specific coordinate representation of the Dirac structure $\mathscr{D}$. For example, if $\mathscr{D}$ is given in matrix kernel representation

$$\begin{aligned} \mathscr{D} = \Big\{ (f_S, e_S, f_R, e_R, f, e) \in \mathscr{X} \times \mathscr{X}^* \times \mathscr{F}_R \times \mathscr{F}_R^* \times \mathscr{F} \times \mathscr{F}^* \mid \\ F_S f_S + E_S e_S + F_R f_R + E_R e_R + F f + E e = 0 \Big\} \end{aligned} \tag{2.116}$$

with

$$\begin{aligned} &(i) \quad E_S F_S^{\mathrm{T}} + F_S E_S^{\mathrm{T}} + E_R F_R^{\mathrm{T}} + F_R E_R^{\mathrm{T}} + E F^{\mathrm{T}} + F E^{\mathrm{T}} = 0 \\ &(ii) \quad \operatorname{rank} \begin{bmatrix} F_S \mid E_S \mid F_R \mid E_R \mid F \mid E \end{bmatrix} = \dim(\mathscr{X} \times \mathscr{F}_R \times \mathscr{F}) \end{aligned} \tag{2.117}$$

then the port-Hamiltonian system is given by the set of equations

$$F_S \dot{x}(t) = E_S \frac{\partial H}{\partial x}(x(t)) + F_R f_R(t) + E_R e_R(t) + F f(t) + E e(t) \tag{2.118}$$

with $f_R(t), e_R(t)$ satisfying for all t the resistive relation

$$R(f_R(t), e_R(t)) = 0 \tag{2.119}$$

Note that, in general, (2.118) consists of differential equations *and* algebraic equations in the state variables x (DAEs), together with equations relating the state variables and their time-derivatives to the external port variables (f, e).

Example 2.10 (1-D mechanical systems). Consider a *spring* with elongation q and energy function $H_s(q)$, which for a linear spring is given as $H_s(q) = \frac{1}{2}kq^2$. Let (v_s, F_s) represent the external port through which energy can be exchanged with the spring, where v_s is equal to the rate of elongation (velocity) and F_s is equal to the elastic force. This port-Hamiltonian system (without dissipation) can be written in kernel representation as

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\dot{q} \\ v_s \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} kq \\ F_s \end{bmatrix} = 0 \tag{2.120}$$

Similarly we can model a *moving mass* m with scalar momentum p and kinetic energy $H_m(p) = \frac{1}{2m}p^2$ as the port-Hamiltonian system

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\dot{p} \\ F_m \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{p}{m} \\ v_m \end{bmatrix} = 0 \tag{2.121}$$

where (F_m, v_m) are respectively the external force exerted on the mass and the velocity of the mass.

The mass and the spring can be *interconnected* to each other using the symplectic gyrator

$$\begin{bmatrix} v_s \\ F_m \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} F_s \\ v_m \end{bmatrix} \tag{2.122}$$

Collecting all equations we have obtained a port-Hamiltonian system with energy variables $x = (q, p)$, total energy $H(q, p) = H_s(q) + H_m(p)$ and with interconnected port variables (v_s, F_s, F_m, v_m). After elimination of the interconnection variables (v_s, F_s, F_m, v_m) one obtains the port-Hamiltonian system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\dot{q} \\ -\dot{p} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} kq \\ \frac{p}{m} \end{bmatrix} = 0 \tag{2.123}$$

which is the ubiquitous mass-spring system. Note that the Dirac structure of this mass-spring system is derived from the Dirac structures of the spring system and the mass system together with their interconnection by means of the symplectic gyrator (which itself defines a Dirac structure). How to systematically derive the resulting interconnected Dirac structure is studied in Sect. 2.5.

In case of a Dirac structure modulated by the energy variables x and the state space $\mathscr{X}$ being a manifold, the flows $f_x = -\dot{x}$ are elements of the tangent space $T_x\mathscr{X}$ at the state $x \in \mathscr{X}$, and the efforts e_x are elements of the co-tangent space $T_x^*\mathscr{X}$.

Still, locally on $\mathscr{X}$, we obtain the kernel representation (2.118) for the resulting port-Hamiltonian system, but now the matrices F_S, E_S, F and E will depend on x.

The important special case of input-state-output port-Hamiltonian systems as treated before

$$\begin{cases} \dot{x} = J(x)\dfrac{\partial H}{\partial x}(x) + g(x)u \\ y = g^{\mathrm{T}}(x)\dfrac{\partial H}{\partial x}(x) \end{cases} \qquad x \in \mathscr{X}$$

can be interpreted as arising from a hybrid input-output representation of the Dirac structure (from e_S, u to f_S, y). If the matrices J, g depend on the energy variables x, then this is again a modulated Dirac structure.

In general, by a combination of the hybrid representation and the constrained input-output representation, it can be shown that, locally, any port-Hamiltonian systems can be represented in the following way

$$\begin{cases} \dot{x} = J(x)\dfrac{\partial H}{\partial x}(x) + g(x)u + b(x)\lambda \\ y = g^{\mathrm{T}}(x)\dfrac{\partial H}{\partial x}(x) \\ 0 = b^{\mathrm{T}}(x)\dfrac{\partial H}{\partial x}(x) \end{cases} \qquad x \in \mathscr{X} \tag{2.124}$$

where $y^{\mathrm{T}}u$ denotes the power at the external port.

Example 2.11 (Coupled masses – Internal constraints). Consider two point masses m_1 and m_2 that are rigidly linked to each other, moving in one dimension. When decoupled, the masses are described by the port-Hamiltonian systems

$$\begin{cases} \dot{p}_i = F_i \\ v_i = \dfrac{p_i}{m_i} \end{cases} \qquad i = 1, 2 \tag{2.125}$$

with F_i denoting the force exerted on mass m_i. Rigid coupling of the two masses is achieved by setting

$$F_1 = -F_2 \qquad\qquad v_1 = v_2 \tag{2.126}$$

This leads to the port-Hamiltonian system

$$\begin{aligned} \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \lambda \\ 0 &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{p_1}{m_1} \\ \frac{p_2}{m_2} \end{bmatrix} \end{aligned} \tag{2.127}$$

where $\lambda = F_1 = -F_2$ now denotes the *internal* constraint force. The resulting interconnected system no longer has external ports. On the other hand, external ports for

the interconnected system can be included by either extending (2.125) to

$$\begin{cases} \dot{p}_i = F_i + F_i^{\text{ext}} \\ v_i = \frac{p_i}{m_i} \\ v_i^{\text{ext}} = \frac{p_i}{m_i} \end{cases} \qquad i = 1, 2 \tag{2.128}$$

with F_i^{ext} and v_i^{ext} denoting the external forces and velocities, or by modifying the interconnection constraints (2.126) to e.g. $F_1 + F_2 + F^{\text{ext}} = 0$ and $v_1 = v_2 = v^{\text{ext}}$, with F^{ext} and v^{ext} denoting the external force exerted on the coupled masses, respectively the velocity of the coupled masses.

2.4.3 Elimination of Lagrangian multipliers and constraints

As shown above, it is relatively easy to eliminate the Lagrange multipliers in any constrained input-output representation of the Dirac structure. As a result, it is also relatively easy to eliminate the Lagrange multipliers in any port-Hamiltonian system. Indeed, consider the port-Hamiltonian system (2.124). The Lagrange multipliers λ can be eliminated by constructing a matrix $b^{\perp}(x)$ of maximal rank such that

$$b^{\perp}(x)b(x) = 0$$

Then, by pre-multiplication with this matrix $b^{\perp}(x)$, one obtains the equations

$$\begin{cases} b^{\perp}(x)\dot{x} = b^{\perp}(x)J(x)\dfrac{\partial H}{\partial x}(x) + b^{\perp}(x)g(x)u \\ y = g^{\mathrm{T}}(x)\dfrac{\partial H}{\partial x}(x) \\ 0 = b^{\mathrm{T}}(x)\dfrac{\partial H}{\partial x}(x) \end{cases} \qquad x \in \mathscr{X} \tag{2.129}$$

without Lagrange multipliers. This is readily seen to be a kernel representation of the port-Hamiltonian system.

Example 2.12 (Example 2.11, continued). Consider the system of two coupled masses in Example 2.11. Pre-multiplication of the dynamic equations by the row vector $\begin{bmatrix} 1 & 1 \end{bmatrix}$ yields the equations

$$\dot{p}_1 + \dot{p}_2 = 0 \qquad\qquad \frac{p_1}{m_1} - \frac{p_2}{m_2} = 0 \tag{2.130}$$

which constitutes a kernel representation of the port-Hamiltonian system, with matrices

$$F = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad\qquad E = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

A more difficult question concerns the possibility to solve for the *algebraic constraints* of a port-Hamiltonian system. This is a problem which is independent of the specific representation of the port-Hamiltonian system. In case of the representation (2.129), the algebraic constraints are given by

$$0 = b^{\mathrm{T}}(x)\frac{\partial H}{\partial x}(x) \tag{2.131}$$

In general, these equations will constrain the state variables x. However, the precise way this takes place very much depends on the properties of the Hamiltonian H as well as of the matrix $b(x)$. For example, if the Hamiltonian H is such that its gradient $\frac{\partial H}{\partial x}(x)$ happens to be contained in the kernel of the matrix $b^{\mathrm{T}}(x)$ for all x, then the algebraic constraints (2.131) actually do not constrain the state variables.

In general, under constant rank assumptions, the set

$$\mathscr{X}_c := \left\{ x \in \mathscr{X} \mid b^{\mathrm{T}}(x)\frac{\partial H}{\partial x}(x) = 0 \right\}$$

will define a sub-manifold of the total state space $\mathscr{X}$, called the *constrained state space*. In order that this constrained state space qualifies as the state space for a port-Hamiltonian system *without* further algebraic constraints, one needs to be able to restrict the dynamics of the port-Hamiltonian system to the constrained state space. This is always possible under the condition that the matrix

$$b^{\mathrm{T}}(x)\frac{\partial^2 H}{\partial x^2}(x)b(x) \tag{2.132}$$

has full rank. Indeed, under this condition, the differentiated constraint equation

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(b^{\mathrm{T}}(x)\frac{\partial H}{\partial x}(x) \right) = * + b^{\mathrm{T}}(x)\frac{\partial^2 H}{\partial x^2}(x)b(x)\lambda \tag{2.133}$$

(with $*$ denoting unspecified terms) can always be uniquely solved for λ, leading to a uniquely defined dynamics on the constrained state space $\mathscr{X}_c$. Using terminology from the theory of DAEs, the condition that the matrix in (2.132) has full rank ensures that the *index* of the DAEs specified by the port-Hamiltonian system is equal to 1, [37]. If the matrix in (2.132) does not have full rank, it may be necessary to further constrain the space $\mathscr{X}_c$ by considering apart from the algebraic constraints (2.131), also their time-derivatives (sometimes called *secondary constraints*).

Example 2.13 (Example 2.11, continued – DAE index). Differentiating the constraint equation $\frac{p_1}{m_1} - \frac{p_2}{m_2} = 0$ and using the dynamical equations $\dot{p}_1 = \lambda$ and $\dot{p}_2 = -\lambda$, one obtains

$$\left(\frac{1}{m_1} + \frac{1}{m_2} \right) \lambda = 0 \tag{2.134}$$

which obviously determines the constraint force λ to be equal to 0. Thus, the index of this system equals one. Defining the total momentum $p = p_1 + p_2$, one trivially obtains the reduced system $\dot{p} = \dot{p}_1 + \dot{p}_2 = 0$.

On the other hand, suppose that the mass m_1 is connected to a linear spring with spring constant k_1 and elongation q_1, and that the mass m_2 is connected to a linear spring with spring constant k_2 and elongation q_2. Then the dynamical equations change into $\dot{p}_1 = -k_1 q_1 + \lambda$ and $\dot{p}_2 = -k_2 q_2 - \lambda$, and differentiation of the constraint $\frac{p_1}{m_1} - \frac{p_2}{m_2} = 0$ leads to

$$-\frac{k_1}{m_1} q_1 + \frac{k_2}{m_2} q_2 + \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \lambda = 0 \tag{2.135}$$

This still determines the constraint force λ as

$$\lambda = \frac{m_1 m_2}{m_1 + m_2} \left(\frac{k_1}{m_1} q_1 - \frac{k_2}{m_2} q_2 \right)$$

and results in the (obvious) dynamical equation for the total momentum p given by

$$\dot{p} = -k_1 q_1 - k_2 q_2 \tag{2.136}$$

Generalizing Example 2.13, let us consider the equations of a mechanical system subject to kinematic constraints as discussed in Sect. 2.2.2. The constrained Hamiltonian equations (2.47) define a port-Hamiltonian system with respect to the Dirac structure $\mathscr{D}$ (in constrained input-output representation):

$$\begin{aligned} \mathscr{D} = \Bigg\{ (f_S, e_S, f_C, e_C) \mid 0 = \begin{bmatrix} 0 & A^{\mathrm{T}}(q) \end{bmatrix} e_S,\ e_C = \begin{bmatrix} 0 & B^{\mathrm{T}}(q) \end{bmatrix} e_S, \\ -f_S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} e_S + \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} f_c,\ \lambda \in \mathbb{R}^k \Bigg\} \end{aligned} \tag{2.137}$$

The algebraic constraints on the state variables (q, p) are thus given as

$$0 = A^{\mathrm{T}}(q) \frac{\partial H}{\partial p}(q, p) \tag{2.138}$$

The *constrained state space* is therefore given as the following subset of the phase space (q, p):

$$\mathscr{X}_c = \left\{ (q, p) \mid A^{\mathrm{T}}(q) \frac{\partial H}{\partial p}(q, p) = 0 \right\} \tag{2.139}$$

We may solve for the algebraic constraints and eliminate the resulting constraint forces $A(q)\lambda$ in the following way. Since rank $A(q) = k$, there exists locally an $n \times (n-k)$ matrix $S(q)$ of rank $n-k$ such that

$$A^{\mathrm{T}}(q) S(q) = 0 \tag{2.140}$$

Now define $\tilde{p} = (\tilde{p}^1, \tilde{p}^2) = (\tilde{p}_1, \ldots, \tilde{p}_{n-k}, \tilde{p}_{n-k+1}, \ldots, \tilde{p}_n)$ as

$$\begin{aligned} \tilde{p}^1 &:= S^{\mathrm{T}}(q)p \\ \tilde{p}^2 &:= A^{\mathrm{T}}(q)p \end{aligned} \qquad \tilde{p}^1 \in \mathbb{R}^{n-k},\ \tilde{p}^2 \in \mathbb{R}^k \tag{2.141}$$

It is readily checked that $(q,p) \mapsto (q,\tilde{p}^1,\tilde{p}^2)$ is a coordinate transformation. Indeed, by (2.140), the rows of $S^{\mathrm{T}}(q)$ are orthogonal to the rows of $A^{\mathrm{T}}(q)$. In the new coordinates, the constrained Hamiltonian system (2.47) takes the form (see [182] for details), $*$ denoting unspecified elements,

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{\tilde{p}}^1 \\ \dot{\tilde{p}}^2 \end{bmatrix} &= \begin{bmatrix} 0_n & S(q) & * \\ -S^{\mathrm{T}}(q) & \left(-p^{\mathrm{T}}[S_i,S_j](q)\right)_{i,j} & * \\ * & * & * \end{bmatrix} \begin{bmatrix} \frac{\partial \tilde{H}}{\partial q} \\ \frac{\partial \tilde{H}}{\partial \tilde{p}^1} \\ \frac{\partial \tilde{H}}{\partial \tilde{p}^2} \end{bmatrix} + \\ &\qquad + \begin{bmatrix} 0 \\ 0 \\ A^{\mathrm{T}}(q)A(q) \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ B_c(q) \\ \overline{B}(q) \end{bmatrix} u \\ A^{\mathrm{T}}(q)\frac{\partial H}{\partial p} &= A^{\mathrm{T}}(q)A(q)\frac{\partial \tilde{H}}{\partial \tilde{p}^2} \ = \ 0 \end{aligned} \tag{2.142}$$

with $\tilde{H}(q,\tilde{p})$ the Hamiltonian H expressed in the new coordinates $q,\tilde{p}$. Here, S_i denotes the i-th column of $S(q)$, $i = 1,\ldots,n-k$, and $[S_i,S_j]$ is the *Lie bracket* of S_i and S_j, in local coordinates q given as (see e.g. [2, 156])

$$[S_i,S_j](q) \ = \ \frac{\partial S_j}{\partial q}(q)S_i(q) - \frac{\partial S_i}{\partial q}S_j(q) \tag{2.143}$$

with $\frac{\partial S_j}{\partial q}$ and $\frac{\partial S_i}{\partial q}$ denoting the $n \times n$ Jacobian matrices.

Since λ only influences the $\tilde{p}^2$-dynamics, and the constraints $A^{\mathrm{T}}(q)\frac{\partial H}{\partial p}(q,p) = 0$ are equivalently given by $\frac{\partial \tilde{H}}{\partial \tilde{p}^2}(q,\tilde{p}) = 0$, the constrained dynamics is determined by the dynamics of q and $\tilde{p}^1$, which serve as coordinates for the constrained state space $\mathscr{X}_c$:

$$\begin{bmatrix} \dot{q} \\ \dot{\tilde{p}}^1 \end{bmatrix} = J_c(q,\tilde{p}^1) \begin{bmatrix} \frac{\partial H_c}{\partial q}(q,\tilde{p}^1) \\ \frac{\partial H_c}{\partial \tilde{p}^1}(q,\tilde{p}^1) \end{bmatrix} + \begin{bmatrix} 0 \\ B_c(q) \end{bmatrix} u, \tag{2.144}$$

where $H_c(q,\tilde{p}^1)$ equals $\tilde{H}(q,\tilde{p})$ with $\tilde{p}^2$ satisfying $\frac{\partial \tilde{H}}{\partial \tilde{p}^2} = 0$, and where the skew-symmetric matrix $J_c(q,\tilde{p}^1)$ is given as the left-upper part of the structure matrix in (2.142), that is

$$J_c(q,\tilde{p}^1) = \begin{bmatrix} O_n & S(q) \\ -S^{\mathrm{T}}(q) & \left(-p^{\mathrm{T}}[S_i,S_j](q)\right)_{i,j} \end{bmatrix}, \tag{2.145}$$

where p is expressed as function of $q,\tilde{p}$, with $\tilde{p}^2$ eliminated from $\frac{\partial \tilde{H}}{\partial \tilde{p}^2} = 0$. Furthermore, in the coordinates $q,\tilde{p}$, the output map is given in the form

$$y = \begin{bmatrix} B_c^{\mathrm{T}}(q) & \overline{B}^{\mathrm{T}}(q) \end{bmatrix} \begin{bmatrix} \frac{\partial \tilde{H}}{\partial \tilde{p}^1} \\ \frac{\partial \tilde{H}}{\partial \tilde{p}^2} \end{bmatrix} \tag{2.146}$$

which reduces on the constrained state space $\mathscr{X}_c$ to

$$y = B_c^{\mathrm{T}}(q) \frac{\partial \tilde{H}}{\partial \tilde{p}^1}(q, \tilde{p}^1) \tag{2.147}$$

Summarizing, (2.144) and (2.147) define a *port-Hamiltonian system* on $\mathscr{X}_c$, with Hamiltonian H_c given by the constrained total energy, and with structure matrix J_c given by (2.145).

Example 2.14 (Example 2.4, continued). Define according to (2.141) new p-coordinates

$$\begin{aligned} p_1 &= p_\varphi \\ p_2 &= p_\theta + p_x \cos\varphi + p_y \sin\varphi \\ p_3 &= p_x - p_\theta \cos\varphi \\ p_4 &= p_y - p_\theta \sin\varphi \end{aligned}$$

The constrained state space $\mathscr{X}_c$ is given by $p_3 = p_4 = 0$, and the dynamics on $\mathscr{X}_c$ is computed as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\varphi} \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} & & & & 0 & \cos\varphi \\ & & & & 0 & \sin\varphi \\ & & O_4 & & 0 & 1 \\ & & & & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -\cos\varphi & -\sin\varphi & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_c}{\partial x} \\ \frac{\partial H_c}{\partial y} \\ \frac{\partial H_c}{\partial \theta} \\ \frac{\partial H_c}{\partial \varphi} \\ \frac{\partial H_c}{\partial p_1} \\ \frac{\partial H_c}{\partial p_2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2.148}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} p_2 \\ p_1 \end{bmatrix}$$

where $H_c(x, y, \theta, \varphi, p_1, p_2) = \frac{1}{2} p_1^2 + \frac{1}{4} p_2^2$.

2.4.4 Port-Hamiltonian systems in canonical coordinates – Casimirs and algebraic constraints

Consider a port-Hamiltonian system *without* resistive ports and external ports (so only containing energy-storage ports). Then, the flows $f \in \mathscr{F}$ are given as the flows f_S of the energy storage, and the efforts $e \in \mathscr{F}^*$ as the efforts e_S. By the canonical coordinate representation, we may consider a basis for $\mathscr{F}$ and a dual basis for $\mathscr{F}^*$

such that the Dirac structure is described by (2.114). In general, however, $\mathscr{F}$ is modulated by the energy variables. In fact, if the state space $\mathscr{X}$ is a manifold, then $\mathscr{F}$ at any point $x \in \mathscr{X}$ will be given as the tangent space $T_x\mathscr{X}$. Furthermore, also the Dirac structure, being given at any point x as a subspace $\mathscr{D}(x) \subset T_x\mathscr{X} \times T_x^*\mathscr{X}$, will be a modulated Dirac structure. If the Dirac structure $\mathscr{D}$ satisfies an additional *integrability condition* then we can choose local coordinates $x = (q,p,r,s)$ for $\mathscr{X}$ (with $\dim q = \dim p$), such that, in the corresponding bases for (f_q, f_p, f_r, f_s) for $T_x\mathscr{X}$ and (e_q, e_p, e_r, e_s) for $T_x^*\mathscr{X}$, the Dirac structure on this coordinate neighborhood is still given by the relations (2.114). For more details regarding the precise form of the integrability conditions see Sect. 2.7.

Substituting the flow and effort relations of the energy storage

$$\begin{aligned} f_q &= -\dot{q} & e_q &= \frac{\partial H}{\partial q} & f_p &= -\dot{p} & e_p &= \frac{\partial H}{\partial p} \\ f_r &= -\dot{r} & e_r &= \frac{\partial H}{\partial r} & f_s &= -\dot{s} & e_s &= \frac{\partial H}{\partial s} \end{aligned}$$

into the canonical coordinate representation (2.114) of the Dirac structure yields the following dynamics:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \\ \dot{r} &= 0 \\ 0 &= \frac{\partial H}{\partial s} \end{aligned} \tag{2.149}$$

The variables q, p are the canonical coordinates known from classical Hamiltonian dynamics. Furthermore, the variables r have the interpretation of conserved quantities or *Casimirs*, see Section Sect. 2.6.2. Indeed, every function $K(r)$ will satisfy the property $\frac{\mathrm{d}}{\mathrm{d}t}K = 0$. Finally, the last equations $\frac{\partial H}{\partial s} = 0$ specify the *algebraic constraints* present in the system.

If the Hamiltonian H satisfies additional *regularity* properties such as the partial Hessian matrix $\frac{\partial^2 H}{\partial s^2}$ being invertible, then by the Implicit Function theorem we may in principle eliminate the algebraic constraints by solving for s as a function $s(q,p,r)$. Then the DAEs (2.149) reduce to the ODEs

$$\begin{aligned} \dot{q} &= \frac{\partial \bar{H}}{\partial p} \\ \dot{p} &= -\frac{\partial \bar{H}}{\partial q} \\ \dot{r} &= 0 \end{aligned} \tag{2.150}$$

where $\bar{H}(q,p,r) := H(q,p,r,s(q,p,r))$.

2.4.5 Well-posedness of port-Hamiltonian systems

As we have seen above, coordinate representations of port-Hamiltonian systems often lead to mixed sets of differential and algebraic equations (DAEs). This implies that the existence of unique solutions for feasible initial conditions is not guaranteed. In particular, problems may arise for port-Hamiltonian systems where the flow variables of the resistive ports are input variables for the dynamics, while the resistive relation is *not* effort-controlled. This is illustrated by the following example taken from [174].

Example 2.15 (Degenerate Van der Pol oscillator). Consider a degenerate form of the V an der Pol oscillator consisting of a 1-F capacitor

$$\dot{Q} = I \qquad\qquad V = Q \tag{2.151}$$

in parallel with a nonlinear resistor with the characteristic:

$$\left\{ (f_R, e_R) = (I,V) \mid V = -\frac{1}{3}I^3 + I \right\} \tag{2.152}$$

This resistive characteristic is clearly *not* voltage-controlled, but instead is current-controlled. Hence the resistive relation can*not* be expressed as an input-output mapping as in (2.31).

As a consequence, the equations (2.151) and (2.152) define an implicitly defined dynamics on the one-dimensional constraint sub-manifold R in (I,V) space given by

$$R = \left\{ (I,V) \,|\, V + \frac{1}{3}I^3 - I = 0 \right\}.$$

Difficulties in the dynamical interpretation arise at the points $\left(-1,-\frac{2}{3}\right)$ and $\left(1,\frac{2}{3}\right)$. At these points $\dot{V}$ is negative, respectively positive (while the corresponding time-derivative of I at these points tends to plus or minus infinity, depending on the direction along which these points are approached). Hence, because of the form of the constraint manifold R it is not possible to "integrate" the dynamics from these points in a continuous manner along R. These points are sometimes called *impasse points*.

For a careful analysis of the dynamics of this system we refer to [174]. In particular, it has been suggested in [174] that a suitable interpretation of the dynamics from these impasse points is given by the following *jump rules*:

$$\left(-1,-\frac{2}{3}\right) \rightarrow \left(2,-\frac{2}{3}\right) \qquad \left(1,\frac{2}{3}\right) \rightarrow \left(-2,\frac{2}{3}\right) \tag{2.153}$$

The resultant trajectory (switching from the region $I \leq -1$ to the region $I \geq 1$) is a 'limit cycle' that is known as a *relaxation oscillation*.

Existence and uniqueness of solutions *is* guaranteed if the resistive relation is well-behaved and the DAEs are of index 1 as discussed in the previous Sect. 2.4.3.

Indeed, consider again the case of a port-Hamiltonian system given in the format

$$\begin{cases} \dot{x} = \big[J(x) - R(x)\big] \dfrac{\partial H}{\partial x}(x) + g(x)u + b(x)\lambda \\ y = g^{\mathrm{T}}(x) \dfrac{\partial H}{\partial x}(x) \qquad\qquad x \in \mathscr{X} \\ 0 = b^{\mathrm{T}}(x) \dfrac{\partial H}{\partial x}(x) \end{cases} \tag{2.154}$$

Imposing the same condition as before in Sect. 2.4.3, namely that the matrix

$$b^{\mathrm{T}}(x) \frac{\partial^2 H}{\partial x^2}(x) b(x) \tag{2.155}$$

has full rank, it can be seen that there is a unique solution starting from every feasible initial condition $x_0 \in \mathscr{X}_c$. Furthermore, this solution will remain in the constrained state space $\mathscr{X}_c$ for all time.

Example 2.16. A somewhat trivial example of a case where multiple solutions arise from feasible initial conditions can be deduced from the example of a linear LC-circuit with standard Hamiltonian $H(Q,\phi) = \frac{1}{2C}Q^2 + \frac{1}{2L}\phi^2$, where the voltage over the capacitor is constrained to be zero:

$$\begin{aligned} \dot{Q} &= \frac{1}{L}\phi + \lambda \\ \dot{\phi} &= \frac{1}{C}Q \\ 0 &= \frac{1}{C}Q \end{aligned} \tag{2.156}$$

Here λ denotes the current through the external port whose voltage is set equal to zero. Since $b^{\mathrm{T}}(x)\frac{\partial^2 H}{\partial x^2}(x)b(x)$ in this case reduces to $\frac{1}{C}$ it follows that there is a unique solution starting from every feasible initial condition. Indeed, the constrained state space $\mathscr{X}_c$ of the above port-Hamiltonian system is simply given by $\{(Q,\phi) \mid Q = 0\}$, while the Lagrange multiplier λ for any feasible initial condition $(0,\phi_0)$ is uniquely determined as $\lambda = -\frac{1}{L}\phi$. Now ever, consider the singular case where $C = \infty$, in which case the Hamiltonian reduces to $H(Q,\phi) = \frac{1}{2L}\phi^2$ and the constraint equation $0 = \frac{1}{C}Q$ becomes vacuous, i.e., there are no constraints anymore. In this case the Lagrange multiplier λ (the current through the external port) is not determined anymore, leading to multiple solutions $(Q(t),\phi(t))$ where $\phi(t)$ is constant (equal to the initial value ϕ_0) while $Q(t)$ is an *arbitrary* function of time.

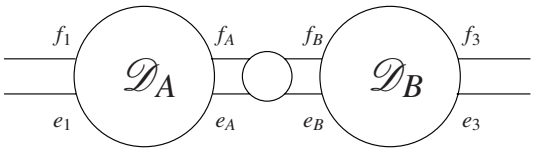

Fig. 2.7 The composition of $\mathscr{D}_A$ and $\mathscr{D}_B$.

2.5 Interconnection of port-Hamiltonian systems

Crucial feature of network modeling, analysis and control is 'interconnectivity' or 'compositionality', meaning that complex systems can be built up from simpler parts, and that certain properties of the complex system can be studied in terms of its constituent parts and the way they are interconnected. The class of port-Hamiltonian systems completely fits within this paradigm, in the sense that the power-conserving interconnection of port-Hamiltonian systems again defines a port-Hamiltonian system. Furthermore, it will turn out that the Hamiltonian of the interconnected system is simply the sum of the Hamiltonians of its parts, while the Dirac structure of the interconnected system is solely determined by the Dirac structures of its components.

2.5.1 Composition of Dirac structures

In this subsection, we investigate the *composition* or *interconnection* properties of Dirac structures. Physically it is clear that the composition of a number of power-conserving interconnections with partially shared variables should yield again a power-conserving interconnection. We show how this can be formalized within the framework of Dirac structures.

First, we consider the composition of *two* Dirac structures with partially shared variables. Once we have shown that the composition of two Dirac structures is again a Dirac structure, it is immediate that the power-conserving interconnection of any number of Dirac structures is again a Dirac structure. Thus consider a Dirac structure $\mathscr{D}_A$ on a product space $\mathscr{F}_1 \times \mathscr{F}_2$ of two linear spaces $\mathscr{F}_1$ and $\mathscr{F}_2$, and another Dirac structure $\mathscr{D}_B$ on a product space $\mathscr{F}_2 \times \mathscr{F}_3$, with also $\mathscr{F}_3$ being a linear space. The linear space $\mathscr{F}_2$ is the space of shared flow variables, and $\mathscr{F}_2^*$ the space of shared effort variables; see Fig. 2.7.

In order to compose $\mathscr{D}_A$ and $\mathscr{D}_B$, a problem arises of *sign* convention for the power flow corresponding to the power variables $(f_2, e_2) \in \mathscr{F}_2 \times \mathscr{F}_2^*$. Indeed, if $\langle e \mid f \rangle$ denotes *incoming* power (see the previous section), then for

$$(f_1, e_1, f_A, e_A) \in \mathscr{D}_A \subset \mathscr{F}_1 \times \mathscr{F}_1^* \times \mathscr{F}_2 \times \mathscr{F}_2^*$$

the term $\langle e_A \mid f_A \rangle$ denotes the incoming power in $\mathscr{D}_A$ due to the power variables $(f_A, e_A) \in \mathscr{F}_2 \times \mathscr{F}_2^*$, while for

$$(f_B, e_B, f_3, e_3) \in \mathscr{D}_B \subset \mathscr{F}_2 \times \mathscr{F}_2^* \times \mathscr{F}_3 \times \mathscr{F}_3^*$$

the term $\langle e_B \mid f_B \rangle$ denotes the incoming power *in* $\mathscr{D}_B$. Clearly, the *incoming* power in $\mathscr{D}_A$ due to the power variables in $\mathscr{F}_2 \times \mathscr{F}_2^*$ should equal the *outgoing* power from $\mathscr{D}_B$. Thus we cannot simply equate the flows f_A and f_B and the efforts e_A and e_B, but instead we define the interconnection constraints as

$$f_A = -f_B \in \mathscr{F}_2 \qquad\qquad e_A = e_B \in \mathscr{F}_2^* \tag{2.157}$$

In bond graph terminology we link the two bonds corresponding to (f_A, e_A) and (f_B, e_B) by a 0-junction. Therefore, the *composition* of the Dirac structures $\mathscr{D}_A$ and $\mathscr{D}_B$, denoted $\mathscr{D}_A \parallel \mathscr{D}_B$, is defined as

$$\mathscr{D}_A \parallel \mathscr{D}_B := \Big\{ (f_1, e_1, f_3, e_3) \in \mathscr{F}_1 \times \mathscr{F}_1^* \times \mathscr{F}_3 \times \mathscr{F}_3^* \mid \exists (f_2, e_2) \in \mathscr{F}_2 \times \mathscr{F}_2^* \\ \text{s.t. } (f_1, e_1, f_2, e_2) \in \mathscr{D}_A \text{ and } (-f_2, e_2, f_3, e_3) \in \mathscr{D}_B \Big\}$$

The fact that the composition of two Dirac structures is again a Dirac structure has been proved in [59, 179]. Here we follow the simpler alternative proof provided in [45] (inspired by a result in [153]), which also allows one to study the *regularity* of the composition, and to obtain explicit representations of the composed Dirac structure.

Theorem 2.1. *Let $\mathscr{D}_A$ and $\mathscr{D}_B$ be Dirac structures (defined with respect to $\mathscr{F}_1 \times \mathscr{F}_1^* \times \mathscr{F}_2 \times \mathscr{F}_2^*$, respectively $\mathscr{F}_2 \times \mathscr{F}_2^* \times \mathscr{F}_3 \times \mathscr{F}_3^*$, and their bi-linear forms). Then $\mathscr{D}_A \parallel \mathscr{D}_B$ is a Dirac structure with respect to the bi-linear form on $\mathscr{F}_1 \times \mathscr{F}_1^* \times \mathscr{F}_3 \times \mathscr{F}_3^*$.*

Proof. Consider $\mathscr{D}_A$ and $\mathscr{D}_B$ defined in matrix kernel representation by

$$\mathscr{D}_A = \Big\{ (f_1, e_1, f_A, e_A) \in \mathscr{F}_1 \times \mathscr{F}_1^* \times \mathscr{F}_2 \times \mathscr{F}_2^* \mid F_1 f_1 + E_1 e_1 + F_{2A} f_A + E_{2A} e_A = 0 \Big\}$$
$$\mathscr{D}_B = \Big\{ (f_B, e_B, f_3, e_3) \in \mathscr{F}_2 \times \mathscr{F}_2^* \times \mathscr{F}_3 \times \mathscr{F}_3^* \mid F_{2B} f_B + E_{2B} e_B + F_3 f_3 + E_3 e_3 = 0 \Big\}$$

In the following we shall make use of the following basic fact from linear algebra.

$$\big[(\exists \lambda \text{ s.t. } A\lambda = b) \big] \Leftrightarrow \big[\forall \alpha \text{ s.t. } \alpha^{\mathrm{T}} A = 0 \Rightarrow \alpha^{\mathrm{T}} b = 0 \big]$$

Note that $\mathscr{D}_A$ and $\mathscr{D}_B$ are alternatively given in matrix image representation as

$$\mathscr{D}_A = \operatorname{Im} \begin{bmatrix} E_1^{\mathrm{T}} \\ F_1^{\mathrm{T}} \\ E_{2A}^{\mathrm{T}} \\ F_{2A}^{\mathrm{T}} \\ 0 \\ 0 \end{bmatrix} \qquad\qquad \mathscr{D}_B = \operatorname{Im} \begin{bmatrix} 0 \\ 0 \\ E_{2B}^{\mathrm{T}} \\ F_{2B}^{\mathrm{T}} \\ E_3^{\mathrm{T}} \\ F_3^{\mathrm{T}} \end{bmatrix}$$

Hence, $(f_1, e_1, f_3, e_3) \in \mathscr{D}_A \parallel \mathscr{D}_B$ if and only if it exists λ_A and λ_B such that

$$\begin{bmatrix} f_1 \\ e_1 \\ 0 \\ 0 \\ f_3 \\ e_3 \end{bmatrix} = \begin{bmatrix} E_1^{\mathrm{T}} & 0 \\ F_1^{\mathrm{T}} & 0 \\ E_{2A}^{\mathrm{T}} & E_{2B}^{\mathrm{T}} \\ F_{2A}^{\mathrm{T}} & -F_{2B}^{\mathrm{T}} \\ 0 & F_3^{\mathrm{T}} \\ 0 & E_3^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \lambda_A \\ \lambda_B \end{bmatrix}$$

which means that for all $[\beta_1, \alpha_1, \beta_2, \alpha_2, \beta_3, \alpha_3]$ such that

$$\begin{bmatrix} \beta_1^{\mathrm{T}} & \alpha_1^{\mathrm{T}} & \beta_2^{\mathrm{T}} & \alpha_2^{\mathrm{T}} & \beta_3^{\mathrm{T}} & \alpha_3^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} E_1^{\mathrm{T}} & 0 \\ F_1^{\mathrm{T}} & 0 \\ E_{2A}^{\mathrm{T}} & E_{2B}^{\mathrm{T}} \\ F_{2A}^{\mathrm{T}} & -F_{2B}^{\mathrm{T}} \\ 0 & F_3^{\mathrm{T}} \\ 0 & E_3^{\mathrm{T}} \end{bmatrix} = 0$$

we have that

$$\beta_1^{\mathrm{T}} f_1 + \alpha_1^{\mathrm{T}} e_1 + \beta_3^{\mathrm{T}} f_3 + \alpha_3^{\mathrm{T}} e_3 = 0.$$

Equivalently, for all $[\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3]$ such that

$$\begin{bmatrix} F_1 & E_1 & F_{2A} & E_{2A} & 0 & 0 \\ 0 & 0 & -F_{2B} & E_{2B} & F_3 & E_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \\ \alpha_3 \\ \beta_3 \end{bmatrix} = 0$$

we have that

$$\beta_1^{\mathrm{T}} f_1 + \alpha_1^{\mathrm{T}} e_1 + \beta_3^{\mathrm{T}} f_3 + \alpha_3^{\mathrm{T}} e_3 = 0 \tag{2.158}$$

which means that for all $(\alpha_1, \beta_1, \alpha_3, \beta_3) \in \mathscr{D}_A \parallel \mathscr{D}_B$, relation (2.158) is equivalent to require that $(f_1, e_1, f_3, e_3) \in (\mathscr{D}_A \parallel \mathscr{D}_B)^{\perp}$. Thus, $\mathscr{D}_A \parallel \mathscr{D}_B = (\mathscr{D}_A \parallel \mathscr{D}_B)^{\perp}$, and so it is a Dirac structure.

In the following theorem, an explicit expression is given for the composition of two Dirac structures in terms of a matrix kernel/image representation.

Theorem 2.2. *Let $\mathscr{F}_i, i = 1,2,3$ be finite-dimensional linear spaces with $\dim \mathscr{F}_i = n_i$. Consider Dirac structures $\mathscr{D}_A \subset \mathscr{F}_1 \times \mathscr{F}_1^* \times \mathscr{F}_2 \times \mathscr{F}_2^*$, $n_A = \dim \mathscr{F}_1 \times \mathscr{F}_2 = n_1 + n_2$, $\mathscr{D}_B \subset \mathscr{F}_2 \times \mathscr{F}_2^* \times \mathscr{F}_3 \times \mathscr{F}_3^*$, $n_B = \dim \mathscr{F}_2 \times \mathscr{F}_3 = n_2 + n_3$, given by relaxed matrix kernel/image representations $(F_A, E_A) = ([F_1 \mid F_{2A}], [E_1 \mid E_{2A}])$, with F_A and E_A $n'_A \times n_A$ matrices, $n'_A \geq n_A$, respectively $(F_B, E_B) = ([F_{2B} \mid F_3], [E_{2B} \mid E_3])$, with F_B and E_B $n'_B \times n_B$ matrices, $n'_B \geq n_B$. Define the $(n'_A + n'_B) \times 2n_2$ matrix*

$$M = \begin{bmatrix} F_{2A} & E_{2A} \\ -F_{2B} & E_{2B} \end{bmatrix} \tag{2.159}$$

and let L_A and L_B be $m \times n'_A$, respectively $m \times n'_B$, matrices ($m := \dim\ker M^T$), with

$$L = \begin{bmatrix} L_A \mid L_B \end{bmatrix} \qquad \ker L = \operatorname{Im} M \tag{2.160}$$

Then

$$F = \begin{bmatrix} L_A F_1 \mid L_B F_3 \end{bmatrix} \qquad E = \begin{bmatrix} L_A E_1 \mid L_B E_3 \end{bmatrix} \tag{2.161}$$

is a relaxed matrix kernel/image representation of $\mathscr{D}_A \parallel \mathscr{D}_B$.

Proof. Consider the notation corresponding to Fig. 2.7 and for any $\lambda_A \in \mathbb{R}^{n'_A}$, $\lambda_B \in \mathbb{R}^{n'_B}$ their *associated* elements in $\mathscr{D}_A$, respectively $\mathscr{D}_B$, given by

$$\begin{bmatrix} f_1 \\ e_1 \\ f_A \\ e_A \end{bmatrix} = \begin{bmatrix} E_1^{\mathrm{T}} \\ F_1^{\mathrm{T}} \\ E_{2A}^{\mathrm{T}} \\ F_{2A}^{\mathrm{T}} \end{bmatrix} \lambda_A \qquad \begin{bmatrix} f_3 \\ e_3 \\ f_B \\ e_B \end{bmatrix} = \begin{bmatrix} E_3^{\mathrm{T}} \\ F_3^{\mathrm{T}} \\ E_{2B}^{\mathrm{T}} \\ F_{2B}^{\mathrm{T}} \end{bmatrix} \lambda_B \tag{2.162}$$

Since

$$\begin{bmatrix} E_{2A}^{\mathrm{T}} \\ F_{2A}^{\mathrm{T}} \end{bmatrix} \lambda_A = \begin{bmatrix} f_A \\ e_A \end{bmatrix} = \begin{bmatrix} -f_B \\ e_B \end{bmatrix} = \begin{bmatrix} -E_{2B}^{\mathrm{T}} \\ F_{2B}^{\mathrm{T}} \end{bmatrix} \lambda_B \Leftrightarrow \begin{bmatrix} \lambda_A \\ \lambda_B \end{bmatrix} \in \ker M^{\mathrm{T}} \tag{2.163}$$

it follows that $(f_1, f_3, e_1, e_3) \in \mathscr{D}_A \parallel \mathscr{D}_B$ if and only if exists $[\lambda_A^{\mathrm{T}}, \lambda_B^{\mathrm{T}}]^{\mathrm{T}} \in \ker M^{\mathrm{T}}$ such that (2.162) holds for some (f_A, e_A) and (f_B, e_B), necessarily satisfying $f_A = -f_B$ and $e_A = e_B$. For any matrix L defined as in (2.160) we can write $[\lambda_A^{\mathrm{T}}, \lambda_B^{\mathrm{T}}]^{\mathrm{T}} \in \ker M^{\mathrm{T}}$ as

$$\begin{bmatrix} \lambda_A \\ \lambda_B \end{bmatrix} = \begin{bmatrix} L_A^{\mathrm{T}} \\ L_B^{\mathrm{T}} \end{bmatrix} \lambda \qquad \lambda \in \mathbb{R}^m \tag{2.164}$$

Substituting (2.164) in (2.162) we obtain the following characterization of $\mathscr{D}_A \parallel \mathscr{D}_B$

$$\mathscr{D}_A \parallel \mathscr{D}_B = \left\{ (f_1, e_1, f_3, e_3) \mid \begin{bmatrix} f_1 \\ e_1 \\ f_3 \\ e_3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} E_1^{\mathrm{T}} \\ F_1^{\mathrm{T}} \end{bmatrix} L_A^{\mathrm{T}} \\ \begin{bmatrix} E_3^{\mathrm{T}} \\ F_3^{\mathrm{T}} \end{bmatrix} L_B^{\mathrm{T}} \end{bmatrix} \lambda,\ \lambda \in \mathbb{R}^m \right\} \tag{2.165}$$

which corresponds to the relaxed matrix image representation given in (2.161).

Remark 2.8. The relaxed kernel/image representation (2.161) can be readily understood by pre-multiplying the equations characterizing the composition of $\mathscr{D}_A$ with $\mathscr{D}_B$

$$\begin{bmatrix} F_1 & E_1 & F_{2A} & E_{2A} & 0 & 0 \\ 0 & 0 & -F_{2B} & E_{2B} & F_3 & E_3 \end{bmatrix} \begin{bmatrix} f_1 \\ e_1 \\ f_2 \\ e_2 \\ f_3 \\ e_3 \end{bmatrix} = 0, \tag{2.166}$$

by the matrix $L := [L_A \,|\, L_B]$. Since $LM = 0$ this results in the relaxed kernel representation

$$L_A F_1 f_1 + L_A E_1 e_1 + L_B F_3 f_3 + L_B E_3 e_3 = 0 \tag{2.167}$$

corresponding to (2.161).

Instead of the canonical interconnection constraints $f_A = -f_B$, $e_A = e_B$ (cf. (2.157)), another standard power-conserving interconnection is the 'gyrative' interconnection

$$f_A = e_B \qquad\qquad f_B = -e_A \tag{2.168}$$

Composition of two Dirac structures $\mathscr{D}_A$ and $\mathscr{D}_B$ by this gyrative interconnection also results in a Dirac structure. In fact, the gyrative interconnection of $\mathscr{D}_A$ and $\mathscr{D}_B$ equals the interconnection $\mathscr{D}_A \parallel \mathscr{I} \parallel \mathscr{D}_B$, where $\mathscr{I}$ is the gyrative (or *symplectic*) Dirac structure

$$f_{IA} = -e_{IB} \qquad\qquad f_{IB} = e_{IA} \tag{2.169}$$

interconnected to $\mathscr{D}_A$ and $\mathscr{D}_B$ via the canonical interconnections $f_{IA} = -f_A, e_{IA} = e_A$ and $f_{IB} = -f_B, e_{IB} = e_B$.

Example 2.17 (Port-Hamiltonian systems with effort or flow constraints). Consider a general port-Hamiltonian system given in kernel representation as

$$F_S \dot{x}(t) = E_S \frac{\partial H}{\partial x}(x(t)) + F_R f_R(t) + E_R e_R(t) + F f(t) + E e(t) = 0 \tag{2.170}$$

with $f_R(t), e_R(t)$ satisfying for all t the resistive relation

$$R(f_R(t), e_R(t)) = 0 \tag{2.171}$$

Suppose the system is constrained by imposing the *effort constraints* $e = 0$. This corresponds to the composition of the Dirac structure of the port-Hamiltonian system with the 'effort-constraint' Dirac structure $\mathscr{D}^{\text{ec}}$ defined as

$$\mathscr{D}^{\text{ec}} = \Big\{ (f,e) \in \mathscr{F} \times \mathscr{F}^* \mid e = 0 \Big\} \tag{2.172}$$

It follows that the matrix M in this case is given as

$$M = \begin{bmatrix} F & E \\ 0 & I \end{bmatrix} \tag{2.173}$$

where the dimensions of the square zero matrix 0 and the identity matrix I are equal to the dimension of $\mathscr{F}$. Hence the matrix $L = \begin{bmatrix} L^{\rm ec} & L' \end{bmatrix}$ should satisfy

$$\begin{bmatrix} L^{\rm ec} & L' \end{bmatrix} \begin{bmatrix} F & E \\ 0 & I \end{bmatrix} = 0 \tag{2.174}$$

It follows that the composed Dirac structure $\mathscr{D}^{\rm ec}_{\rm red}$ corresponding to the effort-constraint $e = 0$ is given by the explicit equations

$$L^{\rm ec} F_S f_S + L^{\rm ec} E_S e_S + L^{\rm ec} F_R f_R + L^{\rm ec} E_R e_R = 0 \tag{2.175}$$

where $L^{\rm ec}$ is a matrix of maximal rank satisfying

$$L^{\rm ec} F = 0 \tag{2.176}$$

It follows that the constrained port-Hamiltonian system resulting from imposing the effort constraints $e = 0$ is given as the port-Hamiltonian system

$$L^{\rm ec} F_S \dot{x} = L^{\rm ec} E_S \frac{\partial H^{\rm ec}_{\rm red}}{\partial x}(x) + L^{\rm ec} F_R f_R + L^{\rm ec} E_R e_R, \tag{2.177}$$

with f_R, e_R satisfying the resistive relation $R(f_R(t), e_R(t)) = 0$. Similarly, the reduced Dirac structure $\mathscr{D}^{\rm fc}_{\rm red}$ corresponding to the *flow-constraints* $f = 0$ is given by the equations

$$L^{\rm fc} F_S f_S + L^{\rm fc} E_S e_S + L^{\rm fc} F_R f_R + L^{\rm fc} E_R e_R = 0 \tag{2.178}$$

where $L^{\rm fc}$ is any matrix of maximal rank satisfying

$$L^{\rm fc} E = 0 \tag{2.179}$$

A similar analysis can be made for any hybrid set of effort and flow constraints of complementarity type

$$e_i = 0 \quad (i \in K) \qquad\qquad f_j = 0 \quad (j \notin K) \tag{2.180}$$

for any subset $K \subset \{1, \cdots, p\}$, where $p = \dim \mathscr{F}$.

Example 2.18 (Feedback interconnection). The standard feedback interconnection of two input-state-output systems can be regarded as an example of a gyrative interconnection as above. Indeed, let us consider two input-state-output systems as in (2.51), for simplicity *without* external inputs d and external outputs z,

$$\Sigma_i : \begin{cases} \dot{x}_i = \left[J_i(x_i) - R_i(x_i) \right] \dfrac{\partial H_i}{\partial x_i}(x_i) + g_i(x_i) u_i \\ y_i = g_i^{\rm T}(x_i) \dfrac{\partial H_i}{\partial x_i}(x_i) \end{cases} \qquad x_i \in \mathscr{X}_i \tag{2.181}$$

for $i = 1, 2$. The standard feedback interconnection

$$u_1 = -y_2 \qquad\qquad u_2 = y_1 \tag{2.182}$$

is equal to the negative gyrative interconnection between the flows u_1, u_2 and the efforts y_1, y_2. The closed-loop system is the port-Hamiltonian system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} J_1(x) - R_1(x_1) & -g_1(x_1) g_2^{\mathrm{T}}(x_2) \\ g_2(x_2) g_1^{\mathrm{T}}(x_1) & J_2(x_2) - R_2(x_2) \end{bmatrix} \begin{bmatrix} \frac{\partial H_1}{\partial x_1}(x_1) \\ \frac{\partial H_2}{\partial x_2}(x_2) \end{bmatrix}$$

with state space $\mathscr{X}_1 \times \mathscr{X}_2$ and Hamiltonian $H_1(x_1) + H(x_2)$.

2.5.2 Regularity of interconnections

In this subsection we study a particular property in the composition of Dirac structures, namely the property that the variables corresponding to the ports through which the connection takes place (the *internal* port variables) are uniquely determined by the values of the *external* port variables[7]. The concept of regularity can thus be regarded as a kind of *observability* property (inferring the values of the internal variables from the values of the external port variables).

Definition 2.3. Given two Dirac structures $\mathscr{D}_A \subset \mathscr{F}_1 \times \mathscr{F}_1^* \times \mathscr{F}_2 \times \mathscr{F}_2^*$ and $\mathscr{D}_B \subset \mathscr{F}_2 \times \mathscr{F}_2^* \times \mathscr{F}_3 \times \mathscr{F}_3^*$. Their composition is said to be *regular* if the values of the port variables in $\mathscr{F}_2 \times \mathscr{F}_2^*$ are uniquely determined by the values of the port variables in $\mathscr{F}_1 \times \mathscr{F}_1^* \times \mathscr{F}_3 \times \mathscr{F}_3^*$; that is, the following implication holds:

$$\left.\begin{array}{l} (f_1, e_1, f_2, e_2) \in \mathscr{D}_A,\ (-f_2, e_2, f_3, e_3) \in \mathscr{D}_B \\ (f_1, e_1, \tilde{f}_2, \tilde{e}_2) \in \mathscr{D}_A,\ (-\tilde{f}_2, \tilde{e}_2, f_3, e_3) \in \mathscr{D}_B \end{array}\right\} \Longrightarrow f_2 = \tilde{f}_2,\ e_2 = \tilde{e}_2 \tag{2.183}$$

The following proposition yields an explicit characterization for regularity in terms of the matrix M defined in the previous theorem.

Proposition 2.2. *The composition of two Dirac structures $\mathscr{D}_A$ and $\mathscr{D}_B$ given in matrix kernel representation by $([F_1 \,|\, F_{2A}], [E_1 \,|\, E_{2A}])$ and $([F_3 \,|\, F_{2B}], [E_3 \,|\, E_{2B}])$, respectively, is a regular interconnection if and only if the $(n_1 + 2n_2 + n_3) \times 2n_2$ matrix M defined in (2.159) is of full rank ($= 2n_2$).*

Proof. Let $(f_1, e_1, f_3, e_3) \in \mathscr{D}_A \parallel \mathscr{D}_B$, and let (f_2, e_2) be such that $(f_1, e_1, f_2, e_2) \in \mathscr{D}_A$, $(f_3, e_3, -f_2, e_2) \in \mathscr{D}_B$. Then, by linearity, it is easy to prove that (f_2', e_2') satisfies $(f_1, e_1, f_2', e_2') \in \mathscr{D}_A$ and $(f_3, e_3, -f_2', e_2') \in \mathscr{D}_B$ if and only if $(\tilde{f}_2, \tilde{e}_2) := (f_2 - f_2', e_2 - e_2')$ satisfies

$$\begin{cases} (0, 0, \tilde{f}_2, \tilde{e}_2) \in \mathscr{D}_A \\ (0, 0, -\tilde{f}_2, \tilde{e}_2) \in \mathscr{D}_B \end{cases}$$

[7] The concept of regularity of compositions of Dirac structures is related to the concept of *regular interconnection* of linear dynamical systems in the behavioral framework [220]; see [45] for details.

In matrix kernel representation, this means that

$$\begin{cases} \left[F_1 \,|\, E_1 \,|\, F_{2A} \,|\, E_{2A}\right] \left[0\ 0\ \tilde{f}_2^{\mathrm{T}}\ \tilde{e}_2^{\mathrm{T}}\right]^{\mathrm{T}} = 0 \\ \left[F_3 \,|\, E_3 \,|\, -F_{2B} \,|\, E_{2B}\right] \left[0\ 0\ \tilde{f}_2^{\mathrm{T}}\ \tilde{e}_2^{\mathrm{T}}\right]^{\mathrm{T}} = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \left[F_{2A} \,|\, E_{2A}\right] \left[\tilde{f}_2^{\mathrm{T}}\ \tilde{e}_2^{\mathrm{T}}\right]^{\mathrm{T}} = 0 \\ \left[F_{2B} \,|\, -E_{2B}\right] \left[\tilde{f}_2^{\mathrm{T}}\ \tilde{e}_2^{\mathrm{T}}\right]^{\mathrm{T}} = 0 \end{cases}$$

which is equivalent to

$$\left[\tilde{f}_2^{\mathrm{T}}\ \tilde{e}_2^{\mathrm{T}}\right]^{\mathrm{T}} \in \ker M$$

Hence $\tilde{f}_2 = 0$ and $\tilde{e}_2 = 0$ if and only if $\ker M = 0$.

Example 2.19 (Example 2.17, continued). Consider again a port-Hamiltonian system in kernel representation. For effort constraints $e = 0$ the matrix M is given as

$$M = \begin{bmatrix} F & E \\ 0 & I \end{bmatrix}$$

which has full rank if and only if $\operatorname{rank} F = \dim \mathscr{F}$. Similarly, the interconnection corresponding to flow constraints $f = 0$ is regular if and only if $\operatorname{rank} E = \dim \mathscr{F}$.

Example 2.20. A typical case of the preceding example is an input-state-output port-Hamiltonian system with output constraints. Indeed, consider an input-state-output port-Hamiltonian system with Dirac structure $\mathscr{D}_A$ the graph of the skew-symmetric map

$$\begin{bmatrix} -J & -g \\ g^{\mathrm{T}} & 0 \end{bmatrix} \tag{2.184}$$

Consider the composition of $\mathscr{D}_A$ with the Dirac structure $\mathscr{D}_B$ corresponding to the zero-output constraint $y = g^{\mathrm{T}} \frac{\partial H}{\partial x} = 0$. In kernel representation $\mathscr{D}_A$ is given as

$$\mathscr{D}_A = \left\{ (f_S, e_S, u, y) \;\middle|\; \begin{bmatrix} I \\ 0 \end{bmatrix} f_S + \begin{bmatrix} J \\ g^{\mathrm{T}} \end{bmatrix} e_S + \begin{bmatrix} g \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -I \end{bmatrix} y = 0 \right\} \tag{2.185}$$

while $\mathscr{D}_B$ is given by

$$\mathscr{D}_B = \left\{ (u, y) \;\middle|\; 0 \cdot u + I \cdot y = 0 \right\} \tag{2.186}$$

Hence the matrix M in this case is given by

$$M = \begin{bmatrix} g & 0 \\ 0 & I \\ 0 & I \end{bmatrix}$$

which has full rank if and only if $\operatorname{rank} g = \dim y$. Hence, the composition is not regular if $\operatorname{rank} g < \dim y$, in which case the input variable u is not uniquely determined. This type of irregularity is common in mechanical systems where dependent kinematic constraints lead to non-uniqueness of the constraint forces. A typical case is a table with four legs standing on the ground.

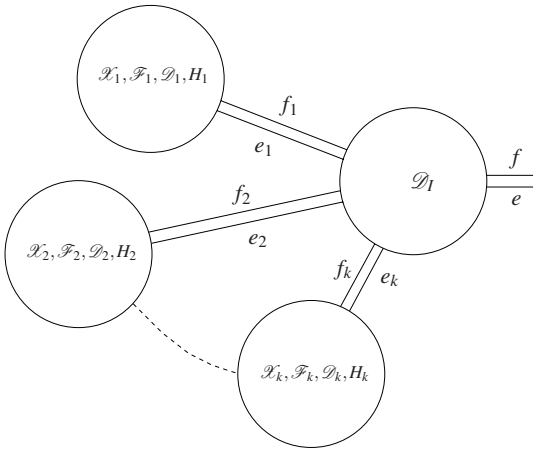

Fig. 2.8 Interconnection of port-Hamiltonian systems

The standard feedback interconnection is an obvious example of a regular interconnection, since the interconnection variables (f_2, e_2) are uniquely determined as being the *outputs* of one of the two interconnected systems.

2.5.3 Interconnection of port-Hamiltonian systems

The result derived in Section 2.5.1 concerning the compositionality of Dirac structures immediately leads to the result that any power-conserving interconnection of port-Hamiltonian systems again defines a port-Hamiltonian system. This can be regarded as a fundamental building block in the theory of port-Hamiltonian systems. The result not only means that the theory of port-Hamiltonian systems is a completely modular theory for modelling, but it also serves as a starting point for design and control.

Consider k port-Hamiltonian systems $(\mathscr{X}_i, \mathscr{F}_i, \mathscr{D}_i, H_i)$, $i = 1, \cdots, k$, interconnected by a Dirac structure $\mathscr{D}_I$ on $\mathscr{F}_1 \times \cdots \times \mathscr{F}_k \times \mathscr{F}$, with $\mathscr{F}$ a linear space of flow port variables, cf. Fig. 2.8. This can be seen to define a port-Hamiltonian system $(\mathscr{X}, \mathscr{F}, \mathscr{D}, H)$, where $\mathscr{X} := \mathscr{X}_1 \times \cdots \times \mathscr{X}_k$, $H := H_1 + \cdots + H_k$, and where the Dirac structure $\mathscr{D}$ on $\mathscr{X} \times \mathscr{F}$ is determined by $\mathscr{D}_1, \cdots, \mathscr{D}_k$ and $\mathscr{D}_I$. Indeed, consider the *product* of the Dirac structures $\mathscr{D}_1, \cdots, \mathscr{D}_k$ on $(\mathscr{X}_1 \times \mathscr{F}_1) \times (\mathscr{X}_2 \times \mathscr{F}_2) \times \cdots \times (\mathscr{X}_k \times \mathscr{F}_k)$, and compose this with the Dirac structure $\mathscr{D}_I$ on $(\mathscr{F}_1 \times \cdots \times \mathscr{F}_k) \times \mathscr{F}$. This yields a total Dirac structure $\mathscr{D}$ modulated by $x = (x_1, \cdots, x_k) \in \mathscr{X} = \mathscr{X}_1 \times \cdots \times \mathscr{X}_k$ which is point-wise given as

$$\mathscr{D}(x_1, \cdots, x_k) \subset T_{x_1}\mathscr{X}_1 \times T^*_{x_1}\mathscr{X}_1 \times \cdots \times T_{x_k}\mathscr{X}_k \times T^*_{x_k}\mathscr{X}_k \times \mathscr{F} \times \mathscr{F}^*$$

Finally we mention that the theory of composition of Dirac structures and the interconnection of port-Hamiltonian systems can be also extended to *infinite-dimensional* Dirac structures and port-Hamiltonian systems [107, 164].

2.6 Analysis of port-Hamiltonian systems

In this section, we will briefly discuss some of the structural properties of port-Hamiltonian systems. More details, and their application for control, can be found in Chapter 5. First, we will discuss the implications of the energy-balance that is underlying the port-Hamiltonian structure. Next, we will discuss the existence of *conserved quantities*, independent of the Hamiltonian, which may be analyzed on the basis of the Dirac structure. Dually, we will briefly discuss the determination of algebraic constraints based on the Dirac structure. Finally, for linear port-Hamiltonian systems without energy dissipation, some issues of realization theory will be addressed.

2.6.1 Passivity

A basic property of any port-Hamiltonian system is its energy balance, cf. (2.35)

$$\frac{\mathrm{d}}{\mathrm{d}t}H = e_R^{\mathrm{T}} f_R + e_C^{\mathrm{T}} f_C + e_I^{\mathrm{T}} f_I \leq e_C^{\mathrm{T}} f_C + e_I^{\mathrm{T}} f_I \tag{2.187}$$

where the inequality holds because the term $e_R^{\mathrm{T}} f_R$ is always less than or equal to zero (by definition of the resistive elements). This implies that any port-Hamiltonian system is *passive* with respect to the *supply rate* $e_C^{\mathrm{T}} f_C + e_I^{\mathrm{T}} f_I$ and storage function H if H qualifies as a storage function, that is, if H is semi-positive definite, i.e. $H(x) \geq 0$ for all x. *In general*, the converse is not true, that is, not every passive system is a port-Hamiltonian system. This is illustrated by the following example.

Example 2.21. Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = x_1^2 x_2$$

which is passive (in fact, loss-less) with respect to the storage function $H(x_1, x_2) = \frac{1}{2} x_1^2 x_2^2$. However, it is easy to see that there does *not* exist a 2×2 matrix $J(x) = -J^{\mathrm{T}}(x)$, with entries depending smoothly on x, such that

$$\begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} = J(x) \begin{bmatrix} x_1 x_2^2 \\ x_1^2 x_2 \end{bmatrix}$$

However, for a *linear* input-state-output system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{2.188}$$

we will show that, under a natural extra condition, every passive system can be written as a port-Hamiltonian system.

We consider linear port-Hamiltonian input-state-output systems *with feed-through term* (see Sect. 2.2.4)

$$\begin{aligned} \dot{x}(t) &= \big(J-R\big)Qx(t) + \big(G-P\big)u(t) \\ y(t) &= \big(G+P\big)^{\mathrm{T}}Qx(t) + \big(M+S\big)u(t) \end{aligned} \tag{2.189}$$

where J is a skew-symmetric $n \times n$ matrix and M is a skew-symmetric $k \times k$ matrix. The Hamiltonian $H(x)$ (the energy of the system) is given by the quadratic function $H(x) = \frac{1}{2}x^{\mathrm{T}}Qx$, where Q is a symmetric $n \times n$ matrix. Furthermore, R is an $n \times n$ symmetric matrix, S is a symmetric $k \times k$ matrix, and G and P are $n \times k$ matrices, satisfying the following condition:

$$\begin{bmatrix} R & P \\ P^{\mathrm{T}} & S \end{bmatrix} \geq 0 \tag{2.190}$$

In particular, if $P = 0$, then this condition reduces to the condition that $R \geq 0$ and $S \geq 0$.

Theorem 2.3. *The following properties hold:*

1. *If the system (2.188) is passive with quadratic storage function $\frac{1}{2}x^{T}Qx$ satisfying $Q \geq 0$, and $Qx = 0$ implies $Ax = 0$ and $Cx = 0$, then (2.188) can be rewritten into the port-Hamiltonian form (2.189).*
2. *If $Q \geq 0$, then the port-Hamiltonian linear system (2.189) is passive.*

Remark 2.9. Note that the condition ($Qx = 0 \Rightarrow Ax = 0,\ Cx = 0$) is automatically satisfied if $Q > 0$.

Proof. Because of the condition ($Qx = 0 \Rightarrow Ax = 0,\ Cx = 0$), it follows from linear algebra that there exists a matrix Σ such that

$$\begin{bmatrix} A & B \\ -C & -D \end{bmatrix} = \Sigma \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \tag{2.191}$$

In fact, if $Q > 0$ then such a Σ is uniquely defined as

$$\Sigma := \begin{bmatrix} AQ^{-1} & B \\ -CQ^{-1} & -D \end{bmatrix} \tag{2.192}$$

Now, passivity of the system (2.188) with quadratic storage function $\frac{1}{2}x^{\mathrm{T}}Qx$ amounts to the differential dissipation inequality

$$x^{\mathrm{T}}Q\dot{x} - u^{\mathrm{T}}y \leq 0 \tag{2.193}$$

for all x, u. Substituting $\dot{x} = Ax + Bu$ and $y = Cx + Du$, and making use of (2.191), this can be rewritten as

$$\begin{bmatrix} x^{\mathrm{T}} & u^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \Sigma \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq 0 \tag{2.194}$$

for all x, u, or equivalently

$$\begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \left(\Sigma + \Sigma^{\mathrm{T}}\right) \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \leq 0 \tag{2.195}$$

It follows from basic linear algebra that we can choose Σ satisfying (2.191) in such a way that

$$\Sigma + \Sigma^{\mathrm{T}} \leq 0 \tag{2.196}$$

Hence, if we write $\Sigma = \bar{J} - \bar{R}$, with $\bar{J} = -\bar{J}^{\mathrm{T}}$ and $\bar{R} = \bar{R}^{\mathrm{T}}$, then $\bar{R} \geq 0$. Now, denote

$$\bar{J} = \begin{bmatrix} J & G \\ -G^{\mathrm{T}} & -M \end{bmatrix} \qquad \bar{R} = \begin{bmatrix} R & P \\ P^{\mathrm{T}} & S \end{bmatrix} \tag{2.197}$$

with $J = -J^{\mathrm{T}}$, $M = -M^{\mathrm{T}}$, $R = R^{\mathrm{T}}$ and $S = S^{\mathrm{T}}$. Then, (2.188) can be written as

$$\begin{bmatrix} \dot{x} \\ -y \end{bmatrix} = \left\{ \begin{bmatrix} J & G \\ -G^{\mathrm{T}} & -M \end{bmatrix} - \begin{bmatrix} R & P \\ P^{\mathrm{T}} & S \end{bmatrix} \right\} \begin{bmatrix} Qx \\ u \end{bmatrix} \tag{2.198}$$

or equivalently

$$\begin{cases} \dot{x} = \big(J - R\big)Qx + \big(G - P\big)u \\ y = \big(G + P\big)^{\mathrm{T}}Qx + (M + S)u, \end{cases} \tag{2.199}$$

which is a system with Hamiltonian dynamics (2.189).

Secondly, we show that port-Hamiltonian linear systems (2.189) are passive with the Hamiltonian $H(x) = \frac{1}{2}x^{\mathrm{T}}Qx$ being a storage function. Along trajectories of the port-Hamiltonian linear system we have (time arguments left out for brevity):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}H(x) &= x^{\mathrm{T}}Q\dot{x} \\ &= y^{\mathrm{T}}u - \begin{bmatrix} (Qx)^{\mathrm{T}} & u^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} R & P \\ P^{\mathrm{T}} & S \end{bmatrix} \begin{bmatrix} Qx \\ u \end{bmatrix} \\ &\leq y^{\mathrm{T}}u \end{aligned} \tag{2.200}$$

2.6.2 Casimirs of port-Hamiltonian systems

Passivity is a key property for stability analysis since the Hamiltonian function H may serve as a Lyapunov function. However, in some cases, the Hamiltonian H does not have a minimum at the equilibrium x^* under study, in which case H alone is not itself a Lyapunov function. A well-known method in Hamiltonian systems, sometimes called the Energy-Casimir method, is to use in the Lyapunov analysis, next to the Hamiltonian function, additional *conserved quantities* (dynamical invariants) which may be present in the system. Indeed, if we may find conserved quantities, then candidate Lyapunov functions can be sought within the class of *combinations* of the Hamiltonian H and those conserved quantities.

Consider a port-Hamiltonian system without control and interaction ports, in which case the energy balance (2.187) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}H = e_R^{\mathrm{T}} f_R \leq 0 \tag{2.201}$$

Consider functions $C : \mathscr{X} \to \mathbb{R}$ such that $\frac{\mathrm{d}}{\mathrm{d}t}C = 0$ along the trajectories of the system. Such a function C is called a *conserved quantity*. Suppose that we can find a conserved quantity $C : \mathscr{X} \to \mathbb{R}$ such that $V := H + C$ has a minimum at the equilibrium x^*. Then we can still infer stability or asymptotic stability by replacing (2.187) by

$$\frac{\mathrm{d}}{\mathrm{d}t}V = e_R^{\mathrm{T}} f_R \leq 0 \tag{2.202}$$

and thus using V as a Lyapunov function.

Functions that are conserved quantities of the system for *every* Hamiltonian are called Casimir functions. It turns out that Casimirs are completely characterized by the Dirac structure of the port-Hamiltonian system. Indeed, a function $C : \mathscr{X} \to \mathbb{R}$ is a Casimir function of the autonomous port-Hamiltonian system (without resistive, control and interaction port) $\Sigma = (\mathscr{X}, H, \mathscr{D})$, if and only if the gradient vector $e = \frac{\partial^{\mathrm{T}} C}{\partial x}$ satisfies

$$e^{\mathrm{T}} f_S = 0, \text{ for all } f_S \text{ for which } \exists e_S \text{ s.t. } (f_S, e_S) \in \mathscr{D} \tag{2.203}$$

Indeed, (2.203) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}C = \frac{\partial^{\mathrm{T}} C}{\partial x}(x(t))\dot{x}(t) = \frac{\partial^{\mathrm{T}} C}{\partial x}(x(t)) f_S = e^{\mathrm{T}} f_S = 0 \tag{2.204}$$

for every port-Hamiltonian system $(\mathscr{X}, H, \mathscr{D})$ with the same Dirac structure $\mathscr{D}$. By the power-conservation property of the Dirac structure, (2.203) is readily seen to be equivalent to the requirement that $e = \frac{\partial^{\mathrm{T}} C}{\partial x}$ satisfies

$$(0, e) \in \mathscr{D}$$

Indeed, let e satisfy (2.203). Then

$$e^{\mathrm{T}} f_S + e_S^{\mathrm{T}} \cdot 0 = 0 \tag{2.205}$$

for all $(f_S, e_S) \in \mathscr{D}$, implying that $(0,e) \in \mathscr{D}^\perp = \mathscr{D}$. Conversely, if $(0,e) \in \mathscr{D}$ then (2.205) holds for all $(f_S, e_S) \in \mathscr{D}$ implying (2.203).

Example 2.22. The well-known Casimir in the case of a spinning rigid body (Example 2.3) is the total angular momentum $p_x^2 + p_y^2 + p_z^2$ (whose vector of partial derivatives is in the kernel of the matrix $J(p)$ in (2.40). Similarly, in the LC-circuit of Example 2.5 the total flux $\phi_1 + \phi_2$ for $u = 0$ is a Casimir.

Example 2.23. Consider a mechanical system with kinematic constraints (2.47) with $u = 0$. Then $(0,e) \in \mathscr{D}$ if and only if

$$0 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} e + \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda \qquad\qquad \begin{bmatrix} 0 & A^{\mathrm{T}}(q) \end{bmatrix} e = 0$$

Partioning $e = \begin{bmatrix} e_q^{\mathrm{T}} & e_p^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ this means that $e_q = A(q)\lambda$, or equivalently, $e_q \in \mathrm{Im} A(q)$. Since in general $A(q)$ is depending on q finding Casimirs now involves an additional *integrability condition*, see also Sect. 2.7. In fact, Casimirs correspond to vectors $e_q \in \mathrm{Im} A(q)$ which additionally can be written as a vector of partial derivatives $\frac{\partial C}{\partial q}(q)$ for some function $C(q)$ (the Casimir). In the case of Example 2.14 it can be verified that this additional integrability condition is *not* satisfied, corresponding to the fact that the kinematic constraints in this example are completely *nonholonomic*. On the other hand, in Example 2.13, where the kinematic constraints are *holonomic*, a Casimir is given by $C(q_1, q_2) = q_1 - q_2$. In general it can be shown [182] that there exist as many independent Casimirs as the rank of the matrix $A(q)$ if and only if the kinematic constraints are holonomic, in which case the Casimirs are equal to the integrated kinematic constraints.

2.6.2.1 Casimirs in the presence of resistive elements

Similarly, we define a Casimir function for a port-Hamiltonian system with dissipation $\Sigma = (\mathscr{X}, H, \mathscr{R}, \mathscr{D})$ to be any function $C : \mathscr{X} \to \mathbb{R}$ satisfying

$$\left(0, e = \frac{\partial^{\mathrm{T}} C}{\partial x}, 0, 0\right) \in \mathscr{D} \tag{2.206}$$

Indeed, this will imply that

$$\frac{\mathrm{d}}{\mathrm{d}t} C = \frac{\partial^{\mathrm{T}} C}{\partial x}(x(t))\dot{x}(t) = \frac{\partial^{\mathrm{T}} C}{\partial x}(x(t)) f_S = e^{\mathrm{T}} f_S = 0 \tag{2.207}$$

for every port-Hamiltonian system $(\mathscr{X}, H, \mathscr{R}, \mathscr{D})$ with the same Dirac structure $\mathscr{D}$. In fact, every element $(0, e, 0, 0) \in \mathscr{D}$ satisfies

$$0 = \ll (0, e, 0, 0), (f_S, e_S, f_R, e_R) \gg = e^{\mathrm{T}} f_S$$

for all $(f_S, e_S, f_R, e_R) \in \mathscr{D}$, thus implying for $e = \frac{\partial C}{\partial x}$ the previous equality (2.207).

At this point, one might suspect that the definition of Casimir function may be *relaxed* by requiring that (2.206) only holds for a *specific* resistive relation

$$R_f f_R + R_e e_R = 0 \tag{2.208}$$

As before, here the square matrices R_f and R_e satisfy the symmetry and semi-positive definiteness condition (see (2.25))

$$R_f R_e^{\mathrm{T}} = R_e R_f^{\mathrm{T}} \geq 0 \tag{2.209}$$

together with the rank condition

$$\operatorname{rank} \left[R_f \,|\, R_e \right] = \dim f_R \tag{2.210}$$

We show that actually this is *not* a relaxation, if the required semi-definite positiveness of the resistive relation is strengthened to positive-definiteness

$$R_f R_e^{\mathrm{T}} = R_e R_f^{\mathrm{T}} > 0 \tag{2.211}$$

In this case the condition for a function to be a Casimir for *one* resistive relation will *imply* that it is a Casimir for *all* resistive relations.

Indeed, let $C : \mathscr{X} \to \mathbb{R}$ be a function satisfying (2.206) for a specific resistive port $\mathscr{R}$ defined by matrices R_f and R_e as above. This means that $e = \frac{\partial C}{\partial x}(x)$ satisfies

$$e^{\mathrm{T}} f_S = 0$$

for all f_S for which there exist e_S, f_R and e_R such that

$$(f_S, e_S, f_R, e_R) \in \mathscr{D} \qquad\qquad R_f f_R + R_e e_R = 0$$

This implies that $(0, e) \in (\mathscr{D} \parallel \mathscr{R})^{\perp}$.

At this point we note that the proof of the composition of two Dirac structures (see Theorem 2.1) resulting in a new Dirac structure, immediately extends to the composition of a Dirac structure and a resistive structure:

Proposition 2.3. *Let $\mathscr{D}$ be a Dirac structure defined with respect to $\mathscr{F}_S \times \mathscr{F}_S^* \times \mathscr{F}_R \times \mathscr{F}_R^*$. Furthermore, let $\mathscr{R}$ be a resistive structure defined with respect to $\mathscr{F}_R \times \mathscr{F}_R^*$ given by*

$$R_f f_R + R_e e_R = 0 \tag{2.212}$$

where the square matrices R_f and R_e satisfy the symmetry and semi-positive definiteness condition

$$R_f R_e^T = R_e R_f^T \geq 0 \tag{2.213}$$

Define the composition $\mathscr{D} \parallel \mathscr{R}$ of the Dirac structure and the resistive structure in the same way as the composition of two Dirac structures. Then

$$(\mathscr{D} \parallel \mathscr{R})^{\perp} = \mathscr{D} \parallel (-\mathscr{R}) \tag{2.214}$$

where $-\mathscr{R}$ denotes the pseudo-resistive structure[8] *given by*

$$R_f f_R - R_e e_R = 0 \tag{2.215}$$

Proof. We follow the same steps as in the above proof that the composition of two Dirac structures is again a Dirac structure, where we take $\mathscr{F}_1 = \mathscr{F}_S$, $\mathscr{F}_2 = \mathscr{F}_R$, and $\mathscr{F}_3$ void. Because of the sign difference in the definition of a resistive structure as compared with the definition of a Dirac structure, we immediately obtain the stated proposition.

Hence $(0,e) \in (\mathscr{D} \parallel \mathscr{R})^\perp = \mathscr{D} \parallel (-\mathscr{R})$, and thus there exist $\tilde{f}_R, \tilde{e}_R$ such that $R_f \tilde{f}_R - R_e \tilde{e}_R = 0$ and satisfying $(0,e,\tilde{f}_R,\tilde{e}_R) \in \mathscr{D}$. Therefore

$$0 = e^{\mathrm{T}} \cdot 0 + \tilde{e}_R^{\mathrm{T}} \tilde{f}_R = \tilde{e}_R^{\mathrm{T}} \tilde{f}_R$$

By writing the pseudo-resistive structure $-\mathscr{R}$ in image representation $\tilde{f}_R = R_e^{\mathrm{T}} \lambda$ and $\tilde{e}_R = R_f^{\mathrm{T}} \lambda$, it follows that

$$\lambda^{\mathrm{T}} R_f R_e^{\mathrm{T}} \lambda = 0$$

and, by the positive-definiteness condition $R_f R_e^{\mathrm{T}} = R_e R_f^{\mathrm{T}} > 0$, this implies that $\lambda = 0$, whence $\tilde{f}_R = \tilde{e}_R = 0$. Hence not only $(0,e,\tilde{f}_R,\tilde{e}_R) \in \mathscr{D}$, but actually $(0,e,0,0) \in \mathscr{D}$, implying that e is the gradient of a Casimir function as defined before.

Of course, the above argument does not fully carry through if the resistive relations are only *semi*-positive definite. In particular, this is the case if $R_f R_e^{\mathrm{T}} = 0$ (implying *zero* energy dissipation), corresponding to the presence of ideal power-conserving constraints. In fact, if $R_f R_e^{\mathrm{T}} = 0$, then the resistive structure reduces to a particular type of Dirac structure.

The fact that a Casimir for one resistive relation is actually a Casimir for *all* resistive relations is closely related to the so-called *dissipation obstacle* for the existence of Casimir functions in the case of input-state-output port-Hamiltonian systems, cf. [160, 161, 180]. In fact, let us consider a Casimir $C : \mathscr{X} \to \mathbb{R}$ for an input-state-output port-Hamiltonian system without external (control and interaction) ports for a specific resistive structure, that is

$$\frac{\partial^{\mathrm{T}} C}{\partial x}(x)\,(J(x) - R(x)) = 0$$

Post-multiplication by $\frac{\partial C}{\partial x}(x)$ immediately implies

$$\frac{\partial^{\mathrm{T}} C}{\partial x}(x)\,(J(x) - R(x))\,\frac{\partial C}{\partial x}(x) = 0$$

Then, by transposition, we obtain a second equation

[8] $-\mathscr{R}$ is called a pseudo-resistive structure since it corresponds to a *negative* instead of a positive resistance.

$$\frac{\partial^{\mathrm{T}} C}{\partial x}(x)\left(-J(x)-R(x)\right)\frac{\partial C}{\partial x}(x)=0$$

Adding these two equations one derives

$$\frac{\partial^{\mathrm{T}} C}{\partial x}(x)R(x)\frac{\partial C}{\partial x}(x)=0$$

which by positive semi-definiteness of $R(x)$ implies

$$\frac{\partial^{\mathrm{T}} C}{\partial x}(x)R(x)=0$$

(and then also $\frac{\partial^{\mathrm{T}} C}{\partial x}(x)J(x)=0$). This so-called dissipation obstacle implies that Casimirs are necessarily independent of those state space coordinates that are directly affected by physical damping. We conclude that the dissipation obstacle for finding Casimirs directly generalizes to port-Hamiltonian systems defined with respect to a general Dirac structure, and that it is equivalent to the observation that a function that is a conserved quantity with respect to one positive definite resistive relation is necessarily conserved for all resistive relations.

2.6.3 Algebraic constraints of port-Hamiltonian systems

Algebraic constraints on the state variables are primarily determined by the Dirac structure, as well as by the Hamiltonian. Indeed, let us first consider a port-Hamiltonian system without external and resistive ports, described by a Dirac structure $\mathscr{D}$ and a Hamiltonian H. Define for every $x\in\mathscr{X}$ the subspace

$$P_{\mathscr{D}}(x):=\left\{\alpha\in T_x^*\mathscr{X}\mid \exists X\in T_x\mathscr{X} \text{ such that } (\alpha,X)\in\mathscr{D}(x)\right\} \tag{2.216}$$

This defines a *co-distribution* on the manifold $\mathscr{X}$. Then the definition of the port-Hamiltonian system implies that

$$\frac{\partial H}{\partial x}(x)\in P_{\mathscr{D}}(x) \tag{2.217}$$

In general, this imposes algebraic constraints on the state variables $x\in\mathscr{X}$. In particular, if the Dirac structure is given in image representation as

$$\mathscr{D}(x)=\left\{(X,\alpha)\in T_x\mathscr{X}\times T_x^*\mathscr{X}\mid X=E^{\mathrm{T}}(x)\lambda,\ \alpha=F^{\mathrm{T}}(x)\lambda\right\} \tag{2.218}$$

then it follows that

$$\frac{\partial H}{\partial x}(x)\in \mathrm{Im}\,F^{\mathrm{T}}(x)$$

In the case of external ports, these algebraic constraints on the state variables x may also depend on the external port variables. A special case arises for resistive ports. Consider a Dirac structure

$$\left\{(X,\alpha,f_R,e_R)\in\mathscr{D}(x)\subset T_x\mathscr{X}\times T_x^*\mathscr{X}\times\mathscr{F}_R\times\mathscr{F}_R^*\right\}$$

with the resistive flow and effort variables satisfying a relation

$$R(f_R,e_R)=0$$

Then the gradient of the Hamiltonian has to satisfy the condition

$$\frac{\partial H}{\partial x}(x)\in\Big\{\alpha\in T_x^*\mathscr{X}\mid\exists X,f_R,e_R\in T_x\mathscr{X}\times\mathscr{F}_R\times\mathscr{F}_R^* \\ \text{s.t. } (X,\alpha,f_R,e_R)\in\mathscr{D}(x),R(f_R,e_R)=0\Big\}$$

Depending on the resistive relation $R(f_R,e_R)=0$ this may again induce algebraic constraints on the state variables x.

2.7 Integrability of modulated Dirac structures

A key issue in the case of modulated Dirac structures is that of *integrability*. Loosely speaking, a Dirac structure is *integrable* if it is possible to find local coordinates for the state space manifold such that, in these coordinates, the Dirac structure becomes a *constant* Dirac structure, that is, it is *not* modulated anymore by the state variables.

First let us consider modulated Dirac structures which are given for every $x\in\mathscr{X}$ as the *graph* of a skew-symmetric mapping $J(x)$ from the co-tangent space $T_x^*\mathscr{X}$ to the tangent space $T_x\mathscr{X}$ (see also the discussion at the end of Sect. 2.1.1). Integrability in this case means that the structure matrix J satisfies the conditions

$$\sum_{l=1}^{n}\left[J_{lj}(x)\frac{\partial J_{ik}}{\partial x_l}(x)+J_{li}(x)\frac{\partial J_{kj}}{\partial x_l}(x)+J_{lk}(x)\frac{\partial J_{ji}}{\partial x_l}(x)\right]=0 \tag{2.219}$$

with $i,j,k=1,\ldots,n$. In this case we may find, by Darboux's theorem (see e.g. [217]) around any point x_0 where the rank of the matrix $J(x)$ is constant, local coordinates $x=(q,p,r)$ in which the matrix $J(x)$ becomes the constant skew-symmetric matrix

$$\begin{bmatrix}0&-I_k&0\\I_k&0&0\\0&0&0\end{bmatrix} \tag{2.220}$$

Such coordinates are called *canonical*. A skew-symmetric matrix $J(x)$ satisfying (2.219) defines a *Poisson bracket* on $\mathscr{X}$, given for every $F,G:\mathscr{X}\to\mathbb{R}$ as

$$\{F,G\} = \frac{\partial^{\mathrm{T}} F}{\partial x} J(x) \frac{\partial G}{\partial x} \tag{2.221}$$

Indeed, by (2.219) the Poisson bracket satisfies the *Jacobi-identity*

$$\{F,\{G,K\}\} + \{G,\{K,F\}\} + \{K,\{F,G\}\} = 0 \tag{2.222}$$

for all functions F, G, K.

The choice of coordinates $x = (q,p,r)$ for the state space manifold also induces a basis for $T_x\mathscr{X}$ and a dual basis for $T_x^*\mathscr{X}$. Denoting the corresponding splitting for the flows by $f = (f_q, f_p, f_r)$ and for the efforts by $e = (e_q, e_p, e_r)$, the Dirac structure defined by J in canonical coordinates is seen to be given by

$$\mathscr{D} = \Big\{ (f_q, f_p, f_r, e_q, e_p, e_r) \mid f_q = -e_p,\ f_p = e_q,\ f_r = 0 \Big\} \tag{2.223}$$

A similar story can be told for the case of a Dirac structure given as the graph of a skew-symmetric mapping $\omega(x)$ from the tangent space $T_x\mathscr{X}$ to the co-tangent space $T_x^*\mathscr{X}$. In this case the integrability conditions take the (slightly simpler) form

$$\frac{\partial \omega_{ij}}{\partial x_k}(x) + \frac{\partial \omega_{ki}}{\partial x_j}(x) + \frac{\partial \omega_{jk}}{\partial x_i}(x) = 0 \qquad i,j,k = 1,\ldots,n \tag{2.224}$$

The skew-symmetric matrix $\omega(x)$ can be regarded as the coordinate representation of a *differential two-form* ω on the manifold $\mathscr{X}$, that is $\omega = \sum_{i=1,j=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}x_j$, and the integrability condition (2.224) corresponds to the *closedness* of this two-form ($\mathrm{d}\omega = 0$). The differential two-form ω is called a *pre-symplectic structure*, and a *symplectic structure* if the rank of $\omega(x)$ is equal to the dimension of $\mathscr{X}$. If (2.224) holds, then again by a version of Darboux's theorem we may find, around any point x_0 where the rank of the matrix $\omega(x)$ is constant, local coordinates $x = (q,p,s)$ in which the matrix $\omega(x)$ becomes the constant skew-symmetric matrix

$$\begin{bmatrix} 0 & I_k & 0 \\ -I_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.225}$$

Such coordinates are again called *canonical*. The choice of coordinates $x = (q,p,s)$ as before induces a basis for $T_x\mathscr{X}$ and a dual basis for $T_x^*\mathscr{X}$. Denoting the corresponding splitting for the flows by $f = (f_q, f_p, f_s)$ and for the efforts by $e = (e_q, e_p, e_s)$, the Dirac structure corresponding to ω in canonical coordinates is seen to be given by

$$\mathscr{D} = \Big\{ (f_q, f_p, f_s, e_q, e_p, e_s) \mid f_q = -e_p,\ f_p = e_q,\ e_s = 0 \Big\} \tag{2.226}$$

In case of a symplectic structure the variables s are absent and the Dirac structure reduces to

$$\mathscr{D} = \Big\{ (f_q, f_p, e_q, e_p) \mid f_q = -e_p,\ f_p = e_q \Big\} \tag{2.227}$$

which is the standard *symplectic gyrator*.

For general Dirac structures, integrability is defined in the following way, [67]

Definition 2.4. A Dirac structure $\mathscr{D}$ on $\mathscr{X}$ is *integrable* if for arbitrary pairs of smooth vector fields and differential one-forms $(X_1,\alpha_1),(X_2,\alpha_2),(X_3,\alpha_3)\in\mathscr{D}$ there holds

$$\langle L_{X_1}\alpha_2 \mid X_3\rangle + \langle L_{X_2}\alpha_3 \mid X_1\rangle + \langle L_{X_3}\alpha_1 \mid X_2\rangle = 0 \tag{2.228}$$

with L_{X_i} denoting the Lie-derivative.

Remark 2.10 (Pseudo-Dirac structures). In the usual definition of Dirac structures on manifolds (see [54, 67]), this *integrability* condition is *included* in the definition. Dirac structures that do *not* satisfy this integrability condition are therefore sometimes called *pseudo*-Dirac structures.

The above integrability condition for Dirac structures generalizes properly the closedness of symplectic forms and the Jacobi identity for Poisson brackets as discussed before. In particular, for Dirac structures given as the graph of a symplectic or Poisson structure, the notion of integrability is equivalent to the Jacobi-identity or closedness condition as discussed above (see e.g. [54, 59, 67] for details).

Note that a *constant* Dirac structure trivially satisfies the integrability condition. Conversely, a Dirac structure satisfying the integrability condition together with an additional constant rank condition can be represented *locally* as a *constant* Dirac structure. The precise form of the constant rank condition can be stated as follows. For any Dirac structure $\mathscr{D}$, we may define the distribution

$$G_{\mathscr{D}}(x) = \left\{ X\in T_x\mathscr{X} \mid \exists\,\alpha\in T_x^*\mathscr{X} \text{ s.t. } (X,\alpha)\in\mathscr{D}(x)\right\}$$

Dually we may define the co-distribution

$$P_{\mathscr{D}}(x) = \left\{ \alpha\in T_x^*\mathscr{X} \mid \exists X\in T_x\mathscr{X} \text{ s.t. } (X,\alpha)\in\mathscr{D}(x)\right\}$$

We call x_0 a *regular* point for the Dirac structure if both the distribution $G_{\mathscr{D}}$ and the co-distribution $P_{\mathscr{D}}$ have constant dimension around x_0.

If the Dirac structure is integrable and x_0 is a regular point, then, again by a version of Darboux's theorem, we can choose local coordinates $x=(q,p,r,s)$ for $\mathscr{X}$ (with $\dim q=\dim p$), such that, in the resulting bases for (f_q,f_p,f_r,f_s) for $T_x\mathscr{X}$ and (e_q,e_p,e_r,e_s) for $T_x^*\mathscr{X}$, the Dirac structure on this coordinate neighborhood is given as (see (2.114))

$$\begin{cases} f_q = -e_p \\ f_p = e_q \\ f_r = 0 \\ e_s = 0 \end{cases} \tag{2.229}$$

Coordinates $x=(q,p,r,s)$ as above are again called *canonical*. Note that the choice of canonical coordinates for a Dirac structure satisfying the integrability condition encompasses the choice of canonical coordinates for a Poisson structure and for a (pre-)symplectic structure as above.

Explicit conditions for integrability of a Dirac structure can be readily stated in terms of a kernel/image representation. Indeed, let

$$\begin{aligned}\mathscr{D} &= \Big\{ (f,e) \mid F(x)f + E(x)e = 0 \Big\} \\ &= \Big\{ (f,e) \mid f = E^{\mathrm{T}}(x)\lambda,\ e = F^{\mathrm{T}}(x)\lambda,\ \lambda \in \mathbb{R}^n \Big\}\end{aligned}$$

Denote the transpose of i-th row of $E(x)$ by $Y_i(x)$ and the transpose of the i-th row of $F(x)$ by $\beta_i(x)$. The vectors $Y_i(x)$ are naturally seen as coordinate representations of *vector fields* while the vectors $\beta_i(x)$ are coordinate representations of *differential forms*. Then integrability of the Dirac structure is equivalent to the condition

$$\big\langle L_{Y_i}\beta_j \mid Y_k \big\rangle + \big\langle L_{Y_j}\beta_k \mid Y_i \big\rangle + \big\langle L_{Y_k}\beta_i \mid Y_j \big\rangle = 0 \tag{2.230}$$

for all indices $i,j,k = 1,\cdots,n$.

Another form of the integrability conditions can be obtained as follows. In [54, 59, 67] it has been shown that a Dirac structure on a manifold $\mathscr{X}$ is integrable if and only if, for all pairs of smooth vector fields and differential one-forms $(X_1,\alpha_1),(X_2,\alpha_2) \in \mathscr{D}$, it holds that

$$\Big([X_1,X_2], i_{X_1}\mathrm{d}\alpha_2 - i_{X_2}\mathrm{d}\alpha_1 + \mathrm{d}\,\langle \alpha_2 \mid X_1 \rangle \Big) \in \mathscr{D} \tag{2.231}$$

Using the definition of the vector fields Y_i and differential forms β_i, $i = 1,\cdots,n$, as above, it follows that the Dirac structure is integrable if and only if

$$\Big([Y_i,Y_j], i_{Y_i}\mathrm{d}\beta_j - i_{Y_j}\mathrm{d}\beta_i + \mathrm{d}\big\langle \beta_j \mid Y_i \big\rangle \Big) \in \mathscr{D} \tag{2.232}$$

for all $i,j = 1,\cdots,n$. This can be more explicitly stated by requiring that

$$F(x)[Y_i,Y_j](x) + E(x)\Big(i_{Y_i}\mathrm{d}\beta_j(x) - i_{Y_j}\mathrm{d}\beta_i(x) + \mathrm{d}\big\langle \beta_j \mid Y_i \big\rangle(x) \Big) = 0 \tag{2.233}$$

for all $i,j = 1,\cdots,n$ and for all $x \in \mathscr{X}$. See for more details [59].

Example 2.24 (Kinematic constraints). Recall from the discussion in Sect. 2.2.2 that the modulated Dirac structure corresponding to an actuated mechanical system subject to kinematic constraints $A^{\mathrm{T}}(q)\dot{q} = 0$ is given by

$$\begin{aligned}\mathscr{D} = \Big\{ (f_S,e_S,f_C,e_C) \mid\ & 0 = \begin{bmatrix} 0 & A^{\mathrm{T}}(q) \end{bmatrix} e_S,\ e_C = \begin{bmatrix} 0 & B^{\mathrm{T}}(q) \end{bmatrix} e_S, \\ & -f_S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} e_S + \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} f_c,\ \lambda \in \mathbb{R}^k \Big\}\end{aligned}$$

Complete necessary and sufficient conditions for integrability of this Dirac structure have been derived in [59]. Here we only state a slightly simplified version of this result, also detailed in [59]. We assume that the actuation matrix $B(q)$ has the special

form (often encountered in examples) where every j-th column ($j = 1, \ldots, m$) is given as

$$\begin{bmatrix} 0 \\ \frac{\partial C_j}{\partial q}(q) \end{bmatrix}$$

for some function $C_j(q)$ only depending on the configuration variables q. In this case, the Dirac structure $\mathscr{D}$ is integrable *if and only if the kinematic constraints are holonomic*.

It has been shown in Sect. 2.4.3 that, after elimination of the Lagrange multipliers and the algebraic constraints, the constrained mechanical system reduces to a port-Hamiltonian system on the constrained sub-manifold defined with respect to a Poisson structure matrix J_c. As has been shown in [182], J_c satisfies the integrability condition (2.219) again if and only if the constraints (2.42) are *holonomic*. In fact, if the constraints are holonomic, then the coordinates s as in (2.220) can be taken to be equal to the 'integrated constraint functions' $\overline{q}_{n-k+1}, \ldots, \overline{q}_n$ of (2.44), and the matrix g_s as in (2.71) is zero.

It can be verified that the structure matrix J_c obtained in Example 2.4, see (2.148), does not satisfy the integrability conditions, in accordance with the fact that the rolling constraints in this example are *non-holonomic*.

2.8 Scattering representation of Dirac structures and port-Hamiltonian systems

In this section, we show how, by using in the total space of port variables $\mathscr{F} \times \mathscr{F}^*$, a *different splitting* than the 'canonical' duality splitting (in flows $f \in \mathscr{F}$ and efforts $e \in \mathscr{F}^*$), we obtain other useful representations of Dirac structures and port-Hamiltonian systems.

2.8.1 What is "scattering"?

Scattering is a well known phenomena in physics and in network and communication theory: when a wave propagating in a material medium encounters discontinuities, its properties (direction, frequency or polarization) are changed, in strict relation with the intrinsic characteristics of the material through which the wave is propagating, [110]. Consequently, it seems to be *natural* that *scattering theory* is concerned with the effect obstacles or inhomogeneities have on an incident waves. There are two types of problems in this area:

- *The direct problem:* this problem is concerned with determining the scattered field from the knowledge of the incident field and the scattering obstacle;

- *The inverse problem:* this problem is, basically, the determination of the shape and/or physical properties of the scatterer from the measurement of the scattered field for a number of incident fields.

As it will be clear later, the results presented in this section can be interpreted as if developed within the direct problem framework.

Roughly speaking, the word *scattering* is related to the propagation of waves and to the change of their properties when *obstacles* are met. Consider a transmission line given by the interconnection of two parts, each of them characterized by different properties (i.e. different impedance): a traveling wave encounters an obstacle when it reaches the interconnection point. The consequence is well-known: the wave splits into reflected and transmitted components. Try to generalize to generic power propagation phenomena: at the interconnection point of physical systems, an amount of traveling power is transmitted, while the remaining part is reflected. Consequently, the total traveling power becomes the sum of two distinct terms, each of them moving in opposite "directions".

Consider, for example, a transmission line for which $C(z) = C$ is the distributed capacitance and by $L(z) = L$ the distributed inductance, with $z \in [0, L]$ being L the length of the line. If $V(z,t)$ and $I(z,t)$ are voltages and currents, it is well known that

$$\begin{cases} C\dfrac{\partial V}{\partial t} = -\dfrac{\partial I}{\partial z} \\ L\dfrac{\partial I}{\partial t} = -\dfrac{\partial V}{\partial z} \end{cases}$$

and, clearly, that

$$\frac{\partial^2 I}{\partial t^2} = LC\frac{\partial^2 I}{\partial z^2} \qquad\qquad \frac{\partial^2 V}{\partial t^2} = LC\frac{\partial^2 V}{\partial z^2}$$

which are both 1-dimensional wave equation. If $v = \frac{1}{\sqrt{LC}}$ denotes the wave speed, we have that

$$I(z,t) = F_1(z+vt) + F_2(z-vt)$$

is a generic solution of the first PDE, where F_1 and F_2 are two waveforms traveling in opposite directions. In particular, F_1 in the negative z direction and F_2 in the positive one. As regard the time evolution of the voltage along the transmission line, it is easy to compute that

$$V(z,t) = \sqrt{\frac{L}{C}}\,[F_1(z+vt) + F_2(z-vt)]$$

where $Z := \sqrt{L/C}$ is the *impedance* of the line. Note that Z can be interpreted as a map from the space of flows (currents) to the space of efforts (voltages). If $N := \sqrt{Z}$, consider the following functions:

$$S^+(z,t) = \frac{N^{-1}}{\sqrt{2}}\left[V(z,t) + ZI(z,t)\right] \quad S^-(z,t) = \frac{N^{-1}}{\sqrt{2}}\left[V(z,t) - ZI(z,t)\right] \tag{2.234}$$

It is easy to compute that $S^+(z,t) = \sqrt{2Z}F_1(z,t)$ and $S^-(z,t) = \sqrt{2Z}F_2(z,t)$, showing that each $S^{+,-}(z,t)$ is only function of a specific wavefront. Moreover, denote by $P(z,t) := V(z,t)I(z,t)$ the traveling power. Then, $P(z,t) = Z[F_1^2(z,t) - F_2^2(z,t)]$ or, in terms of S^+ and S^-

$$P(z,t) = \frac{1}{2}\left\|S^+(z,t)\right\|^2 - \frac{1}{2}\left\|S^-(z,t)\right\|^2 \tag{2.235}$$

Relation (2.235) shows that the change of variable (2.234) is able to reveal two distinct power flows propagating inside the transmission line. Moreover, each new variable is related to the power traveling in a specific direction: S^+ and S^- are called *scattering variables* while (2.234) is the *scattering map*. Moreover, the result expressed by (2.235) is known as *scattering power decomposition*.

At this point, we can imagine if it is possible to extend these classical results on wave propagation to generic power propagation phenomena. In order to understand that this connection is possible, let us recall that the interaction between physical systems is an exchange of power, the amount of which can be calculated by means of intrinsic operations on the so-called power variables, defined on the ports of the systems themselves. Clearly, it is on these ports that the interaction takes place. As in wave propagation it is possible to operate on the state variables in order to obtain a couple of new state variables related to the *transmitted* and *reflected* power, it is possible to show that, once an orientation is fixed for each power port of the system, it is possible to split the total power flow into two distinct ones by means of a generalization of (2.234). In this respect, the interconnection of systems can be described in terms of these new port variables.

2.8.2 Scattering representation of ports and Dirac structures

Consider the space of port variables given in general form as $\mathscr{F} \times \mathscr{F}^*$, for some finite-dimensional linear space $\mathscr{F}$. The duality product $\langle e \mid f \rangle$ defines the instantaneous *power* of the signal $(f,e) \in \mathscr{F} \times \mathscr{F}^*$. It has been shown that the basic idea of a *scattering representation* is to rewrite the power as the *difference* between two non-negative terms, that is, the difference between an *incoming* power and an *outgoing* power. This is accomplished by the introduction of new coordinates for the total space $\mathscr{F} \times \mathscr{F}^*$, based on the bi-linear form introduced before

$$\ll (f_1,e_1),(f_2,e_2) \gg = \langle e_1 \mid f_2 \rangle + \langle e_2 \mid f_1 \rangle \tag{2.236}$$

on $\mathscr{F} \times \mathscr{F}^*$ (see Sect. 2.1), with $(f_i,e_i) \in \mathscr{F} \times \mathscr{F}^*$, $i = 1,2$. From a matrix representation of $\ll \cdot,\cdot \gg$, it immediately follows that $\ll \cdot,\cdot \gg$ is an *indefinite* symmet-

ric bi-linear form, which has n singular values $+1$ and n singular values -1, with $n = \dim \mathscr{F}$.

A[9] pair of subspaces $\Sigma^+, \Sigma^- \subset \mathscr{F} \times \mathscr{F}^*$ is called a pair of *scattering subspaces* if

(i) $\Sigma^+ \oplus \Sigma^- = \mathscr{F} \times \mathscr{F}^*$;
(ii) $\ll \sigma_1^+, \sigma_2^+ \gg > 0$ for all $\sigma_1^+, \sigma_2^+ \in \Sigma^+$ unequal to 0, and $\ll \sigma_1^-, \sigma_2^- \gg < 0$ for all $\sigma_1^-, \sigma_2^- \in \Sigma^-$ unequal to 0;
(iii) $\ll \sigma^+, \sigma^- \gg = 0$ for all $\sigma^+ \in \Sigma^+, \sigma^- \in \Sigma^-$.

It is readily seen that any pair of scattering subspaces (Σ^+, Σ^-) satisfies

$$\dim \Sigma^+ = \dim \Sigma^- = \dim \mathscr{F}$$

The collection of pairs of scattering subspaces can be characterized as follows.

Lemma 2.1. *Let (Σ^+, Σ^-) be a pair of scattering subspaces. Then there exists an invertible linear map $R : \mathscr{F} \to \mathscr{F}^*$ with*

$$\langle (R + R^*) f \mid f \rangle > 0 \tag{2.237}$$

for all $0 \neq f \in \mathscr{F}$, such that

$$\begin{aligned} \Sigma^+ &:= \left\{ (R^{-1} e, e) \in \mathscr{F} \times \mathscr{F}^* \mid e \in \mathscr{F}^* \right\} \\ \Sigma^- &:= \left\{ (-f, R^* f) \in \mathscr{F} \times \mathscr{F}^* \mid f \in \mathscr{F} \right\} \end{aligned} \tag{2.238}$$

Conversely, for any invertible linear map $R : \mathscr{F} \to \mathscr{F}^*$ satisfying (2.237) the pair (Σ^+, Σ^-) defined in (2.238) is a pair of scattering subspaces.

Proof. Let (Σ^+, Σ^-) be a pair of scattering subspaces. Since $\ll \cdot, \cdot \gg$ is positive definite on Σ^+, while it is zero on $\mathscr{F} \times 0$ and $0 \times \mathscr{F}$, we can write Σ^+ as in (2.238) for some invertible linear map R. Checking positive-definiteness of $\ll \cdot, \cdot \gg$ on Σ^+ then yields (2.237). Similarly, $\Sigma^- \cap (\mathscr{F} \times 0) = 0, \Sigma^- \cap (0 \times \mathscr{F}^*) = 0$. Orthogonality of Σ^- with respect to Σ^+ (condition *(iii)*) implies that Σ^- is given as in (2.238). Conversely, a direct computation shows that (Σ^+, Σ^-) defined in (2.238) for R satisfying (2.237) defines a pair of scattering subspaces.

The fundamental relation between the representation in terms of power vectors $(f, e) \in \mathscr{F} \times \mathscr{F}^*$ and the scattering representation is given by the following. Let (Σ^+, Σ^-) be a pair of scattering subspaces. Then every pair of power vectors $(f, e) \in \mathscr{F} \times \mathscr{F}^*$ can be also represented as

$$(f, e) = \sigma^+ + \sigma^- \tag{2.239}$$

for a uniquely defined $\sigma^+ \in \Sigma^+, \sigma^- \in \Sigma^-$, called the *wave vectors*. Using orthogonality of Σ^+ w.r.t. Σ^- it immediately follows that for all $(f_i, e_i) = \sigma_i^+ + \sigma_i^-$, $i = 1, 2$:

[9] In general, there exist many pairs of scattering subspaces.

$$\ll (f_1,e_1),(f_2,e_2) \gg = \left\langle \sigma_1^+, \sigma_2^+ \right\rangle_{\Sigma^+} - \left\langle \sigma_1^-, \sigma_2^- \right\rangle_{\Sigma^-} \tag{2.240}$$

where $\langle \cdot,\cdot \rangle_{\Sigma^+}$ denotes the inner product on Σ^+ defined as the restriction of $\ll \cdot,\cdot \gg$ to Σ^+, and $\langle \cdot,\cdot \rangle_{\Sigma^-}$ denotes the inner product on Σ^- defined as *minus* the restriction of $\ll \cdot,\cdot \gg$ to Σ^-. Taking $f_1 = f_2 = f$, $e_1 = e_2 = e$ and thus $\sigma_1^+ = \sigma_2^+ = \sigma^+$, $\sigma_1^- = \sigma_2^- = \sigma^-$, leads to

$$\langle e \mid f \rangle = \frac{1}{2} \ll (f,e),(f,e) \gg = \frac{1}{2} \left\langle \sigma^+, \sigma^+ \right\rangle_{\Sigma^+} - \frac{1}{2} \left\langle \sigma^-, \sigma^- \right\rangle_{\Sigma^-} \tag{2.241}$$

Equation (2.241) yields the following interpretation of the wave vectors. The vector σ^+ can be regarded as the *incoming wave* vector, with half times its norm being the *incoming power*, and the vector σ^- is the *outgoing wave* vector, with half times its norm being the *outgoing power*. Note the similarities with respect to (2.235).

Applied to a port-Hamiltonian system, this leads to the following. Let $\mathscr{F} \times \mathscr{F}^*$ be the space of external port variables f,e, *not* including the internal port variables f_S, e_S, and consider a scattering representation as above. Then the power-balance $\frac{\mathrm{d}}{\mathrm{d}t} H = \langle e \mid f \rangle$ changes into

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{1}{2} \left\langle \sigma^+, \sigma^+ \right\rangle_{\Sigma^+} - \frac{1}{2} \left\langle \sigma^-, \sigma^- \right\rangle_{\Sigma^-} \tag{2.242}$$

expressing that the increase in internal energy is equal to the energy of the incoming wave minus the energy of the outgoing wave.

Let $\mathscr{D} \subset \mathscr{F} \times \mathscr{F}^*$ be a Dirac structure, that is, $\mathscr{D} = \mathscr{D}^\perp$ with respect to $\ll \cdot,\cdot \gg$. What is its representation in wave vectors? Since $\ll \cdot,\cdot \gg$ is zero restricted to $\mathscr{D}$, it follows that for every pair of scattering subspaces (Σ^+, Σ^-)

$$\mathscr{D} \cap \Sigma^+ = 0 \qquad\qquad \mathscr{D} \cap \Sigma^- = 0 \tag{2.243}$$

and hence (see [180] for more information) $\mathscr{D}$ can be represented as the graph of an invertible linear map $\mathscr{O} : \Sigma^+ \to \Sigma^-$, that is:

$$\mathscr{D} = \left\{ (\sigma^+, \sigma^-) \mid \sigma^- = \mathscr{O}\sigma^+, \ \ \sigma^+ \in \Sigma^+ \right\} \tag{2.244}$$

Furthermore, by (2.240), $\left\langle \sigma_1^+, \sigma_2^+ \right\rangle_{\Sigma^+} = \left\langle \mathscr{O}\sigma_1^+, \mathscr{O}\sigma_2^+ \right\rangle$ for every $\sigma_1^+, \sigma_2^+ \in \Sigma^+$, and thus

$$\mathscr{O} : (\Sigma^+, \langle \cdot,\cdot \rangle_{\Sigma^+}) \to (\Sigma^-, \langle \cdot,\cdot \rangle_{\Sigma^-}) \tag{2.245}$$

is a unitary map (isometry). Conversely, every unitary map $\mathscr{O}$ as in (2.245) defines a Dirac structure by (2.244). Thus, for every pair of scattering subspaces (Σ^+, Σ^-), we have a one-to-one correspondence between unitary maps (2.245) and Dirac structures $\mathscr{D} \subset \mathscr{F} \times \mathscr{F}^*$.

This is an appealing result since maps are often easier to deal with than relations. This is illustrated by Sect. 2.8.4 where it will be shown how the composition of two Dirac structures can be obtained by a special product (called the Redheffer product) of the two associated unitary maps.

2.8.3 Inner product scattering representations

A particular useful class of scattering subspaces (Σ^+,Σ^-) are those defined by an invertible map $R:\mathscr{F}\to\mathscr{F}^*$ satisfying (2.237) such that $R=R^*$. Indeed, in this case R is determined by the *inner product* on $\mathscr{F}$ defined as

$$\langle f_1,f_2\rangle_R := \langle Rf_1 \mid f_2\rangle = \langle Rf_2 \mid f_1\rangle \tag{2.246}$$

or equivalently by the inner product on $\mathscr{F}^*$ defined as

$$\langle e_1,e_2\rangle_{R^{-1}} := \langle e_2 \mid R^{-1}f_1\rangle = \langle e_1 \mid R^{-1}f_2\rangle \tag{2.247}$$

In this case, we may define an explicit representation of the pair of scattering subspaces (Σ^+,Σ^-) as follows. Define for every $(f,e)\in\mathscr{F}\times\mathscr{F}^*$ the pair (s^+,s^-) as

$$s^+ := \frac{1}{\sqrt{2}}(e+Rf)\in\mathscr{F}^* \qquad\qquad s^- := \frac{1}{\sqrt{2}}(e-Rf)\in\mathscr{F}^* \tag{2.248}$$

See the similarities with respect to (2.234). Let (s_i^+,s_i^-) correspond to (f_i,e_i), $i=1,2$. Then, by direct computation,

$$\begin{aligned} 2\langle s_1^+,s_2^+\rangle_{R^{-1}} &= \langle e_1,e_2\rangle_{R^{-1}}+\langle f_1,f_2\rangle_R+\ll (f_1,e_1),(f_2,e_2)\gg \\ 2\langle s_1^-,s_2^-\rangle_{R^{-1}} &= \langle e_1,e_2\rangle_{R^{-1}}+\langle f_1,f_2\rangle_R-\ll (f_1,e_1),(f_2,e_2)\gg \end{aligned} \tag{2.249}$$

Hence, if $(f_i,e_i)\in\Sigma^+$, or equivalently $s_i^-=e_i-Rf_i=0$, then

$$\langle s_1^+,s_2^+\rangle_{R^{-1}} = \ll (f_1,e_1),(f_2,e_2)\gg$$

while if $(f_i,e_i)\in\Sigma^-$, or equivalently $s_i^+=e_i+Rf_i=0$, then

$$\langle s_1^-,s_2^-\rangle_{R^{-1}} = -\ll (f_1,e_1),(f_2,e_2)\gg$$

Thus the mappings

$$\begin{aligned} \sigma^+=(f,e)\in\Sigma^+ &\longmapsto s^+=\frac{1}{\sqrt{2}}(e+Rf)\in\mathscr{F}^* \\ \sigma^-=(f,e)\in\Sigma^- &\longmapsto s^-=\frac{1}{\sqrt{2}}(e-Rf)\in\mathscr{F}^* \end{aligned} \tag{2.250}$$

are isometries with respect to the inner products on Σ^+ and Σ^-, and the inner product on $\mathscr{F}^*$ defined by (2.247). Hence we may identify the wave vectors (σ^+,σ^-) with (s^+,s^-).

Remark 2.11. Note that by (2.249) the pair of scattering subspaces (Σ^+,Σ^-) corresponding to R may be elegantly characterized as

$$\Sigma^+ = \left\{ (f,e) \in \mathscr{F} \times \mathscr{F}^* \mid \ll (f,e),(f,e) \gg = \langle e,e \rangle_{R^{-1}} + \langle f,f \rangle_R \right\}$$
$$\Sigma^- = \left\{ (f,e) \in \mathscr{F} \times \mathscr{F}^* \mid \ll (f,e),(f,e) \gg = -\langle e,e \rangle_{R^{-1}} - \langle f,f \rangle_R \right\}$$

This is the starting point taken in [197, 198].

Let us now consider the representation of a Dirac structure $\mathscr{D}$ in terms of the wave vectors (s^+, s^-) (see also the treatment in [180, Sect. 4.3.3]). For every Dirac structure $\mathscr{D} \subset \mathscr{F} \times \mathscr{F}^*$ there exist linear mappings $F : \mathscr{F} \to \mathscr{V}$ and $E : \mathscr{F}^* \to \mathscr{V}$ satisfying (see 2.109)

$$\begin{aligned} &(i) \quad EF^* + FE^* = 0, \\ &(ii) \quad \operatorname{rank}(F+E) = \dim \mathscr{F}, \end{aligned} \tag{2.251}$$

where $\mathscr{V}$ is a linear space with the same dimension as $\mathscr{F}$. Thus, for any $(f,e) \in \mathscr{D}$, the wave vectors (s^+, s^-) defined by (2.250) are given as

$$s^+ = \frac{1}{\sqrt{2}}(F^*\lambda + RE^*\lambda) = \frac{1}{\sqrt{2}}(F^* + RE^*)\lambda$$
$$s^- = \frac{1}{\sqrt{2}}(F^*\lambda - RE^*\lambda) = \frac{1}{\sqrt{2}}(F^* - RE^*)\lambda$$

with $\lambda \in \mathscr{V}^*$. The mapping $F^* + RE^*$ is invertible. Indeed, suppose that $(F^* + RE^*)(\lambda) = 0$. By (2.251-*(i)*) also $EF^*\lambda + FE^*\lambda = 0$. It follows that $ERE^*\lambda + FR^{-1}F^*\lambda = 0$, and hence by positive-definiteness of R and (2.251-*(ii)*) $\lambda = 0$. Therefore

$$s^- = (F^* - RE^*)(F^* + RE^*)^{-1}s^+ \tag{2.252}$$

Hence the unitary map $\mathscr{O} : \mathscr{F}^* \to \mathscr{F}^*$ associated with the Dirac structure (recall that we identify Σ^+ and Σ^- with $\mathscr{F}^*$ by (2.250)) is given as

$$\mathscr{O} = (F^* - RE^*)(F^* + RE^*)^{-1} \tag{2.253}$$

By adding $EF^* + FE^* = 0$ it follows that

$$(FR^{-1} + E)(F^* + RE^*) = (FR^{-1} - E)(F^* - RE^*)$$

and hence also

$$\mathscr{O} = (FR^{-1} - E)^{-1}(FR^{-1} + E) \tag{2.254}$$

since, similarly as above, it can be shown that $FR^{-1} - E$ is invertible. Therefore

$$\begin{aligned} \mathscr{O}^*R^{-1}\mathscr{O} &= (F+ER)^{-1}(F-ER)R^{-1}(FR^{-1}-E)^{-1}(FR^{-1}+E) \\ &= (F+ER)^{-1}(F-ER)(F-ER)^{-1}(FR^{-1}+E) \\ &= (F+ER)^{-1}(FR^{-1}+E) = R^{-1} \end{aligned}$$

showing indeed (as proved before by general considerations) that $\mathscr{O} : \mathscr{F}^* \to \mathscr{F}^*$ is a unitary mapping.

Given a kernel/image representation (F,E) for a Dirac structure $\mathscr{D}$, that is, $\mathscr{D} = \ker(F+E)$, it is obvious that for any invertible map $C : \mathscr{V} \to \mathscr{V}'$ also $\mathscr{D} = \ker C(F+E) = \ker(CF+CE) = \ker(F'+E')$. Hence, there are infinitely many (F,E) pairs representing $\mathscr{D}$ in kernel/image representation, corresponding to only one $\mathscr{O}$ map in the chosen scattering representation. The following theorem establishes the characterization of the set of (F,E) pairs representing $\mathscr{D}$ based on this unique scattering representation of $\mathscr{D}$.

Theorem 2.4. *Consider any inner product R on $\mathscr{F}$ and the resulting scattering representation. The set of (F,E) pairs representing a given Dirac structure $\mathscr{D}$ on $\mathscr{F} \times \mathscr{F}^*$ in kernel/image representation is given by*

$$\Big\{(F,E) \mid F = X(\mathscr{O}+I)R,\ E = X(\mathscr{O}-I), X : \mathscr{F}^* \to \mathscr{V} \text{ invertible}\Big\} \tag{2.255}$$

where $\mathscr{O}$ is the scattering representation of $\mathscr{D}$.

Proof. Obviously, any (F,E) pair corresponding to $\mathscr{D}$ can be expressed as

$$\begin{cases} F = (A+B)R \\ E = A-B \end{cases} \tag{2.256}$$

in terms of the pair of the mappings A,B given by

$$\begin{cases} A = \frac{1}{2}(FR^{-1}+E) \\ B = \frac{1}{2}(FR^{-1}-E) \end{cases} \tag{2.257}$$

By (2.254) the mappings A and B are invertible, while $\mathscr{O} = B^{-1}A$. Hence, substituting $A = B\mathscr{O}$ in (2.256), F and E can be expressed as

$$\begin{cases} F = B(\mathscr{O}+I)R \\ E = B(\mathscr{O}-I) \end{cases} \tag{2.258}$$

Taking $C = B^{-1}$ in (2.258), the following 'canonical' kernel representation for $\mathscr{D}$ is found

$$\begin{cases} F' = (\mathscr{O}+I)R \\ E' = \mathscr{O}-I \end{cases} \tag{2.259}$$

immediately yielding the parametrization of $\mathscr{D}$ given in (2.255).

Remark 2.12. Using the property $\mathscr{O}^*R^{-1}\mathscr{O} = R^{-1}$ one may also rewrite (2.259) as

$$\begin{cases} F' = R((\mathscr{O}^*)^{-1}+I) \\ E' = R((\mathscr{O}^*)^{-1}-I)R^{-1} \end{cases} \tag{2.260}$$

and then obtain the alternative parametrization

$$\left\{(F,E) : F = X((\mathscr{O}^*)^{-1}+I),\ E = X((\mathscr{O}^*)^{-1}-I)R^{-1}\right\} \tag{2.261}$$

Remark 2.13. Equation (2.253) together with (2.255) allow bi-directional conversion between scattering and kernel/image representations.

2.8.4 Interconnection in scattering representation

Recall that interconnection in power vector representation is simply given by the interconnection constraints (0-junction)

$$f_A = -f_B \in \mathscr{F} \qquad\qquad e_A = e_B \in \mathscr{F}^* \tag{2.262}$$

Now consider the scattering representation of the power vectors (f_A, e_A) with respect to an inner product R_A as given by the wave vectors

$$s_A^+ := \frac{1}{\sqrt{2}}(e_A + R_A f_A) \in \mathscr{F}^* \qquad s_A^- := \frac{1}{\sqrt{2}}(e_A - R_A f_A) \in \mathscr{F}^* \tag{2.263}$$

and analogously the scattering representation of the power vectors (f_B, e_B) with respect to another inner product R_B, given by

$$\begin{aligned} s_B^+ &:= \frac{1}{\sqrt{2}}(e_B + R_B f_B) \in \mathscr{F}^* \\ s_B^- &:= \frac{1}{\sqrt{2}}(e_B - R_B f_B) \in \mathscr{F}^* \end{aligned} \tag{2.264}$$

Then, the interconnection constraints (2.262) on the power vectors yield the following interconnection constraints on the wave vectors

$$s_A^+ - s_B^- = \frac{1}{\sqrt{2}}(R_A - R_B)f \qquad s_B^+ - s_A^- = \frac{1}{\sqrt{2}}(R_A - R_B)f \tag{2.265}$$

together with

$$s_A^+ - s_A^- = \sqrt{2}R_A f \qquad\qquad s_B^- - s_B^+ = \sqrt{2}R_B f \tag{2.266}$$

leading to

$$s_A^+ - s_B^- = s_B^+ - s_A^- \qquad R_A^{-1}(s_A^+ - s_A^-) + R_B^{-1}(s_B^+ - s_B^-) = 0 \tag{2.267}$$

As described in Fig. 2.9, the first equation of (2.267) can be interpreted as a *power balance* of the wave vectors. Indeed, in our convention for power flow, s^+ are *incoming* wave vectors for the system and thus *outgoing* wave vectors for the point of interconnection, while s^- are *outgoing* wave vectors for the system and thus *incoming* wave vectors for the point of interconnection. Hence the first equation of

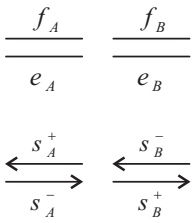

Fig. 2.9 Ingoing / outgoing wave vectors.

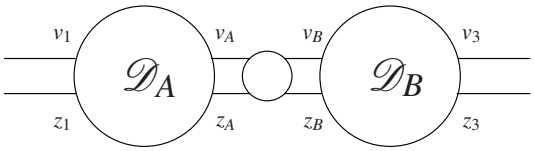

Fig. 2.10 Interconnection of $\mathscr{D}_A$ and $\mathscr{D}_B$ using wave vectors.

(2.267) states that the loss (= difference) between the outgoing wave vector s_A^+ and the incoming wave vector s_B^- is equal to the loss between the outgoing wave vector s_B^+ and the incoming wave vector s_A^-. The second equation expresses a balance between the loss as seen from A and the loss as seen from B.

The scattering at A is said to be *matching* with the scattering at B if $R_A = R_B$. In this case (2.262) is equivalent to the following interconnection constraints between the wave vectors:

$$s_A^+ = s_B^- \qquad\qquad s_B^+ = s_A^-, \tag{2.268}$$

simply expressing that the outgoing wave vector for A equals the incoming wave vector for B, and conversely.

In the rest of this section, we restrict ourselves to the matching case $R_A = R_B = R$. Also, in order to simplify computations, we consider a coordinate representation such that R is given by the identity matrix (= Euclidean inner product). Furthermore, for ease of notation we denote s_A^+, s_B^+ by v_A, v_B and s_B^-, s_A^- by z_A, z_B. Thus we consider the composition as in Fig. 2.10 of two Dirac structures $\mathscr{D}_A, \mathscr{D}_B$ by the interconnection equations (in scattering representation)

$$v_A = z_B \qquad\qquad z_A = v_B \tag{2.269}$$

By redrawing Fig. 2.10 in standard *feedback interconnection* form as in Fig. 2.11, it is readily seen that this corresponds to the well-known *Redheffer star product* (see e.g. [169]) of $\mathscr{O}_{\mathscr{A}}$ and $\mathscr{O}_{\mathscr{B}}$.

Proposition 2.4. *The scattering representation of $\mathscr{D}_A \parallel \mathscr{D}_B$ is given by $\mathscr{O}_A \star \mathscr{O}_B$, with the unitary mappings $\mathscr{O}_A$ and $\mathscr{O}_B$ being the scattering representation of $\mathscr{D}_A$ and $\mathscr{D}_B$ respectively, and $\star$ denoting the Redheffer star product.*

Note that this immediately yields that the Redheffer star product of two unitary mappings is again a unitary mapping, since $\mathscr{D}_A \parallel \mathscr{D}_B$ is again a Dirac structure. Explicit formulas for $\mathscr{O}_A \star \mathscr{O}_B$ tend to become rather involved. One way is to employ

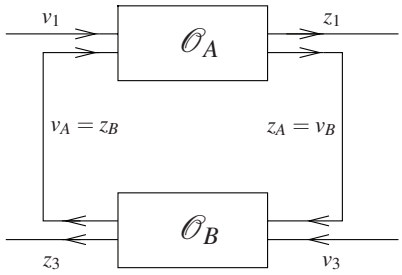

Fig. 2.11 Fig. 2.10 redrawn as the Redheffer star product of $\mathscr{O}_{\mathscr{A}}$ and $\mathscr{O}_{\mathscr{B}}$.

the (relaxed) kernel/image representation of $\mathscr{D}_A \parallel \mathscr{D}_B$ obtained in Sect. 2.5.1 and then to use the conversion (from power vectors to wave vectors, and conversely) formulas obtained in Sect. 2.8.3.

In case the interconnection is *regular*, a direct formula for $\mathscr{O}_A \star \mathscr{O}_B$ is straightforward. Indeed, denote the decomposition of $\mathscr{O}_A$ and $\mathscr{O}_B$ by

$$\begin{aligned} \begin{bmatrix} z_1 \\ z_A \end{bmatrix} &= \mathscr{O}_A \begin{bmatrix} v_1 \\ v_A \end{bmatrix} = \begin{bmatrix} \mathscr{O}^A_{11} & \mathscr{O}^A_{12} \\ \mathscr{O}^A_{21} & \mathscr{O}^A_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_A \end{bmatrix} \\ \begin{bmatrix} z_3 \\ z_B \end{bmatrix} &= \mathscr{O}_B \begin{bmatrix} v_3 \\ v_B \end{bmatrix} = \begin{bmatrix} \mathscr{O}^B_{33} & \mathscr{O}^B_{32} \\ \mathscr{O}^B_{23} & \mathscr{O}^B_{22} \end{bmatrix} \begin{bmatrix} v_3 \\ v_B \end{bmatrix} \end{aligned} \tag{2.270}$$

Using (2.269) and (2.270), the following expressions are obtained

$$\begin{aligned} (I - \mathscr{O}^A_{22}\mathscr{O}^B_{22})z_A &= \mathscr{O}^A_{21}v_1 + \mathscr{O}^A_{22}\mathscr{O}^B_{23}v_3 \\ (I - \mathscr{O}^B_{22}\mathscr{O}^A_{22})z_B &= \mathscr{O}^B_{23}v_3 + \mathscr{O}^B_{22}\mathscr{O}^A_{21}v_1 \end{aligned}$$

Regularity of the interconnection is equivalent to the property that z_A and z_B are uniquely determined by the above two equations, and hence is equivalent to the invertibility of the matrices $(I - \mathscr{O}^A_{22}\mathscr{O}^B_{22})$ and $(I - \mathscr{O}^B_{22}\mathscr{O}^A_{22})$. If this is the case, then one can directly express z_A and z_B in terms of v_1 and v_3, and substitute these expressions in (2.270), thus leading to an explicit formula for $\mathscr{O}_A \star \mathscr{O}_B$. Explicit formulas may be found in [44].

Chapter 3
Port-Based Modeling in Different Domains

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Abstract In this Chapter we present some detailed examples of modelling in several domains using port and port-Hamiltonian concepts, as have been presented in the previous chapters. We start with the electromechanical domain in Sect. 3.1, while in Sect. 3.2 it is shown how port-Hamiltonian systems can be fruitfully used for the structured modelling of robotics mechanisms. In Sect. 3.3, it is show how to model simple elastic systems either in the Lagrangian and Hamiltonian framework, while, in Sect. 3.4, an expressions of the models representing momentum, heat and mass transfer as well as chemical reactions within homogeneous fluids in the port-based formalism is proposed. To this end, the entropy balance and the associated source terms are systematically written in accordance with the principle of irreversible thermodynamics. Some insights are also given concerning the constitutive equations and models allowing to calculate transport and thermodynamic properties. As it will be shown, for each physical domain, these port-based models can be translated into bond-graph models, in the case of distributed as well as lumped parameters models.

3.1 Modeling of electrical systems

Electromechanical energy conversion has already been discussed in Sect. 1.9.3, and, in particular, the constraints imposed by energy conservation on the constitutive laws of the ports, Maxwell's relations, have been derived. As the name indicates, electromechanical systems (EMS) bridge the gap between the electrical and mechanical domains. In practice, on the electrical side one has an electric circuit of a very special class, what is called an *electronic power converter*, which, if the system is working as an electrical motor, takes the electrical energy from some source and provides a suitable voltage to the EMS so that the desired mechanical speed is reached; likewise, if the system acts as a generator, the power converter transforms the raw electrical energy into a form adapted for immediate use, storage or transportation. The main characteristic of electronic power converters is that they are *variable structure systems* (VSS). They contain a number of switches and diodes, of

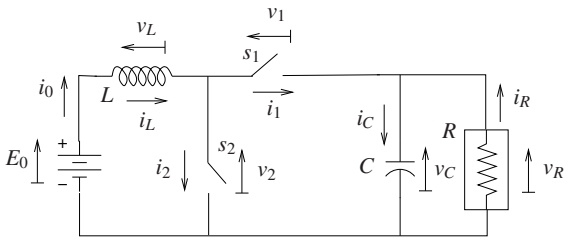

Fig. 3.1 A functional description of the boost converter.

which the former can be opened or closed in a periodic manner by a suitable control algorithm, in order to effect the necessary electrical energy conversion.

Since electronic power converters are so important for EMS, and also for many other applications, such as portable equipment, energy supply systems in the aerospace industry, or uninterruptible power supply systems, we present first an explicit example of modelling of a power converter in the port-Hamiltonian framework. Next we discuss in detail the port-Hamiltonian description of a general EMS, and use it to describe an elementary electromagnet. Finally we couple both systems and display the complete port-Hamiltonian structure.

Although modelling of VSS in the port-Hamiltonian framework is straightforward, numerical simulation can be quite complex and time-intensive, due to the abrupt structure changes. Approximate, smooth models can be obtained from a VSS, using suitable averages of the state variables and the control signals. For completeness, we also present the simplest form of this averaging theory, which yields models which can be easily implemented in bond graph theory.

3.1.1 Electronic power converter circuits

Fig. 3.1 shows a functional model[1] of the boost (or elevator) converter (the detailed electronics of how the switches are implemented is not shown). The switches s_1 and s_2 are complementary: when s_1 is closed ($s_1 = 1$), s_2 is open ($s_2 = 0$), and vice-versa. Thus, the different circuit topologies can be described with a single boolean variable $S = s_2$.

The port Hamiltonian modeling of electric circuits can be done in a systematic way using tools from graph theory [145], but since we are dealing here with a circuit of very small size we will adopt a more pedestrian approach and concentrate on the problems presented by the switches, using the ideas of [74]. A more in-deep conceptual analysis of the switches can be found in [63, 73, 82]. The Hamiltonian dynamical variables of the boost converter are the magnetic flux at the coil, ϕ_L, and the charge of the capacitor, q_C. Hence we have two one-dimensional Hamiltonian

[1] In a real setup, one of the switches (s_1) is replaced by a diode. This may cause, under the appropriate conditions, the apparition of the so-called discontinuous conduction modes, which this simplified model cannot support.

subsystems, with a global Hamiltonian $H_B = H_C + H_L$,

$$\frac{\mathrm{d}q_C}{\mathrm{d}t} = i_C \qquad\qquad v_C = \frac{\partial H_B}{\partial q_C}, \tag{3.1}$$

and

$$\frac{\mathrm{d}\phi_L}{\mathrm{d}t} = v_L \qquad\qquad i_L = \frac{\partial H_B}{\partial \phi_L} \tag{3.2}$$

connected by Kirchhoff's laws

$$\begin{aligned} i_L &= i_1 + i_2 \\ i_1 &= i_C - i_R \\ v_2 + v_L &= E_0 \\ v_C + v_1 &= v_2 \\ v_C &= v_R \\ i_0 - i_L &= 0 \end{aligned} \tag{3.3}$$

These 6 independent relations define a Dirac structure in $\mathbb{R}^{12}$, the space of the efforts and flows of the 6 interconnected electrical elements (the two switches, the capacitor, the inductor, the load and the voltage source).

Here we treat the switches as ports, with their correspondent effort and flow variables. For the time being we do not terminate the resistive port, i.e. we do not use $v_R = -Ri_R$ (the minus sign is necessary since we are adopting an input power convention for the rest of the system, hence an output power one for the resistor; we could get rid of this nuisance by introducing auxiliary variables at the resistive port). Using (3.1) and (3.2), the first four equations of (3.3) can be written as

$$\begin{aligned} \frac{\partial H_B}{\partial \phi_L} &= i_1 + i_2 \\ i_1 &= \frac{\mathrm{d}q_C}{\mathrm{d}t} - i_R \\ v_2 + \frac{\mathrm{d}\phi_L}{\mathrm{d}t} &= E_0 \\ \frac{\partial H_B}{\partial q_C} + v_1 &= v_2 \end{aligned} \tag{3.4}$$

The second and third equations in (3.4) yield a Hamiltonian system with four inputs and $J = R = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} q_C \\ \phi_L \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \\ i_R \\ E_0 \end{bmatrix} \tag{3.5}$$

Next we will use the constraints imposed by the switches to absorb the ports s_1 and s_2 into the Hamiltonian structure:

- $S = 0 \Rightarrow s_1 = 1,\ s_2 = 0 \Rightarrow v_1 = 0,\ i_2 = 0;$
- $S = 1 \Rightarrow s_1 = 0,\ s_2 = 1 \Rightarrow i_1 = 0,\ v_2 = 0.$

Hence, when $S = 1$ we already have the values of the port variables i_1, v_2 in (3.5), while if $S = 0$, using the first and fourth equations in (3.4),

$$i_1 = \frac{\partial H_B}{\partial \phi_L} \qquad\qquad v_2 = \frac{\partial H_B}{\partial q_C}$$

We can put together both results as

$$i_1 = (1-S)\frac{\partial H_B}{\partial \phi_L} \qquad\qquad v_2 = (1-S)\frac{\partial H_B}{\partial q_C}. \tag{3.6}$$

Now

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} q_C \\ \phi_L \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} (1-S)\frac{\partial H_B}{\partial \phi_L} \\ (1-S)\frac{\partial H_B}{\partial q_C} \\ i_R \\ E_0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1-S \\ -(1-S) & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_B}{\partial q_C} \\ \frac{\partial H_B}{\partial \phi_L} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_R \\ E_0 \end{bmatrix} \end{aligned} \tag{3.7}$$

which is a port Hamiltonian system with outputs

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \frac{\partial H_B}{\partial q_C} \\ \frac{\partial H_B}{\partial \phi_L} \end{bmatrix} = \begin{bmatrix} v_C \\ i_L \end{bmatrix} = \begin{bmatrix} v_R \\ i_0 \end{bmatrix} \tag{3.8}$$

Finally, we may terminate the resistive port using

$$i_R = -\frac{v_R}{R} = -\frac{v_C}{R} = -\frac{1}{R}\frac{\partial H_B}{\partial q_C}$$

and get our final port Hamiltonian representation of the boost converter with resistive load

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} q_C \\ \phi_L \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 1-S \\ -(1-S) & 0 \end{bmatrix} - \begin{bmatrix} 1/R & 0 \\ 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} \frac{\partial H_B}{\partial q_C} \\ \frac{\partial H_B}{\partial \phi_L} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} E_0 \tag{3.9}$$

with natural output

$$y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \frac{\partial H_B}{\partial q_C} \\ \frac{\partial H_B}{\partial \phi_L} \end{bmatrix} = i_L = i_0. \tag{3.10}$$

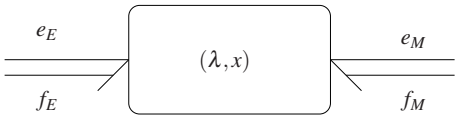

Fig. 3.2 A generalized electromechanical system.

Notice that the interconnection structure J is modulated by the boolean variable S. Designing a control for this system means choosing S as a function of the state variables.

3.1.2 Electromechanical energy conversion in the port-Hamiltonian framework

As explained Sect. 1.9.3, and in particular in Example 1.1, electrical domain systems with constitutive relations depending on geometric parameters develop additional mechanical ports through which power can flow and be exchanged with the electrical ports. Here we will cast the expressions for the constitutive laws of the ports into a Hamiltonian form. Consider the system displayed in Fig. 3.2. There are n_E generalized electrical ports (e_E, f_E) and n_M generalized mechanical ones (e_M, f_M), and the state variables are denoted by $\lambda \in \mathbb{R}^{n_E}$, $x \in \mathbb{R}^{n_M}$. Note that here we use a magnetic and translation mechanics notation, although the ports can be of any nature.

The equations of motion and the constitutive relations of the ports of this system, namely $\dot{\lambda} = e_E$, $\dot{x} = f_M$, $f_E = \frac{\partial H_E}{\partial \lambda}$, $e_M = \frac{\partial H_E}{\partial x}$, where $H_E = H_E(\lambda, x)$ is the energy function, can be expressed in explicit port-Hamiltonian form as:

$$\begin{bmatrix} \dot{\lambda} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} e_E \\ f_M \end{bmatrix} \tag{3.11}$$

$$\begin{bmatrix} f_E \\ e_M \end{bmatrix} = \begin{bmatrix} \frac{\partial H_E}{\partial \lambda} \\ \frac{\partial H_E}{\partial x} \end{bmatrix} \tag{3.12}$$

This is just the purely electromagnetic part of an electromechanical system. In fact, the electromechanical system always contains some mechanical inertia, independently of whether the port is connected to other systems or not. To model this, consider a generalized mechanical element with n_I ports (e_I, f_I) and state variables $p \in \mathbb{R}^{n_I}$. The dynamical equations of the element, $\dot{p} = e_I$, $f_I = I^{-1} p$, are written in port-Hamiltonian form as

$$\dot{p} = e_I \tag{3.13}$$

$$f_I = \frac{\partial H_I}{\partial p} \tag{3.14}$$

with $H_I(p) = p^{\mathrm{T}} I^{-1} p$.

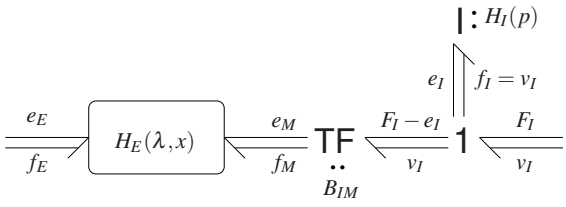

Fig. 3.3 Bond graph of a generalized electromechanical system with mechanical inertia included.

This purely mechanic part can be coupled to the electromagnetic part and to the rest of the system (if any), by means of

$$e_I = -B_{IM} e_M + F_I \tag{3.15}$$

$$f_M = B_{IM}^{\mathrm{T}} f_I \tag{3.16}$$

$$f_I = v_I \tag{3.17}$$

where the mechanical ports of the inertia element have been split into one contribution from the electromagnetic part, (e_M, f_M), and the connection to other subsystems, (F_I, v_I), with $F_I, v_I \in \mathbb{R}^{n_I}$. The matrix B_{IM} takes into account the fact that the mechanical ports may be connected to the electromagnetic part in a nontrivial way (or the fact that $n_I \neq n_M$), and the minus sign in (3.15) reflects Newton's third law (e_M is the force *on* the electromagnetic part, so a minus sign must be introduced to get the force on the mechanical element). Notice that the above relations define a Dirac structure in $\mathbb{R}^{n_M+2n_I} \times \mathbb{R}^{n_M+2n_I}$ with coordinates $(e_M, -f_M, -e_I, f_I, F_I, v_I)$, since the $n_M + 2n_I$ equations are clearly independent and can be written as

$$\underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}}_{F} \begin{bmatrix} -f_M \\ -e_I \\ v_I \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & B_{IM}^{\mathrm{T}} & 0 \\ -B_{IM} & 0 & I \\ 0 & -I & 0 \end{bmatrix}}_{E} \begin{bmatrix} e_M \\ f_I \\ F_I \end{bmatrix} = 0$$

with $EF^{\mathrm{T}} + FE^{\mathrm{T}} = E + E^{\mathrm{T}} = 0$. Notice that the two minus signs in $-f_M$ and $-e_I$ correspond to power flowing into the mechanical port of the electromagnetic subsystem and power flowing into the mechanical inertia, respectively, so that

$$F_I^{\mathrm{T}} v_I = e_I^{\mathrm{T}} f_I + e_M^{\mathrm{T}} f_M$$

The bond graph corresponding to the whole system is displayed in Fig. 3.3, where the power flow conventions can be clearly appreciated.

From (3.11), (3.12), (3.13), (3.14), (3.15), (3.16) and (3.17), one can express the equations of motion for the state variables in terms of the external inputs (e_E, F_I), and obtain also the corresponding outputs (f_E, v_I). Indeed, eliminating the internal port variables (e_M, f_M) and (e_I, f_I), one gets

$$\begin{aligned}
\dot{\lambda} &= e_E \\
\dot{x} &= B_{IM}^{\mathrm{T}} \frac{\partial H_I}{\partial p} \\
\dot{p} &= -B_{IM} \frac{\partial H_E}{\partial x} + F_I \\
f_E &= \frac{\partial H_E}{\partial \lambda} \\
v_I &= \frac{\partial H_I}{\partial p}
\end{aligned}$$

This can be given a port-Hamiltonian form, with total Hamiltonian

$$H_{EM}(\lambda, x, p) = H_E(\lambda, x) + H_I(p), \tag{3.18}$$

and

$$\begin{bmatrix} \dot{\lambda} \\ \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & B_{IM}^{\mathrm{T}} \\ 0 & -B_{IM} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_{EM}}{\partial \lambda} \\ \frac{\partial H_{EM}}{\partial x} \\ \frac{\partial H_{EM}}{\partial p} \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} e_E \\ F_I \end{bmatrix} \tag{3.19}$$

$$\begin{bmatrix} f_E \\ v_I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \frac{\partial H_{EM}}{\partial \lambda} \\ \frac{\partial H_{EM}}{\partial x} \\ \frac{\partial H_{EM}}{\partial p} \end{bmatrix} \tag{3.20}$$

Many electromechanical systems of interest admit this explicit port-Hamiltonian form, including dc motors, levitating systems, elementary electromagnets (which will be presented in detail next) or microelectromechanical devices (MEMS). Alternating current machines can also be written in port-Hamiltonian form. However, several coordinate transformations are used in the electrical engineering literature to simplify the complex, geometry dependent constitutive relations involved in most of the cases. It turns out that, after carrying out those transformations, the system is still in port-Hamiltonian form, although with nontrivial, state dependent interconnection matrices. We will not pursue this here, but the interested reader is referred, for instance, to [16].

3.1.3 Elementary electromagnet

Fig. 3.4 shows an elementary electromagnet, a magnetic system with a moving part so that the flux linkage λ through the coil depends on a geometry variable, the "air gap" x. This can be written in the general form of electromechanical systems described above, with $n_E = 1$ and $n_M = n_I = 1$, $B_{IM} = 1$, $v_I = v$, $F_I = F$, $e_E = E$, $f_E = i$, and $H_I(p) = p^2/(2m)$. We just have to specify $H_E(\lambda, x)$, which we will deduce next from first principles, under suitable simplifications.

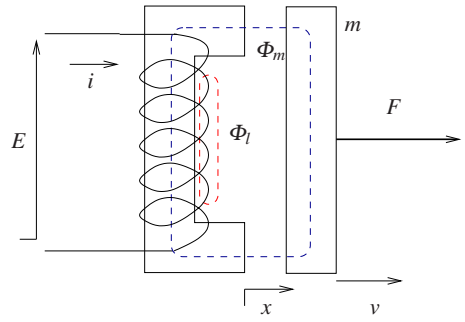

Fig. 3.4 An elementary electromagnet: a magnetic system with a moving part.

The flux linkage λ can be computed from the number of turns, N, and the magnetic induction flux, Φ, as

$$\lambda = N\Phi.$$

In turn, Φ has a leakage, Φ_l, and a magnetizing, Φ_m, parts, $\Phi = \Phi_l + \Phi_m$, which can be computed in terms of the reluctance of the respective paths:

$$\Phi_l = \frac{Ni}{\mathscr{R}_l} \qquad\qquad \Phi_m = \frac{Ni}{\mathscr{R}_m}.$$

The reluctance of the magnetizing path has a fixed contribution, the part of the iron path, and a variable one, the part of the air gap:

$$\mathscr{R}_m = \frac{l_i}{\mu_{ri}\mu_0 A_i} + \frac{2x}{\mu_0 A_g},$$

where μ_{ri}, the relative magnetic permeability of the iron core, is of the order of 10^3. Assuming that the sections of the iron and air gap paths are the same, $A_i = A_g = A$, one gets

$$\mathscr{R}_m = \frac{1}{\mu_0 A}\left(\frac{l_i}{\mu_{ri}} + 2x\right).$$

The relation between the current and the flux linkage can finally be written as

$$\lambda = \left(\frac{N^2}{\mathscr{R}_l} + \frac{N^2}{\mathscr{R}_m}\right) i = (L_l + L_m) i$$

with

$$L_m = \frac{N^2}{\mathscr{R}_m} = \frac{N^2\mu_0 A}{\frac{l_i}{\mu_{ri}} + 2x} \equiv \frac{b}{c + 2x}.$$

Assembling these results, we can obtain the constitutive relation at the electrical port

$$i(\lambda, x) = \left(a + \frac{b}{c + 2x}\right)^{-1} \lambda, \tag{3.21}$$

where $a = L_l = N^2/\mathscr{R}_l$. As explained elsewhere in this book, the constitutive laws of a multi-port must obey Maxwell's reciprocity relations, which in this case read

$$\frac{\partial F}{\partial \lambda} = \frac{\partial i}{\partial x}. \tag{3.22}$$

One gets then from (3.21)

$$\frac{\partial F}{\partial \lambda} = \frac{2b}{(2ax+ac+b)^2}\lambda,$$

from which

$$F(\lambda,x) = \frac{b\lambda^2}{(2ax+ac+b)^2}. \tag{3.23}$$

Finally, $H_E(\lambda,x)$ can be computed from

$$H_E(\lambda,x) = \int_{(0,0)}^{(\lambda,x)} \left(i(\tilde{\lambda},\tilde{x})\mathrm{d}\tilde{\lambda} + F(\tilde{\lambda},\tilde{x})\mathrm{d}\tilde{x} \right),$$

or just using the result for linear magnetic materials. Either way, one gets

$$H_E(\lambda,x) = \frac{1}{2}\frac{c+2x}{2ax+ac+b}\lambda^2. \tag{3.24}$$

The port-Hamiltonian structure is thus

$$\begin{aligned} \begin{bmatrix} \dot{\lambda} \\ \dot{x} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_{EM}}{\partial \lambda} \\ \frac{\partial H_{EM}}{\partial x} \\ \frac{\partial H_{EM}}{\partial p} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E \\ F \end{bmatrix} \\ \begin{bmatrix} i \\ v \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial H_{EM}}{\partial \lambda} \\ \frac{\partial H_{EM}}{\partial x} \\ \frac{\partial H_{EM}}{\partial p} \end{bmatrix} \end{aligned} \tag{3.25}$$

where

$$H_{EM}(\lambda,x,p) = \frac{1}{2}\frac{c+2x}{2ax+ac+b}\lambda^2 + \frac{1}{2m}p^2.$$

Notice that, just replacing F with a constant gravitational force and expressing it as a gradient of the gravitational energy, which can then be added to H_{EM}, one obtains the model of the magnetically levitating ball of Example 2.2, albeit without the dissipation term.

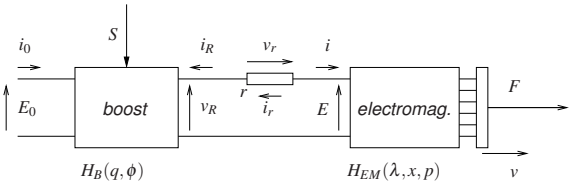

Fig. 3.5 Coupling of the boost and the electromagnet.

3.1.4 Coupling of the boost converter and the electromagnet

As a final example, we connect the boost converter, without the resistive termination, to the electromagnet, inserting a resistor r in series connection between them, as shown in Fig. 3.5, where we have renamed the boost variables to (q,ϕ). The series resistor obeys (we adopt an input power convention for this one)

$$v_r = ri_r \tag{3.26}$$

and where the interconnecting Dirac structure is provided by Kirchhoff's laws

$$i_R = i_r \qquad i_r = -i \qquad E = v_R + v_r$$

From the first output relation of the boost subsystem, $v_R = \frac{\partial H_B}{\partial q}$, and the first one of the electromagnet, $i = \frac{\partial H_{EM}}{\partial \lambda}$, together with (3.26) and Kirchhoff's laws, one can express the internal port variables as

$$E = v_R + v_r = \frac{\partial H_B}{\partial q} + ri_r = \frac{\partial H_B}{\partial q} - ri = \frac{\partial H_B}{\partial q} - r\frac{\partial H_{EM}}{\partial \lambda}$$
$$i_R = -i = -\frac{\partial H_{EM}}{\partial \lambda}$$

Substituting these into (3.7) and (3.25), if $H(q,\phi,\lambda,x,p) = H_B(q,\phi) + H_{EM}(\lambda,x,p)$ one gets:

$$\begin{bmatrix} \dot{q} \\ \dot{\phi} \\ \dot{\lambda} \\ \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1-S & 0 & 0 & 0 \\ -1+S & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial \phi} \\ \frac{\partial H}{\partial \lambda} \\ \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} -\frac{\partial H}{\partial \lambda} \\ E_0 \\ \frac{\partial H}{\partial q} - r\frac{\partial H}{\partial \lambda} \\ 0 \\ F \end{bmatrix}$$

This can be rewritten in explicit port-Hamiltonian form, with inputs E_0 and F and outputs i_0 and v, as

$$\begin{bmatrix} \dot{q} \\ \dot{\phi} \\ \dot{\lambda} \\ \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1-S & -1 & 0 & 0 \\ -1+S & 0 & 0 & 0 & 0 \\ 1 & 0 & -r & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial \phi} \\ \frac{\partial H}{\partial \lambda} \\ \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E_0 \\ F \end{bmatrix} \tag{3.27}$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial \phi} \\ \frac{\partial H}{\partial \lambda} \\ \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{bmatrix} \tag{3.28}$$

3.1.5 Variable structure systems

Assume a VSS system such that the change in the state variables is small over the time length of an structure change, or such that one is not interested about the fine details of the variation. Then one may try to formulate a dynamical system for the time average of the state variables (*state space averaging*, or SSA)

$$\langle x \rangle(t) = \frac{1}{T} \int_{t-T}^{t} x(\tau)\, \mathrm{d}\tau, \tag{3.29}$$

where T is the period, assumed constant, of a cycle of structure variations. Let our VSS system be described in explicit port Hamiltonian form

$$\dot{x} = [J(S,x) - R(S,x)] \frac{\partial H}{\partial x}(x) + g(S,x)u, \tag{3.30}$$

where S is a (multi)-index, with values on a finite, discrete set, enumerating the different structure topologies. For notational simplicity, we will assume from now on that we have a single index (corresponding to a single switch, or a set of switches with a single degree of freedom) and that $S \in \{0,1\}$. Hence, we have two possible dynamics, which we denote as

$$\begin{aligned} S = 0 &\Rightarrow \dot{x} = \big[J_0(x) - R_0(x)\big] \frac{\partial H}{\partial x}(x) + g_0(x)u, \\ S = 1 &\Rightarrow \dot{x} = \big[J_1(x) - R_1(x)\big] \frac{\partial H}{\partial x}(x) + g_1(x)u. \end{aligned} \tag{3.31}$$

Note that controlling the system means choosing the value of S as a function of the state variables, and that u is, in most cases, just a constant external input. Moreover, from (3.29) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle x \rangle(t) = \frac{x(t) - x(t-T)}{T}. \tag{3.32}$$

Now the central assumption of the SSA approximation method is that for a given structure we can substitute $x(t)$ by $\langle x\rangle(t)$ in the right-hand side of the dynamical equations, so that (3.31) become

$$\begin{aligned} S=0 &\Rightarrow \dot{x} \approx \big[J_0(\langle x\rangle) - R_0(\langle x\rangle)\big]\frac{\partial H}{\partial x}(\langle x\rangle) + g_0(\langle x\rangle)u, \\ S=1 &\Rightarrow \dot{x} \approx \big[J_1(\langle x\rangle) - R_1(\langle x\rangle))\big]\frac{\partial H}{\partial x}(\langle x\rangle) + g_1(\langle x\rangle)u. \end{aligned} \tag{3.33}$$

The rationale behind this approximation is that $\langle x\rangle$ does not have time to change too much during a cycle of structure changes. We assume also that the length of time in a given cycle when the system is in a given topology is determined by a function of the state variables or, in our approximation, a function of the averages, $t_0(\langle x\rangle)$, $t_1(\langle x\rangle)$, with $t_0 + t_1 = T$. Since we are considering the right-hand sides in (3.33) constant over the time scale of T, we can integrate the equations to get[2]:

$$\begin{aligned} x(t) = x(t-T) &+ t_0(\langle x\rangle)\left[\big[J_0(\langle x\rangle) - R_0(\langle x\rangle)\big]\frac{\partial H}{\partial x}(\langle x\rangle) + g_0(\langle x\rangle)u\right] \\ &+ t_1(\langle x\rangle)\left[\big[J_1(\langle x\rangle) - R_1(\langle x\rangle)\big]\frac{\partial H}{\partial x}(\langle x\rangle) + g_1(\langle x\rangle)u\right]. \end{aligned}$$

Using (3.32) we get the SSA equations for the variable $\langle x\rangle$ (which we rewrite again as x to simplify the notation):

$$\begin{aligned} \dot{x} &= d_0(x)\left[\big[J_0(\langle x\rangle) - R_0(\langle x\rangle)\big]\frac{\partial H}{\partial x}(\langle x\rangle) + g_0(\langle x\rangle)u\right] + \\ &\quad + d_1(x)\left[\big[J_1(\langle x\rangle) - R_1(\langle x\rangle)\big]\frac{\partial H}{\partial x}(\langle x\rangle) + g_1(\langle x\rangle)u\right] \\ &= \Big\{\big[d_0(x)J_0(x) + d_1(x)J_1(x)\big] - \big[d_0(x)R_0(x) + d_1(x)R_1(x)\big]\Big\}\frac{\partial H}{\partial x}(x) + \\ &\quad + \big[d_0(x)g_0(x) + d_1(x)g_1(x)\big]u, \end{aligned} \tag{3.34}$$

where

$$d_{0,1}(\langle x\rangle) = \frac{t_{0,1}(\langle x\rangle)}{T}, \tag{3.35}$$

with $d_0 + d_1 = 1$. In the power converter literature d_1 (or d_0, depending on the switch configuration) is referred to as the *duty cycle*. Equation (3.34) is again a port-Hamiltonian system, with interconnection, dissipation and port matrices given by combinations of the individual topology matrices. Notice that this is a smooth system, whose numerical implementation is much easier than the original VSS. Discussion of these kind of averaged systems in the bond graph formalism can be found in [63]. In fact, this analysis can be extended to the case when higher order harmonics, and not just the zeroth order one considered here, are introduced; the result, as shown in [17], is again a system in port-Hamiltonian form.

[2] We also assume that u does not vary over this time scale; in fact u is constant in many applications.

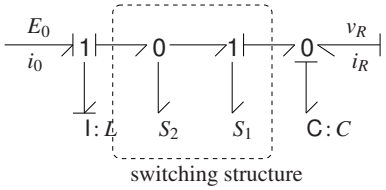

Fig. 3.6 The bond graph of the boost converter.

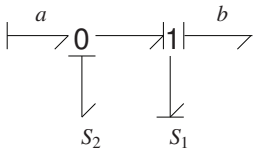

Fig. 3.7a Causality assignments of the switching structure of the boost converter: S_2 closed and S_1 open.

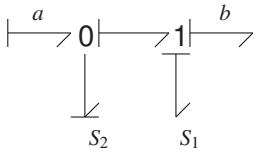

Fig. 3.7b Causality assignments of the switching structure of the boost converter: S_2 open and S_1 closed.

As an example, we can retake the boost converter discussed previously. In the above notation, for the case of an open load port, one has

$$J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad J_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad g_0 = g_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and no dissipation, i.e. $R_0 = R_1 = 0$. Putting this into (3.34) and denoting $\mu = t_0(x)/T$, one gets

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} q_C \\ \phi_L \end{bmatrix} = \begin{bmatrix} 0 & \mu \\ -\mu & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_B}{q_C} \\ \frac{\partial H_B}{\phi_L} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} ci_R \\ E_0 \end{bmatrix} \tag{3.36}$$

This has exactly the same form that the exact, non-smooth model (3.7), with $\mu = 1 - S$, except for the fact that all the state variables are averages, and μ can take values in the continuum $[0, 1]$. It is instructive to derive these results from a bond graph approach. Indeed, the bond graph of the boost converter, Fig. 3.1, considering the two switches as open ports, is displayed in Fig. 3.6. The interior bonds of the switching structure have no causality assignment, because it depends on the state of the switches. In fact, there are two possibilities, as shown in Figures 3.7a and 3.7b. Notice that the two possible interior assignments have the same output causality. In the first case represented in Fig. 3.7a, one has $v_2 = 0$ and $i_1 = 0$, since S_2 acts a zero voltage source while S_1 is a zero current source; the situation is reversed in the second case, illustrated in Fig. 3.7b. Writing down the bond graph equations for both cases, one arrives at the input/output relations

$$e_a = \mu e_b \qquad\qquad f_b = \mu f_a$$

with $\mu = 0$ in the first case, and $\mu = 1$ in the second one. This corresponds to the relations of an ideal transformer, and the associated bond graph is displayed in

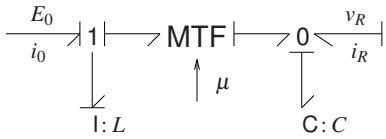

Fig. 3.8 The bond graph of the boost converter with a transformer instead of the two switches.

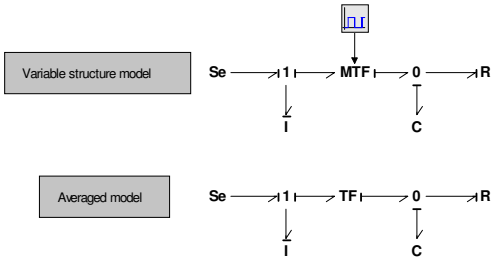

Fig. 3.9 Variable structure and averaged 20-sim models for the boost converter.

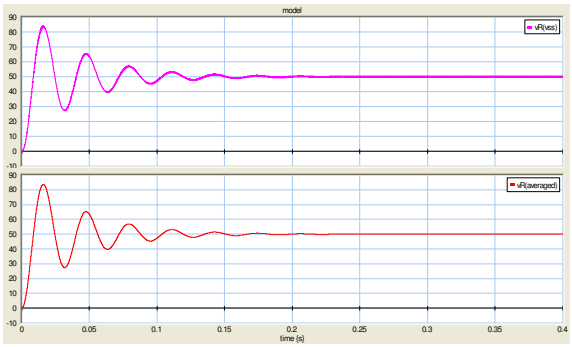

Fig. 3.10 Load voltage waveform for the VSS and averaged models.

Fig. 3.8. If one writes the equations associated to this bond graph, the same form as in (3.36) is obtained. From the point of view of the bond graph, μ could be a boolean variable $\mu \in \{0,1\}$ or a continuous one $\mu \in [0,1]$. However, if the later option is taken the averaged model is recovered. Fig. 3.9 shows the 20-sim schemes for both models, with the same system parameters, and Fig. 3.10 shows the load voltage for the corresponding simulations, for system parameters $C = 0.002$, $L = 0.002$, $E_0 = 20$, and $R = 10$ (in SI units). For the variable structure model, a square periodic signal with $T = 0.0004$ s and duty-cycle $\mu = 0.6$ is injected into the MTF, while for the averaged model the parameter transformer is $\mu = 0.6$. The averaged model yields, indeed, the averages of the state variables of the VSS model; on closer inspection it can be seen that the later has, however, a small ripple due to the commutation, not present in the averaged model simulation. Notice that, in both cases, the asymptotic output voltage is $E_0/(1-\mu) = 50$ V, as expected for a boost converter.

3.2 Modeling of mechanical systems

3.2.1 Short introduction and motivations

In Chapter 1, we have introduced the basic notions of port-Hamiltonian systems. In this part we will show how it possible to use these techniques for the structured modeling of robotics mechanisms. If we would just start modeling the motion of a point mass, due to the geometry of the space it moves in, this could be done by just considering usual coordinates. The port variables would then be the usual vector forces and velocities and no extra structure would be necessary. To be more specific, a particle mass is easy to describe because its configuration can be associated to a point of the three-dimensional Euclidean space. After having chosen coordinates, each point can be associated to a triple of real numbers in $\mathbb{R}^3$; but the most important thing is that the algebraic and topological properties of $\mathbb{R}^3$ correspond to real physical properties of the motion of the particle: forces can be added; velocity vectors too; magnitudes of vectors correspond to magnitudes of forces and velocities; "orthogonality" of a force and velocity vectors gives zero power; the velocity and acceleration vectors are the time derivatives of the points position vector; Newton's Laws link a three-dimensional force vector to a three-dimensional acceleration vector, through the *apparently* scalar quantity "mass", for point masses, as well as spherically-symmetric rigid bodies such as planets and canon balls.

In contrast to the simplicity of the point mass motion properties, the motion and the dynamics of a rigid body are much more complex. A rigid body is composed of an infinite number of point masses, which are constrained not to move with respect to each other. It turns out that the dimension of the space necessary to describe the configuration of a rigid body is six: three dimensions for orientation, and three for translation. The force-acceleration relation is now a full six-by-six matrix, and not a scalar anymore. Moreover, the acceleration involved in this dynamic relation is *not* just the second-order time derivative of the position/orientation vector of the rigid body.

Even the short overview above should make clear that it is wrong to treat the six position/orientation coordinates of a rigid body in the same way as one treats the three position coordinates of a point: the *geometrical properties* of rigid bodies are fundamentally different from the geometrical properties of point masses. For example, if one continuously increases one of these six numbers (i.e., one that corresponds to orientation representation), the rigid body arrives at the *same* configuration after every rotation over 360 degrees. This "curvature" property does not occur when one indefinitely increases any of the three coordinates of a point configuration. Locally (i.e., in the neighborhood of a specific configuration) it *is* possible to describe a configuration using six real numbers, but this description is not an *intrinsic property* of the motion. (An intuitive definition of an "intrinsic property" is: any property that does not change if one changes the coordinate representation.)

A lot of powerful tools are available which allow to describe the motion of rigid bodies in a geometrical and global way. These methods are related to the geometry

of *lines* and *screws* and to the differential geometric concept of a *Lie group* (see Sect. B.1.4). Furthermore, the concept of a *Lie group* is the key structure which will allow to describe interconnection of rigid bodies by expressing common variables on which the interconnection is based. In a nutshell: it's not because one can use n numbers as coordinates on a given space, that the objects in that space have exactly the same properties as the n-tuples in $\mathbb{R}^n$!

3.2.2 Configuration and twist of a rigid body

We will now start introducing the concepts which are needed to handle multi-body systems.

3.2.2.1 Describing configurations

In case of a point mass moving in the Euclidean space $\mathbb{E}$, once an orthonormal reference frame Ψ has been chosen, we can associate with a bijective relation three numbers to it: $(x,y,z) \in \mathbb{R}^3$. Velocities with respect to an observer not moving with respect to Ψ and expressed in the same reference frame will be simply equal to $(\dot{x},\dot{y},\dot{z}) \in \mathbb{R}^3$.

For rigid bodies this will be more complicated. In principle, a rigid body configuration is a six dimensional space (three translation and three rotations), but due to the topology of the space of rotations, there does not exist six global coordinates. In what follows, we will associate the configuration of a rigid body to a matrix which is called a homogeneous matrix once a reference has been chosen. First, for reasons based on projective geometry which can be further read in [196], it is convenient to describe the coordinate of a point using a four dimensional vector in which the first three components are the usual ones and the last is the scalar 1. For a point p expressed in an orthonormal frame Ψ_i, its coordinates will then be a vector of the following form:

$$P^i = \begin{bmatrix} x_i & y_i & z_i & 1 \end{bmatrix}^{\mathrm{T}} \tag{3.37}$$

If we consider a second orthonormal reference frame Ψ_j, the same point will have a similar representation with different numbers if Ψ_i and Ψ_j do not coincide. It would be possible to see that the change of coordinates would be given by

$$P^j = H_i^j P^i \tag{3.38}$$

with

$$H_i^j = \begin{bmatrix} R_i^j & p_i^j \\ 0_3 & 1 \end{bmatrix} \tag{3.39}$$

where $R_i^j \in SO(3)$ is an orthonormal matrix, i.e. with determinant equal to 1 and such that $R^{\mathrm{T}} = R^{-1}$, and $p_i^j \in \mathbb{R}^3$.

If we now consider Ψ_i fixed to a body I and Ψ_j fixed to a body J, it is possible to describe their relative configuration by the relative configuration of the two frames which is represented by H_i^j using their change of coordinates. It could also easily be seen that H_i^j beside representing the change of coordinates of the same physical point from frame Ψ_i to frame Ψ_j, it also coincides with the physical motion which brings Ψ_j to Ψ_i for points expressed in either of the two frames; note that the inverse direction of the indices is not a typo. To be more precise, if we consider that a rigid body motion would bring frame Ψ_j to Ψ_i and this motion would also bring a point p to a point q, we would have that $q^i = H_i^j p^i$ and $q^j = H_i^j q^j$. For what it will follow later, it is important to notice that $(H_i^j)^{-1} = H_j^i$.

3.2.2.2 Relative instantaneous motions and twists

If a representation of a velocity of a point mass would be just the time derivative of its coordinate, we could consider as velocity of a rigid body the time derivative of the matrix H representing the relative configuration of the body with respect to another frame, the observer. This is in principle correct, but there are a number of problems related to this approach:

1. $\dot{H}$ has many more elements than necessary to express the six dimensional infinitesimal motion;
2. From the information of $\dot{H}$ would not be possible to have an idea of the relative motion without knowing H;
3. If we wanted to interconnect two bodies A and B using port variables, the velocity $\dot{H}$ could not be used since each of the bodies would have a different configuration H and a different $\dot{H}$. This would correspond to two vectors belonging to two different tangent spaces in a differential geometric context and therefore no natural operation can be performed among these vectors.

The solution to all the previous problems can be achieved by using the intrinsic structure of what is called a Lie group for the group of motions represented by the matrices $H \in SE(3)$. Properly speaking, the Lie Group $SE(3)$ is more general than a Lie group of matrices, but for what it will presented it is didactically sensible to describe what follows using matrix Lie groups. A Lie group is both a manifold, smooth nonlinear structure locally bijective to $\mathbb{R}^n$, and a group, it has a special point called the identity and an operation which allows to compose elements of the manifold, satisfying associativity and having for each element an inverse. This structure allows to associate to each element $(H, \dot{H}) \in T\,SE(3)$ belonging to the vector space $T_H SE(3)$ tangent to the manifold $SE(3)$ at the point $H \in SE(3)$ two unique vectors in the tangent space $se(3) := T_I SE(3)$ at the identity of the group, which in our case corresponds to the 4 identity matrix $I \in SE(3)$. These two vectors have a clear geometrical interpretation and they are called the *left* and *right translation* of $(H, \dot{H})$ to the Lie algebra $se(3)$.

Such a structure allows to naturally find unique representatives of velocities in a common space called a Lie algebra, allowing circumventing all previously de-

scribed problems. This is a generalization of what is done for rotations in which the angular velocity vector can be easily and effectively used to describe rotational motions in a coordinate free, body independent way; the role of angular velocity for general rigid motions will be a twist. It could be shown that, following the notation previously introduced we can indicate with $H_i^j(t) \in SE(3)$ the smoothly time varying homogeneous matrix which can be used to change the homogeneous coordinates of a point P from its representation in the two relatively moving frames Ψ_i to Ψ_j. After we introduce a third frame Ψ_k, we could also consider the changes of coordinate between any of these frames and it could be proven that ${}^k\tilde{T}_i^j := H_j^k \dot{H}_i^j H_k^i$ has always the following form:

$$ {}^k\tilde{T}_i^j := \begin{bmatrix} \tilde{\omega} & v \\ 0_3 & 0 \end{bmatrix} \tag{3.40} $$

where for any three dimensional vector ω, $\tilde{\omega}$ is the unique skew-symmetric matrix such that for each x, $\omega \times x = \tilde{\omega} x$. In case $k = j$, the corresponding matrix is called the *right translation* to the Lie algebra and in case $k = i$ the *left translation* to the Lie algebra for reasons which can be found in any reference on Lie groups[3]. It can be seen that in (3.40), only the six scalars of ω and v are independent, and therefore we can also define the equivalent six dimensional vector representation

$$ {}^kT_i^j = \left[\omega^{\mathrm{T}} \; v^{\mathrm{T}}\right]^{\mathrm{T}} $$

This vector, which is called a twist, is a real geometrical object which describes the relative instantaneous motion of the body rigidly connected to frame Ψ_i with respect to the body rigidly connected to frame Ψ_j expressed numerically as a vector in the frame Ψ_k.

It is possible to see that we can change coordinates of a twist using what is called the Adjoint representation of the group which is represented by a 6×6 matrix:

$$ {}^kT_i^j = \mathrm{Ad}_{H_l^k} \, {}^lT_i^j \tag{3.41} $$

where

$$ \mathrm{Ad}_{H_l^k} = \begin{bmatrix} R_l^k & 0 \\ \tilde{p}_l^k R_l^k & R_l^k \end{bmatrix} \tag{3.42} $$

R_l^k is the rotation sub-matrix of H_l^k and $\tilde{p}_l^k$ is the skew-symmetric representation of the position sub-vector of H_l^k.

For rotations, we can associate a vector called the angular velocity, which geometrically completely expresses the instantaneous rotation of a body independently of its pose at a certain instant of time. the direction of the angular rotation vector, represents the instantaneous axis of rotation and direction (clockwise or anticlock-

[3] In Lie group theory, ${}^k\tilde{T}_i^j$ is introduced as the Adjoint transformation of the left or right translation, for example ${}^k\tilde{T}_i^j = \mathrm{Ad}_{H_j^k}\left(R_{H_j^i}(H_i^j, \dot{H}_i^j)\right)$ where $R_{H^{-1}}$ indicated the right translation of the vector $(H, \dot{H})$ to the Lie algebra $se(3)$.

wise around the axis) and its magnitude the angular velocity. In the same way, as shown by the Mozzi theorem [150], a twist is represented by a geometrical object which is called a screw since it is represented by an axis, a pitch, a direction and a magnitude. Mozzi's theorem says that we can always write a twist as:

$$\begin{bmatrix} \omega \\ v \end{bmatrix} = ||\omega|| \underbrace{\begin{bmatrix} \hat{\omega} \\ r \wedge \hat{\omega} \end{bmatrix}}_{rotation} + \alpha \underbrace{\begin{bmatrix} 0 \\ \hat{\omega} \end{bmatrix}}_{translation} \tag{3.43}$$

where ω and v are three dimensional vectors numerically expressed in k. The vector

$$\begin{bmatrix} \hat{\omega} \\ r \wedge \hat{\omega} \end{bmatrix}$$

represents a geometrical line passing through the point r and directed along ω. This line representation called Plücker coordinates has many advantages with respect to others like the fact that its representation has numerically a one to one relation to lines in the projective 3-space. The scalar coefficient α is called the pitch. The theorem basically says that each motion instantaneously can be seen as a screw motion: an instantaneous rotation around an axis, represented as just described, and an infinitesimal translation along this axis. The ratio of these two motions is given by the pitch coefficient α.

An essential feature of twists is that they are independent of the pose of a body and can therefore be used to describe relative motions of any body and are the key to define power ports in rigid multi-body mechanical systems. Twists play the role of flows and once we introduced the dual efforts, we will have the essential components to define a power port for rigid multi-body systems.

3.2.2.3 The dual of twists: wrenches

Since the space $se(3)$ of twists is a vector space, we can directly define the dual space $se^*(3)$ which will be a dual Lie algebra whose elements are called wrenches. Clearly, dimensionally, these elements are of the same size of twists (six dimensional) and the dual product of wrenches and twists will be a scalar representing the power exchanged by the wrench (generalization of the force) with the relative motion represented by the twist. A wrench in vector form can be indicated with

$${}^{k}W_{i}^{j} = \begin{bmatrix} \tau^{\mathrm{T}} & f^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

and represents the wrench applied by i on j as a vector expressed in frame Ψ_k. Sometimes the presence of the index i may not be necessary. As a direct consequence of (3.41) and the dual nature of a wrench with respect to a twist, the change of coordinates of a wrench is expressed by:

$$\left({}^{k}W_i^{j}\right)^{t} = \mathrm{Ad}^{t}_{H_k^l}\left({}^{l}T_i^{j}\right)^{t} \tag{3.44}$$

To understand why such a quantity can be seen as the generalization of a force, the theorem of Pointsot can be used. This theorem states that any system of forces applied to a rigid body, can be simplified as the resultant of one force along a specific line in space and a torque oriented in the same direction. Similarly to (3.43), this dual theorem can be expressed as:

$$\begin{bmatrix}\tau\\ f\end{bmatrix} = ||f|| \underbrace{\begin{bmatrix}r\wedge f\\ f\end{bmatrix}}_{force} + \alpha \underbrace{\begin{bmatrix}f\\ 0\end{bmatrix}}_{torque} \tag{3.45}$$

where the first vector indicates the linear force along the line oriented along f and passing through the point r and the second vector indicate the torque oriented in the same direction of f, but such that its application point does not matter being a torque. The ratio of these two magnitudes is expressed by the scalar pitch α.

3.2.2.4 Screw power ports

It is now possible to introduce the coordinate-free concept of a screw power-port. A screw power port is the pair of a twist and a wrench

$$\left({}^{k}T_i^{j},\ {}^{k}W_{\cdot}^{i}\right) \in se(3)\times se^{*}(3)$$

and can be used as a mean to interconnect multi-body systems as we will see in more details later.

3.2.3 Rigid body dynamics

Using the concept of screws and Lie groups, we can introduce the inertia tensor which represents the complete inertial properties of a rigid body. The inertia tensor is a metric, positive definite quadratic form, which associates the kinetic co-energy to a certain twist of a rigid body:

$$H^{*}({}^{k}T_i^{0}) = \frac{1}{2}\left({}^{k}T_i^{0}\right)^{t}\,{}^{k}I^{i}\,{}^{k}T_i^{0} \tag{3.46}$$

where ${}^{k}I^{i}$ represents the inertia tensor of body i expressed in frame k and it is a 6×6 matrix corresponding to a quadratic form in $se(3)$. Using the inertia tensor we can also define what is called the screw-momenta which is a co-vector (belonging to $se^{*}(3)$) representing the 6-dimensional momenta of a rigid body:

$$\left({}^{k}M^{i}\right)^{t} := {}^{k}I^{i}\ {}^{k}T_{i}^{0} \tag{3.47}$$

where ${}^{k}M^{i}$ indicates the screw-momenta of body i expressed in frame k. Using the expression of momenta we can now also write an expression for the kinetic energy as a quadratic form on $se^{*}(3)$:

$$H({}^{k}M^{i}) = \frac{1}{2}\,{}^{k}M^{i}\left({}^{k}I^{i}\right)^{-1}\left({}^{k}M^{i}\right)^{t} \tag{3.48}$$

Newton's law for a point mass says that in an inertial frame the time change of momenta of a point mass is equal to the total force applied to it, i.e. $\dot{p}^{0} = F^{0}$. It could be shown that integrating this equation for a complete rigid body, it generalizes to:

$${}^{0}\dot{M}^{i} = {}^{0}W_{\cdot}^{i} \tag{3.49}$$

where ${}^{0}M^{i}$ is the momenta of body i expressed numerically in the inertial frame Ψ_0 and ${}^{0}W_{\cdot}^{i}$ the total wrench applied to body i expressed in frame Ψ_0.

Since the time derivative of the momenta is equal to a wrench, both terms of the equation transform identically with changes of coordinates and this is why the wrench is a co-vector. If we write the previous equation as an equality of column vectors, we have

$$\left({}^{0}\dot{M}^{i}\right)^{t} = \left({}^{0}W_{\cdot}^{i}\right)^{t},$$

It can be seen that by changing the coordinates to a frame Ψ_k rigid with the body, the previous expression becomes:

$$({}^{k}\dot{M}^{i})^{t} = \mathrm{ad}^{\mathrm{T}}_{{}^{k}T_{i}^{0}}({}^{k}M^{i})^{t} + ({}^{k}W_{\cdot}^{i})^{t}. \tag{3.50}$$

where

$$\mathrm{ad}^{\mathrm{T}}_{{}^{k}T_{i}^{0}} = \begin{bmatrix} -\tilde{\omega} & -\tilde{v} \\ 0 & -\tilde{\omega} \end{bmatrix}$$

and ω and v are the vectors composing ${}^{k}T_{i}^{0}$ and introduced in (3.40). After some calculations, it could be seen that

$$\mathrm{ad}^{\mathrm{T}}_{{}^{i}T_{i}^{0}}({}^{i}M^{i})^{t} = ({}^{i}M^{i}\wedge)\ {}^{i}T_{i}^{0} \tag{3.51}$$

where

$$({}^{i}M^{i}\wedge) := \begin{bmatrix} {}^{i}\tilde{M}^{i}{}_{\omega} & {}^{i}\tilde{M}^{i}{}_{v} \\ {}^{i}\tilde{M}^{i}{}_{v} & 0 \end{bmatrix} \tag{3.52}$$

and ${}^{i}\tilde{M}^{i}{}_{\omega}$ and ${}^{i}\tilde{M}^{i}{}_{v}$ are the skew-symmetric form of three-vectors corresponding to respectively the first and last three components of ${}^{i}M^{i}$.

By looking closer at (3.50) it can be seen that the configuration of the rigid body does not appear in the equation at all. The equation is useful because in body fix coordinates the expression of the inertia tensor which we will need in order to calculate ${}^{k}M^{i}$ is a constant matrix which can be easily calculated. The equation (3.50)

is only dependent on the momenta and the effect of an applied wrench on it. This equation is all we need if we want to study a single rigid body independently. On the other hand, if we want to calculate dynamics of interconnected mechanisms, each of the quantities, like applied wrenches between bodies, will have to be described in the same coordinate systems. We need therefore a common reference frame in which we can take operations among tensors. Furthermore, if we want to consider gravity, its direction and effect will change in body coordinates and therefore it is necessary to keep track of the pose of a rigid body. We can tackle all these problems by considering in the open model of a rigid body a potential energy function of the configuration of the rigid body with respect to a common, inertial coordinate system.

3.2.3.1 Potential energy

The configuration of a rigid body is homeomorphic to $SE(3)$ and can be associated to a homogeneous matrix $H_i^0 \in SE(3)$ as seen previously. This means that any potential energy function of a rigid body will be expressed by a function of the form:

$$V : SE(3) \to \mathbb{R} \tag{3.53}$$

Normally, we would calculate the corresponding force field of a potential energy by taking the differential of the function, but in the case of a rigid body this is not direct since the argument of the function is a matrix of a specific form: the arguments are a set of variables on which constraints hold. A way to tackle this would be to find a minimal parametrization of $SE(3)$ which would then allow to take the differential in the usual way. This approach would be strictly local and not general. A much more elegant and effective approach is instead the usage of exponential coordinates of the group $SE(3)$ which, for matrix Lie groups, corresponds to the matrix exponential:

$$\phi_i^0 \mapsto H_i^0 = e^{\tilde{\phi}_i^0} \tag{3.54}$$

where the tilde operation in the argument of the exponential corresponds to the same tilde operation as for twists. The entity ϕ_i^0 is geometrically also belonging to $se(3)$ and it corresponds to what is called the *finite twist* (rather than infinitesimal). By means of the exponential map, we can express a potential function using a minimal set of coordinates. In this way we could write a corresponding potential function to (3.53) as $V_\phi : se(3) \to \mathbb{R}$, with

$$\phi_i^0 \mapsto V(\log(\phi_i^0)) \tag{3.55}$$

where log is a periodic function which can be calculated easily with the techniques presented in [213].

For what follows, it is necessary to know the map which relates the time derivatives of the exponential coordinates ϕ_i^0 to the instantaneous twist ${}^iT_i^0$ in the following way:

$$ {}^{i}T_i^0 = K(\phi_i^0)\dot{\phi}_i^0 \tag{3.56} $$

where could be proven that

$$ K(\phi_i^0) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \mathrm{ad}_{\phi_i^0}^k \tag{3.57} $$

and can be calculated easily using the techniques introduced in [213]. From (3.56), the adjoint relation follows directly:

$$ \gamma = K^t(\phi_i^0)\ {}^{i}W_{\cdot}^{i} \tag{3.58} $$

where γ is a dual of the time derivative of the exponential coordinates like is the case for

$$ \frac{\partial V_\phi}{\partial \phi_i^0} \in se^*(3) $$

We can therefore calculate the wrench which the potential energy V_ϕ generates as:

$$ W = K^{-t}(\phi_1^0)\frac{\partial V_\phi}{\partial \phi_i^0} \in se^*(3) \tag{3.59} $$

We can now finally give a port-Hamiltonian expression of the dynamics of a rigid body. The Hamiltonian will be a sum of the kinetic and potential energy and will be:

$$ H({}^{k}M^i, \phi_i^0) = \frac{1}{2}{}^{k}M^i \left({}^{k}I^i\right)^{-1} \left({}^{k}M^i\right)^t + V_\phi(\phi_i^0) \tag{3.60} $$

and the port Hamiltonian equation become:

$$ \begin{bmatrix} \dot{\phi}_i^0 \\ \left({}^{k}\dot{M}^i\right)^t \end{bmatrix} = \begin{bmatrix} 0 & K^{-1}(\phi_i^0) \\ -K^{-t}(\phi_i^0) & ({}^{i}M^i\wedge) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \phi_i^0} \\ \frac{\partial H}{\partial {}^{k}M^i} \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} {}^{i}W_{\cdot}^{i} \tag{3.61} $$

$$ {}^{i}T_i^0 = \begin{bmatrix} 0\ I \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \phi_i^0} \\ \frac{\partial H}{\partial {}^{k}M^i} \end{bmatrix} \tag{3.62} $$

or expliciting all ports, a single matrix representing the interconnection structure and changing the coordinates of the interconnection port we obtain:

$$ \begin{bmatrix} \dot{\phi}_i^0 \\ \left({}^{k}\dot{M}^i\right)^t \\ -\ {}^{0}T_i^0 \end{bmatrix} = \begin{bmatrix} 0 & K^{-1}(\phi_i^0) & 0 \\ -K^{-t}(\phi_i^0) & {}^{i}M^i\wedge & \mathrm{Ad}^t_{e^{\phi_i^0}} \\ 0 & -\mathrm{Ad}_{e^{\phi_i^0}} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \phi_i^0} \\ \frac{\partial H}{\partial {}^{k}M^i} \\ {}^{0}W_{\cdot}^{i} \end{bmatrix} \tag{3.63} $$

In the previous equation, the matrix represents the Dirac structure composed of three ports:

$$ \left(\dot{\phi}_i^0, \frac{\partial H}{\partial \phi_i^0}\right) $$

corresponding to the flow of potential energy,

$$\left({}^{k}\dot{M}^{i}, \frac{\partial H}{\partial {}^{k}M^{i}}\right)$$

corresponding to the flow of kinetic energy and

$$\left({}^{0}T_{i}^{0},\ {}^{0}W_{\cdot}^{i}\right)$$

to the power port which can be used to interconnect and interact with the rigid body.

3.2.4 Rigid mechanisms: interconnections of rigid bodies

We will now discuss the topology of a mechanism and describe how to represent that in an elegant and effective way using the network structure represented by a Dirac structure. We will start looking at the constraints between pairs of rigid bodies.

3.2.4.1 Kinematic pairs

Consider now two rigid bodies whose relative motion is constraint. If we call these bodies i and j, we can describe the allowed motions by a subspace of $se(3)$ to which the twist ${}^{0}T_{i}^{j}$ should belong. Let's call this subspace $\mathscr{T}_A$. This subspace will in general be function of the relative configuration H_i^j and possibly of time even if we will not specifically show that in the following equations for the sake of clarity. Suppose that this subspace is of dimension $n < 6$. In this case there will be n linear independent vectors whose span will coincide with $\mathscr{T}_A$. We will indicate these twists as $T_A^1, T_A^2, \ldots, T_A^n$. Due to these degrees of freedom, there will also be n linear independent wrenches which can be applied between the two bodies which can transfer energy to the relative motion. We will indicate these wrenches $W_A^1, W_A^2, \ldots, W_A^n$ and the subspace they span with $\mathscr{W}_A$. It is very important to realize that we cannot in general define $\mathscr{W}_A$ uniquely once $\mathscr{T}_A$ is known or the other way around due to the absence of a unique metric in $se(3)$. Nevertheless, we could use bi-invariant forms for this purpose like the hyperbolic form.

What we can do in a unique way is to define the dual $6-n$ dimensional spaces $\mathscr{W}_C := \mathscr{T}_A^{\perp}$ corresponding to the constraint wrenches (any of these wrenches will not transfer power $W_C^i T_A^j = 0\ \forall i, j$) and $\mathscr{T}_F := \mathscr{W}_A^{\perp}$ corresponding to the forbidden motions (any applied wrench will not transfer energy to these directions $W_A^i T_F^j = 0\ \forall i, j$). We can then define the non-singular 6×6 matrix

$$W^{\mathrm{T}} := \left[W_A^1\ \ldots\ W_A^n\ W_C^1\ \ldots\ W_C^{6-n}\right] \tag{3.64}$$

It is now possible to consider for a kinematic pair k

$$W_k = W \begin{bmatrix} \tau \\ \lambda \end{bmatrix} \tag{3.65}$$

and dually

$$\begin{bmatrix} \dot{q} \\ v \end{bmatrix} = W^t T_k \tag{3.66}$$

where τ indicates the free torques which can be applied by external motors or actuators on the kinematic pair in order to generate or control motion in $\dot{q}$ and λ are indicating the Lagrangian multipliers which should be such to keep $v = 0$.

We can finally write (3.65) and (3.66) together to show the Dirac structure causally:

$$\begin{bmatrix} W_k \\ \dot{q} \\ v \end{bmatrix} \begin{bmatrix} 0 & W \\ -W^t & 0 \end{bmatrix} \begin{bmatrix} T_k \\ \tau \\ \lambda \end{bmatrix} \tag{3.67}$$

where the port (T_k, W_k) can be used to interconnect with the mechanism as it will be seen later, the port $(\dot{q}, \tau)$ can be used to drive and control the kinematic pair and therefore to supply or subtract energy to and from the mechanism and finally (v, λ) will be a port through which no power should be transferred since λ should be such that v would always be equal to 0.

3.2.4.2 Mechanism topology

Given a mechanism composed of rigid bodies, it is possible to find a network description of the mechanism similarly with what happens with electrical circuits, but now with complex six dimensional motions and time varying constraints. Suppose to have a mechanism composed of m rigid bodies indexed from 0 to $m-1$ and such that the body 0 corresponds to a inertial base. Suppose that among those bodies we have n nodic elements that constraint relative motions of pair of bodies. We can describe this topology with a graph $G_P = (V_P, E_P)$ called *primary graph* in which V_P is the set of m rigid bodies and E_P the set of n edges corresponding to the pair constraints. We can define an *incidence matrix* $B \in \mathbb{R}^{m \times n}$ in which each column corresponding to an edge has a 1 in the row corresponding to the body to which the edge points and a -1 in the row corresponding to the body from which the edge stems from and zero everywhere else. We can then define a second graph $G_L = (V_L, E_L)$ called the *Lagrangian tree* which is composed of $m-1$ edges stemming from each of the bodies which are not the inertial frames and all going toward the vertex 0. The graph which is obtained combining the primary and Lagrangian graph is called the *port connection graph*.

We can then define the fundamental $n \times (m+n)$ loop matrix

$$C := \begin{bmatrix} B^{\mathrm{T}} & I_{n \times n} \end{bmatrix} \tag{3.68}$$

Each of the rows of this matrix represents a loop of the port connection graph which can be obtained considering an edge of E_P and two edges of E_L which close the

edge of E_P via the body 0. To understand what this actually means we can see that the following equation holds:

$$C \begin{bmatrix} {}^0T_1^0 \\ \vdots \\ {}^0T_m^0 \\ T_1 \\ \vdots \\ T_n \end{bmatrix} = 0 \tag{3.69}$$

where T_i indicates the relative twist corresponding to the i-th kinematic pair. This is therefore a kind of Kirchhoff law which sums the relative twist around a loop to zero as it is done in electrical circuits for the sum of relative potentials around a mesh.

Dually we can introduce the fundamental $m \times (n+m)$ *cut-set matrix*

$$Q := \begin{bmatrix} I_{m\times m} & -B \end{bmatrix} \tag{3.70}$$

Each row i of this matrix represents a cut-set corresponding to the sum of wrenches on a body i. Such a cut-set has always one edge corresponding to the Lagrangian tree which will represent the total wrench which will change the momenta of body i and a number of other edges belonging to G_P corresponding to all wrenches the kinematic pairs can apply to body i. This is similar to the Kirchhoff current law at a node in an electrical circuit with the difference that in mechanics the non-nodicity of inertial elements requires explicitly the presence of the Lagrangian tree. With the usage of the fundamental cut-set tree, we obtain:

$$Q \begin{bmatrix} {}^0W_T^1 \\ \vdots \\ {}^0W_T^m \\ W_1 \\ \vdots \\ W_n \end{bmatrix} = 0 \tag{3.71}$$

where ${}^0W_T^i$ indicates the total wrench applied at body i and expressed in frame 0.

It can be easily seen that $CQ^t = 0$ and this actually corresponds to the Teleggen theorem well known in electrical networks. In order to see that, a vector of twists satisfying $C\bar{T} = 0$ can be also expressed using a image representation instead than a kernel representation:

$$\bar{T} = C^{\mathrm{T}}\alpha \qquad \forall \alpha$$

along a similar line of reasoning, any wrench set satisfying the network constrains can be expressed using an image representation with

$$\bar{W} = Q^{\mathrm{T}}\gamma \qquad \forall \gamma$$

If we now want to calculate the net power going into the network, we can just get the dual product of $\bar{W}$ and $\bar{T}$:

$$P = \bar{T}^t W = \alpha^t \underbrace{C Q^t}_{=0} \gamma = 0 \tag{3.72}$$

which indeed proves zero net power flow for any independent choices of $\bar{T}$ and $\bar{W}$ as known in Tellegen's theorem. We can also rearrange equations (3.71) and (3.69) to get to a general causal expression of the Dirac structure:

$$\begin{bmatrix} {}^0W_T^1 \\ \vdots \\ {}^0W_T^m \\ -T_1 \\ \vdots \\ -T_n \end{bmatrix} = \begin{bmatrix} 0 & B \\ -B^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} {}^0T_1^0 \\ \vdots \\ {}^0T_m^0 \\ W_1 \\ \vdots \\ W_n \end{bmatrix} \tag{3.73}$$

It would then finally be possible to couple all ports $({}^0T_1^0,\ {}^0W_T^i)$ to the interconnection port of the rigid body model (3.63) and to connect the ports corresponding to each (T_k, W_k) to the corresponding port of a kinematic pair using the expression in (3.67).

In this way we obtain a model of the dynamics of the complete mechanism as the interconnection of various Dirac structures and port-Hamiltonian subsystem. This way of modeling is structured and very suitable for computer support. Furthermore, the usage of coordinate-free concepts ensures that the analysis is global and singularity free.

3.2.5 Flexible mechanisms

The modeling techniques presented in the previous sections can be generalized in order to define a systematic procedure, based on port concepts, for modeling and simulating mechanical systems with rigid *and* flexible links. The mathematical description of the whole mechanical system results from the interconnection of simpler components (e.g. rigid bodies, flexible links and kinematic pairs). Using the nonlinear model of a flexible link in distributed port-Hamiltonian form presented in [128] and, briefly, in Sect. 4.3.2, it is not necessary, in the definition of the dynamic model, to simplify the elastic and nonlinear effects present in the flexible parts, and therefore also mechanism with large deflections can be easily handled. This approach differs from what is illustrated in the next section, in which a more rigorous way for describing elastic systems is discussed. Moreover, the modularity of the approach can also be exploited for simplifying the simulation of such dynamical systems, even for control applications. In fact, if an object-oriented software package for model-

ing physical system is adopted, beside the mathematical derivation of the model also the numerical simulation of complex mechanisms can be carried out simply by port interconnection, thus leaving the solution of the causality of each sub-system to the simulation package. Among these software packages, one can mention the implementations of the Modelica language [78, 146] as Dymola [70] or Open Modelica [204], or the 20-sim package.

This is something completely different from what can be found in literature, where several methodological approaches for the definition of dynamic models of multi-body mechanical systems, taking possibly into account both rigid and flexible links, [9, 18, 26, 61, 62, 152, 190, 191, 199, 205] can be found. Moreover, a number of software packages, e.g. [4, 20, 50, 108, 194, 203] are also currently available for their numerical simulation. In case of flexible systems, these modeling approaches usually relay on finite dimensional approximations of the flexible link dynamics (e.g. modal expansion, finite elements or floating frame of reference) or on a simplification of the (nonlinear) elastic behavior of flexible links (Timoshenko or Euler-Bernoulli theory), and therefore they do not easily allow the description of mechanisms characterized by large deformations. Moreover, even if the simulation package is able to deal with large deformations and nonlinear effects (e.g. ABAQUS [194], ANSYS Multiphysics [4], or COMSOL Multiphysics [50]), in general it is not a trivial task to include the presence of state-feedback controllers. In fact, this requires the development of proper spatial discretization techniques for the elastic dynamics that are able to deal with time-varying boundary conditions, such as the torques applied at the extremities of each flexible link. These limitation are not present within the port Hamiltonian framework. Refer to [124, 128] for further information.

3.3 Modeling of simple elastic systems

3.3.1 Introduction

Simple elastic structures like strings, beams or membranes and plates are basic elements for many engineering fields. Roughly spoken, their mathematical models are approximations of certain equations of linearized elasticity. The mathematical models of elasticity, like other model in physics consist of two types of equations. The balance and/or conservation equations express that certain physical quantities or their sum are preserved. Typical representatives are the conservation of mass, charge, linear momentum, etc. The constitutive relations describe the behavior of the materials, typical representatives are Hook's law or friction relations. Within this setting one assumes that balance equations are never violated, whereas constitutive relations are often approximately known only.

Mathematical models of elastic structures are based on the conservation of mass and the balance of linear momentum and momentum of momentum, see

[133, 210, 221]. To simplify this complicated set of partial differential equations, one makes the strong constitutive assumption of the symmetry of the stress. This relation guarantees that balance of momentum of momentum is fulfilled, and we have to take conservation of mass and balance of linear momentum into account only. Additionally in simple elasticity, one assumes the existence of the stored energy function to express certain constitutive relations. If this function exists and the symmetry of stress is met, then the derived mathematical models have the structure of a Lagrangian or Hamiltonian system for a certain choice of the coordinates. The Lagrangian or Hamiltonian structure of these models is preserved by their linearization.

The models of structures, like beams or plates etc., with a small extension in one direction compared to the others, can be approximated by reduced models with less spatial variables. This way, the models of beams, plates, etc. are derived by the reduction of the linearized equations of simple elasticity. Of course, if the reduction process preserves the Hamiltonian or Lagrangian structure, then the resulting models will have this structure, too. Proceeding this way we show in an exemplary fashion how the Lagrangian or Hamiltonian formulation of the Euler Bernoulli beam can be derived in a straightforward manner. Furthermore, we use the simple example of the motion of a planar rigid body to illustrate the presented methods and ideas. But it is worth to mention that the presented methods can be applied to other mechanical structures like the Timoshenko beam (see [206] and Sect. 4.3.1), the Kirchhoff or Midline plate [123, 147], shell or membranes in an analogous manner.

3.3.2 Simple elasticity

The geometry of the general equations of elasticity are far beyond this contribution. Therefore, we confine ourselves to the case of simple elasticity, where the existence of the so called stored energy function is assumed. Furthermore, we describe the motion in an inertial frame with Euclidean coordinates and trivial metric. This choice is essential, since the following considerations are valid only in these coordinates. The Lagrangian and the Eulerian description are the most popular ones in continuum mechanics. Since we consider elastic bodies, we choose the Lagrangian description, which allows us to take into account the constitutive relations, which describe the behavior of material. Furthermore, we confine ourselves to the time invariant case, but we permit inputs like force or stress fields.

From now on, we use the standard tensor notation to keep formulas as short as possible and apply Einstein's convention for sums. Whenever the range of an index $i = 1, \ldots, n$ is clear, we us the abbreviation

$$a^i b_i = \sum_{i=1}^{n} a^i b_i$$

Furhtermore, to avoid mathematical subleties we assume that all functions are sufficiently often continously differentiable, and that all regions for the integration are sufficiently nice.

We will consider functions, which depend on the time t and on the spatial coordinates X^I, $I = 1,\dots,p$. Let x^i, $i = 1,\dots,q$ denote the dependent coordinates, then $x^i = x^i(t,X)$ assigns the functions $x^i(t,X)$ to the coordinate x^i. Where no confusion occurs, we use the same symbol for the coordinate x^i and for the assigned function $x^i = x^i(t,X)$. Since we will deal with several higher order derivatives of these functions, we use the abbreviations

$$\partial_I = \frac{\partial}{\partial X^I} \qquad \partial_t = \frac{\partial}{\partial t} \qquad \partial_{IJ} = \partial_I \partial_J \qquad \partial_{tI} = \partial_t \partial_I$$

etc. We need also the derivative coordinates of first order x^i_t, x^i_I and higher order $x^i_{M\dots}$ with the unordered multi index $M = m_1,\dots,m_k,\dots,m_r$, $m_k \in \{t\} \cup \{1,\dots,p\}$, where $\#M = r$ is the order of the derivative. This notation is motivated by the assignment $x^i_M = \partial_M x^i(t,X)$, $\partial_M = \partial_{m_1} \cdots \partial_{m_r}$. We will also use the conventions $x^i_M = x^i$, $\partial_M x^i(t,X) = x^i(t,X)$ for $\#M = 0$.

Let us consider a function $f(t,X,x_M)$, $0 \le \#M \le m$. We say that the total derivative $(d_I f)(t,X,x_N)$, $0 \le \#N \le m+1$ of f in the direction of I is the unique function $(d_I f)$, which meets

$$\partial_I f\,(t,X,\partial_M x\,(t,X)) = (d_I f)\,(t,X,\partial_N x\,(t,X)) \tag{3.74}$$

Obviously, the differential operator d_I or the total derivative into the direction of I is given by

$$d_I = \partial_I + x^i_{M,I}\partial^M_i = \partial_I + \sum_{\#M \ge 0} x^i_{M,I}\partial^M_i \qquad \partial^M_i = \frac{\partial}{\partial x^i_M}$$

with $I \in \{t\} \cup \{1,\dots,p\}$.

3.3.2.1 Motion and coordinates

In general we need three coordinate systems for the modeling of an elastic body, the configuration space $\mathscr{C}$, where physics takes place, the reference space $\mathscr{R}$, where we do bookkeeping, and the more abstract space $\mathscr{G}$, which is used to parameterize maps from $\mathscr{R}$ to $\mathscr{C}$ by its generalized coordinates. Here, we choose $\mathscr{C} = \mathbb{R}^n$, $\mathscr{R} = \mathbb{R}^n$ and assume that $\mathscr{C}$, $\mathscr{R}$ are equipped with the Euclidean coordinates (x^i), (X^I), wit $i,I = 1,\dots,n$[4]. In addition $\mathscr{C}$ is an inertial space. The position of a mass point is given by $X \in \mathscr{B} \subset \mathscr{R}$, where $\mathscr{B}$ denotes the set of all mass points of the elastic body. A motion is a map

[4] Of course, values of $n \in \{1,2,3\}$ are of physical interest only. Since the following considerations are indpendent of choice of p, the value of p is unspecified.

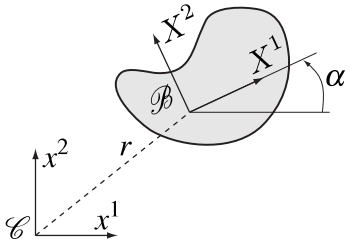

Fig. 3.11 Motion of a planar rigid body.

$$x^i = \phi^i(t,X) \qquad i = 1,\ldots,n \tag{3.75}$$

which assigns the position x of a mass point X at time t. We assume that we can invert ϕ such that $X = \phi^{-1}(t,x)$ is met. Fig. 3.11 illustrates the configuration and reference space $\mathscr{C} = \mathbb{R}^2$, $\mathscr{R} = \mathbb{R}^2$ together with map (3.75) for the example of a planar rigid body.

Throughout this contribution we equip the Euclidean space $\mathscr{C}$ with the trivial metric. Let (v^i), (w^i) be two elements of the tangent space $T\mathscr{C}$ of $\mathscr{C}$, then their product is given by

$$(v,w) = v^i g_{ij} w^j \tag{3.76}$$

with

$$g_{ij} = \delta_{ij}$$

being δ_{ij} with the Kronecker symbol. The choice of the trivial metric is essential to simplify the following. Before we can proceed with the balance laws, we introduce the spatial velocity V and the velocity vector field v,

$$V^i = \partial_t \phi^i(t,X) \qquad v^i = V^i \circ \phi^{-1}(t,x) \tag{3.77}$$

In addition we derive from (3.75) the so called deformation gradient

$$\left[J^i_I\right] = \left[\partial_I \phi^i\right] \tag{3.78}$$

The inverse of $[J^i_I]$ is denoted by $[\bar{J}^I_i]$.

To parameterize the map (3.75) we choose further coordinates $\tilde{X}^{\tilde{I}}$ and $\bar{x}^{\bar{i}}$, with $\tilde{I} = 1,\ldots,n$ and $\bar{i} = 1,\ldots,m$, and assume that the functions $\bar{x}^{\bar{i}}(t,\bar{X})$ depend on t and $\bar{X}^{\bar{I}} = \tilde{X}^{\bar{I}}$ only, with $\bar{I} = 1,\ldots,\bar{n}$, or equivalently they are independent of $\hat{X}^{\hat{I}} = \tilde{X}^{\hat{I}}$, with $\hat{I} = \bar{n}+1,\ldots,n$. A parameterization of (3.75) is given by the $2n$ functions ψ^i and φ^I, i.e.:

$$\begin{aligned} x^i &= \psi^i(X,\bar{x}_M) \qquad \#\bar{M} \geq 0 \\ X^I &= \varphi^I\left(\tilde{X}\right) \end{aligned} \tag{3.79}$$

such that the map φ is invertible and

$$\phi^i(t,X) = \psi^i(X,\partial_{\bar{M}}\bar{x}(t,\bar{X}))$$

is met for $\bar{x}^{\bar{i}} = \bar{x}^{\bar{i}}(t,\bar{X})$. Therefore, we choose the coordinates $(\bar{X}^{\bar{I}},\bar{x}^{\bar{i}})$ for $\mathscr{G}$ and add the required derivative coordinates $\bar{x}^{\bar{i}}_{\bar{M}}$.

Let us take a look at the parametrization of the motion of the planar rigid body of Fig. 3.11. With $X^{\hat{I}} = \hat{X}^{\hat{I}}$, with $\hat{I} = 1,2$, $\bar{x}^{\hat{i}} = r^{\hat{i}}$, with $\hat{i} = 1,2$, and $\bar{x}^3 = \alpha$ one possibility is given by the well known relations

$$x^i = R^i_{\hat{I}}(\alpha)\hat{X}^{\hat{I}} + \delta^i_{\bar{i}} r^{\bar{i}} \qquad R = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \tag{3.80}$$

for $i = 1,2$ with the rotary matrix R and the Kronecker symbol δ. Obviously, the coordinates (r,α) describe locally $\mathbb{R}^2 \times SO^2$ with the one dimensional rotary group SO^2.

3.3.2.2 Conservation of mass and balance of momentum

Let $\rho(t,x)$ denote the mass density in $\mathscr{C}$ and let $\mathscr{D}$ be an arbitrary subset $\mathscr{D} \subset \mathscr{B}$ such that we can integrate over $\mathscr{D}$. With the total time derivative d_t, and the volume forms $\mathrm{d}x = \mathrm{d}x^1 \cdots \mathrm{d}x^n$, $\mathrm{d}X = \mathrm{d}X^1 \cdots \mathrm{d}X^n$ on $\mathscr{C}$ and $\mathscr{R}$ we derive the identity

$$d_t \int_{\phi(t,\mathscr{D})} \rho(t,x)\,\mathrm{d}x = d_t \int_{\mathscr{D}} \rho(t,\phi(t,X))\,|J(t,X)|\,\mathrm{d}X = \int_{\mathscr{D}} \partial_t \rho_{\mathscr{R}}(t,X)\,\mathrm{d}X$$

with

$$\rho_{\mathscr{R}}(t,X) = \rho(t,\phi(t,X))\,|J(t,X)|$$

where $|J(t,X)|$ denotes the determinant of J. Obviously, conservation of mass implies

$$\int_{\mathscr{D}} \partial_t \rho_{\mathscr{R}}(t,X)\,\mathrm{d}X = 0 \tag{3.81}$$

To proceed with the balance of linear momentum, we make the strong constitutive assumption of the symmetry of the Cauchy stress tensor $\sigma(t,x)$, see [133]. This assumption implies that the balance of momentum of momentum is met. Let $\partial\mathscr{D}$ denote the boundary of $\mathscr{D}$ and

$$\partial_i \rfloor \mathrm{d}x = (-1)^{(i-1)}\,\mathrm{d}x^1 \cdots \widehat{\mathrm{d}x^i} \cdots \mathrm{d}x^n$$

where the term $\widehat{\mathrm{d}x^i}$ is omitted, be the i^{th} surface element, then balance of linear momentum is given by

$$\begin{aligned} d_t \int_{\phi(t,\mathscr{D})} v^i(t,x)\,\rho(t,x)\,\mathrm{d}x &= \int_{\phi(t,\mathscr{D})} f^i(t,x)\,\mathrm{d}x + \int_{\phi(t,\partial\mathscr{D})} \sigma^{ij}(t,x)\,\partial_j \rfloor \mathrm{d}x \\ &= \int_{\phi(t,\mathscr{D})} \left(f^i(t,x) + \partial_j \sigma^{ij}(t,x)\right)\mathrm{d}x \end{aligned}$$

with the body forces f. Using (3.81) we get

$$d_t \int_{\phi(t,\mathscr{D})} v^i(t,x)\,\rho(t,x)\,\mathrm{d}x = \int_{\mathscr{D}} \rho_{\mathscr{R}}(X)\,\partial_t V^i(t,X)\,\mathrm{d}X$$

and derive in a similar manner the force field F in Lagrangian description and the first Piola Kirchoff stress tensor P, see [133]:

$$\begin{aligned} F^i(t,X) &= f^i(t,\phi(t,X))\,|J(t,X)| \\ P^{iJ}(t,X) &= \sigma^{ij}(t,\phi(t,X))\,\bar{J}^J_j(t,X)\,|J(t,X)| \end{aligned}$$

such that the relations

$$\begin{aligned} \int_{\phi(t,\mathscr{D})} f^i(t,x)\,\mathrm{d}x &= \int_{\mathscr{D}} F^i(t,X)\,\mathrm{d}X \\ \int_{\phi(t,\partial\mathscr{D})} \sigma^{ij}(t,x)\,\partial_j \rfloor \mathrm{d}x &= \int_{\partial\mathscr{D}} P^{iJ}(t,X)\,\partial_J \rfloor \mathrm{d}X \end{aligned} \tag{3.82}$$

are met. Summarizing, we may write the balance of linear momentum in Lagrangian description in the form

$$\int_{\mathscr{D}} \rho_{\mathscr{R}}(X)\,\partial_t V^i(t,X)\,\mathrm{d}X = \int_{\mathscr{D}} \left(F^i(t,X) + \partial_I P^{iI}(t,X)\right)\mathrm{d}X \tag{3.83}$$

3.3.2.3 Equations of motion

The equations (3.81), (3.83) are incomplete, since the constitutive relations are missing. To overcome this problem, we have to parameterize the map (3.75) by generalized coordinates. The simplest choice is $\bar{X}^I = X^I$, with $I = 1,\ldots,n$ for the independent, and $\bar{x}^i = x^i$, with $i = 1,\ldots,n$ for the dependent spatial coordinates. Since the generalized coordinates $\bar{X}$ and $\bar{x}$ conincide with the coordinates X and x of $\mathscr{C}$ and $\mathscr{R}$ we suppress the accent here. If (3.83) holds for every "nice" subset $\mathscr{D} \subset \mathscr{B}$, then we may conclude that

$$\rho_{\mathscr{R}} x^i_{tt} = \left(F^i + d_I P^{iI}\right) \tag{3.84}$$

is met. For the present we allow that P^{iI} and F^i may depend on t, X and x_M, with $\#M \geq 0$. Therefore, we have to use the total derivative d_I, see (3.74), instead of the partial derivative ∂_I, like in (3.83). Multiplication of (3.84) with $(x^j_t g_{ij})$ and integration over $\mathscr{D}$ leads to

$$\begin{aligned} \int_{\mathscr{D}} \rho_{\mathscr{R}} x^j_t g_{ij} x^i_{tt}\mathrm{d}X &= \int_{\mathscr{D}} \left(x^j_t g_{ij} F^i + x^j_t g_{ij} d_I P^{iI}\right)\mathrm{d}X \\ \int_{\mathscr{D}} \left(\rho_{\mathscr{R}} x^j_t g_{ij} x^i_{tt} + d_I\left(x^j_t g_{ij}\right)P^{iI}\right)\mathrm{d}X &= \int_{\mathscr{D}} \left(x^j_t g_{ij} F^i + d_I\left(x^j_t g_{ij} P^{iI}\right)\right)\mathrm{d}X \end{aligned}$$

or

$$\int_{\mathscr{D}} \left(d_t \left(\frac{\rho_{\mathscr{R}}}{2} x_t^i g_{ij} x_t^j \right) + \frac{1}{2} d_t \left(x_I^i g_{ij} x_J^j \right) S^{IJ} \right) \mathrm{d}X =$$

$$= \int_{\mathscr{D}} \left(d_t \left(\frac{\rho_{\mathscr{R}}}{2} x_t^i g_{ij} x_t^j \right) + \frac{1}{2} d_t \left(C_{IJ} \right) S^{IJ} \right) \mathrm{d}X =$$

$$= \int_{\mathscr{D}} x_t^i g_{ij} F^j \mathrm{d}X + \int_{\partial\mathscr{D}} x_t^i g_{ij} P^{jI} \partial_I \rfloor \mathrm{d}X$$

with

$$C_{IJ} = x_I^i g_{ij} x_J^j \qquad P^{iJ} = x_I^i S^{IJ} \tag{3.85}$$

The Cauchy Green deformation tensor C, see [133], is symmetric by construction, whereas the symmetry of the second Piola Kirchhoff tensor S, see [133], is a consequence of the symmetry of the Cauchy stress tensor σ. In simple elasticity one assumes $S^{IJ} = S^{IJ}(X,C)$ and the existence of the stored energy function $e_E(X,C)$ such that

$$2 \frac{\partial}{\partial C_{IJ}} e_E \left(X, C \right) = S^{IJ} \left(X, C \right) \tag{3.86}$$

is met. This is possible only, if S is symmetric. In this case the balance of energy is given by

$$\int_{\mathscr{D}} d_t \left(e_K + e_E \right) \mathrm{d}X = \int_{\mathscr{D}} x_t^i g_{ij} F^j \mathrm{d}X + \int_{\partial\mathscr{D}} x_t^i g_{ij} P^{jI} \partial_I \rfloor \mathrm{d}X \tag{3.87}$$

with the kinetic energy density

$$e_K \left(X, x_t \right) = \frac{\rho_{\mathscr{R}}}{2} x_t^i g_{ij} x_t^j \tag{3.88}$$

Obviously, the coordinates (t, X^I, x_M^i), with $M = m_1, \ldots, m_r$, $m_k \in \{t\} \cup \{1, \ldots, n\}$ and $0 \leq \#M \leq 2$, are necessary to model a simple elastic system, and the equations of motion (3.84) are partial differential equations of second order. In addition, a simple elastic body allows two types of ports defined by the pairs $((x_t, F), (x_t, P))$ distributed over $\mathscr{B}$ and $\partial\mathscr{B}$. It is worth mentioning that $((x_t^i), (g_{ij} F^j))$ are elements of two linear spaces dual to each other. The analogous property is met by $((x_t^i), (g_{ij} P^{jI}))$. One can use these ports to connect the body to other systems in a power preserving manner. If sliding of the ports is permitted then the relations $x_t^i = \tilde{x}_t^i$, $F^i = -\tilde{F}^i$ and $x_t^i = \tilde{x}_t^i$, $P^{iI} = -\tilde{P}^{iI}$ must be met, where $\tilde{\cdot}$ refers to the second system.

3.3.3 The Hamiltonian and Lagrangian picture

To show that the equations (3.84) are of the Lagrangian type, it is sufficient to find a Lagrangian density $l(X, x_M^i)$, with $0 \leq \#M \leq 1$ such that the equations

$$\delta_i l + g_{ij} F^j = 0 \tag{3.89}$$

with the variational derivative

$$\delta_i = \sum_{\#M \geq 0} (-1)^{\#M} d_M \partial_i^M \tag{3.90}$$

coincide with the equations (3.84), where $d_M = d_{m_1} \cdots d_{m_r}$. Obviously, the choice

$$l = e_K - e_E$$

with the kinetic energy of (3.88) and the stored energy function of (3.86) solves this problem because of

$$\delta_i l = -d_t \partial_i^t e_K + d_I \left(\frac{1}{2} S^{JK} \partial_i^I C_{JK} \right) = -\rho_{\mathscr{R}} x_{tt}^i + d_I \left(2 x_K^i \frac{1}{2} S^{IK} \right)$$

Furthermore, if the force field F meets the condition $\delta_{ij} F^j = -\partial_i e_F(X,x)$, then it can be included in the Lagrangian by $l = e_K - e_E - e_F$.

The Lagrangian equations (3.89) are implicit equations in general. Furthermore time and spatial variables are handled on the same footing, their different nature is expressed in the boundary conditions only. In the Hamiltonian picture the equations are explicit equations solved with respect to the time derivatives of the dependent variables. To derive them, we introduce the generalized momenta p_i by the Legendre transformation

$$p_i = \partial_i^t l = \rho_{\mathscr{R}} g_{ij} x_t^j \tag{3.91}$$

and proceed with the coordinates $(t, X^I, x^i, p_i, x_t^i, p_{i,t}, x_{\bar{M}}^i)$, where we have that $M = m_1, \ldots, m_k, \ldots, m_r$, $m_k \in \{1, \ldots, n\}$ and $1 \leq \#M \leq 2$. The Hamiltonian density h is given by

$$h\left(X, x^i, p_i, x_M^i\right) = \left(p^i \partial_i^t l - l\right) = e_K\left(X, g^{ij} p_j / \rho_{\mathscr{R}}\right) + e_E \tag{3.92}$$

with $g^{ik} g_{kj} = \delta_j^i$, and the Hamiltonian equations take the form

$$x_t^i = \bar{\delta}^i h \qquad\qquad p_{i,t} = -\bar{\delta}_i h + g_{ij} F^j \tag{3.93}$$

with the variational derivatives

$$\bar{\delta}^i = \partial^i = \frac{\partial}{\partial p_i} \qquad\qquad \bar{\delta}_i = \sum_{\#M \geq 0} (-1)^{\#M} d_M \partial_i^M$$

in the new coordinates. Again, the Hamiltonian equations are partial differential equations of second order with respect to the spatial derivatives, but of first order only with respect to the time derivatives. The boundary conditions for both, the Lagrangian or Hamiltonian equations, follow from the same consideration like above.

3.3.4 The linearized scenario

To find the linearized equations of simple elasticity, we have to consider the relations (3.84), (3.86). Let us consider the equation (3.84) first. Now it is appropriate to introduce the Lagrangian strain tensor E:

$$2E_{IJ} = C_{IJ} + \delta_{IJ}$$

and displacement coordinates, see [133, 210, 221], by

$$x^i = \delta^i_I X^I + u^i$$

In the case of small strain one replaces E by the linearized small strain tensor ε:

$$2\varepsilon_{IJ} = u^i_I g_{ij} \delta^j_J + \delta^i_I g_{ij} u^j_J \tag{3.94}$$

Furthermore, we set $P^{iJ} \approx \delta^i_I S^{IJ}(X, \varepsilon)$. In this case the equations (3.84) simplify to

$$\rho_{\mathscr{R}} u^i_{tt} = \left(F^i + d_I \delta^i_J S^{JI}\right) \tag{3.95}$$

It is worth mentioning that nonlinear constitutive equations are still possible provided the strain remains small. Often one assumes that the stored energy function e_E takes the quadratic form

$$e_E(X, \varepsilon) = \frac{1}{2} \varepsilon_{IJ} E^{IJKL}(X) \varepsilon_{KL} \tag{3.96}$$

with $E = E^{JIKL} = E^{IJLK} = E^{KLIJ}$ then (3.86) simplifies to

$$S^{IJ} = \partial_{\varepsilon_{IJ}} e_E = E^{IJKL}(X) \varepsilon_{KL} \tag{3.97}$$

which is nothing else than Hook's law. Following these considerations and the previous section, it is straightforward to derive the Lagrangian or Hamiltonian equations for the linearized scenario.

3.3.5 Reduction

Often the equations (3.84) can be approximated by simpler ones. Examples are beams or plates, where the extension of the structure in a certain direction differs significantly from the ones in the others. The reduction under consideration here is based on the Lagrangian (3.89) or the Hamiltonian (3.91) description of (3.84). The main idea is quite simple. With the parameterization (3.79) one determines the Lagrangian $\bar{l}$ or Hamiltonian density $\bar{h}$ in the generalized coordinates and gets the equations of motion from this functions.

Let us discuss the reduction based on the Lagrangian equations first. Given (3.79) we extend these relations to the derivative coordinates by

$$x_t^i = d_{\bar{I}}\psi^i \qquad\qquad x_I^i = d_{\bar{I}}\psi^i D_I^{\tilde{I}} \tag{3.98}$$

where $\bar{D} = [\bar{D}_I^{\tilde{I}}]$ denotes the inverse of the Jacobian $D = [\partial_{\bar{I}}\varphi^I]$. The total derivatives $d_{\bar{I}}$ in the new coordinates $(t, \overline{X}^{\bar{I}}, \bar{x}^i_{\bar{M}})$ are given by

$$\begin{aligned} d_{\bar{I}} &= \partial_t + \bar{x}^{\tilde{I}}_{\bar{M},t}\partial_{\tilde{I}}^{\bar{M}} \\ d_{\bar{I}} = d_{\bar{I}} &= \partial_{\bar{I}} + \bar{x}^{\tilde{I}}_{\bar{M},\bar{I}}\partial_{\tilde{I}}^{\bar{M}} && \bar{I} = 1,\ldots,\bar{n} \\ d_{\hat{I}} &= \partial_{\hat{I}} && \hat{I} = \bar{n}+1,\ldots n \end{aligned}$$

Proceeding with (3.98) in an analogous manner we derive the relations for higher order relatives. Now the simplified Lagrangian density $\bar{l}$ follows as

$$\bar{l}\,(t,\bar{X},\bar{x}_{\bar{M}}) = \int_{\widehat{\mathscr{X}}(\bar{X})} l\,(t,X,x_M)\,|D|\,\mathrm{d}\hat{X}^{\hat{n}+1}\cdots\mathrm{d}\hat{X}^n \tag{3.99}$$

where we have to plug in the functions for X and x_M according to (3.79) and (3.98), and the relations for the higher order derivatives. The domain $\widehat{\mathscr{X}}(\bar{X})$ of integration meets $\mathscr{B} = \widehat{\mathscr{X}}(\overline{\mathscr{X}})$ as well as $\bar{X}_a \neq \bar{X}_b$ implies $\widehat{\mathscr{X}}(\bar{X}_a) \cap \widehat{\mathscr{X}}(\bar{X}_b) = \emptyset$. To derive the force field $\bar{F}_{\tilde{i}}$ from (3.89) we need the tangent map of (3.79) given by

$$\dot{x}^i = \partial_{\tilde{i}}^{\bar{M}}\psi^i\dot{\bar{x}}^{\tilde{i}}_{\bar{M}} = \partial_{\tilde{i}}^{\bar{M}}\psi^i d_{\bar{M}}\dot{\bar{x}}^{\tilde{i}} \tag{3.100}$$

The functions $\bar{F}_{\tilde{i}}$, also called generalized forces, follow from the relation

$$\int_{\overline{\mathscr{X}}}\left(\int_{\widehat{\mathscr{X}}(\bar{X})}\partial_{\tilde{i}}^{\bar{M}}\psi^i\left(d_{\bar{M}}\dot{\bar{x}}^{\tilde{i}}\right)g_{ij}F^j\,|D|\,\mathrm{d}\widehat{X} - \dot{\bar{x}}^{\tilde{i}}\bar{F}_{\tilde{i}}\right)\mathrm{d}\overline{X} = 0 \tag{3.101}$$

which is supposed to hold for arbitrary $\dot{\bar{x}}^{\tilde{i}} = \dot{\bar{x}}^{\tilde{i}}(t,\bar{X})$, which vanishes on the boundary $\partial\overline{\mathscr{X}}$. It is worth mentioning that (3.100) contains the differential operators $d_{\bar{M}}$. This is just the point, where the boundary conditions come into the play. Because of the complexity of this problem, we will consider special cases in the examples only. Finally, the Lagrangian equations follow as

$$\delta_{\tilde{i}}\bar{l} + \overline{F}_{\tilde{i}} = 0 \tag{3.102}$$

To derive the reduced equations in the Hamiltonian picture, one can apply the Legendre transformation to (3.99). Provided this transformation exists and is a point transformation. Then one derives the Hamiltonian equations in a straightforward manner. Annother way is to look for relations for the generalized momenta $\bar{p}_{\tilde{i}}$. We require that

$$\int_{\overline{\mathscr{X}}}\left(\int_{\hat{\mathscr{X}}(\bar{X})} p_i\dot{x}^i\,|D\varphi|\,\mathrm{d}\widehat{X} - \bar{p}_{\tilde{i}}\dot{\bar{x}}^{\tilde{i}}\right)\mathrm{d}\overline{X} = 0 \tag{3.103}$$

is met for p_i from (3.91), $\dot{x}^i$ from (3.100) and arbitrary functions for $\dot{\bar{x}}^{\bar{i}}$, which vanish on the boundary $\partial\overline{\mathscr{X}}$. If it is possible to determine the generalized momenta, then one can apply reduction procedure to the Hamiltonian density (3.92) analogously to the one for the Lagrangian density described above. To complete this section, we have to show, how the boundary conditions for the reduced model are derived. Because of the complexity of this problem we will discuss this in the example of Sect. 3.3.6 only.

3.3.5.1 The rigid body

Let us consider again the planar rigid body of Fig. 3.11, which is influenced by a force field F now. From (3.80), (3.98) we derive the relations

$$x_t^i = R_{\hat{I}}^i \Omega_{\hat{J}}^{\hat{I}} \alpha_t \hat{X}^{\hat{J}} + \delta_{\bar{i}}^i r_t^{\bar{i}} \qquad\qquad \Omega = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Now it is well known, that the Lagrangian L takes the simple form

$$\begin{aligned} L(r,\alpha,r_t,\alpha_t) &= \frac{1}{2}\int_{\mathscr{B}} \left(R_{\hat{I}}^i \Omega_{\hat{J}}^{\hat{I}} \alpha_t \hat{X}^{\hat{J}} + \delta_{\bar{i}}^i r_t^{\bar{i}}\right) g_{ij} \left(R_{\hat{K}}^j \Omega_{\hat{L}}^{\hat{K}} \alpha_t \hat{X}^{\hat{L}} + \delta_{\bar{j}}^j r_t^{\bar{j}}\right) \rho_{\mathscr{R}} \mathrm{d}\hat{X} \\ &= \frac{1}{2}\left(\theta\,(\alpha_t)^2 + M r_t^{\bar{i}} \delta_{\bar{i}\bar{j}} r_t^{\bar{j}}\right) \end{aligned}$$

with the rotational inertia θ, and mass M,

$$\theta = \int_{\mathscr{B}} X^I \delta_{IJ} X^J \rho_{\mathscr{R}} \mathrm{d}\hat{X} \qquad\qquad M = \int_{\mathscr{B}} \rho_{\mathscr{R}} \mathrm{d}\hat{X} \tag{3.104}$$

provided the point $\hat{X}^I = 0$ is the center of gravity such that

$$\int_{\mathscr{B}} \hat{X}^{\hat{I}} \rho_{\mathscr{R}} \mathrm{d}\hat{X} = 0$$

is met. With the tangent map, see (3.100)

$$\dot{x}^i = R_{\hat{I}}^i \Omega_{\hat{J}}^{\hat{I}} \dot{\alpha} \hat{X}^{\hat{J}} + \delta_{\bar{i}}^i \dot{r}^{\bar{i}}$$

and from, see (3.101):

$$\int_{\mathscr{B}} \left(\left(R_{\hat{I}}^i \Omega_{\hat{J}}^{\hat{I}} \dot{\alpha} \hat{X}^{\hat{J}} + \delta_{\bar{i}}^i \dot{r}^{\bar{i}}\right) g_{ij} F^j - \dot{\alpha}\bar{T} - \bar{F}_{\bar{i}} \dot{r}^{\bar{i}}\right) \mathrm{d}\hat{X} = 0$$

we derive the generalized forces $\bar{F}_{\bar{i}}, \bar{T}$ as

$$\bar{F}_{\bar{i}} = \int_{\mathscr{B}} \delta_{\bar{i}}^i g_{ij} F^j \mathrm{d}\hat{X} \qquad\qquad \bar{T} = \int_{\mathscr{B}} R_{\hat{I}}^i \Omega_{\hat{J}}^{\hat{I}} \hat{X}^{\hat{J}} g_{ij} F^j \mathrm{d}\hat{X}$$

and finally the equations of motion

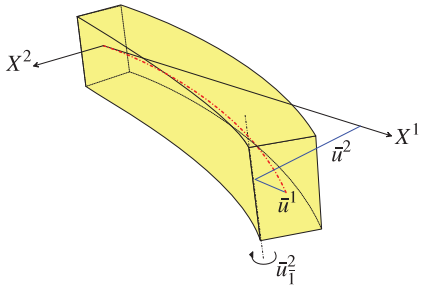

Fig. 3.12 Euler Bernoulli beam.

$$\theta\alpha_{tt} = \bar{T} \qquad\qquad M\delta_{\bar{i}\bar{j}}r^{\bar{j}}_{tt} = \bar{F}_{\bar{i}}$$

Since the derivation of the Hamiltonian counterpart to these equations is straightforward, it will be omitted here.

3.3.6 The Euler-Bernoulli beam

Beams are examples of two dimensional structures with different extensions in X^1 and X^2 directions. We choose a simple elastic material with the stored energy function, see (3.94) and (3.96):

$$e_E = \frac{1}{2}\left(a\left((\varepsilon_{11})^2 + (\varepsilon_{22})^2\right) + 2b\varepsilon_{11}\varepsilon_{22} + c\,(\varepsilon_{12})^2\right)$$

with $a, c \in \mathbb{R}^+$, $b \in \mathbb{R}$, $a > b$. Now, we apply the reduction, see (3.79) and [221]:

$$X^1 = \bar{X} \qquad X^2 = \hat{X} \qquad u^1 = \bar{u}^1 - \hat{X}\bar{u}^2_{\bar{1}} \qquad u^2 = \bar{u}^2 \tag{3.105}$$

according to the assumptions for the Euler-Bernoulli beam, see also Fig. 3.12. Following the consideration from above we derive the additional relations

$$\begin{aligned} u^1_t &= \bar{u}^1_t - \hat{X}\bar{u}^2_{t\bar{1}} & u^2_t &= \bar{u}^2_t \\ \varepsilon_{11} &= u^1_1 = \bar{u}^1_{\bar{1}} - \hat{X}\bar{u}^2_{\bar{1}\bar{1}} & 2\varepsilon_{12} &= u^1_2 + u^2_1 = 0 \qquad \varepsilon_{22} = u^2_2 = 0 \end{aligned}$$

The Lagragian density $\bar{l} = \bar{l}(\bar{X}, \bar{u}^1_t, \bar{u}^2_t, \bar{u}^1_{\bar{1}}, \bar{u}^2_{\bar{1}\bar{1}}, \bar{u}^2_{t\bar{1}})$ follows according to (3.99) as

$$\begin{aligned} \bar{l} &= \frac{1}{2}\int_{h/2}^{h/2}\left(\rho_{\mathscr{R}}\left(\left(\bar{u}^1_t - \bar{u}^2_{t\bar{1}}\hat{X}\right)^2 + \left(\bar{u}^2_t\right)^2\right) - a\left(\bar{u}^1_{\bar{1}} - \hat{X}\bar{u}^2_{\bar{1}\bar{1}}\right)^2\right)\mathrm{d}\hat{X} \\ &= \frac{h}{2}\left(\rho_{\mathscr{R}}\left(\left(\bar{u}^1_t\right)^2 + \underbrace{\frac{h^2}{12}\left(\bar{u}^2_{t\bar{1}}\right)^2}_{\approx 0} + \left(\bar{u}^2_t\right)^2\right) - a\left(\left(\bar{u}^1_{\bar{1}}\right)^1 + \frac{h^2}{12}\left(\bar{u}^2_{\bar{1}\bar{1}}\right)^2\right)\right) \end{aligned} \tag{3.106}$$

Now we use (3.101) to derive the reduced field $\bar{F}$. From the relations

$$\int_0^L \left(\int_{-h/2}^{h/2} \left(F^1 \left(\dot{\hat{u}}^1 - \hat{X} \dot{\bar{u}}^2_{\bar{1}} \right) + F^2 \dot{\hat{u}}^2 \quad \right) \mathrm{d}\hat{X} - \bar{F}_{\bar{i}} \dot{\bar{u}}^{\bar{i}} \right) \mathrm{d}\bar{X} = 0$$

$$\int_0^L \left(\dot{\bar{u}}^1 \int_{-h/2}^{h/2} F^1 \mathrm{d}\hat{X} + \dot{\bar{u}}^2 \int_{-h/2}^{h/2} \left(d_{\bar{1}} F^1 \hat{X} + F^2 \right) \mathrm{d}\hat{X} - \bar{F}_{\bar{i}} \dot{\bar{u}}^{\bar{i}} \right) \mathrm{d}\bar{X} = \underbrace{\dot{\bar{u}}^2 \int_{-h/2}^{h/2} F^1 \hat{X} \mathrm{d}\hat{X} \Big|_0^L}_{=0}$$

and the Lagrangian (3.106) one gets the inhomogenous wave equation

$$h \left(\rho_{\mathscr{R}} \bar{u}^1_{tt} - a \bar{u}^1_{\bar{1}\bar{1}} \right) = \bar{F}_1 = \int_{-h/2}^{h/2} F^1 \mathrm{d}\hat{X} \tag{3.107}$$

for $\bar{u}^1$ and the well known beam equation

$$h \left(\rho_{\mathscr{R}} \left(\bar{u}^2_{tt} - \underbrace{\frac{h^2}{12} \bar{u}^2_{tt\bar{1}\bar{1}}}_{\approx 0} \right) + a \frac{h^2}{12} \bar{u}^2_{\bar{1}\bar{1}\bar{1}\bar{1}} \right) = \bar{F}_2 = \int_{-h/2}^{h/2} \left(d_{\bar{1}} F^1 \hat{X} + F^2 \right) \mathrm{d}\hat{X} \tag{3.108}$$

for $\bar{u}^2$, provided the underbraced term is neglected in (3.106) and (3.108).

Finally, the evaluation of the boundary term of the energy relation (3.87) for $\bar{X} \in \{0, L\}$ with the lenght L of the beam to

$$\begin{aligned} \int_{-h/2}^{h/2} \left(\bar{u}^1_t S^{11} + \bar{u}^2_t S^{21} \right) \mathrm{d}\hat{X} &= \bar{u}^1_t \bar{u}^1_{\bar{1}} \int_{-h/2}^{h/2} a \mathrm{d}\hat{X} - \bar{u}^2_t \bar{u}^2_{\bar{1}\bar{1}} \int_{-h/2}^{h/2} a \hat{X} \mathrm{d}\hat{X} \\ &= \left(\bar{u}^1_t \left(ah \bar{u}^1_{\bar{1}} \right) - \bar{u}^2_t \left(\bar{u}^2_{\bar{1}\bar{1}} \frac{ah^2}{2} \right) \right) \end{aligned}$$

Therefore, we may introduce two ports at eauch boundary build up by the pairs $(\bar{u}^1_t, ah\bar{u}^1_{\bar{1}})$ and $(\bar{u}^2_t, \bar{u}^2_{\bar{1}\bar{1}} ah^2/2)$, wich can be used to connect the Euler Bernoulli beam to other structures.

Let us now take the Hamiltonian point of view. We determine the generalized momenta $\bar{p}_1$ and $\bar{p}_2$. According to (3.103), from

$$\int_0^L \left(\int_{-h/2}^{h/2} \left(\left(\bar{u}^1_t - \hat{X} \bar{u}^2_{\bar{t}\bar{1}} \right) \left(\dot{\bar{u}}^1 - \hat{X} \dot{\bar{u}}^2_{\bar{1}} \right) + \bar{u}^2_t \dot{\bar{u}}^2 \right) \rho_{\mathscr{R}} \mathrm{d}\hat{X} - \bar{p}_{\bar{i}} \dot{\bar{u}}^{\bar{i}} \right) \mathrm{d}\bar{X} = 0$$

$$\int_0^L \left(\left(\bar{u}^1_t \dot{\bar{u}}^1 + \frac{h^2}{12} \bar{u}^2_{t\bar{1}} \dot{\bar{u}}^2_{\bar{1}} + \bar{u}^2_t \dot{\bar{u}}^2 \right) \rho_{\mathscr{R}} \mathrm{d}\hat{X} - \bar{p}_{\bar{i}} \dot{\bar{u}}^{\bar{i}} \right) \mathrm{d}\bar{X} = 0$$

$$\int_0^L \left(\left(\bar{u}^1_t \dot{\bar{u}}^1 + \left(\bar{u}^2_t - \frac{h^2}{12} \bar{u}^2_{t\bar{1}\bar{1}} \right) \dot{\bar{u}}^2 \right) \rho_{\mathscr{R}} \mathrm{d}\hat{X} - \bar{p}_{\bar{i}} \dot{\bar{u}}^{\bar{i}} \right) \mathrm{d}\bar{X} = \underbrace{-\frac{h^2}{12} \bar{u}^2_{t\bar{1}} \dot{\bar{u}}^2 \Big|_0^L}_{=0}$$

the relations

$$\bar{p}_1 = \rho_{\mathscr{R}} h \bar{u}_t^1 \qquad \bar{p}_2 = \rho_{\mathscr{R}} h \left(\bar{u}_t^2 - \frac{h^2}{12} \bar{u}_{t\bar{1}\bar{1}}^2 \right)$$

Obviously, it is straightforward to derive $\bar{p}_2$ as a function of $\bar{u}_t^2$ and its derivatives. But the determination of the inverse map requires the solution of a differential equation. Therefore the Legendre trasformation fails to be a point transformation. Therefore, we stop here and consider the simplified Lagrangian (3.106). Now, the determination of the generalized momenta $\bar{p}_1, \bar{p}_2$ is straightforward and the Hamiltonian density is given by

$$\bar{h} = \bar{p}_1 \bar{u}_t^1 + \bar{p}_2 \bar{u}_t^2 - \bar{l} = \frac{1}{2h\rho_{\mathscr{R}}} \left((\bar{p}_1)^2 + (\bar{p}_2)^2 \right) + \frac{h}{2} a \left(\left(\bar{u}_{\bar{1}}^1 \right)^2 + \frac{h^2}{12} \left(\bar{u}_{\bar{1}\bar{1}}^2 \right)^2 \right)$$

With the variational derivatives, see also (3.93), one derives the Hamiltonian equations as

$$\bar{u}_t^1 = \delta^{\bar{1}} \bar{h} = \frac{1}{h\rho_{\mathscr{R}}} \bar{p}_1 \qquad \bar{p}_{1,t} = -\delta_{\bar{1}} \bar{h} = ah\bar{u}_{\bar{1}}^{\bar{1}}$$

$$\bar{u}_t^2 = \delta^{\bar{2}} \bar{h} = \frac{1}{h\rho_{\mathscr{R}}} \bar{p}_2 \qquad \bar{p}_{2,t} = -\delta_{\bar{2}} \bar{h} = -ah\bar{u}_{\bar{1}\bar{1}\bar{1}}^2$$

Of course, one can start with the Hamiltonian (3.92) and apply the reduction procedure. If one neglects the terms $\approx h^2$ in kinetic energy, then one derives the same set of equations.

3.3.7 Summary

The mathematical modeling of elastic structures can be significantly simplified by the use of differential geometric methods. Starting with the fundamental conservation and balance principles, one has to parameterize certain maps to bring the constitutive equations into the play. If one assumes the existence of the stored energy function in the presented manner, then one deals with simple elasticity. A further consequence of this assumption is, that one can rewrite the equations of motion in a Hamiltonian or Lagrangian manner. This fact is often used to derive simpler models, where the simplification is archived by adding holonomic constraints. Of course, one can also linearize the equations of motion. Exemplarily, these approaches has been presented for the rigid body and for the Euler Bernoulli beam such that the simplified equations of motion are derived by a systematic reduction procedure from the general ones. Since one can apply the presented methods to other mechanical structures like the Timoshenko beam (see [206] and Sect. 4.3.1), the Kirchhoff or Mindlin plate [123, 147], shell or membranes in an analogous manner, it is obvious, how their Lagrangian or Hamiltonian description can be achieved.

3.4 Port-based modelling and irreversible thermodynamics

Aim of this section is to express models of physico-chemical systems involving momentum, heat and mass transfer as well as chemical reactions in the port-based formalism. To this end, the entropy balance and the associated source terms will be systematically written in accordance with the principle of irreversible thermodynamics. Some insights will been given concerning the constitutive equations and models allowing to calculate transport and thermodynamic properties. These port-based models can be translated into bond-graph models, in the case of distributed as well as lumped parameter systems. Examples of this are reported in the concluding part of the section. The meaning of all the symbols appearing in this section has been reported in Appendix C.

3.4.1 Basic concepts

A thermodynamic system is a piece of matter containing a sufficiently high number of elementary particles (atoms, molecules, ions etc) so that macroscopic variables like pressure P, temperature T, mass density ρ, mass concentrations ρ_i can make sense. The energy of such a thermodynamic system is defined as the internal energy U. The other thermodynamic functions like enthalpy H, Gibbs free energy G are defined with respect to U as Legendre transforms (see Appendix B, Sect. B.2.6).

Some of the thermodynamic variables correspond to quantities subject to balance equations. These variables are extensive variables in the sense that they are dependent on the size of the system under consideration. For such variables, one defines specific mass or molar variables as well as fluxes, these concepts being necessary to derive balance equations. Some of them will lead to the definition of flow variables as they are defined in the port-based approach. As far as specific mass or molar variables are concerned, they can be considered as intensive variables in the sense that they are independent of the size of the system under consideration. Other intensive variables are not defined as specific variables. They will prove to be effort variables allowing the description of equilibrium situations (see Appendix B, Sect. B.2.2).

As far as only chemical reactions are considered, the total mass is conservative. This means that mass is neither destroyed nor created during the processes under consideration. In the case of distributed parameter systems, the total mass balance is as follows:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{v} \tag{3.109}$$

where ρ is the mass density and $\mathbf{v}$ the fluid velocity.

When balances are established for distributed parameter systems, the conservation equations can be written under two forms [21]. If Y is a scalar quantity and y the corresponding quantity per unit mass, the partial derivative

$$\frac{\partial y}{\partial t}$$

is the time variation of y at a given point. In a frame moving according to the fluid velocity $\mathbf{v}$,

$$\frac{\mathrm{D}y}{\mathrm{D}t}$$

is the time derivative of y "following the motion" [21]. The relation between the two is obtained by applying the chain rule of derivation:

$$\frac{\mathrm{D}y}{\mathrm{D}t} = \frac{\partial y}{\partial t} + \mathbf{v} \cdot \nabla y \tag{3.110}$$

From that point, two forms of balance equations can be derived.

In a fixed frame, the balance of Y takes the general form:

$$\frac{\partial \rho y}{\partial t} = -\nabla \cdot \mathbf{f}_y + \sigma_y \tag{3.111}$$

where $\mathbf{f}_y$ is the flux of Y per unit of surface area and σ_y a possible volumetric source term. This source term can be either the result of a true destruction or creation of Y or the expression of a transfer with the surrounding that is expressed as a quantity by time and volume unit. In a frame moving according to $\mathbf{v}$, the total mass balance (3.109) being given, the Y balance becomes:

$$\rho \frac{\mathrm{D}y}{\mathrm{D}t} = -\nabla \cdot (\mathbf{f}_y - \rho \mathbf{v} y) + \sigma_y = -\nabla \cdot \mathbf{f}_y^R + \sigma_y \tag{3.112}$$

The source term is supposed to remain unchanged but the flux of Y is now a relative flux with respect to the convected one $\mathbf{f}_y^R = \mathbf{f}_y - \rho \mathbf{v} y$. The total mass balance can also be given in a frame following the fluid motion:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \nabla \cdot \mathbf{v} \tag{3.113}$$

or by using the specific volume $\mathrm{v} = \frac{1}{\rho}$:

$$\rho \frac{\mathrm{D}\mathrm{v}}{\mathrm{D}t} = \nabla \cdot \mathbf{v} \tag{3.114}$$

If y is one of the components of a vector, equations (3.111) and (3.112) have their counterparts for the vector $\mathbf{y}$. In a fixed frame, the balance equation is:

$$\frac{\partial \rho \mathbf{y}}{\partial t} = -\nabla \cdot \Phi_y + \sigma_y \tag{3.115}$$

while in a frame moving according to $\mathbf{v}$:

$$\rho \frac{\mathrm{D}\mathbf{y}}{\mathrm{D}t} = -\nabla \cdot (\Phi_y - \rho \mathbf{v} \mathbf{y}) + \sigma_y = -\nabla \cdot \Phi_y^R + \sigma_y \tag{3.116}$$

where Φ_y and Φ_y^R are tensors of order 2 representing respectively the absolute and relative flux of the **y** quantity and σ_y a vectorial source term per unit of volume.

3.4.2 Distributed parameter systems

In order to calculate output variables that can be measured like pressure, temperature and composition, the classical approach consists in deriving a balance equation for the internal energy. The internal energy balance equation has to be coupled to other balance equations (material and momentum). As far the port-based approach is concerned, the internal energy balance is replaced by the entropy balance.

3.4.2.1 Balance equations

The following material can be found in many textbooks (see for example, [21]). We restrict ourselves to an homogeneous fluid (in the sense that it is under the form of only one phase) subject to simultaneous mass, momentum and heat transfer. Chemical reactions are also supposed to occur in the system. The derivation of the equations is performed on a mass basis.

Two forms of energy have to be considered. The energy of the matter as a whole and the energy of the matter as a collection of elementary particles (atoms, molecules, ions ...). The total energy per mass unit is then given by:

$$\hbar = \frac{v^2}{2} + u \tag{3.117}$$

that one can differentiate:

$$\mathrm{d}\hbar = \mathbf{v} \cdot \mathrm{d}\mathbf{p} + \mathrm{d}u \tag{3.118}$$

where **p** is the momentum per mass unit. If one assumes that $\hbar$ is a conserved quantity, in the sense that it is never destroyed nor produced but only transformed from one form to another, it is possible to derive a balance equation for the internal energy. To this end, equation (3.118) is assumed to be valid for the substantial time derivatives:

$$\frac{\mathrm{D}\hbar}{\mathrm{D}t} = \mathbf{v} \cdot \frac{\mathrm{D}\mathbf{p}}{\mathrm{D}t} + \frac{\mathrm{D}u}{\mathrm{D}t} \tag{3.119}$$

so that

$$\rho \frac{\mathrm{D}\hbar}{\mathrm{D}t} = \rho \mathbf{v} \cdot \frac{\mathrm{D}\mathbf{p}}{\mathrm{D}t} + \rho \frac{\mathrm{D}u}{\mathrm{D}t} = \rho \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{v^2}{2}\right) + \rho \frac{\mathrm{D}u}{\mathrm{D}t} \tag{3.120}$$

is also valid.

Let us denote ρ_i as the mass concentration of component i in a mixture containing N species and $\omega_i = \frac{\rho_i}{\rho}$ its mass fraction. If one or more chemical reactions occur in the system, they will produce or consume component i: σ_i is the corresponding net source term expressed in mass of component i per unit of volume and time.

According to equation (3.111), the mass balance of component i is as follows:

$$\frac{\partial \rho \omega_i}{\partial t} = -\nabla \cdot \mathbf{f}_i + \sigma_i \qquad i = 1, \ldots, N \tag{3.121}$$

where $\mathbf{f}_i = \rho_i \mathbf{v}_i$ is the mass flux of component i per unit of surface area defined with respect to a fixed frame. If R chemical reactions occur in the system, σ_i is given by:

$$\sigma_i = \sum_{k=1}^{R} \bar{M}_i \nu_i^k r_k \tag{3.122}$$

where ν_i^k is the stoechiometric coefficient of component i when it is involved in the k^{th} reaction. It is more convenient to express r_k, the rate of the k^{th} reaction per unit of volume, on a molar basis so that $\bar{M}_i$, the molar mass of component i, has been included in equation (3.122) to express σ_i. As far as the total mass is conservative, by summing equation (3.121) over all components, one recovers the total mass balance (3.109). $\mathbf{v}$, which turns to be the mass average velocity in the fixed frame, is defined by:

$$\left(\sum_{i=1}^{N} \rho_i \right) \mathbf{v} = \rho \mathbf{v} = \sum_{i=1}^{N} \rho_i \mathbf{v}_i \tag{3.123}$$

In a moving frame defined with respect to $\mathbf{v}$, according to (3.112), the component i mass balance becomes:

$$\rho \frac{\mathrm{D}\omega_i}{\mathrm{D}t} = -\nabla \cdot (\mathbf{f}_i - \rho \mathbf{v} \omega_i) + \sigma_i = -\nabla \cdot \mathbf{f}_i^R + \sigma_i \tag{3.124}$$

The relative flux $\mathbf{f}_i^R$ per unit of surface area is the mass diffusion flux defined with respect to the mass average velocity $\mathbf{v}$.

Since $\mathbf{p}$ is the momentum per mass unit, it is clear that $\mathbf{p} \equiv \mathbf{v}$ but, contrary to the classical presentation, we distinguish here the velocity as an effort variable from the momentum as a flow variable. In a fixed frame, the momentum balance equation is as follows:

$$\frac{\partial \rho \mathbf{p}}{\partial t} = -\nabla \cdot \Phi_p + \sigma_p \tag{3.125}$$

The source term $\sigma_p = \sum_{i=1}^{N} \rho_i \mathbf{g}_i$ is due to the action of the external body force $\mathbf{g}_i$ exerted per mass unit on component i. σ_p is then an exchange of momentum with the surrounding. Φ_p is a second order tensor allowing to represent the momentum flux per unit of surface area. The same momentum balance can be considered in the moving frame according to (3.116):

$$\rho \frac{\mathrm{D}\mathbf{p}}{\mathrm{D}t} = -\nabla \cdot (\Phi_p - \rho \mathbf{v} \mathbf{p}) + \sigma_p = -\nabla \cdot \Phi_p^R + \sigma_p \tag{3.126}$$

where $\rho \mathbf{v} \mathbf{p}$ is the convected momentum, while $\Phi_p^R = \Phi_p - \rho \mathbf{v} \mathbf{p}$ is the momentum flux defined with respect to the moving frame. This tensor can be split into two terms:

$$\Phi_p^R = P\mathbf{I} + \tau \tag{3.127}$$

where P is the pressure and τ the viscous part of the momentum flux (or shear stress tensor). In (3.127), we have assumed that the fluid under consideration is a non-elastic one [60].

3.4.2.2 The classical approach based on the internal energy balance

According to the first principle of thermodynamic, the total energy of the system being considered as a conserved quantity, the source term that will appear in its balance will necessary be only due to the action of external body forces – i.e. an exchange of energy with the surrounding:

$$\frac{\partial \rho\hbar}{\partial t} = -\nabla\cdot\mathbf{f}_\hbar + \sigma_\hbar = -\nabla\cdot\mathbf{f}_\hbar + \sum_{i=1}^{N}\mathbf{f}_i\cdot\mathbf{g}_i \tag{3.128}$$

and in the moving frame according to $\mathbf{v}$:

$$\rho\frac{\mathrm{D}\hbar}{\mathrm{D}t} = -\nabla\cdot(\mathbf{f}_\hbar - \rho\mathbf{v}\hbar) + \sigma_\hbar = -\nabla\cdot\mathbf{f}_\hbar^R + \sum_{i=1}^{N}\mathbf{f}_i\cdot\mathbf{g}_i \tag{3.129}$$

By combining (3.119), (3.126) and (3.129), one obtain:

$$-\nabla\cdot\mathbf{f}_\hbar^R + \sum_{i=1}^{N}\mathbf{f}_i\cdot\mathbf{g}_i = -\mathbf{v}\cdot\left[\nabla\cdot\Phi_p^R\right] + \sum_{i=1}^{N}\mathbf{v}\cdot\rho_i\mathbf{g}_i + \rho\frac{\mathrm{D}u}{\mathrm{D}t} \tag{3.130}$$

According to the following relations:

$$\sum_{i=1}^{N}(\mathbf{f}_i - \mathbf{v}\cdot\rho_i)\cdot\mathbf{g}_i = \sum_{i=1}^{N}(\mathbf{f}_i - \rho\mathbf{v}\cdot\omega_i)\cdot\mathbf{g}_i = \sum_{i=1}^{N}\mathbf{f}_i^R\cdot\mathbf{g}_i \tag{3.131}$$

$$\mathbf{v}\cdot\left[\nabla\cdot\Phi_p^R\right] = \nabla\cdot\left[\Phi_p^R\cdot\mathbf{v}\right] - \Phi_p^R : \nabla\mathbf{v} \tag{3.132}$$

(3.130) can be regarded as follows:

$$\rho\frac{\mathrm{D}u}{\mathrm{D}t} = -\nabla\cdot\left(\mathbf{f}_\hbar^R - \left[\Phi_p^R\cdot\mathbf{v}\right]\right) - \Phi_p^R : \nabla\mathbf{v} + \sum_{i=1}^{N}\mathbf{f}_i^R\cdot\mathbf{g}_i \tag{3.133}$$

according to the general form (3.112):

$$\rho\frac{\mathrm{D}u}{\mathrm{D}t} = -\nabla\cdot\mathbf{f}_u^R + \sigma_u \tag{3.134}$$

with

$$\begin{cases} \mathbf{f}_{\dot{h}}^R = \mathbf{f}_u^R + \left[\Phi_p^R \cdot \mathbf{v}\right] \\ \sigma_u = -\Phi_p^R : \nabla\mathbf{v} + \sum_{i=1}^{N} \mathbf{f}_i^R \cdot \mathbf{g}_i = -P\nabla\cdot\mathbf{v} - \tau : \nabla\mathbf{v} + \sum_{i=1}^{N} \mathbf{f}_i^R \cdot \mathbf{g}_i \end{cases} \tag{3.135}$$

The internal energy is not a conserved quantity since the source term σ_u contains the positive term $-\tau : \nabla\mathbf{v}$ due to viscous dissipation. It is the expression of the transformation of mechanical energy into internal energy.

3.4.2.3 The port-based approach for modelling

The port-based approach formulates the entropy balance instead of the energy balance: it is based on the thermodynamic of irreversible processes concepts [60]. The main goal of this derivation is to obtain the expression of a source term since, according to the second principle of thermodynamic, the entropy is non-conservative.

The initial form of the Gibbs equation is concerned with macroscopic systems assumed to be at equilibrium [173]. As far as the internal energy of a macroscopic system U is considered as a function of its entropy S, its volume V and of the mass of each component M_i, the Gibbs equation gives the differential of U:

$$\begin{cases} \mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V + \sum_{i=1}^{N} \mu_i \mathrm{d}M_i \\ \mu_i = \left(\dfrac{\partial U}{\partial M_i}\right)_{S,V,M_{k\neq i}} = \left(\dfrac{\partial G}{\partial M_i}\right)_{P,T,M_{k\neq i}} \\ \quad = \left(\dfrac{\partial H}{\partial M_i}\right)_{P,T,M_{k\neq i}} - T\left(\dfrac{\partial S}{\partial M_i}\right)_{P,T,M_{k\neq i}} = h_i - Ts_i \end{cases} \tag{3.136}$$

where μ_i is the chemical potential of component i and $G = U + PV - TS = H - TS$ the Gibbs free energy (cf. Appendix B, Sect. B.2.6); s_i and h_i are respectively the partial entropy and enthalpy per mass unit. It can be seen that a fundamental assumption has been made in (3.136). Energy-conjugated variables associated to heat, space and mass transfer have been postulated for systems at equilibrium, respectively (T,S), $(-P,V)$ and (μ_i, M_i). As far as distributed parameter systems are concerned, a local version of (3.136) has to be derived.

According to the definitions of the quantities per mass unit $u = \frac{U}{M}$, $\mathrm{v} = \frac{V}{M}$, $s = \frac{S}{M}$ and the mass fraction $\omega_i = \frac{M_i}{M}$, with $M = \sum_{i=1}^{N} M_i$, (3.136) can be written as follows:

$$\mathrm{d}(Mu) = T\mathrm{d}(Ms) - P\mathrm{d}(M\mathrm{v}) + \sum_{i=1}^{N} \mu_i \mathrm{d}(M\omega_i) \tag{3.137}$$

After differentiation of each term, (3.137) becomes:

$$M\mathrm{d}u = M\left(T\mathrm{d}s - P\mathrm{d}\mathrm{v} + \sum_{i=1}^{N} \mu_i \mathrm{d}\omega_i\right) + \mathrm{d}M\left(\sum_{i=1}^{N} \mu_i\omega_i - (u + P\mathrm{v} - Ts)\right) \tag{3.138}$$

According to the definition of G and to (3.136), one can easily derive the differential of G:

$$\mathrm{d}G = V\mathrm{d}P - S\mathrm{d}T + \sum_{i=1}^{N} \mu_i \mathrm{d}M_i \tag{3.139}$$

where G is a function P, T and it is a first order homogenous function with respect to M_i (cf. Appendix B, Sect. B.2.1) so that Euler theorem can be applied [173]:

$$G = \sum_{i=1}^{N} M_i \left(\frac{\partial G}{\partial M_i} \right)_{P,T,M_{k\neq i}} = \sum_{i=1}^{N} M_i \mu_i = U + PV - TS \tag{3.140}$$

A similar equation can be written with the specific quantities by dividing (3.140) by M:

$$g = \frac{G}{M} = \sum_{i=1}^{N} \omega_i \mu_i = u + P\mathrm{v} - Ts \tag{3.141}$$

By combining (3.138) and (3.141), the local form of the Gibbs equation is as follows:

$$\mathrm{d}u = T\mathrm{d}s - P\mathrm{d}\mathrm{v} + \sum_{i=1}^{N} \mu_i \mathrm{d}\omega_i \tag{3.142}$$

If now one considers (3.118) and (3.142), a set of specific energy-conjugated variables $(\mathbf{v}, \mathbf{p})$, (T, s), $(-P, \mathrm{v})$ and (μ_i, ω_i) can be defined for a thermodynamic system at equilibrium and subject to heat, mass, momentum transfer and chemical reactions:

$$\mathrm{d}\hbar = \mathbf{v} \cdot \mathrm{d}\mathbf{p} + T\mathrm{d}s - P\mathrm{d}\mathrm{v} + \sum_{i=1}^{N} \mu_i \mathrm{d}\omega_i \tag{3.143}$$

This set of energy-conjugated variables is associated to the total energy of the system. As far as one is now interested in deriving a balance equation for the entropy, the balance equation for the internal energy as it has been derived above can be used.

When a system in not at equilibrium, its state variables vary with space and time. Irreversible phenomena and entropy production are partly due to these spatial variations. However, one can consider that at a sufficiently small scale, equilibrium is reached at each time. Balances are then considered in the frame following the fluid motion and the substantial times derivatives $\frac{\mathrm{D}u}{\mathrm{D}t}$, $\frac{\mathrm{D}\mathrm{v}}{\mathrm{D}t}$, $\frac{\mathrm{D}\omega_i}{\mathrm{D}t}$ and $\frac{\mathrm{D}s}{\mathrm{D}t}$ are assumed to satisfy the local form of the Gibbs equation (3.142):

$$\rho \frac{\mathrm{D}s}{\mathrm{D}t} = \rho \left(\frac{\mathrm{D}u}{\mathrm{D}t} + P\frac{\mathrm{D}\mathrm{v}}{\mathrm{D}t} - \sum_{i=1}^{N} \mu_i \frac{\mathrm{D}\omega_i}{\mathrm{D}t} \right) \frac{1}{T} \tag{3.144}$$

Once the expression of $\rho \frac{\mathrm{D}s}{\mathrm{D}t}$ is derived by using (3.144), the volumetric entropy production as well as the relative entropy flux $\mathbf{f}_s^R = \mathbf{f}_s - \rho \mathbf{v} s$ are obtained by identification to the general form of a balance equation (3.112).

Let us recall the internal energy balance, the mass balance of component i as well as the total mass balance as they have been already derived in the moving frame:

$$\begin{cases} \rho \dfrac{\mathrm{D}u}{\mathrm{D}t} = -\nabla \cdot (\mathbf{f}_u - \rho \mathbf{v} u) + \sigma_u = -\nabla \cdot \mathbf{f}_u^R + \sigma_u \\ \rho \dfrac{\mathrm{D}\omega_i}{\mathrm{D}t} = -\nabla \cdot (\mathbf{f}_i - \rho \mathbf{v} \omega_i) + \sigma_i = -\nabla \cdot \mathbf{f}_i^R + \sigma_i \\ \rho \dfrac{\mathrm{D}\mathrm{v}}{\mathrm{D}t} = -\nabla \cdot \mathbf{v} \end{cases} \tag{3.145}$$

By combining (3.144) and (3.145), the entropy balance expression with respect to the moving frame is as follows:

$$\rho \frac{\mathrm{D}s}{\mathrm{D}t} = -\frac{1}{T} \nabla \cdot \mathbf{f}_u^R + \frac{1}{T} \sigma_u + \frac{P}{T} \nabla \cdot \mathrm{v} + \sum_{i=1}^{N} \frac{\mu_i}{T} \nabla \cdot \mathbf{f}_i^R - \sum_{i=1}^{N} \frac{\mu_i \sigma_i}{T} \tag{3.146}$$

To transform (3.146) into an equation having the general form of a balance equation, we use the following relations:

$$\begin{cases} \nabla \cdot \left(\dfrac{\mathbf{f}_u^R}{T} \right) = \dfrac{1}{T} \nabla \cdot \mathbf{f}_u^R + \mathbf{f}_u^R \cdot \nabla \left(\dfrac{1}{T} \right) \\ \nabla \cdot \left(\dfrac{\mu_i \mathbf{f}_i^R}{T} \right) = \dfrac{\mu_i}{T} \nabla \cdot \mathbf{f}_i^R + \mathbf{f}_i^R \cdot \nabla \left(\dfrac{\mu_i}{T} \right) \end{cases} \tag{3.147}$$

According to the expression of σ_u (see (3.135):

$$\sigma_u = -P \nabla \cdot \mathbf{v} - \tau : \nabla \mathbf{v} + \sum_{i=1}^{N} \mathbf{f}_i^R \cdot \mathbf{g}_i \tag{3.148}$$

the entropy balance becomes:

$$\begin{aligned} \rho \frac{\mathrm{D}s}{\mathrm{D}t} = & -\nabla \cdot \left(\frac{\mathbf{f}_u^R - \sum_{i=1}^{N} \mu_i \mathbf{f}_i^R}{T} \right) - \sum_{i=1}^{N} \mathbf{f}_i^R \cdot \nabla \left(\frac{\mu_i}{T} \right) - \sum_{i=1}^{N} \frac{\mu_i \sigma_i}{T} \\ & + \mathbf{f}_u^R \cdot \nabla \left(\frac{1}{T} \right) - \frac{\tau : \nabla \mathbf{v}}{T} + \frac{\sum_{i=1}^{N} \mathbf{f}_i^R \cdot \mathbf{g}_i}{T} \end{aligned} \tag{3.149}$$

The relative entropy flux per unit of surface area and the volumetric entropy production are then as follows:

$$\begin{cases} \mathbf{f}_s^R = \dfrac{\mathbf{f}_u^R - \sum_{i=1}^{N} \mu_i \mathbf{f}_i^R}{T} \\ \sigma_s = \mathbf{f}_u^R \cdot \nabla\left(\dfrac{1}{T}\right) - \dfrac{1}{T}\sum_{i=1}^{N} \mathbf{f}_i^R \cdot \left[T\nabla\left(\dfrac{\mu_i}{T}\right) - \mathbf{g}_i\right] - \\ \qquad\qquad - \sum_{i=1}^{N} \dfrac{\mu_i \sigma_i}{T} - \dfrac{\tau : \nabla \mathbf{v}}{T} \geq 0 \end{cases} \tag{3.150}$$

The relative flux of entropy is given by

$$\mathbf{f}_s^R = \frac{\mathbf{f}_q}{T} + \sum_{i=1}^{N} \mathbf{f}_i^R s_i$$

where $\mathbf{f}_q$ is the heat flux per unit of surface area by conduction. Then, according to the definition of the chemical potential (see (3.136)), the relative internal energy flux per unit of surface area can be expressed as follows:

$$\mathbf{f}_u^R = \mathbf{f}_q + \sum_{i=1}^{N} \mathbf{f}_i^R h_i \tag{3.151}$$

According to the relation $\frac{\partial}{\partial T}\left(\frac{\mu_i}{T}\right) = -\frac{h_i}{T^2}$ [173],

$$\nabla\left(\frac{\mu_i}{T}\right) = \frac{(\nabla \mu_i)_T}{T} - \frac{h_i}{T^2}\nabla T$$

so that the volumetric entropy production can also be expressed as follows:

$$\sigma_s = -\frac{\mathbf{f}_q}{T^2} \cdot \nabla T - \frac{1}{T}\sum_{i=1}^{N} \mathbf{f}_i^R \cdot [(\nabla \mu_i)_T - \mathbf{g}_i] - \sum_{i=1}^{N} \frac{\mu_i \sigma_i}{T} - \frac{\tau : \nabla \mathbf{v}}{T} \geq 0 \tag{3.152}$$

According to the second principle of thermodynamic, the source term σ_s correspond to a true creation of entropy. The first term is the entropy production due to heat conduction, the second one the entropy production due to diffusion and external body forces, the third one the entropy production due to chemical reactions and the fourth one the entropy production due to viscous effects.

The absolute temperature T is the power conjugated variable associated to σ_s since $T\sigma_s$ is a volumetric power. It represents the power that is locally dissipated due to irreversible processes. Another expression of this quantity can be derived from (3.150) by using the equality $\nabla\left(\frac{\mu_i}{T}\right) = \frac{1}{T}\nabla\mu_i + \mu_i \nabla\left(\frac{1}{T}\right)$. The dissipated power is then as follows:

$$T\sigma_s = -\mathbf{f}_s^R \cdot \nabla T - \sum_{i=1}^{N} \mathbf{f}_i^R \cdot (\nabla \mu_i - \mathbf{g}_i) - \sum_{i=1}^{N} \mu_i \sigma_i - \tau : \nabla \mathbf{v} \geq 0 \tag{3.153}$$

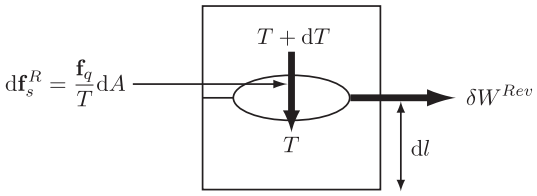

Fig. 3.13 Micro Carnot engine.

The significance of this dissipated power is more general then the ordinary dissipation due to friction or viscous effects. To understand this significance, we only consider the first term of the dissipated power $\mathbf{f}_s^R \cdot \nabla T$, and we assume that the entropy flux is only due to heat transfer by conduction, i.e. $\mathbf{f}_s^R = \frac{\mathbf{f}_q}{T}$.

Let us consider a piece of matter of volume $\mathrm{d}V$ at temperature T surrounded by a piece of matter at $T + \mathrm{d}T$ (see Fig. 3.13) and denote with $\mathrm{d}A$ the area of the contact surface between the two pieces. Let us imagine that a "micro Carnot engine" can be placed between the two pieces of matter. The power δW^{Rev} that this engine can produce reversibly is given by

$$\delta W^{\mathrm{Rev}} = \mathrm{d}\mathbf{f}_s^R(T + \mathrm{d}T - T) = \mathrm{d}\mathbf{f}_s^R \mathrm{d}T = \mathbf{f}_s^R \mathrm{d}A\mathrm{d}T$$

and by unit of volume

$$\delta w^{\mathrm{Rev}} = \mathbf{f}_s^R \mathrm{d}T \frac{\mathrm{d}A}{\mathrm{d}V} \propto \mathbf{f}_s^R \frac{\mathrm{d}T}{\mathrm{d}l}$$

If now the "micro Carnot engine" is removed, $\delta w^{\mathrm{Rev}} \propto \mathbf{f}_s^R \frac{\mathrm{d}T}{\mathrm{d}l}$ is lost and one can see that $\delta w^{\mathrm{Rev}} \propto \mathbf{f}_s^R \frac{\mathrm{d}T}{\mathrm{d}l}$ and $T\sigma_s = -\mathbf{f}_s^R \cdot \nabla T$ are similar. The minus sign of the expression of $T\sigma_s$ is due to the fact that $\mathbf{f}_s^R$ and ∇T are in the opposite direction. Finally, the dissipated power is a power that should have been produced reversibly by using appropriate systems.

Once the entropy production has been derived, one can reformulate the general equations according to the port-based approach. The flow variables balances are as follows in a fixed frame:

$$\begin{cases} \dfrac{\partial \rho \mathbf{p}}{\partial t} = -\nabla \cdot \Phi_p + \sigma_p \\ \dfrac{\partial \rho \omega_i}{\partial t} = -\nabla \cdot \mathbf{f}_i + \sigma_i \\ \dfrac{\partial \rho s}{\partial t} = -\nabla \cdot \left(\dfrac{\mathbf{f}_q}{T} + \displaystyle\sum_{i=1}^{N} \mathbf{f}_i s_i \right) + \sigma_s \end{cases} \tag{3.154}$$

According to the fact that $\sum_{i=1}^{N} \sigma_i = 0$, the total mass balance is implicitly satisfied:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{v} \tag{3.155}$$

Similarly, according to the expressions of σ_i, σ_s and σ_p, the total energy balance is also satisfied by using (3.143):

$$\frac{\partial \rho \hbar}{\partial t} = -\nabla \cdot \mathbf{f}_\hbar + \sum_{i=1}^{N} \mathbf{f}_i \cdot \mathbf{g}_i \tag{3.156}$$

The flux of $\hbar$ per unit of surface area is then as follows:

$$\mathbf{f}_\hbar = \rho \mathbf{v} \left(\frac{v^2}{2} + u \right) + \mathbf{f}_q + P\mathbf{v} + \tau \cdot \mathbf{v} + \sum_{i=1}^{N} \mathbf{f}_i^R h_i \tag{3.157}$$

This expression can be rearranged in order to exhibit the enthalpy per mass unit $h = u + P\mathrm{v} = u + \frac{P}{\rho}$:

$$\mathbf{f}_\hbar = \rho \mathbf{v} \left(\frac{v^2}{2} + h \right) + \mathbf{f}_q + \tau \cdot \mathbf{v} + \sum_{i=1}^{N} \mathbf{f}_i^R h_i = \rho \mathbf{v} \frac{v^2}{2} + \sum_{i=1}^{N} \mathbf{f}_i h_i + \mathbf{f}_q + \tau \cdot \mathbf{v} \tag{3.158}$$

The two first terms of (3.158)) represent the energy transported by the matter while the two last ones the energy flux due to heat conduction and to the work exerted by the shear stress tensor.

If the forces per mass unit $\mathbf{g}_i$ can be derived from a time independent potential Ψ_i so that:

$$\begin{cases} \mathbf{g}_i = -\nabla \Psi_i \\ \dfrac{\partial \Psi_i}{\partial t} = 0 \end{cases} \tag{3.159}$$

the total energy $\hbar'$ of the system (see (3.143)) is modified according to the following equation:

$$\begin{cases} \hbar' = \dfrac{v^2}{2} + u + \displaystyle\sum_{i=1}^{N} \omega_i \Psi_i = \frac{v^2}{2} + u + \Psi \\ \mathrm{d}\hbar = \mathbf{v} \cdot \mathrm{d}\mathbf{p} + T\mathrm{d}s - P\mathrm{d}\mathrm{v} + \displaystyle\sum_{i=1}^{N} \mu_i \mathrm{d}\omega_i + \sum_{i=1}^{N} \Psi_i \mathrm{d}\omega_i + \sum_{i=1}^{N} \omega_i \mathrm{d}\Psi_i \end{cases} \tag{3.160}$$

A new set of energy-conjugated variables $(\mathbf{v}, \mathbf{p})$, (T, s), $(-P, \mathrm{v})$, (μ_i, ω_i), (Ψ_i, ω_i) and (ω_i, Ψ_i) is defined while the total energy balance is given by:

$$\rho \frac{\mathrm{D}\hbar'}{\mathrm{D}t} = -\nabla \cdot \mathbf{f}_{\hbar'}^R \tag{3.161}$$

The source term has disappeared since the exchange energy is now expressed as the variation of a potential energy. As far as a balance equation for the quantity

$$\rho \frac{\mathrm{D}}{\mathrm{D}t} \left(\frac{v^2}{2} + \Psi \right) = \rho \left(\mathbf{v} \cdot \frac{\mathrm{D}\mathbf{p}}{\mathrm{D}t} + \sum_{i=1}^{N} \Psi_i \frac{\mathrm{D}\omega_i}{\mathrm{D}t} + \sum_{i=1}^{N} \omega_i \frac{\mathrm{D}\Psi_i}{\mathrm{D}t} \right)$$

can be derived by using the component i balance (see (3.124)), an internal energy balance equation similar to (3.134) and (3.135) can be obtained:

$$\rho \frac{\mathrm{D}u}{\mathrm{D}t} = -\nabla \cdot \mathbf{f}_u^R + \sigma_u \tag{3.162}$$

with

$$\begin{cases} \mathbf{f}_{h'}^R = \mathbf{f}_u^R + \left[\Phi_p^R \cdot \mathbf{v} \right] + \sum_{i=1}^{N} \mathbf{f}_i^R \Psi_i \\ \sigma_u = -\Phi_p^R : \nabla \mathbf{v} + \sum_{i=1}^{N} \mathbf{f}_i^R \cdot \mathbf{g}_i = -P \nabla \cdot \mathbf{v} - \tau : \nabla \mathbf{v} + \sum_{i=1}^{N} \mathbf{f}_i^R \cdot \mathbf{g}_i \end{cases} \tag{3.163}$$

In this derivation, it has been assumed that:

- according to (3.159), the substantial time derivative of Ψ_i is reduced to

$$\frac{\mathrm{D}\Psi_i}{\mathrm{D}t} = \mathbf{v} \cdot \nabla \Psi_i = -\mathbf{v} \cdot \mathbf{g}_i$$

- the chemical reactions do not modify the potential energy, i.e. $\sum_{i=1}^{N} \Psi_i \sigma_i = 0$.

Neither σ_u nor the entropy balance are modified and the system represented by (3.160) can be seen as a port-based one in the same manner then previously.

3.4.3 Lumped parameter systems

In many circumstances, the derivation of models is based on the definition of a network of lumped parameter sub-systems (cf. Chapter 1). The state variables are supposed to be spatially uniform in each element of the network. In chemical engineering, a very famous element of such a network is the so-called CSTR (Continuous Stirred Tank Reactor) that is highly used for the modelling of chemical reactors [115]. Once the network is spatially defined, the balance equations are established for each element of the network. The total energy that is generally considered is reduced to the internal energy as far as the effects due to viscous dissipation are generally negligible with respect to those due to heat transfer or chemical reactions. A noticeable exception is the case of highly viscous fluids like polymers for example where the viscous dissipation has to be taken into account (see for example [48]). As in the case of distributed parameter systems, the objective of a model is to calculate the output variables P, T and the composition and the two approaches can be used to define state variables. The classical one consists in manipulating material and energy balances while the port-based one is based on material and entropy balances.

3.4.3.1 The classical approach

We will establish the balances on a molar basis so that $\bar{h}_i$ is the molar partial enthalpy of component i while $\bar{h}$ is the molar enthalpy of the mixture. Let us consider one lumped parameter system as an element of a network of compartments. This element may exchange matter and energy with its neighbours through links designated by the l index. It contains N_i mole of component i and its internal energy is U. The energy and component i balances are written as follows:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = F_q + F_w + \sum_l F_l \bar{h}_l = F_q + F_w + \sum_{i,l} F_{il} \bar{h}_{il} \tag{3.164}$$

$$\frac{\mathrm{d}N_i}{\mathrm{d}t} = \sum_l F_l \chi_{il} + \sigma_i V = \sum_l F_{il} + \sigma_i V \tag{3.165}$$

where F_q and F_w are respectively the total heat flux and the total power that are exchanged by the system through its boundary, while F_{il} and χ_{il} are respectively the molar flow rate and the molar fraction of component i through the link l, while F_l is the corresponding total molar flow rate.

3.4.3.2 The port-based approach

The Gibbs equation (3.136) is supposed to be valid for the system at uniform pressure, temperature and composition so that one can derive the entropy balance from (3.136), (3.164) and (3.165):

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{1}{T}\left[F_q + F_w + \sum_{i,l} F_{il}\bar{h}_{il} - \sum_i \bar{\mu}_i \left(\sum_l F_{il} + \sigma_i V\right)\right] + \frac{P}{T}\frac{\mathrm{d}V}{\mathrm{d}t} \tag{3.166}$$

According to the definition of the chemical potential per mole unit $\bar{\mu}_i = \bar{h}_i - T\bar{s}_i$, (3.166) can be rearranged in order to exhibit a source term and exchanged terms:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \sum_{i,l} F_{il}\bar{s}_{il} + \frac{F_q + F_w}{T} + \frac{P}{T}\frac{dV}{dt} + \sum_{i,l} F_{il}\left(\frac{\bar{h}_{il} - T\bar{s}_{il}}{T} - \frac{\bar{\mu}_i}{T}\right) - \sum_{i=1}^{N} \bar{\mu}_i \sigma_i V \tag{3.167}$$

Let us consider for example that the heat flux is exchanged with a heat source at T_{ext} and that the volume V varies in contact with a pressure source at P_{ext} with $F_w = -P_{\mathrm{ext}}\frac{\mathrm{d}V}{\mathrm{d}t}$. Equation (3.167) can be given under the following form:

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \displaystyle\sum_{i,l} F_{il}\bar{s}_{il} + \frac{F_q}{T_{\text{ext}}} + \sigma_s V \\ \sigma_s = \dfrac{1}{V}\left[\left(\dfrac{F_q}{T} - \dfrac{F_q}{T_{\text{ext}}}\right) + \dfrac{P - P_{\text{ext}}}{T}\dfrac{dV}{dt} + \right. \\ \qquad\qquad \left. + \displaystyle\sum_{i,l} F_{il}\left(\frac{\bar{h}_{il} - T\bar{s}_{il}}{T} - \frac{\bar{\mu}_i}{T}\right) - \sum_{i=1}^{N} \bar{\mu}_i \sigma_i V\right] \end{cases} \tag{3.168}$$

The first term in the entropy production is due irreversible heat transfer, the second one is due to mechanical friction, the third one to the irreversible mixing of the inlet fluxes of matter with the matter contained in the system and finally the fourth one is due to chemical reactions.

3.4.4 Constitutive equations

A model only based on balance equations cannot be used if constitutive equations are not available (cf. Appendix B, Sect. B.2.8). These relations constitute a model for the matter properties – i.e. transport, thermodynamic properties (see for example [167]) and chemical reaction rates. In order to calculate these properties of the matter, a great number of models are available, more or less complicated according to the situation. Classical textbooks or references are devoted to this question, which is probably the most important and difficult one in chemical engineering [167, 173, 209].

3.4.4.1 Thermodynamic properties

One has to relate the specific internal energy, entropy, enthalpy of the system to P, T and the composition. As far as the total mass balance is involved, one has also to relate the specific volume or the mass or molar density to the same variables. A thermodynamic model is a set of relations allowing to calculate all these properties (see for example [173, 209]. These relations have been derived by considering equilibrium situations. According to the principle of local equilibrium, they are also assumed to apply at a point, even if the system under consideration is not at equilibrium as a whole. According to the classical way to proceed, we present very briefly this question by separating the case of pure components from the case of mixtures.

Equations of state are generally used to model the specific volume v of a fluid. They are given under the general form $\varphi(P, \mathrm{v}, T, \theta_1, \theta_2, \dots) = 0$, where θ_1, θ_2, ... are parameters. Some of these equations of state can be applied for both liquid and gaseous phases while others are devoted to only one phase. All the properties are calculated by using equation of states. As far as u is concerned, the following expression is derived where u is primarily considered as a function of v and T:

$$\mathrm{d}u = c_{\mathrm{v}}(\mathrm{v},T)\mathrm{d}T + \left(T\frac{\partial P}{\partial T} - P\right)\mathrm{dv} \tag{3.169}$$

where $c_{\mathrm{v}}(\mathrm{v},T)$ is the heat capacity at constant volume. In many applications, it is better to consider P and T as the output variables so that the enthalpy $h = u + P\mathrm{v}$ is more convenient. The following relation is then derived:

$$\mathrm{d}h = c_P(P,T)\mathrm{d}T + \left(\mathrm{v} - T\frac{\partial \mathrm{v}}{\partial T}\right)\mathrm{d}P \tag{3.170}$$

where $c_P(P,T)$ is the heat capacity at constant pressure. If the port-based approach is used, the specific entropy s can be expressed by using the following relations:

$$\begin{cases} \mathrm{d}s = \dfrac{c_{\mathrm{v}}(\mathrm{v},T)}{T}\mathrm{d}T + \dfrac{\partial P}{\partial T}\mathrm{dv} \\ \mathrm{d}s = \dfrac{c_p(P,T)}{T}\mathrm{d}T - \dfrac{\partial \mathrm{v}}{\partial T}\mathrm{d}P \end{cases} \tag{3.171}$$

Equations (3.169), (3.170) and (3.171) have to be integrated along calculations paths. To do so, an arbitrary origin has to be defined for u or h and s.

Equations of state can also be used for mixtures. They are given under the general form $\varphi(P,\mathrm{v},T,\theta_{1m},\theta_{2m},\dots) = 0$, where the parameters θ_{1m}, θ_{2m}, ... depend on the composition. As far as the other properties are concerned, in order to be able to define an arbitrary origin for them, one has to consider each component separately and define its contribution to a given property. Let us for example consider the case of the enthalpy H. H is primarily considered as a function of P, T and N_i, the number of mole of component i. As it is a first order homogenous function with respect to N_i, Euler theorem can be applied [173] (cf. Appendix B, Sect. B.2.1):

$$\begin{cases} H(P,T,N_i) = \displaystyle\sum_i N_i \bar{h}_i(P,T,\chi_i) \\ \chi_i = \dfrac{N_i}{\sum_{i=1}^{N} N_i} \end{cases} \tag{3.172}$$

By definition, $\bar{h}_i = \left(\frac{\partial H}{\partial N_i}\right)_{P,T,N_{k\neq i}}$ is the partial molar enthalpy. Within the framework of the port-based approach, it is also better to consider S as a function of P, T and N_i so that Euler theorem can also be applied:

$$S(P,T,N_i) = \sum_{i=1}^{N} N_i \bar{s}_i(P,T,\chi_i) \tag{3.173}$$

with $\bar{s}_i = \left(\frac{\partial S}{\partial N_i}\right)_{P,T,N_{k\neq i}}$ the partial molar entropy.

Another way to model a mixture is to model the excess Gibbs molar free energy $\Delta g^{\mathrm{e}}(P,T,\chi_i)$. This quantity is the difference between g, the Gibbs molar free energy

of a mixture and g^{id}, the Gibbs molar free energy of the same mixture considered as an ideal solution [173, 209].

As far as reactive systems are concerned, the origin for the calculation of $\bar{h}_i$ is the so-called standard enthalpy of formation. This quantity has been chosen to be 0 for atoms or simple components under their more stable state at ambient conditions (H_2, O_2 for example).

3.4.4.2 Transport properties and chemical reaction rates

These properties allow expressing the fluxes as functions of the effort variables according to the expression of the total entropy production (see (3.152)). The main properties that are required are viscosity, thermal conductivity and diffusion coefficients. These coefficients allow to relate respectively the shear stress tensor to the velocity gradient, the heat flux by conduction to the temperature gradient and the fluxes of component i to the chemical potential gradient. Furthermore, source terms in the material balances are given by the rates of chemical reactions. Finally, one has also to consider coupled phenomena if necessary. Let us first consider the way such coupling may occur.

According to the Curie symmetry principle applied to isotropic systems, all the fluxes are not functions of all the effort variables but only of those having the same tensorial order [60]. The total entropy production can be split into three terms, each of them being positive. In order to express the entropy production according to these three terms, one has to consider the entropy production associated to viscous effects. The quantity $\tau : \nabla\mathbf{v}$ can be expressed as follows provided that τ is symmetric [60]:

$$\tau : \nabla\mathbf{v} = \tilde{\tau} : \left(\tilde{\nabla}\mathbf{v}\right)^s + \frac{1}{3}\mathrm{Tr}(\tau)\nabla\cdot\mathbf{v} \tag{3.174}$$

To derive (3.174), τ has been decomposed as follows:

$$\tau = \tilde{\tau} + \frac{1}{3}\mathrm{Tr}(\tau)\mathbf{I}, \qquad \text{with } \mathrm{Tr}(\tilde{\tau}) = 0 \tag{3.175}$$

Similarly, $\nabla\mathbf{v}$ has been decomposed according to:

$$\nabla\mathbf{v} = \left(\tilde{\nabla}\mathbf{v}\right) + \frac{1}{3}\mathrm{Tr}(\nabla\mathbf{v})\mathbf{I} = \left(\tilde{\nabla}\mathbf{v}\right)^s + \left(\tilde{\nabla}\mathbf{v}\right)^a + \frac{1}{3}\left(\nabla\cdot\mathbf{v}\right)\mathbf{I}, \text{ with } \mathrm{Tr}(\tilde{\nabla}\mathbf{v}) = 0 \tag{3.176}$$

where $\left(\tilde{\nabla}\mathbf{v}\right)^s$ and $\left(\tilde{\nabla}\mathbf{v}\right)^a$ are respectively the symmetric and anti-symmetric parts of $\tilde{\nabla}\mathbf{v}$. By combining (3.152) and (3.174), the entropy production can be expressed as follows:

$$\sigma_s = \underbrace{-\sum_{i=1}^{N}\frac{\mu_i\sigma_i}{T} - \frac{\frac{1}{3}\mathrm{Tr}(\tau)\nabla\cdot\mathbf{v}}{T}}_{\sigma_s^{\mathrm{Sc}}} \underbrace{-\frac{\mathbf{f}_q}{T^2}\nabla T - \frac{1}{T}\sum_{i=1}^{N}\mathbf{f}_i^R\left((\nabla\mu_i)_T - \mathbf{g}_i\right)}_{\sigma_s^{\mathrm{Vect}}} \\ \underbrace{-\frac{\tilde{\tau}:\left(\tilde{\nabla}\mathbf{v}\right)^s}{T}}_{\sigma_s^{\mathrm{Tens}}} \geq 0 \tag{3.177}$$

where σ_s^{Sc}, σ_s^{Vect} and σ_s^{Tens} are respectively the scalar, vectorial and tensorial contributions to the entropy production. Coupling between phenomena occurs only within these three sets of phenomena.

Within the so-called "Linear thermodynamics of irreversible processes", the relations between the flow and effort variables are expressed linearly. This does not mean that the resulting models are linear. They are generally non-linear firstly because thermodynamic and transport properties are functions of the state variables as well as the chemical rates [60] and secondly because of the coupling between the phenomena.

As far as we know, coupling between chemical reactions and scalar viscous effects are generally not considered [21] so that we express separately $\mathrm{Tr}(\tau)$ as a function of $\nabla\cdot\mathbf{v}$ on the one hand and σ_i as functions of μ_i on the other hand.

The flow variable $\frac{1}{3}\mathrm{Tr}(\tau)$ is assumed to be a function of the effort variable $\nabla\cdot\mathbf{v}$ only. If the system is not to far from equilibrium, a linear relation can be postulated:

$$\frac{1}{3}\mathrm{Tr}(\tau) = -\kappa(\nabla\cdot\mathbf{v}) \tag{3.178}$$

where κ is the dilatational or volume viscosity, which is independent of $\nabla\cdot\mathbf{v}$ within the framework of linear irreversible thermodynamic. This situation corresponds to the so-called Newtonian fluid but κ depends on P, T and the composition of the fluid.

Beyond the linear thermodynamics of irreversible processes, the case of non-linear chemical kinetics has to be carefully considered. As a matter of fact, the definition of effort and flow variables as well as the relation between them is not evident. For the sake of simplicity, we restrict ourselves to the case of thermally activated chemical reactions. When a mixture is subject to chemical reactions, some species are consumed, some others are produced. Global or apparent chemical reaction rates can be fitted to experimental results but such an approach is empirical. A deep understanding of a global chemical transformation is based on a decomposition of such a global chemical process into independent elementary processes [176]. It is assumed that for an elementary process, the species are really brought into contact one with the others in order the atoms initially present in the reactants can be redistributed in the products. Such elementary processes are assumed to involve a low number of reactants, mainly, one, two or three (mono-, bi- or tri-molecular processes) and their orders are assumed to be equal to the stoechiometric coefficients.

Let us consider the entropy production due to chemical reactions, i.e. $\sigma_s^{\text{Reac}} = -\sum_{i=1}^{N} \frac{\mu_i \sigma_i}{T}$. We assume that the overall chemical process has been decomposed according to a set of R independent elementary processes so that σ_i is expressed by (3.122). Then, σ_s^{Reac} can be expressed as follows:

$$\sigma_s^{\text{Reac}} = -\frac{1}{T} \sum_{i=1}^{N} \sum_{k=1}^{R} \left(\nu_i^k \mu_i \bar{M}_i \right) r_k = \frac{1}{T} \sum_{k=1}^{R} A_k r_k \tag{3.179}$$

where

$$A_k = -\sum_{i=1}^{N} \nu_i^k \bar{\mu}_i = -\Delta_r G_k \tag{3.180}$$

where A_k is the affinity or the opposite of the Gibbs free energy $\Delta_r G_k$ of the k-th chemical reaction and $\bar{\mu}_i = \mu_i \bar{M}_i$ the chemical potential per mole unit. In (3.179) and (3.180), the stoechiometric coefficients ν_i^k are considered to be positive for a product and negative for a reactant. As pointed out in [11, 163] the rate of a chemical reaction, and particularly the rate of an elementary process, is a function of the concentrations. Consequently, it can be given as a function of the affinity, as it is suggested by (3.179), only for an equilibrated reaction and near the equilibrium position [60]. In this case, the overall chemical process is described within the framework of the linear thermodynamic of irreversible processes. Otherwise, one has to consider separately the forward and reverse directions (see also [114]). In order to illustrate this point, let us consider the case of an elementary chemical process described by the following stoechiometric equation:

$$\sum_{i=1}^{N} \nu_i^f Re_i \quad \leftrightarrow \quad \sum_{i=1}^{B} \nu_i^r Pr_i \tag{3.181}$$

This process is supposed to lead to a possible equilibrium. In order to split more easily a chemical process into forward and the reverse directions, we have defined positive stoechiometric coefficients ν_i^f and ν_i^r respectively for those directions. As far as the process (3.181) is assumed to be an elementary one, its rate can be expressed as follows:

$$r = r_f - r_i = k_f \prod_{i=1}^{N} C_i^{\nu_i^f} - k_r \prod_{i=1}^{N} C_i^{\nu_i^r} \tag{3.182}$$

where k_f and k_r are the rate constants that are commonly considered as functions of the temperature according to the Arrhenius relation:

$$k_f = k_f^0 \mathrm{e}^{-\frac{E_f}{RT}} \qquad k_r = k_r^0 \mathrm{e}^{-\frac{E_r}{RT}} \tag{3.183}$$

In order to express the rate of the process as a function of the chemical potentials, one has to invert the relation between $\bar{\mu}_i$ and C_i. A thermodynamic model is then necessary. The rather simple but frequently case considered in chemical kinetics has been treated in [11, 163] by assuming the case of an ideal solution. These authors have considered a less usual reference for the expression of $\bar{\mu}_i$ (the component i at

unit concentration) then the one ordinary used in thermodynamic. However, their approach can be used and generalized to non-ideal solutions. As a matter of fact, a more usual but not unique way to derive a calculation path for the chemical potential is to consider $\bar{\mu}_i^*(P,T)$, the chemical potential of the pure component i in the same physical conditions as the reference [173]. The chemical potential is then expressed as follows:

$$\begin{cases} \bar{\mu}_i(P,T,\chi_1,\ldots,\chi_{N-1}) = \bar{\mu}_i^*(P,T) + RT\ln(a_i) \\ \qquad\qquad\qquad\qquad = \bar{\mu}_i^*(P,T) + RT\ln(\gamma_i(P,T,\chi_1,\ldots,\chi_{N-1})\,\chi_i) \\ \qquad\qquad\qquad \chi_i = \dfrac{C_i}{C} \end{cases} \tag{3.184}$$

where C is the total molar concentration, a_i the activity of i with respect to the state of pure component and γ_i the activity coefficient. γ_i has to be calculated from a thermodynamic excess model. The simplest model is the ideal solution where $\gamma_i^{\mathrm{id}} = 1$. By combining (3.182) and (3.184), one can find that:

$$r = r_f - r_r = \frac{k_f C^{\sum_{i=1}^N \nu_i^f} \mathrm{e}^{-\frac{A_f^*}{RT}} \mathrm{e}^{\frac{A_f}{RT}}}{\prod_{i=1}^N (\gamma_i)^{\nu_i^f}} - \frac{k_r C^{\sum_{i=1}^N \nu_i^r} \mathrm{e}^{-\frac{A_r^*}{RT}} \mathrm{e}^{\frac{A_r}{RT}}}{\prod_{i=1}^N (\gamma_i)^{\nu_i^r}} \tag{3.185}$$

where

$$A_f = \sum_{i=1}^N \nu_i^f \bar{\mu}_i \quad A_f^* = \sum_{i=1}^N \nu_i^f \bar{\mu}_i^* \quad A_r = \sum_{i=1}^N \nu_i^r \bar{\mu}_i \quad A_r^* = \sum_{i=1}^N \nu_i^r \bar{\mu}_i^* \tag{3.186}$$

A_f and A_r are respectively the forward and reverse affinities while A_f^* and A_r^* are the corresponding quantities calculated at the reference state. The same flow variable r is associated to the two effort variables A_f and A_r so that the entropy production due the elementary process (3.181) is given by:

$$\sigma_s^{\mathrm{Reac}} = \frac{A_f r - A_r r}{T} = \frac{Ar}{T} \tag{3.187}$$

Let us notice that, as far as the process (3.181) can lead to an equilibrium, the latter can be predicted directly from thermodynamic by using the condition:

$$A^{\mathrm{eq}} = -\sum_{i=1}^N \nu_i^k \bar{\mu}_i = A_f^{\mathrm{eq}} - A_r^{\mathrm{eq}} = 0 \tag{3.188}$$

This equilibrium condition also corresponds to the fact that:

$$r^{\mathrm{eq}} = r_f^{\mathrm{eq}} - r_r^{\mathrm{eq}} = 0 \tag{3.189}$$

Consequently, k_f and k_r must satisfy a condition that can be derived by combining (3.184), (3.185), (3.188) and (3.189):

$$k_f \frac{(C^{\mathrm{eq}})^{\sum_{i=1}^{N} \nu_i^f} \, \mathrm{e}^{-\frac{A_f^*}{RT}}}{\prod_{i=1}^{N} \left(\gamma_i^{\mathrm{eq}}\right)^{\nu_i^f}} = k_r \frac{(C^{\mathrm{eq}})^{\sum_{i=1}^{N} \nu_i^r} \, \mathrm{e}^{-\frac{A_r^*}{RT}}}{\prod_{i=1}^{N} \left(\gamma_i^{\mathrm{eq}}\right)^{\nu_i^r}} = \kappa^{\mathrm{eq}} \tag{3.190}$$

Finally, if the process is considered to be close to the equilibrium conditions, one can linearise (3.185) by evaluating

$$k_f \frac{C^{\sum_{i=1}^{N} \nu_i^f} \, \mathrm{e}^{-\frac{A_f^*}{RT}}}{\prod_{i=1}^{N} (\gamma_i)^{\nu_i^f}} \qquad \text{and} \qquad k_r \frac{C^{\sum_{i=1}^{N} \nu_i^r} \, \mathrm{e}^{-\frac{A_r^*}{RT}}}{\prod_{i=1}^{N} (\gamma_i)^{\nu_i^r}}$$

at the equilibrium conditions according to (3.190) and by linearizing the results with respect to A_f and A_r. The linearized rate expression of the process (3.181) is then given by:

$$r = r^{\mathrm{eq}} + \tilde{r} = \tilde{r} = \kappa^{\mathrm{eq}} \frac{\mathrm{e}^{\frac{A_f^{\mathrm{eq}}}{RT}}}{RT} A \tag{3.191}$$

according to the linear thermodynamic of irreversible processes.

As far as elementary processes are concerned, a model has been derived by [11, 84] to explain the dependence of chemical reaction rate constants with temperature as given by the Arrhenius relation: this model is based on the concept of activated complex. We give a very simplified presentation of this model since it involves highly detailed calculations based on molecular theories. To this end, we consider the case of an elementary bi-molecular process in the forward direction:

$$A + B \;\; \rightarrow \;\; Pr \tag{3.192}$$

If such a process is elementary, it is assumed that the atoms composing A and B are really brought into contact: this elementary amount of matter is called the activated complex and is noted $(AB)^{\mp}$. This activated complex is considered to be an ordinary molecule possessing usual thermodynamic properties. However, at the molecular level, one direction of vibration leads to the decomposition of the activated complex proportionally to its concentration so that the activated complex is "treated as a molecule with one degree of vibrational freedom less than normal" (cf. [84], p. 402). In order to derive the rate of the elementary process (3.192), the activated complex $(AB)^{\mp}$ is considered as if it were in equilibrium with the reactants, according to the following equivalent process:

$$(a) \quad A + B \;\leftrightarrow\; (AB)^{\mp} \qquad\qquad (b) \quad (AB)^{\mp} \;\rightarrow\; Pr \tag{3.193}$$

The rate r_f of the process [84, 109] is given by the decomposition rate of $(AB)^{\mp}$ since the quasi-stationary state principle is applied to this entity. From statistical thermodynamic arguments, it is shown that r_f is given by:

$$r_f = \frac{\bar{k}T}{\bar{h}} C_{(AB)^{\mp}} \tag{3.194}$$

where $\bar{k}$ and $\bar{h}$ are respectively the Boltzmann and Planck constants. The derivation of (3.194) is based on the assumption that the rate of decomposition of the activated complex is related the frequency of one vibrational degree of freedom inside this complex. If now the activated complex and the reactants A and B are assumed as if they were at equilibrium, one can expresse $C_{(AB)^{\mp}}$ as a function of C_A and C_B. If $\bar{\mu}_A$, $\bar{\mu}_B$ and $\bar{\mu}_{(AB)^{\mp}}$ are respectively the chemical potential per mole unit of A, B and $(AB)^{\mp}$, the chemical equilibrium condition:

$$\bar{\mu}_{(AB)^{\mp}} = \bar{\mu}_A + \bar{\mu}_B \tag{3.195}$$

can be used to derive the equilibrium constant of the process (3.193)(a) $K_{\mp}$, which is defined by:

$$RT \ln\left(\frac{a_{(AB)^{\mp}}}{a_A a_B}\right) = RT \ln K_{\mp} = -\left(\bar{\mu}^0_{(AB)^{\mp}} - \left(\bar{\mu}^0_A + \bar{\mu}^0_B\right)\right) = -\Delta G^0_{\mp} \tag{3.196}$$

As far as $K_{\mp}$ is a function of the mixture composition, the rate equation (3.194) can be formally expressed as a function of $K_{\mp}$ through its dependence on $C_{(AB)^{\mp}}$. The way this dependency will be expressed depends on the thermodynamic model that is used to represent the mixture. As far as the chemical rates are functions of the molar concentrations:

$$r_f = k_f C_A C_B \tag{3.197}$$

by combining (3.194), (3.196) and (3.197), the rate constant can be expressed as follows:

$$k_f = \frac{\bar{k}T}{\bar{h}} K_{\mp} \frac{\dfrac{C_{(AB)^{\mp}}}{a_{(AB)^{\mp}}}}{\dfrac{C_A}{a_A}\dfrac{C_B}{a_B}} = \frac{\bar{k}T}{\bar{h}} \frac{C_{(AB)^{\mp}}}{C_A C_B} \frac{a_A a_B}{a_{(AB)^{\mp}}} \mathrm{e}^{\frac{-\Delta G^0_{\mp}}{RT}} \tag{3.198}$$

After some tedious calculations [57], it can be shown that the entropy production due to vectorial phenomena can be expressed as follows:

$$\sigma_s^{\text{Vect}} = -\frac{\mathbf{f}_q}{T} \cdot \nabla \ln(T) - \frac{1}{T} \sum_{i=1}^{N} \mathbf{f}_i^R \cdot \mathbf{e}_i \tag{3.199}$$

The effort variables $\mathbf{e}_i$ per unit of mass are given by the following relation:

$$\mathbf{e}_i = \frac{CRT}{\rho_i} \mathbf{d}_i = T\nabla\left(\frac{\mu_i}{T}\right) + h_i \nabla \ln(T) - \frac{\nabla P}{\rho} - \mathbf{g}_i + \frac{\sum_{j=1}^{N} \rho_j \mathbf{g}_j}{\rho} \tag{3.200}$$

According to their definitions, $\mathbf{e}_i$ and $\mathbf{f}_i^R$ are not independent since:

$$\sum_{i=1}^{N} \mathbf{f}_i^R = 0 \qquad \sum_{i=1}^{N} \mathbf{d}_i = 0 \tag{3.201}$$

Within the framework of linear irreversible thermodynamic, the flow variables $\mathbf{f}_i^R$ and $\mathbf{f}_q$ are linearly expressed as functions of the effort variables $\mathbf{e}_i$ and $\nabla \ln(T)$. The relation between $\mathbf{f}_q$ and $\nabla \ln(T)$ is analogous to the Fourier relation while the relations between $\mathbf{f}_i^R$ and $\mathbf{e}_i$ is related to classical isothermal diffusion. However, coupling phenomena can exist. The thermal effort $\nabla \ln(T)$ can generate mass fluxes: this is the Soret effect. The Dufour effect is the generation of a heat flux $\mathbf{f}_q$ due to the diffusive effort variables. As far as the flow variables $\mathbf{f}_i^R$ are concerned, the Maxwell-Stefan approach consisting in expressing the effort variables as function of the fluxes [201] is more and more used in chemical engineering. The advantage of this approach is that the physical interpretation of the diffusion coefficients is easiest since they are analogous to drag or friction coefficients. Since coupling between $\nabla \ln(T)$ and $\mathbf{f}_i^R$ is considered, the so-called generalized Maxwell-Stefan equations are derived as follows.

For the sake of simplicity, let us first consider the case of isothermal systems and let us consider the forces acting on component i. A driving force $\mathbf{e}_i$ is assumed to compensate exactly a drag force that is due to the presence of the other components j. This drag force is proportional to the relative velocity of component i with respect to component j. It is usual to express this momentum balance by using $\mathbf{d}_i$ instead of $\mathbf{e}_i$ as follows:

$$\begin{cases} \mathbf{d}_i = \sum\limits_{j \neq i} \dfrac{\chi_i \chi_j}{D_{ij}} (\mathbf{v}_j - \mathbf{v}_i) = \sum\limits_{j \neq i} \dfrac{\chi_i \chi_j}{D_{ij}} ((\mathbf{v}_j - \mathbf{v}) - (\mathbf{v}_i - \mathbf{v})) \\ \quad = \sum_{j \neq i} \frac{\chi_i \chi_j}{D_{ij}} \left(\frac{\mathbf{f}_j^R}{\rho_j} - \frac{\mathbf{f}_i^R}{\rho_i} \right) \\ D_{ij} = D_{ji} \end{cases} \tag{3.202}$$

If now the Soret effect is included, one define D_j^{T} as the thermal diffusion coefficient for component j, a modified relative velocity is introduced so that the force equilibrium becomes:

$$\begin{cases} \mathbf{d}_i = \sum\limits_{j \neq i} \dfrac{\chi_i \chi_j}{D_{ij}} \left[\left(\dfrac{\mathbf{f}_j^R}{\rho_j} + \dfrac{D_j^{\mathrm{T}}}{\rho_j} \nabla \ln(T) \right) - \left(\dfrac{\mathbf{f}_i^R}{\rho_i} + \dfrac{D_i^{\mathrm{T}}}{\rho_i} \nabla \ln(T) \right) \right] \\ \sum\limits_{j=1}^{N} D_j^{\mathrm{T}} = 0 \end{cases} \tag{3.203}$$

Equation (3.203) is an implicit relation between the effort and the flow variables that has to be inverted in order to express the flow variables as functions of the effort variables [201]. As far the heat flux is concerned, its expression is as follows [57]:

$$\mathbf{f}_q = -\left[\lambda_0 + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{CRT}{\rho_i} D_i^{\mathrm{T}} \frac{\chi_i \chi_j}{D_{ij}} \left(\frac{D_j^{\mathrm{T}}}{\rho_j} - \frac{D_i^{\mathrm{T}}}{\rho_i}\right)\right] \nabla \ln(T) - \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{CRT}{\rho_i} D_i^{\mathrm{T}} \frac{\chi_i \chi_j}{D_{ij}} \left(\frac{\mathbf{f}_j^R}{\rho_j} - \frac{\mathbf{f}_i^R}{\rho_i}\right) \quad (3.204)$$

The quantity

$$\lambda_0 + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{CRT}{\rho_i} D_i^{\mathrm{T}} \frac{\chi_i \chi_j}{D_{ij}} \left(\frac{D_j^{\mathrm{T}}}{\rho_j} - \frac{D_i^{\mathrm{T}}}{\rho_i}\right)$$

is commonly related to the thermal conductivity λ as it has been introduced by Fourier [21]:

$$\lambda = \frac{\lambda_0 + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{CRT}{\rho_i} D_i^{\mathrm{T}} \frac{\chi_i \chi_j}{D_{ij}} \left(\frac{D_j^{\mathrm{T}}}{\rho_j} - \frac{D_i^{\mathrm{T}}}{\rho_i}\right)}{T} \quad (3.205)$$

As far as tensorial irreversible processes are concerned, there is only one process described here so that no coupling has to be considered. The flow variable $\tilde{\tau}$ is only function of the effort variable $\left(\tilde{\nabla}\mathbf{v}\right)^s$. A linear relation can approximate this function if the system is not to far from equilibrium:

$$\tilde{\tau} = -2\eta \left(\tilde{\nabla}\mathbf{v}\right)^s \quad (3.206)$$

where $\tilde{\tau}$ is assumed to be a symmetric tensor. η is the viscosity of the fluid only depending on P, T and the composition for a Newtonian fluid. One can find for example in [167] the way to calculate η for fluids in many situations.

3.4.5 Port-based modelling examples

Many approaches of computer aided modelling in process engineering are based on the formulation of balance equations (including energy balance), constitutive equations and constraint equations due to the environment. Such approaches have been used to develop structured modelling methodologies for distributed parameter systems such as [83,132]. Even if these structured modelling approaches can be viewed as classical ones in process (and chemical) engineering, we will rather apply here on two examples the port-based methodology previously developed in Chapter 1. The two chosen examples, a batch gas phase chemical reactor and an adsorption column, will highlight the advantages of the approach. In both cases, the port-based approach is applied, starting from Gibbs equation, using internal energy, material and entropy balance equations, deriving constitutive equations in terms of port variables from classical thermodynamic assumptions, in order to obtain a port-based model of the processes in the form of a generalized Bond Graph.

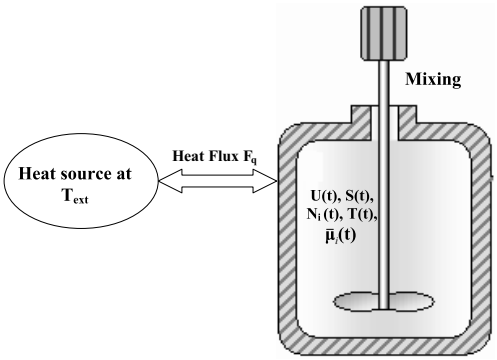

Fig. 3.14 A closed constant volume chemical reactor.

In the reactor example, physical variables are considered as spatially uniformly distributed. As a consequence, a lumped port-based model is derived. It is shown that in this model, two irreversible processes are sources of entropy production: the heat transfer and the chemical reaction.

In the adsorption column case, an isothermal model is considered. The model focuses on mass transfer phenomena (adsorption and diffusion). It follows that a distributed parameters model is derived which points up these transport phenomena are the sources of entropy production. These phenomena (adsorption, diffusion and dispersion) occur at three different scales. Therefore, the model of the adsorption column also provides a nice example of an application of the port-based methodology to multi-scale modelling.

In both examples, the port-based models and their bond graph formulations appear to be easy to re-use and to connect to other process sub-models. This property comes from the use of port variables and is obvious for the lumped parameter model. However, in the adsorption column example, the model is stated independently of the chosen boundary conditions for describing the environment. This "free-boundary" formulation allows to connect the model of the adsorption column to other plant sub-models, if any. This property is a critical advantage of the port-based model on classical partial differential balance equations models. Further readings about modelling and about method of spatial discretization associated to this port based approach can be found in [12–15, 52, 71].

3.4.5.1 Modelling of a batch gas phase chemical reactor

The topic of chemical reactors modelling is very important in chemical engineering (see for example the classical textbook [115]). The situations are very numerous and complex and we give here a rather simple example to illustrate the question of classical - versus port-based approach of their modelling. More complex situations have been described elsewhere ([53]). Let us consider a closed constant volume gas phase chemical reactor as represented on Fig. 3.14. Such a system is a batch reactor because the production is not continuous.

We assume that the system is perfectly mixed: this means that the state variables are uniform. The reactor is initially fed with the reactants and only one reaction is supposed to occur. This reaction is defined by the stoichiometric equation:

$$\sum_{i=1}^{N} \nu_i B_i = 0 \tag{3.207}$$

where B_i are considered chemical species, N their number and ν_i the stoichiometric coefficients with $\nu_i > 0$ for a product and $\nu_i < 0$ for a reactant. During the reaction progress, the reactor is in thermal contact with the surrounding at T_{ext}.

The model is based on components i and internal energy balances (see for example [131]):

- Component i balance:

$$\frac{\mathrm{d}N_i}{\mathrm{d}t} = \nu_i r V \ \forall i \in \{1, \ldots, N\} \tag{3.208}$$

  where N_i are the numbers of moles of component i in the mixture, V is the total reactor volume and r the reaction rate. Let us notice that, due to the fact that there is only one reaction, the state of the system is completely defined by only one variable. One can choose the number of moles of one reactant or product. It is also common to define the extent $\chi(t)$ such that

$$N_i(t) = N_i(0) + \nu_i \chi(t) \tag{3.209}$$

  so that the equations (3.208) become:

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} = rV \tag{3.210}$$

- Internal energy balance:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = F_q \tag{3.211}$$

  where U is the total internal energy and F_q the total heat flux.

One is interested in calculating the evolution of the number of moles N_i as well as the temperature and the pressure. The question of properties then arises. At first, one has to choose a thermodynamic model for the mixture in order to express U and P as functions of P, T and N_i. To calculate the reaction progress, an expression of the chemical rate has to be known as well as an expression of the heat flux F_q. The model will then take the following form:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N} N_i \bar{u}_i(P,T,\chi_i)\right) &= F_q = \alpha A(T_{ext}-T) \\
\frac{\mathrm{d}\chi}{\mathrm{d}t} &= rV \\
N_i(t) &= N_i(0) + \nu_i \chi(t) \\
\varphi(P,V,T,N_i) &= 0
\end{aligned} \tag{3.212}$$

where $\bar{u}_i$ denotes the component i specific internal energy, α a heat transfer coefficient (per surface unit) and A the surface area for the conduction. The map φ is given as a thermodynamic state equation for the mixture (see previously). For instance, if the gas can be assumed to be an ideal gas mixture, the thermodynamic model is very simple:

$$\begin{aligned}
\bar{u}_i(P,T,x_i) &= \bar{u}_i^{ig}(T) \\
d\bar{u}_i^{ig} &= \bar{c}_{v,i}^{ig}(T)\, dT \\
PV &= \left(\sum_{i=1}^{N} N_i\right) RT
\end{aligned} \tag{3.213}$$

where the superscript ig stands for ideal gas and where $\bar{c}_{v,i}^{ig}(T)$ is the constant volume specific heat capacity for component i. The model can then be expressed according to the time variation of the temperature:

$$\begin{aligned}
\left(\sum_{i=1}^{N} N_i \bar{c}_{v,i}^{ig}(T)\right)\frac{\mathrm{d}T}{\mathrm{d}t} &= \alpha A(T_{ext}-T) - \left(\sum_{i=1}^{N} \nu_i \bar{u}_i^{ig}(T)\right) rV \\
\frac{\mathrm{d}\chi}{\mathrm{d}t} &= rV \\
N_i(t) &= N_i(0) + \nu_i \chi(t) \\
PV &= \left(\sum_{i=1}^{N} N_i\right) RT
\end{aligned} \tag{3.214}$$

where $\Delta_r U := \sum_{i=1}^{N} \nu_i \bar{u}_i^{ig}(T)$ is called the internal energy of reaction.

The Gibbs equation (cf. Sect. 3.4.2.3) applied to the mixture on a molar basis in the case of a constant volume system gives:

$$\mathrm{d}U = T\,\mathrm{d}S + \sum_{i=1}^{N} \bar{\mu}_i \mathrm{d}N_i \tag{3.215}$$

where $\bar{\mu}_i$ is the chemical potential for the component i in the mixture. In order to derive the entropy balance, one combines the internal energy balance (3.211), the Gibbs equation (3.215), the material balances (3.208) and the entropy balance:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \underbrace{\frac{F_q}{T_{ext}}}_{f_{heat}} + \Sigma_s \tag{3.216}$$

where Σ_s denotes the entropy production per volume unit. Doing so, one finds the entropy production:

$$\Sigma_s = F_q \left(\frac{1}{T} - \frac{1}{T_{ext}} \right) - \frac{\sum_{i=1}^{N} \nu_i \bar{\mu}_i}{T} rV \tag{3.217}$$

Two irreversible processes are sources of entropy production:

- The entropy production due to heat transfer:

$$\Sigma_{heat} = F_q \left(\frac{1}{T} - \frac{1}{T_{ext}} \right)$$

- The entropy production due to the chemical reaction:

$$\Sigma_{reac} = -\frac{\sum_{i=1}^{N} \nu_i \bar{\mu}_i}{T} rV$$

The port-based model is then in its final form:

$$\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}t} &= \frac{\alpha A (T_{ext} - T)}{T} - \frac{\sum_{i=1}^{N} \nu_i \bar{\mu}_i}{T} rV \\ \frac{\mathrm{d}\chi}{\mathrm{d}t} &= rV \\ N_i(t) &= N_i(0) + \nu_i \chi(t) \\ \frac{\mathrm{d}U}{\mathrm{d}t} &= T \frac{\mathrm{d}S}{\mathrm{d}t} + \sum_{i=1}^{N} \bar{\mu}_i \frac{\mathrm{d}N_i}{\mathrm{d}t} \end{aligned} \tag{3.218}$$

In order to compute the pressure and the temperature, one has to relate these variables to the entropy and the number of moles which is unusual because according to the Gibbs equation, one should use relations of the following form:

$$\begin{aligned} T &= T(S, V, N_i) \\ P &= P(S, V, N_i) \\ \bar{\mu}_i &= \bar{\mu}_i(S, V, N_i) \end{aligned} \tag{3.219}$$

If such relations were available, the model (3.218) could be expressed in an integral causality form. In fact, the thermodynamic models are generally available in the literature under the following form:

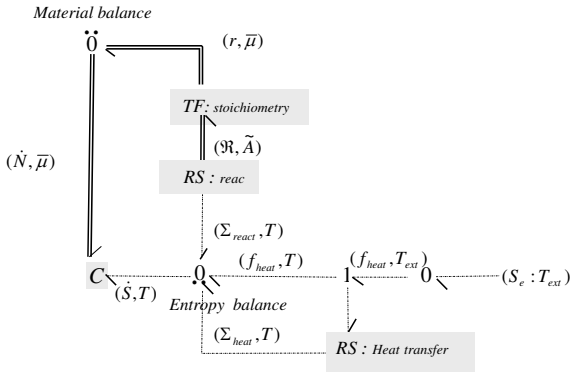

Fig. 3.15 Bond graph representation of the batch reactor.

$$\begin{aligned} S(P,T,N_i) &= \sum_{i=1}^{N} N_i \bar{s}_i(P,T,\chi_i) \\ \bar{\mu}_i &= \bar{\mu}_i(P,T,\chi_i) \\ \varphi(P,V,T,\chi_i) &= 0 \end{aligned} \tag{3.220}$$

where $\bar{s}_i$ denotes the specific molar entropy of component i. For instance, in the case of an ideal gas mixture, this model reduces to

$$\begin{aligned} S^{ig}(P,T,N_i) &= \sum_{i=1}^{N} N_i \bar{s}_i^{ig}(P,T,\chi_i) \\ \bar{\mu}_i(P,T,\chi_i) &= \bar{\mu}_i^{ig*}(P,T) + RT\ln\chi_i \\ PV &= \left(\sum_{i=1}^{N} N_i\right) RT \end{aligned} \tag{3.221}$$

with $\bar{s}_i^{ig}(P,T,x_i) = \bar{s}_i^{ig*}(P,T) - R\ln\chi_i$ and where the exponent * stands for properties of the pure gas.

Finally when dealing with chemical reactions and nonequilibrium thermodynamics, the vector of thermodynamic affinities $\widetilde{A}$ naturally shows up instead of chemical potentials. For thermodynamic equilibrium affinity A is defined as follows: $-\sum_{i=1}^{n} \nu_i \bar{\mu}_i$. In the case of nonequilibrium thermodynamics, in order to obtain structured model we have to split affinity in two components: the first one corresponds to reactants and the other one to products.

The power conjugate flux associated to $\widetilde{A}$ is

$$\Re = \begin{bmatrix} rV \\ -rV \end{bmatrix}$$

The bond graph model of the batch reactor is given Fig. 3.15.

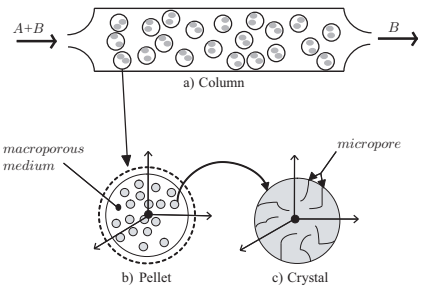

Fig. 3.16 Schematic representation of an adsorption column.

3.4.5.2 Bond graph modelling of an adsorption column

The aim of this example is to present a port-based distributed parameter model of the mass transfer phenomena in an adsorption column. This model is built using Bond Graph language. The main phenomena occurring in the column (dispersion, diffusion) are represented by a dissipative part and an instantaneous power conserving structure named *Stokes-Dirac* structure. This structure is the basis of the infinite dimensional port Hamiltonian models formulation proposed in Chapter 4. These models represent *reversible* systems such as the lossless transmission line, the vibrating string or the eulerian fluid problem. They are hamiltonian with respect to this geometric power conserving structure which is based on Stokes' theorem. This structure then represents a canonical interdomain coupling between two physical domains for reversible systems.

We shall show, in an analogous way, that the port-based model of an adsorption process may be decomposed into a Dirac structure associated with the balance equations and some closure relations applied to some ports of the Dirac structure. These closure equations are the constitutive equations representing storage with the thermodynamical properties or dissipation of energy with the phenomenological model of diffusion. We suggest the reader refer to [12–15, 52, 71] for further details about the port based model.

The adsorption is the phenomenon of deposit of molecules from fluid phase onto solid surface. The adsorption process is based on the ability of a solid to preferentially adsorb constituents present in a fluid phase in order to separate them. This separation is essentially based on the difference of properties that rules the behavior of each constituent in the fluid mixture. For instance, consider a binary gas mixture where one constituent, say A, is adsorbed faster than the other one, say B. Then, when this mixture is supplied at the inlet of the column, component A is adsorbed and the outlet gas is enriched with component B during some transition time.

The central part of a plant associated with the separation process by adsorption is constituted by columns packed with adsorbent pellets, themselves constituted by crystals of solid (Fig. 3.16) (see [96, 172]). In our case, zeolite is used as adsorbent medium, so the description of the mass transfer phenomena may be decomposed considering three scales: namely the column scale, the pellet scale and the crystal scale (respectively called extragranular, macroporous and microporous scale), as it

is represented in Fig. 3.16. This is a *classical* approach in Chemical Engineering, justified by the physical sizes of crystals, pellets and the column. For instance according to [58], the radius of a crystal is of an order of magnitude of 1 μm and the radius of a pellet approximately 0.8 *mm* for 13X CECA zeolite. This difference in the size of crystals, pellets and the column gives rise to several levels of porosity and so different resistance to the mass transfer in the adsorption column. *This is why the adsorption process is a multi scale process*. The complexity of modeling such process is that the mass transfer is modeled by partial differential equations in each level.

As already mentioned in Sect. 3.4.1, the system variables are divided into extensive and intensive variables, depending on whether their values depend on the "size" of the system or not. In analogy with mechanical systems, the thermodynamic force is always an intensive variable and the displacement is always an extensive variable, yielding an extensive energy transfer. The internal energy of a system is then expressed in terms of products of pairs of conjugate variables such as (pressure P, volume V), (temperature T, entropy S) and (chemical potential μ_i, mole number n_i (for species i)). In fact all thermodynamic potentials are expressed in terms of pairings of conjugate variables [53]. In the framework of nonequilibrium thermodynamics, let us consider the simple open one phase system with p species. The internal energy U of this system is a function of the extensive variables V, S and n_i. The coupling of the energy with these extensive variables and the expression of the intensive one can be given by the differential of the fundamental equation also called Gibbs equation (3.136), here reported for clarity:

$$\mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V + \sum_{i=1}^{p} \mu_i \mathrm{d}n_i \tag{3.222}$$

with $T = \frac{\partial U}{\partial S}$, $P = -\frac{\partial U}{\partial V}$ and $\mu_i = \frac{\partial U}{\partial n_i}$. The internal energy corresponds to the total energy of the physical system under consideration and is subject to a conservation law.

When systems are considered at constant pressure and temperature, it is common to deal with the Gibbs free energy $G = U + PV - TS$ which is function of mole numbers, T and P. Since the two variables T and P are constant, only the material domain can be represented. So only the pair of power conjugate variables $\left(\mu_i, \frac{\partial n_i}{\partial t}\right)$ are considered.

In the context of distributed parameter systems with three-dimensional spatial domain, it leads to express the time variation of the Gibbs free energy G over a subvolume Ω of the spatial domain as: $G = \int_\Omega \sum_i \mu_i c_i$ where c_i is the molar density of the species i ($^{\mathrm{mol}}/_{\mathrm{m}^3}$). The variables μ_i and c_i are energy conjugated since their product over the spatial domain yields the energy over the spatial domain.

According to the concepts presented in Chapter 4 (e.g., see Definition 4.3), we have to distinguish between the differential forms of different degrees defined on the spatial domain. The state variables are the molar densities ($^{\mathrm{mol}}/_{\mathrm{m}^3}$). They are 3-forms on the considered spatial domain, as well as the molar density time derivative. Their

evaluation over any sub-volume gives a mole number or $^{mol}/_{s}$. The effort variables, intensive ones, are the chemical potentials ($^{J}/_{mol}$) that are 0-forms. On the other hand, the port variables at the spatial boundary of the domain are the molar flux ($^{mol}/_{m^2 \cdot s}$), which are 2-forms, and the chemical potential. We define the molar flux as a 2-form that can be evaluated on any surface of the boundary of the considered domain and the chemical potential as a function that can be evaluated on any point of this boundary. These two latter variables are power conjugated.

Remark 3.1. In the sequel, for simplicity, we shall consider an isothermal isobaric model of the adsorption column. We shall assume that the mixture injected at the inlet of the column is composed of an inert gas and of one adsorbable gas (penetrating the crystals). The mixture is supposed to behave like an ideal gas in the extragranular and macroporous scales. The well known Langmuir model is used for the adsorption equilibrium. Moreover we shall consider the port-based model in spherical coordinates in the microporous and macroporous scales and we shall use the assumption of spherical symmetry for reducing the spatial domains from $\mathbb{R}^3$ to $\mathbb{R}$. z and x will denote the radial coordinate in the microporous scale and in the macroporous scale respectively. In the extra-granular scale, we shall use cylindrical coordinates (r, θ, l). We shall suppose a symmetry about l and an homogeneity with respect to r. This will reduce the spatial domain to $\mathbb{R}$ in this scale.

It is now necessary to give the constitutive equations representing the dissipative phenomena in each scale of the adsorption column model. These closure equations are presented in appropriate coordinates for each scale. These expressions link the physical molar flux to the driving forces (the gradient of chemical potential). Moreover as we previously noticed, these variables are power conjugate through the integral over the considered volume. But since symmetry assumption are made and the coordinate systems are reduced, they are linked through the integral over the considered interval. The integral over the two other variables are implicitly included in the molar flux. It corresponds to taking as the natural pairing of conjugate variables the 0-form representing the linear molar flux and the gradient of chemical potential that are 1-forms. In a same way, it can be seen that the state variable is then a linear density ($^{mol}/_{m}$) that is a 1-form. In what follows the index i can take two values: 1 will refer to the inert gas and 2 to the adsorbable gas.

In the crystal scale, the assumptions reduce the spatial domain to $Z = [0, R_c] \subset \mathbb{R}$ where R_c is the mean radius of crystals and z is the coordinate. The Maxwell-Stefan's model [201], which expresses the diffusion of p species by setting that the driving force is the chemical potential gradient $\frac{\partial \mu_i^{ads}}{\partial z}$, is used to model the diffusion in the crystal scale. We assume that each molecule which lies in the microporous scale is adsorbed. This means that in the adsorbed phase, there is no possibility of two different molecules undergoing counter-exchange at an adsorption site [201]. In our case, the Maxwell-Stefan's equation is given by:

$$\widehat{N}_2^{ads} = 4\pi z^2 N_2^{ads} = -\frac{\widehat{q_2}^{ads} D_2^s}{RT} \frac{\partial \mu_i^{ads}}{\partial z} \tag{3.223}$$

for the adsorbable species. $\widehat{q_2}^{ads} = 4\pi z^2 q_2^{ads}$ is the linear molar density ($^{mol}/_{m}$) and q_2^{ads} the molar density ($^{mol}/_{m^3}$), respectively. D_2^s is the Maxwell-Stefan diffusivity between the adsorbable species 2 and the solid, R is the ideal gas constant and T the temperature. The constitutive relation (3.223) characterizes the dissipation element which relates the pairing of conjugate variables

$$\left(\widehat{N}_2^{ads},\ \frac{\partial \mu_2^{ads}}{\partial z}\right)$$

as previously defined.

Considering spherical coordinates in the pellet scale and spherical symmetry lead to consider the spatial domain as $X = [0, R_p] \subset \mathbb{R}$ where R_p is the mean radius of pellets and x is the coordinate. We also use Maxwell-Stefan's law for modelling the diffusion in the macroporous scale. In the pellet, we consider only the friction between the gas molecules, the Maxwell-Stefan constitutive relations of diffusion are written as:

$$\begin{aligned} -\frac{\widehat{c}_1^{mac}}{RT}\frac{\partial \mu_1^{mac}}{\partial x} &= \frac{y_2 \hat{N}_1^{mac} - y_1 \hat{N}_2^{mac}}{D_{1,2}} \\ -\frac{\widehat{c}_2^{mac}}{RT}\frac{\partial \mu_2^{mac}}{\partial x} &= \frac{y_1 \hat{N}_2^{mac} - y_2 \hat{N}_1^{mac}}{D_{1,2}} \end{aligned} \tag{3.224}$$

where $y_i = \frac{\widehat{c}_i^{mac}}{\widehat{c}_T^{mac}}$ is the molar fraction of species i in the macroporous scale, $\widehat{c}_T^{mac}$ is the total linear density, $\widehat{N}_i^{mac}$ is the molar flux of species i. $\widehat{c}_i^{mac}$ is the linear molar concentration of species i ($^{mol}/_{m}$) with $\widehat{c}_i^{mac} = 4\pi x^2 c_i^{mac}$. $D_{1,2}$ is the Maxwell-Stefan diffusivity between 1 and 2 ($^{m^2}/_{s}$). This is the *classical* diffusion equation [21, 201]. It is important to notice that, contrary to the case in the microporous medium, this equation does not express explicitly the molar flux as a function of the chemical potential gradient. We obtain an implicit relation to describe the dissipative element.

At the extragranular scale, the spatial domain is reduced to $L = [0, L] \subset \mathbb{R}$ with L the length of the column. l is the coordinate at this scale. The mass transfer phenomenon is slightly different from the two first scales. The mass transfer in this scale is governed by convection and dispersion. The convective flux is given by:

$$\widehat{N}_{i\ conv}^{ext} = \pi R_{cl}^2 c_i^{ext} v = \widehat{c}_i^{ext} v \qquad \text{for } i = 1, 2 \tag{3.225}$$

with v is the mean fluid velocity, c_i^{ext} is the molar density of species i ($^{mol}/_{m^3}$), $\widehat{c}_i^{ext}$ is the linear concentration of species i ($^{mol}/_{m}$) where $\widehat{c}_i^{ext} = \pi R_{cl}^2 c_i^{ext}$ and R_{cl} is the radius of the column section.

The dispersion is due to flow inhomogeneity. It is represented by means of an axial dispersion parameter D_{ax} and its corresponding flux expression is analogous to the Fick's relation [21]. The constitutive relation that gives the dispersive flux as function of the gradient of the chemical potential, at constant temperature and pressure, is given by:

$$\widehat{N}_{i\ disp}^{ext} = -\frac{\widehat{c}_T^{ext} D_{ax}}{RT}\, e^{\frac{\mu_i^{ext} - \mu_i^0(T,P)}{RT}}\, \frac{\partial \mu_i^{ext}}{\partial l} \tag{3.226}$$

where $\mu_i^0(T,P_0)$ is the reference chemical potential and $\widehat{c}_T^{ext}$ the total linear molar density.

To complete the model of adsorption in each scale we shall add the constitutive equations defining the thermodynamic properties of the mixture in each scale. These closure properties define the energy storing elements C. This leads to express intensive variables (the chemical potential at each scale) as functions of the extensive variables.

In the adsorbed phase at microporous scale, we assume that only the component indexed by 2 is diffusing at the micropores. Moreover we use the Langmuir model to describe the adsorption equilibria. We obtain the following closure equation defining the thermodynamic properties of the mixture in the microporous medium.

$$\mu_2^{ads} = \mu_2^0(T,P) + RT \ln\left(\frac{1}{P\,k}\frac{\widehat{q}_2^{ads}}{\widehat{q}_s^{ads} - \widehat{q}_2^{ads}}\right) \tag{3.227}$$

This closure equation expresses the chemical potential μ_2^{ads}, in the adsorbed phase, of the components 2 at some temperature T and pressure P. $\mu_2^0(T,P)$ denotes the chemical potential of pure component 2 at standard state and k is a function of the temperature T and the Langmuir coefficient b given by $k = \frac{b}{RT}$. $\widehat{q}_s^{ads}$ represents the linear saturation concentration.

In the macroporous and the extragranular media the mixture is a gaseous phase assumed to be an ideal gas. The constitutive equations defining the thermodynamical properties in the two scales is the classical expression of the chemical potential for an ideal gas. So in the macroporous medium (respectively in the extragranular) we have:

$$\mu_i^{mac} = \mu_i^0(T,P) + RT \ln\left(\frac{\widehat{c}_i^{mac}}{\widehat{c}_T^{mac}}\right) \qquad \mu_i^{ext} = \mu_i^0(T,P) + RT \ln\left(\frac{\widehat{c}_i^{ext}}{\widehat{c}_T^{ext}}\right) \tag{3.228}$$

In the remaining part of this section, we shall propose the port-based model describing the mass transfer in the three scales identified in the adsorption column previously described. We shall show that the port-based model at each scale of the adsorption process may be decomposed into a Dirac structure associated with the conservation laws and some constitutive equations coupled to some ports of the Dirac structure. Each one of these constitutive equations represents an energetic phenomenon and the *Stokes-Dirac structure* represents the coupling between these energetic phenomena and also with the external environment (the boundaries of each scale). These port-based models shall also be represented in the bond graph language [53, 98] admitting a slight extension as their port variables are now differential forms. Also the interconnections between the microporous-macroporous and the macroporous-extragranular scales are formulated as power conserving interconnection structures.

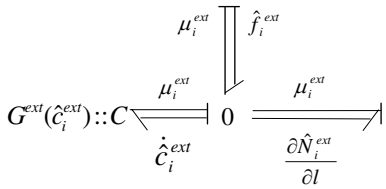

Fig. 3.17 Bond Graph representation of the mass balance in the Extragranular scale.

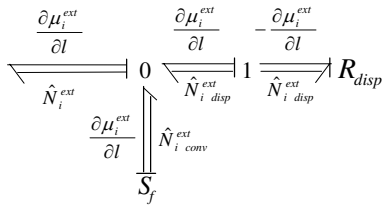

Fig. 3.18 Bond Graph representation of the convective and dispersive fluxes.

In the column scale, the model variables are defined on the spatial domain $L \in \mathbb{R} = [0, L]$. The *dynamic model* in the column scale is then given by the mass balance equation including a distributed source term [21]:

$$\frac{\partial \widehat{c}_i^{ext}}{\partial t} = -\mathrm{div}(\widehat{N}_i^{ext}) + \widehat{f}_i^{ext} \tag{3.229}$$

where $\widehat{N}_i^{ext} = \widehat{N}_{i\ conv}^{ext} + \widehat{N}_{i\ disp}^{ext}$ and $\widehat{f}_i^{ext} = \pi R_{cl}^2 f_i^{ext}$ is the flow of species i per unit of lenght. This is a distributed source accounting for the molar flow coming out of a macroporous medium at a point l of the spatial domain L. In the bond graph, this mass balance is represented by the 0-junction connected to the energy storing element C as shown in Fig. 3.17.

The convection and the dispersion phenomena that generate the flux $\widehat{N}_i^{ext}$ in this scale are represented in the Fig. 3.18. The R_{disp} element represents the dispersion phenomenon with the constitutive relation (3.226).

Let us briefly show how the Stoke-Dirac structure representing the interconnection structure between storage and dissipative part of our subsystems appears in this model. The variation of the total Gibbs energy into the spatial domain is given by:

$$\int_L \mathrm{d}\widehat{g}^{ext} = \int_L \left(\sum_{i=1}^{2} \mathrm{d}\left(\widehat{N}_i^{ext}(t,z) \mu_i^{ext}(t,z) \right) \right)$$

$\widehat{g}^{ext}$, the Gibbs power flux on the boundary of the domain, is a 0-form and $\mathrm{d}\widehat{g}^{ext}$, the linear power density, is a 1-form, on the spatial domain. Using integration by parts this relation leads to the well known Stoke's Theorem:

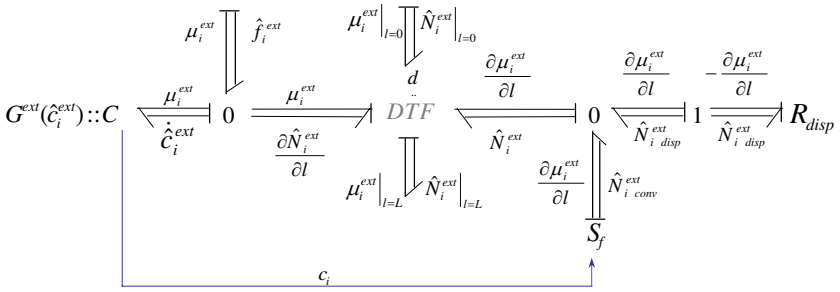

Fig. 3.19 Bond Graph model in the Extragranular scale.

$$\sum_{i=1}^{2}\left(\int_L \mu_i^{ext}(t,z)\mathrm{d}\widehat{N}_i^{ext}(t,z)+\int_L \widehat{N}_i^{ext}(t,z)\mathrm{d}\mu_i^{ext}(t,z)\right)= \\ =\sum_{i=1}^{2}\int_{\partial L}\widehat{N}_i^{ext}(t,z)\mu_i^{ext}(t,z) \quad (3.230)$$

where $\sum_{i=1}^{2}\int_{\partial L}\widehat{N}_i^{ext}(t,z)\mu_i^{ext}(t,z)$ is the total power flux at the boundary ∂L. Now let us define two sets of power conjugate variables,

$$\begin{bmatrix} e_{i1} \\ f_{i1} \end{bmatrix} = \begin{bmatrix} \mu_i^{ext}(t,z) \\ \mathrm{d}\widehat{N}_i^{ext}(t,z) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} e_{i2} \\ f_{i2} \end{bmatrix} = \begin{bmatrix} \mathrm{d}\mu_i^{ext}(t,z) \\ \widehat{N}_i^{ext}(t,z) \end{bmatrix}$$

and the associated boundary conditions

$$\begin{bmatrix} \mu_i^{ext}|_0 \\ -\widehat{N}_i^{ext}|_0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mu_i^{ext}|_L \\ -\widehat{N}_i^{ext}|_L \end{bmatrix}$$

Finally with the constitutive relation for the dispersion (3.226), the Dirac structure is finally obtained:

$$\begin{bmatrix} e_{i2} \\ f_{i1} \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e_{i1} \\ f_{i2} \end{bmatrix}$$

and by the choice of the boundary conditions such that the variation of the internal energy is only due to power flow at the boundary (cf. (3.230)).

The interconnection between the first part, representing the energy storage, and the second part, representing the convection and the dispersion, and also the boundary conditions is represented by the element DTF that symbolizes the Dirac structure. The complete bond graph model is then given in Fig. 3.19. We note that as we consider a fluid moving with a constant velocity, so the Sf represents an energy source coming from another energetic domain.

The mass transfer model in the pellet, macroporous scale, is similar to the model of the adsorption process in the column scale. The variables are defined on the spatial domain $\{l\}\times X=[0,R_p]\subset\mathbb{R}$. This means that the macroporous medium must be indexed by the point of the spatial domain $L=[0,L]\subset\mathbb{R}$. For the sake of clar-

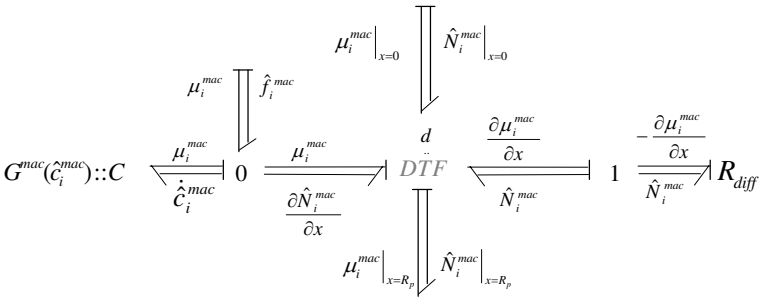

Fig. 3.20 Bond Graph model of the diffusion at the macroporous scale

ity, we will omit this index in the remaining of the section. The *dynamic model* of diffusion process in the pellet is then given by the balance equation:

$$\frac{\partial \widehat{c}_i^{mac}}{\partial t} = -\mathrm{div}(\widehat{N}_i^{mac}) + \widehat{f}_i^{mac} \tag{3.231}$$

where $\widehat{N}_i^{mac}$ is the linear molar flux given by the constitutive equation for diffusion (3.224) and $\widehat{f}_i^{mac}$ is the flow of species i per unit of domain. This term is a distributed source accounting for the molar flow coming out of a microporous medium at a point x in the spatial domain X. The model is given in Fig. 3.20 where the dissipative element R_{diff} is the diffusion model. Its constitutive equation is given in (3.224). The C element represents the storage phenomenon, its constitutive equation is given in the first part of (3.228). The DTF is the Stokes-Dirac structure associated with this scale.

The model at the crystal microporous scale, is similar to the two previous scales. The variables are defined on the spatial domain $\{l\} \times \{x\} \times Z \subset L \times X \times Z$. This means that the microporous medium is indexed by the point of the macroporous spatial domain X which it self indexed by a point in the spatial domain L. For simplicity these indexes will be omitted. The *dynamic model* in the microporous medium is then simply given by the mass balance equation:

$$\frac{\partial \widehat{q}_2^{ads}}{\partial t} = -\mathrm{div}(\widehat{N}_2^{ads}) \tag{3.232}$$

and the bond graph model is represented in Fig. 3.21, where the R_{ads} element is the diffusion model that represents the dissipative phenomenon. Its constitutive equation is given in (3.223). The C element represents the storage phenomenon, its constitutive equation is given in the second part of (3.228). The DTF is the Stokes-Dirac structure related with this scale.

Before illustrating the interconnection structures between the three scales, we start presenting the coupling between the macroporous and the extragranular scales. The hypothesis of separation of the two scales amounts to the following assumptions: we assume that in a slice of fluid there is a sufficient number of pellets of

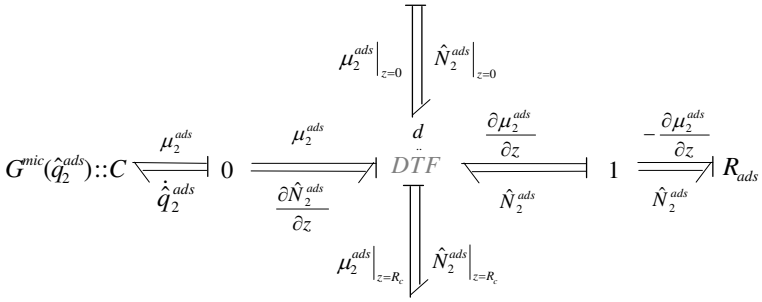

Fig. 3.21 Bond Graph representation of the adsorption process at the microporous scale

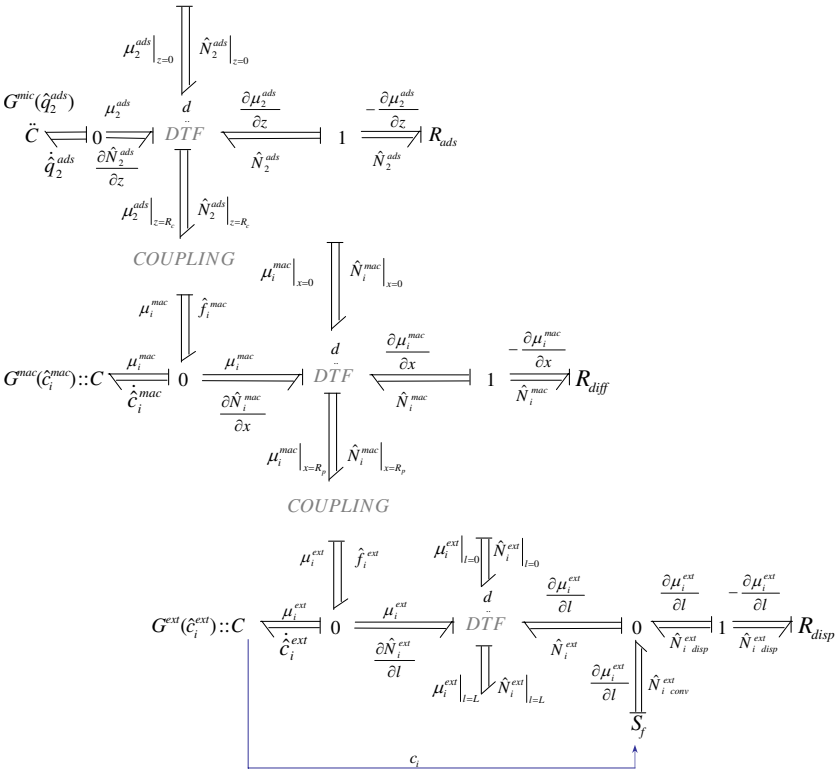

Fig. 3.22 The complete Bond Graph model of the adsorption column.

much smaller size so that a pellet is abstracted to a point. The *concentration* of pellets in the extragranular fluid is denoted by $c_{pellet}(l)$ where l is in L the spatial domain of the extragranular scale. At a point $l_0 \in L$, is attached a spatial domain isomorphic to some domain X and indexed by l_0. Thus the domain of the set of pellets in the fluid is $L \times X$.

Furthermore we use two assumptions, to couple the macroporous and the extragranular scales by relating firstly the intensive variables consisting of the chemical potential $\mu_i^{mac}(l,x)_{|x=R_p}$ of the macroporous scale restricted to the boundary $x = R_p$ of its domain, and the chemical potential $\mu_i^{ext}(l)$ at the extragranular scale at the point $l \in L$. Secondly a coupling relation is defined on the conjugated extensive variables, the volumetric density flux variable at the extragranular scale $\widehat{f}_i^{ext}(l)$ and the flux variable of macroporous scale $\widehat{N}_i^{mac}(l,x)_{|x=R_p}$ restricted to the boundary of its domain.

The coupling relation between the intensive variables is derived from the assumption of local equilibrium at the interphase between the macroporous and the extragranular fluid. This leads to the equation:

$$\mu_i^{mac}(l,R_p) = \mu_i^{ext}(l) \tag{3.233}$$

The coupling relation between the extensive variables expresses the continuity of molar flux exchanged between between the two scales at the point $l \in L$:

$$\widehat{f}_i^{ext}(l) + \widehat{N}_i^{mac}(l,R_p) \cdot c_{pellet}(l) = 0 \tag{3.234}$$

It can be shown that these relations define an interconnection power continuous structure.

The coupling between the macroporous and microporous scales will be identical as the coupling between the extragranular and the macroporous scales, so we have the following equations that relates the intensive and extensive variables:

$$\mu_i^{ads}(x,R_c) = \mu_i^{mac}(x) \tag{3.235}$$

$$\widehat{f}_i^{mac}(x) + \widehat{N}_i^{ads}(x,R_c) \cdot c_{crystal}(x) = 0 \tag{3.236}$$

where $c_{crystal}(x)$ is the *concentration* of crystals in the pellet.

The complete Bond graph model is represented in Fig. 3.22. In this model are represented the models of the three scales and the interconnection between the scales.

Chapter 4
Infinite-Dimensional Port-Hamiltonian Systems

A. Macchelli, B. M. Maschke

Abstract This chapter presents the formulation of distributed parameter systems in terms of port-Hamiltonian system. In the first part it is shown, for different examples of physical systems defined on one-dimensional spatial domains, how the Dirac structure and the port-Hamiltonian formulation arise from the description of distributed parameter systems as systems of conservation laws. In the second part we consider systems of two conservation laws, describing two physical domains in reversible interaction, and it is shown that they may be formulated as port-Hamiltonian systems defined on a canonical Dirac structure called canonical Stokes-Dirac structure. In the third part, this canonical Stokes-Dirac structure is generalized for the examples of the Timoshenko beam, a nonlinear flexible link, and the ideal compressible fluid in order to encompass geometrically complex configurations and the convection of momentum.

4.1 Modelling origins of boundary port-Hamiltonian systems

The aim of this section is to introduce the main concepts and the origin of boundary port Hamiltonian systems that extend the port Hamiltonian formulation from lumped parameter systems defined in Chapter 2 to distributed parameter systems. Dynamic models of distributed parameter systems are defined by considering not only the time but also the space as independent parameters on which the physical quantities are defined. They allow to model objects such as vibrating strings or plates, transmission lines or electromagnetic fields and mass and heat transfer phenomena in tubular reactors or in the heart of fuel cells. In this section we shall use a formulation of distributed parameter systems in terms of systems of conservation laws as they arise in their mathematical analysis [85, 189] or in terms of systems of balance equations as they arise in models of heat and mass transfer phenomena [21]. We shall show how the port Hamiltonian formulation arises from the combination of systems of conservation laws with the axioms of irreversible thermodynamics [42, 97, 168]. The first subsection recalls briefly the concepts of con-

servation law and the axioms of Thermodynamics on the example of a the heat conduction in a cylindrical rod. The second subsection shows how one may define a canonical Dirac structure for reversible physical systems such as the transmission line, the vibrating string, consisting in two coupled conservation laws by applying an unusual thermodynamic perspective to these systems. The third section shows that Stokes-Dirac structures also arise in dissipative phenomena and how one may define dissipative boundary port Hamiltonian systems on the examples of the heat conduction in a cylindrical rod, the lossy transmission line and the vibrating string with structural damping. Finally it should be mentioned that in this section we shall consider for the sake of simplicity, only 1-dimensional spatial domain, i.e. systems defined on some interval in $\mathbb{R}$ and postpone the general case to the next section.

4.1.1 Conservation law and irreversible thermodynamics

Let us recall briefly on the example of the heat conduction, the main concepts which we shall use to define port Hamiltonian systems for distributed parameter systems. For a detailed lecture on the foundations of irreversible thermodynamics the reader is referred to [42, 60, 168], [77, chap. 6.3] and for application to modelling of heat and mass transfer phenomena to [21].

In this section we shall consider the heat diffusion in some 1-dimensional medium (for instance a rod with cylindrical symmetry) and denote its spatial domain by the interval $Z = [a,b] \subset \mathbb{R}$. The time interval on which the variables of the system are defined is denoted by $I \ni t$. We assume the medium to be undeformable (i.e. its deformations are neglected) and consider only one physical domain, the thermal domain and its dynamics.

The first step consist in writing a *conservation law of the conserved quantity*, here the conservation of the density of internal energy, denoted by $u(t,z)$, a extensive thermodynamic variable of the medium:

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial z} J_Q \tag{4.1}$$

where $J_Q(t,z)$ is the *flux variable*, here the heat flux across the section at z. The heat flux itself arises from the thermodynamic non-equilibrium and is defined by some phenomenological law, for instance defined according to Fourier's law by:

$$J_Q(t,z) = -\lambda(T,z)\,\frac{\partial}{\partial z} T(t,z) \tag{4.2}$$

where $\lambda(T,z)$ denotes the heat conduction coefficient and T denotes the temperature of the medium, the intensive thermodynamic variable of the thermal domain.

Actually the axioms of the Irreversible Thermodynamics near equilibrium, decompose the preceding relation by saying that the flux variable is a function of the *thermodynamic driving force* $F(t,z) = \frac{\partial}{\partial z} T(t,z)$ which characterizes the non-

equilibrium condition. However the conservation law (4.1) and the phenomenological law (4.2) should be completed by a relation between the driving force F and the conserved quantity u; this relation is given by the *thermodynamical properties* of the medium which is characterized by some thermodynamical potential.

The thermodynamical properties are given by Gibbs relation [60, 168] which, under the assumption that there is no exchange of matter and that the volume of the medium is constant, reduces to:

$$\mathrm{d}u = T\,\mathrm{d}s \tag{4.3}$$

where s is the entropy of the medium which is also an extensive variable. Due to the irreversibility of thermodynamic processes, the temperature is strictly positive ($T > 0$) [43] in such a way that one may choose equivalently the internal energy or the entropy as thermodynamical potential [6, 72, 76, 151].

The first choice is to choose the internal energy $u = u(s)$ as thermodynamic potential and in this case Gibbs relation defines the temperature as intensive variable conjugated to the (extensive variable) entropy by: $T = \frac{\mathrm{d}u}{\mathrm{d}s}(s)$. This leads to write a the following entropy balance equation (also called conservation law with source term) which is also called *Jaumann's entropy balance* [21, 94]:

$$\frac{\partial s}{\partial t} = -\frac{1}{T}\,\frac{\partial}{\partial z} J_Q = -\frac{\partial}{\partial z} J_S + \sigma \tag{4.4}$$

where J_S denotes the flux of entropy through the section at the point z:

$$J_S = \frac{1}{T} J_Q = -\lambda(T,z)\,\frac{1}{T}\,\frac{\partial}{\partial z} T(t,z) \tag{4.5}$$

and σ denotes the *irreversible entropy creation* and is given by:

$$\sigma = -\frac{1}{T}\,\frac{\partial T}{\partial z}\, J_S = \frac{\lambda(T)}{T^2}\,\left(\frac{\partial T}{\partial z}\right)^2 \geq 0 \tag{4.6}$$

Finally the flux of entropy may be written as a function characterizing the (irreversible) phenomenon of heat conduction $J_S = \frac{\lambda(t,z)}{T} F$ in terms of the generating force $F(t,z) = -\frac{\partial}{\partial z}\frac{\mathrm{d}u}{\mathrm{d}s}(s)$, which itself depends on the differential of the internal energy function characterizing the thermodynamic properties of the medium.

The second choice is to chose the entropy $s = s(u)$ as thermodynamic potential and in this case Gibbs relation defines the inverse of the temperature as the intensive variable conjugated to the (extensive variable) internal energy by: $\frac{1}{T} = \frac{\mathrm{d}s}{\mathrm{d}u}(u)$. And the heat flux J_Q of the energy balance equation (4.1) may be expressed as function

$$J_Q = \lambda(T,z)\,T^2\,F' \tag{4.7}$$

of the driving force characterizing the heat conduction phenomenon

$$F'\left(\frac{1}{T}\right) = \frac{\partial}{\partial z}\frac{\mathrm{d}s}{\mathrm{d}u}(u) \tag{4.8}$$

which depends on the differential of the entropy function characterizing the thermodynamic properties of the medium.

In conclusion the thermodynamic axioms (near equilibrium) define dynamical systems as conservation laws – such as (4.1) or (4.4) containing a source term – completed with the definition of the flux variable by an irreversible phenomenological law, expressing the flux variable as a function of the generating force which is the spatial derivative of the differential of some thermodynamical potential characterizing the thermodynamic (or equilibrium) properties of the system.

Note 4.1. Let us recall in this note the far more usual expression of heat conduction which however does not retain the structure of a conservation law. One uses then the calorimetric property which, assuming a constant volume, i.e. neglecting the deformation of the medium, reduces to:

$$\mathrm{d}u = c_V(T)\,\mathrm{d}T \tag{4.9}$$

where $c_V(T)$ is the heat capacitance of the medium. By substituting this relation in the conservation law (4.1), one obtains the so-called *heat equation* which is then written as:

$$c_V(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(\lambda(T,z)\,\frac{\partial}{\partial z}T\right) \tag{4.10}$$

which does not retain the structure of a conservation law.

4.1.2 Reversible physical systems of two coupled conservation laws

In order to define a port-Hamiltonian formulation for infinite dimensional systems, we shall apply the thermodynamic analysis to reversible physical systems. This perspective is inspired by the bond graph approach to unify the description of reversible and irreversible physical systems as it has been developed under the name of *thermodynamic* or *generalized bond graphs* [30] and port Hamiltonian systems [144]. In this perspective, electromagnetic or elasto-dynamic systems are considered as two physical domain coupled by a reversible inter-domain coupling. In bond graph terms this coupling is represented for lumped parameter systems by a gyrator (see Chapter 1). For distributed parameter systems, one may define some analogous canonical inter-domain coupling which however does not correspond to a symplectic gyrator but an extension defined in [142]. In this section we shall use the thermodynamic perspective for reversible physical systems, in such a way to make appear a canonical Dirac structure associated with some canonical inter-domain coupling.

4.1.2.1 The lossless transmission line

Consider an ideal lossless transmission line with $Z = [a,b] \subset \mathbb{R}$ and let us write the model using a bond graph approach by using the energy variables, the charge and

magnetic flux density, as state variables. Denote the charge density by $Q(t,z)$ and the magnetic flux by $\varphi(t,z)$ on which one may write the following two conservation laws:

$$\begin{aligned}\frac{\partial Q}{\partial t} &= -\frac{\partial I}{\partial z}\\ \frac{\partial \varphi}{\partial t} &= -\frac{\partial V}{\partial z}\end{aligned} \tag{4.11}$$

Hence the current is the *flux variable* for the electrical domain and the voltage is the *flux variable* for the magnetic domain of the two conservation laws. In this case the flux variables are obvious functions of the co-energy variables which are the derivative of the electro-magnetic energy density:

$$\mathscr{H}(Q,\phi) = \frac{1}{2}\left[\frac{Q^2(t,z)}{C(z)} + \frac{\varphi^2(t,z)}{L(z)}\right]$$

leading to the matricial expression:

$$\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} V \\ I \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathscr{H}}{\partial Q} \\ \frac{\partial \mathscr{H}}{\partial \phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{Q}{C} \\ \frac{\varphi}{L} \end{bmatrix} \tag{4.12}$$

More precisely the flux variable of the electrical domain is identical with the co-energy variable of the magnetic domain and vice versa. This is precisely the canonical inter-domain coupling expressed by the symplectic gyrator and suggested in [30]. This relation is the pendant of the phenomenological law for irreversible systems as in (4.2) or (4.5), for reversible systems. However there is a main difference in the sense that it expresses a canonical coupling by a anti-diagonal matrix which has no parameters and shall play a fundamental role in defining the port Hamiltonian formulation. Finally the electro-magnetic energy plays the role of the thermodynamical potential.

4.1.2.2 Hamiltonian formulation of the transmission line

In a first instance let us use the formulation of the dynamics of the transmission line as a system of two conservation laws (4.11) combined with the definition of the flux variables (4.12) in order to define a Hamiltonian formulation [157]. Therefore we have firstly to recall the definition of the variational derivative of a functional [157, chap. 4].

Definition 4.1. Consider a functional H defined by

$$H[x] = \int_a^b \mathscr{H}\left(z, x, x^{(1)}, ..., x^{(n)}\right) \mathrm{d}z$$

for any smooth real function $x(z)$, $z \in Z$ where the integrand $\mathscr{H}$ is a smooth function of x and its derivatives up to some order n. The *variational derivative* of the functional H is denoted by $\frac{\delta H}{\delta x}$ and is the only function that satisfy for every $\varepsilon \in \mathbb{R}$ and smooth real function $\delta x(z)$, $z \in Z$, such that their derivatives satisfy $\delta x^{(i)}(a) = \delta x^{(i)}(b) = 0$, $i = 0, ..., n$:

$$H[x + \varepsilon \delta x] = H[x] + \varepsilon \int_a^b \frac{\delta H}{\delta x}\, \delta x \, \mathrm{d}z + O(\varepsilon^2) \tag{4.13}$$

In the case when $\mathscr{H}$ does depend only the function x and not its derivatives, then the variational derivative is simply obtained by derivation of the integrand, i.e.:

$$\frac{\delta H}{\delta x} = \frac{\mathrm{d}\mathscr{H}}{\mathrm{d}x}.$$

This is precisely the case of the total electro-magnetic energy of the transmission line:

$$H(Q, \varphi) = \int_a^b \frac{1}{2} \left[\frac{Q^2(t,z)}{C(z)} + \frac{\varphi^2(t,z)}{L(z)} \right] \mathrm{d}z \tag{4.14}$$

The co-energy variables are then interpreted as the variational derivatives of the *total* electromagnetic energy.

In order to use matrix notations, let us denote the vector of energy variables by:

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} Q \\ \phi \end{bmatrix}$$

Combining (4.11) with (4.12), one may write:

$$\frac{\partial \alpha}{\partial t} = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta H}{\delta \alpha_1} \\ \frac{\delta H}{\delta \alpha_2} \end{bmatrix} \tag{4.15}$$

This system is an *infinite-dimensional Hamiltonian system* defined with respect to the matrix differential operator:

$$\mathscr{J} = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{bmatrix} \tag{4.16}$$

and generated by the Hamiltonian function H, [157, chap. 6]. Therefore one has to check that the matrix differential operator satisfies two properties: skew-symmetry and the Jacobi identities (see Sect. 2.7 in Chapter 2, and [157, chap. 6]). Consider to vectors of smooth functions

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \qquad\qquad e' = \begin{bmatrix} e'_1 \\ e'_2 \end{bmatrix}$$

and *assume* that they satisfy: $e(a) = e(b) = e'(a) = e'(b) = 0$. By simple integration by parts, one checks the skew symmetry:

$$\begin{aligned}\int_a^b \left(e^t \mathscr{J} e' + e'^t \mathscr{J} e\right) \mathrm{d}z &= \int_a^b e_1 \left[\left(-\frac{\partial}{\partial z} e_2'\right) + e_2 \left(-\frac{\partial}{\partial z} e_1'\right) + \right. \\ &\qquad \left. + e_1 \left(-\frac{\partial}{\partial z} e_2'\right) + e_2 \left(-\frac{\partial}{\partial z} e_1'\right)\right] \mathrm{d}z \\ &= -\left[e_1 e_2' + e_2 e_1'\right]_a^b \\ &= 0\end{aligned} \tag{4.17}$$

Furthermore $\mathscr{J}$ is a constant coefficient differential operator hence it satisfies the Jacobi identities [157, chap. 6].

In the same way as in the finite-dimensional case, the Hamiltonian structure results in an additional conservation law, namely the *conservation of energy*. Indeed:

$$\frac{\mathrm{d}}{\mathrm{d}t} H = \int_a^b \left[\frac{\delta H}{\delta \alpha_1}, \frac{\delta H}{\delta \alpha_2}\right] \frac{\partial \alpha}{\partial t} \mathrm{d}z = \int_a^b \left[\frac{\delta H}{\delta \alpha_1}, \frac{\delta H}{\delta \alpha_2}\right] \mathscr{J} \begin{bmatrix} \frac{\delta H}{\delta \alpha_1} \\ \frac{\delta H}{\delta \alpha_2} \end{bmatrix} \mathrm{d}z = 0 \tag{4.18}$$

by skew symmetry of the operator $\mathscr{J}$.

Considering the transmission line, this might be easily recovered as the two co-energy variables are the currents and the voltages at both ends of the line. And the assumption of Hamiltonian formulation, more precisely for the skew-symmetry of the differential operator $\mathscr{J}$, is that there are identically zero; hence there might be no energy exchange at the boundaries of the line with the rest of the circuit and the energy of the line is conserved. However this is a very particular situation which is not suitable neither from a modular point of view as for control. This is the major motivation for introducing port variables and extend the Hamiltonian formulation to a boundary port Hamiltonian system.

4.1.2.3 Definition of a Stokes Dirac structure extending the canonical differential operator $\mathscr{J}$

However, for functions (i.e. state variables) that are not zero at the boundary of the spatial domain, the matrix differential operator is no more skew-symmetric and some boundary terms appear according to the second line in (4.17). In terms of physical modelling this implies that the energy is not conserved, but obeys the following *energy balance equation*:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} H &= \frac{\delta H}{\delta \alpha_1}(t,a) \frac{\delta H}{\delta \alpha_2}(t,a) - \frac{\delta H}{\delta \alpha_1}(t,b) \frac{\delta H}{\delta \alpha_2}(t,b) \\ &= V(t,a) I(t,a) - V(t,b) I(t,b)\end{aligned} \tag{4.19}$$

This equation (4.19) just says that the variation of energy is equal to the flow of energy per time unit ingoing the system through the boundary of its spatial domain.

This suggests to introduce the restriction of the co-energy variables, the current $I(t,z)$ and the voltage $V(t,z)$ to the boundary $\partial Z = \{a,b\}$ of the spatial domain

as external variables, that will be called port variables, and extend the Hamiltonian system (4.15) in order to encompass the power flow through the boundary [143,184]. Therefore let us complete the Hamiltonian system (4.15) by the definition of two *port boundary variables* as follows:

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \left.\begin{bmatrix} \frac{\delta H}{\delta \alpha_2} \\ \frac{\delta H}{\delta \alpha_1} \end{bmatrix}\right|_{a,b} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left.\begin{bmatrix} \frac{\delta H}{\delta \alpha_1} \\ \frac{\delta H}{\delta \alpha_2} \end{bmatrix}\right|_{a,b} \tag{4.20}$$

where

$$\left.\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}\right|_{a,b} = \begin{bmatrix} e_1(a) & e_1(b) & e_2(a) & e_2(b) \end{bmatrix}^{\mathrm{T}}$$

and 1 denotes the identity matrix of appropriate dimension. In the sequel we shall show that (4.15) and (4.20) define a port Hamiltonian system defined with respect to a Dirac structure called Stokes-Dirac structure which we shall define now.

Let us define firstly the *space of flow variables* composed of triples of two smooth real function on $Z = [a,b]$ and a boundary function defined on $\delta Z = \{a,b\}$:

$$\mathscr{F} = \left\{ f = \begin{bmatrix} f_1 \\ f_2 \\ f_\partial \end{bmatrix} \in C^\infty([a,b]) \times C^\infty([a,b]) \times \mathbb{R}^{\{a,b\}} \right\} \tag{4.21}$$

and the *space of effort variables*, which is defined in an analogous way:

$$\mathscr{E} = \left\{ e = \begin{bmatrix} e_1 \\ e_2 \\ e_\partial \end{bmatrix} \in C^\infty([a,b]) \times C^\infty([a,b]) \times \mathbb{R}^{\{a,b\}} \right\} \tag{4.22}$$

endowed with the following *non-degenerated bi-linear product or pairing*:

$$\left\langle \begin{bmatrix} e_1 \\ e_2 \\ e_\partial \end{bmatrix} \mid \begin{bmatrix} f_1 \\ f_2 \\ f_\partial \end{bmatrix} \right\rangle = \int_a^b (e_1 f_1 + e_2 f_2)\,\mathrm{d}z + e_\partial(b) f_\partial(b) - e_\partial(a) f_\partial(a) \tag{4.23}$$

Now one may define a Dirac structure derived from the formulation of the transmission line in the following way, accordingly to the definition of Sect. 2.1.2 in Chapter 2.

Proposition 4.1. *The linear subset $\mathscr{D} \subset \mathscr{F} \times \mathscr{E}$ defined by:*

$$\mathscr{D} = \left\{ \left(\begin{bmatrix} f_1 \\ f_2 \\ f_\partial \end{bmatrix}, \begin{bmatrix} e_1 \\ e_2 \\ e_\partial \end{bmatrix} \right) \in \mathscr{F} \times \mathscr{E} \mid \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \right.$$
$$\left. \text{and } \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}(a,b) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}(a,b) \right\} \tag{4.24}$$

is a Dirac structure with respect to the symmetric pairing:

$$\ll \begin{bmatrix} f \\ e \end{bmatrix}, \begin{bmatrix} f' \\ e' \end{bmatrix} \gg = \langle e \mid f' \rangle + \langle e' \mid f \rangle, \qquad \begin{bmatrix} f \\ e \end{bmatrix}, \begin{bmatrix} f' \\ e' \end{bmatrix} \in \mathscr{F} \times \mathscr{E} \tag{4.25}$$

where $\langle \cdot \mid \cdot \rangle$ pairing is defined in (4.23).

Proof. Let us firstly prove $\mathscr{D} \subset \mathscr{D}^{\perp}$. And consider two pairs of flow and effort variables belonging to the Stokes-Dirac structure $\mathscr{D}$:

$$\begin{bmatrix} f \\ e \end{bmatrix}, \begin{bmatrix} f' \\ e' \end{bmatrix} \in \mathscr{D}.$$

Then:

$$\begin{aligned}
\ll \begin{bmatrix} f \\ e \end{bmatrix}, \begin{bmatrix} f' \\ e' \end{bmatrix} \gg = & \int_a^b \left(e_1 \, f_1' + e_2 \, f_2' + e_1' \, f_1 + e_2' \, f_2 \right) \mathrm{d}z + \\
& + e_\partial(b) \, f_\partial'(b) - e_\partial(a) \, f_\partial'(a) + e_\partial'(b) \, f_\partial(b) - e_\partial'(a) \, f_\partial(a) \\
= & - \int_a^b \left(e_1 \, \frac{\partial e_2'}{\partial z} + e_2 \, \frac{\partial e_1'}{\partial z} + e_1' \, \frac{\partial e_2}{\partial z} + e_2' \, \frac{\partial e_1}{\partial z} \right) \mathrm{d}z + \\
& + e_1(b) \, e_2'(b) - e_1(a) \, e_2'(a) + e_1'(b) \, e_2(b) - e_1'(a) \, e_2(a) \\
= & - \int_a^b \left(\frac{\partial}{\partial z} \left(e_1 \, e_2' \right) + \frac{\partial}{\partial z} \left(e_2 \, e_1' \right) \right) \mathrm{d}z + \\
& + e_1(b) \, e_2'(b) - e_1(a) \, e_2'(a) + e_1'(b) \, e_2(b) - e_1'(a) \, e_2(a) \\
= & \, 0
\end{aligned}$$

Secondly let us prove the converse, i.e. $\mathscr{D}^{\perp} \subset \mathscr{D}$. Consider a pair of flow and effort variables $(f, e) \in \mathscr{F} \times \mathscr{E}$ such that for any element of the Stokes-Dirac structure $(f', e') \in \mathscr{D}$, the symmetric pairing is zero, i.e.:

$$\ll \begin{bmatrix} f \\ e \end{bmatrix}, \begin{bmatrix} f' \\ e' \end{bmatrix} \gg = 0$$

Observe that in the definition of the Stokes-Dirac structure, the choice of the effort variables in the domain, i.e. $(e_1, e_2) \in C^\infty([a,b]) \times C^\infty([a,b])$ is completely free. Therefore choose for the elements of $\mathscr{D}$, in a first instance for the effort e_2', the null function $e_2'(z) = 0$, $z \in Z$ and for the effort variable e_1' a non null function which vanishes at the boundary of the spatial domain $\delta Z = \{a, b\}$: $e_1'(a) = e_1'(b) = 0$. Then the co-isotropy condition becomes:

$$
\begin{aligned}
0 = &\ll \begin{bmatrix} f \\ e \end{bmatrix}, \begin{bmatrix} f' \\ e' \end{bmatrix} \gg \\
&= \int_a^b \Big(e_1\, f_1' + e_2\, f_2' + e_1'\, f_1 + e_2'\, f_2 \Big) \mathrm{d}z + \\
&\quad + e_\partial(b)\, e_2'(b) - e_\partial(a)\, e_2'(a) + e_1'(b)\, f_\partial(b) - e_1'(a)\, f_\partial(a) \\
&= -\int_a^b \left(e_2\, \frac{\partial e_1'}{\partial z} + e_1'\, f_1 \right) \mathrm{d}z \\
&= \int_a^b e_1' \left(\frac{\partial e_2}{\partial z} - f_1 \right) \mathrm{d}z - \Big[e_1'\, e_2 \Big]_a^b \\
&= \int_a^b e_1' \left(\frac{\partial e_2}{\partial z} - f_1 \right) \mathrm{d}z
\end{aligned}
$$

which is satisfied for any smooth function on Z and vanishing at $\delta Z = \{a,b\}$. Hence $\frac{\partial e_2}{\partial z} - f_1 = 0$. By symmetry one obtains that: $\frac{\partial e_1}{\partial z} - f_2 = 0$, hence the element (f,e) satisfies the first relation in (4.24). Now relax the conditions on the boundary and assume that $e_1'(a) = 0$ but $e_1'(b) \neq 0$, then the isotropy condition becomes:

$$
\begin{aligned}
0 = &\ll \begin{bmatrix} f \\ e \end{bmatrix}, \begin{bmatrix} f' \\ e' \end{bmatrix} \gg \\
&= \int_a^b e_1' \left(\frac{\partial e_2}{\partial z} - f_1 \right) \mathrm{d}z - \Big[e_1'\, e_2 \Big]_a^b + e_1'(b)\, f_\partial(b) \\
&= e_1'(b)\, \left(f_\partial(b) - e_2(b) \right)
\end{aligned}
$$

which is satisfied for any real number $e_1'(b)$, hence $f_\partial(b) - e_2(b) = 0$. By symmetry one obtains that the element (f,e) satisfies the second relation in (4.24). Hence:

$$
\begin{bmatrix} f \\ e \end{bmatrix} \in \mathscr{D}.
$$

Remark 4.1. The construction of the port variables and the Dirac structure associated with the differential operator $\frac{\partial}{\partial z}$, has been extended to higher order linear skew-symmetric operators in [113, 127].

4.1.2.4 Boundary port Hamiltonian system

Now we may define a port Hamiltonian system with respect to this Dirac structure in the following way, completely analogous to the definition for finite dimensional systems in Sect. 2.2.

Definition 4.2. A *boundary port Hamiltonian system* with state variables

$$
\alpha(t) = \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \end{bmatrix} \in C^\infty([a,b]) \times C^\infty([a,b])
$$

and port variables

$$\begin{bmatrix} f_\delta(t) \\ e_\delta(t) \end{bmatrix} \in \mathbb{R}^{\{a,b\}} \times \mathbb{R}^{\{a,b\}}$$

generated by the Hamiltonian functional

$$H[\alpha] = \int_a^b \mathscr{H}(z, \alpha)\, \mathrm{d}z$$

with $\mathscr{H}$ a smooth function, with respect to the Stokes-Dirac structure (4.24) is defined by:

$$\left(\begin{bmatrix} \frac{\partial \alpha_1(t)}{\partial t} \\ \frac{\partial \alpha_2(t)}{\partial t} \\ f_\partial \end{bmatrix}, \begin{bmatrix} \frac{\delta H}{\delta \alpha_1} \\ \frac{\delta H}{\delta \alpha_2} \\ e_\partial \end{bmatrix} \right) \in \mathscr{D} \tag{4.26}$$

Coming back to the motivating example, the transmission line, identifying

$$\alpha(t) = \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \end{bmatrix} = \begin{bmatrix} Q \\ \phi \end{bmatrix}$$

then the two conservation laws (4.11) together with the definition of the co-energy variables appearing in (4.12) and of the port variables (4.20), maybe formulated as a boundary port Hamiltonian system generated by the total electro-magnetic energy (4.14) with respect to the Stokes-Dirac structure (4.24). It may be also noticed that the pairing (4.23) consists in two integral terms corresponding to the electric and the magnetic power in the spatial domain Z plus two terms corresponding to the electromagnetic power at both boundary points of the domain.

4.1.2.5 The vibrating string as a boundary port Hamiltonian systems

In this paragraph shall formulate the vibrating string as a port Hamiltonian system by formulating it firstly as a system of two conservation laws, considering it in some sense as the mechanical analogue of the transmission line. Consider a vibrating string defined on some real interval $Z = [a,b] \subset \mathbb{R}$ and denote its displacement by $u(t,z)$ and the velocity by $v(t,z) = \frac{\partial}{\partial t} u(t,z)$. In the same manner as for the transmission line, we shall use as state variables the *strain* $\alpha_1 = \varepsilon = \frac{\partial u}{\partial z}$ and the *momentum* $\alpha_2 = p = \mu v$ where μ denotes the mass density. Denote the state vector by:

$$\alpha(z,t) = \begin{bmatrix} p \\ \varepsilon \end{bmatrix} \tag{4.27}$$

In these variable the total energy, i.e. the sum of the elastic and the kinetic energies, is written:

$$H_0(\alpha) = \int_a^b \frac{1}{2} \left(T\, \alpha_1^2 + \frac{1}{\mu} \alpha_2^2 \right) \mathrm{d}z \tag{4.28}$$

Notice that the energy functional depends only on the state variables and *not* their spatial derivatives. Furthermore, the co-energy variables are the *velocity* $\beta_1 = \frac{\delta H_0}{\delta \alpha_1} = T\varepsilon$ and the *stress* $\beta_2 = \frac{\delta H_0}{\delta \alpha_2} = \frac{p}{\mu}$.

The model of the vibrating string may be expressed by the system of two conservation laws:

$$\frac{\partial \alpha}{\partial t} = \begin{bmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{bmatrix} \frac{\delta H_0}{\delta \alpha} \tag{4.29}$$

where the first one corresponds to a kinematic identity and the the second one corresponds to Newton's law. In this case the fluxes are expressed as a function of the generating forces $\frac{\delta H_0}{\delta \alpha}$ by:

$$\beta = \begin{bmatrix} -\frac{\partial H_0}{\partial \varepsilon} \\ -\frac{\partial H_0}{\partial p} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta H_0}{\delta \alpha_1} \\ \frac{\delta H_0}{\delta \alpha_2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} T\varepsilon \\ \frac{p}{\mu} \end{bmatrix} \tag{4.30}$$

It is interesting to notice that the Hamiltonian formulation (4.29), up to a sign, is precisely the one of the transmission line (4.15). The port boundary variables are now equal to the restriction of the flux variables (the co-energy variables) at the boundary of the spatial domain:

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} \beta_2 \\ \beta_1 \end{bmatrix} \Bigg|_{a,b} \tag{4.31}$$

that is the velocity and the stress at the boundary points. It may be shown in precisely the same way as in the proof of Proposition 4.1 that the relations (4.29) and (4.31) define a Dirac structure. In summary, by using as state variables the energy variables of the physical system, that is the variables on which the elementary conservation laws applies and such that the energy is a function of them and not their derivatives, one may express the dynamics of the vibrating string as a port Hamiltonian system defined on a Dirac structure which is canonical in the sense that it express simultaneously, the conservation laws, the inter-domain coupling and the interaction through the boundary of the system, independently of its energy properties.

Note 4.2. It should be noted that usually the model of the vibrating string is expressed in terms of the geometric state variables the displacement u and the velocity v. The time variation of the state may be expressed as a function of the variational derivative of the total energy as previously. Indeed define the total energy as: $H(x) = U(u) + K(v)$ where U denotes the elastic potential energy and K denotes the kinetic energy of the string. The elastic potential energy is a function of the *strain* $\varepsilon(t,z) = \frac{\partial u}{\partial z}$:

$$U(u) = \int_a^b \frac{1}{2} T \left(\frac{\partial u}{\partial z} \right)^2 \mathrm{d}z \tag{4.32}$$

where T denotes the elasticity modulus. The kinetic energy K is the following function of the velocity $v(z,t) = \frac{\partial u}{\partial t}$:

$$K(v) = \int_a^b \frac{1}{2}\,\mu v(z,t)^2\,\mathrm{d}z \tag{4.33}$$

Then the system's dynamics may be expressed as follows:

$$\frac{\partial x}{\partial t} = \begin{bmatrix} 0 & \frac{1}{\mu} \\ -\frac{1}{\mu} & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{bmatrix} \tag{4.34}$$

where according to the definition of the variational derivative given in Definition 4.1 one computes:

$$\frac{\delta H}{\delta u} = \frac{\delta U}{\delta u} = -\frac{\partial}{\partial z}\left(T\,\frac{\partial u}{\partial z}\right) \tag{4.35}$$

which is the elastic force and

$$\frac{\delta H}{\delta v} = \frac{\delta K}{\delta v} = \mu\, v \tag{4.36}$$

which is the momentum. The equation (4.34) defines again a Hamiltonian system with respect to a skew-symmetric operator which is now a skew-symmetric matrix with constant coefficients. This matrix is anti-diagonal and skew-symmetric and resembles to canonical symplectic matrix associated with the expression of a canonical inter-domain coupling between the elastic energy and the kinetic energy. However this matrix depends now on some parameter (the mass density) which actually defines the kinetic energy of the string; this goes against the idea of compositional modeling as the definition of energy and the inter-domain coupling are now coupled with the definition of this matrix. Furthermore the Hamiltonian system (4.34) is *not* expressed as a system of conservation laws contrary to (4.15). Instead of being a simplification of the formulation, this reveals to be a drawback for the case when there is some energy flow through the boundary of the spatial domain. Indeed if there is some energy flow through the boundary, the *variational derivative has to be completed by a boundary term as the Hamiltonian functional depends on the spatial derivative of the state*. Indeed, the variation of the elastic-potential energy becomes (using simply integration by parts):

$$U(u+\varepsilon\eta) = U(u) - \varepsilon\int_a^b \frac{\partial}{\partial z}\left(T\,\frac{\partial u}{\partial z}\right)\eta\,\mathrm{d}z + \varepsilon\left[\eta\,\left(T\,\frac{\partial u}{\partial z}\right)\right]_a^b + O(\varepsilon^2) \tag{4.37}$$

Using (4.34) and (4.37) the energy balance equation becomes:

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}t} &= \int_a^b \left(\frac{\delta H}{\delta u}\frac{\partial u}{\partial t} + \frac{\delta H}{\delta v}\frac{\partial v}{\partial t}\right)\mathrm{d}z + \left[\frac{\partial u}{\partial t}\,\left(T\,\frac{\partial u}{\partial z}\right)\right]_a^b \\ &= \left[v\,\left(T\,\frac{\partial u}{\partial z}\right)\right]_a^b \end{aligned} \tag{4.38}$$

Hence one may introduce the following two boundary variables which are conjugated with respect to the balance equation (4.38):

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} v \\ T\ \frac{\partial u}{\partial z} \end{bmatrix}\Bigg|_{a,b} \tag{4.39}$$

which have a non trivial relation with the variational derivatives (4.35) and (4.36) (i.e. the co-energy variables) depending on the parameter of the energy functional U in (4.32). The structure matrix of the Hamiltonian formulation (4.34) is constant and symplectic; it cannot be augmented in order to account for the boundary energy flows. The power continuity properties (attached to a Dirac structure including boundary variables) are not visible in this formulation, contrary to the port Hamiltonian formulation.

4.1.3 Dirac structures underlying dissipative physical systems

In this section we shall show that Dirac structures not only arise in the frame of the Hamiltonian formulation of reversible systems but also in models of dissipative systems of conservation laws. This leads to the definition of dissipative port boundary Hamiltonian systems. As motivating examples, we shall first consider a single conservation law expressing a purely irreversible process: the heat conduction. Then we shall consider again a single conservation law which represents a fluid flow and which may be formulated as a dissipative port boundary Hamiltonian system.

4.1.3.1 Heat conduction in a cylindrical rod

Consider again the heat diffusion in some 1-dimensional non-deformable medium presented in Sect. 4.1.1 choose as state variable the internal energy u and as generating function, the entropy function $s(u)$, hence the entropy conjugated variable is $T = \frac{\mathrm{d}u}{\mathrm{d}s}(s)$. In this case the dynamical model consists in the energy balance equation (4.1), the law of fluxes (4.7) expressing the flux J_Q as a linear function of the driving force defined in (4.8). Now group the energy balance equation (4.1) and the definition of the driving force (4.8) in a single relation:

$$\begin{bmatrix} \frac{\partial u}{\partial t} \\ -F' \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}s}{\mathrm{d}u} \\ J_Q \end{bmatrix} \tag{4.40}$$

One recognizes the canonical differential operator $\mathscr{J}$ defined in (4.16). Hence one may consider the Stokes-Dirac structure extending this operator and defined in Proposition 4.1 with port boundary variables:

$$\begin{bmatrix} f_\delta \\ e_\delta \end{bmatrix}(a,b) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}s}{\mathrm{d}u} \\ J_Q \end{bmatrix}(a,b) \tag{4.41}$$

which are the reciprocal of the temperature $\frac{\mathrm{d}s}{\mathrm{d}u}(u) = \frac{1}{T}$ and the heat flow J_Q at the boundaries. The relations (4.40) and (4.41) defining the Stokes-Dirac structure may be completed with the irreversible law of fluxes (4.7) (i.e. $J_Q = \lambda(T,z)\,T^2\,F'$) and one obtains the complete dynamical system describing the heat conduction in the rod.

In summary, considering the energy balance equation and the definition of the driving force as two adjoint relations, one reveals the Stokes-Dirac structure also for the purely irreversible phenomenon of heat conduction. However the partial differential equation obtained by eliminating the heat flux J_Q and the generating force F' dynamical system:

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial z}\left[\lambda(T,z)\,T^2\left(-\frac{\partial}{\partial z}\frac{\mathrm{d}s}{\mathrm{d}u}\right)\right]$$

which is from parabolic type rather then hyperbolic like in for the reversible systems. The system is of course irreversible (and dissipative) as may be seen from the entropy balance equation deduced from the definition of the Stokes-Dirac structure:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \int_a^b J_Q F'\,\mathrm{d}z - \left[J_Q\frac{1}{T}\right]_a^b = \underbrace{\int_a^b \lambda\,(T,z)\,T^2\,F'^2\,\mathrm{d}z}_{\sigma\geq 0} - \left[J_Q\frac{1}{T}\right]_a^b$$

with positive entropy creation term σ due to the positivity of the heat conduction coefficient λ. It is interesting to note that, in this formulation the dual product has the physical dimension of a time variation of entropy [76].

4.1.3.2 The lossy transmission line

Consider a transmission line where Ohm' law is now taking into account in the dielectric. In this case the model presented in Sect. 4.1.2.1 becomes, denoting the charge density by $Q(t,z)$ and the magnetic flux by $\varphi(t,z)$:

$$\begin{aligned}\frac{\partial Q}{\partial t} &= -\frac{\partial I}{\partial z}\\ \frac{\partial \varphi}{\partial t} &= -\frac{\partial V}{\partial z} - RI\end{aligned} \tag{4.42}$$

where R is Ohm' constant. By considering distributed port variables denoted by $(f_R(t,z), e_R(t,z))$ and extending the Stokes-Dirac structure of Proposition 4.1 to the following vector space:

$$\mathscr{D}_{ext} = \left\{ \left(\begin{bmatrix} f_1 \\ f_2 \\ f_\delta \\ f_R \end{bmatrix}, \begin{bmatrix} e_1 \\ e_2 \\ e_\delta \\ e_R \end{bmatrix} \right) \in \mathscr{F} \times C^\infty([a,b]) \times \mathscr{E} \times C^\infty([a,b]) \mid \right.$$
$$\begin{bmatrix} f_1 \\ f_2 \\ f_R \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & 0 \\ -\frac{\partial}{\partial z} & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_R \end{bmatrix}$$
$$\left. \text{and } \begin{bmatrix} f_\delta \\ e_\delta \end{bmatrix} = \begin{bmatrix} 0\ 1\ 0 \\ 1\ 0\ 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_R \end{bmatrix} \right|_{a,b} \Bigg\} \quad (4.43)$$

which may be proven [112, 211] to be a Dirac structure with respect to the non-degenerated bi-linear product:

$$\left\langle \begin{bmatrix} f_1 \\ f_2 \\ f_\delta \\ f_R \end{bmatrix} \mid \begin{bmatrix} e_1 \\ e_2 \\ e_\delta \\ e_R \end{bmatrix} \right\rangle_{ext} = \int_a^b (e_1\, f_1 + e_2\, f_2 + f_R\, e_R)\, \mathrm{d}z +$$
$$+ e_\partial(b)\, f_\partial(b) - e_\partial(a)\, f_\partial(a) \quad (4.44)$$

the dynamical system is expressed by the implicit dynamical system with Hamiltonian functional (4.14):

$$\left(\begin{bmatrix} \frac{\partial Q}{\partial t} \\ \frac{\partial \phi}{\partial t} \\ f_\delta \\ f_R \end{bmatrix}, \begin{bmatrix} \frac{\delta H}{\delta Q} \\ \frac{\delta H}{\delta \phi} \\ e_\delta \\ R\, f_R \end{bmatrix} \right) \in \mathscr{D}_{ext}$$

In [112, 211, 212] it is shown how the system may be reduced (by eliminating the variable I_R) to a *dissipative* boundary port Hamiltonian system expressed with respect to a differential operator which is the sum of a (formally) skew-symmetric and symmetric operator appearing in the balance equation (4.42):

$$\begin{bmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & -R \end{bmatrix}$$

and how one may define port boundary variables associated with this operator.

4.1.3.3 Vibrating string with dissipation

Consider again the model of the vibrating string presented in Sect. 4.1.2.5 and consider now that it is subject also to some structural dissipation [211, chap. 6] which is expressed by the force

$$e_R = \frac{\partial}{\partial z}\left[b\left(\frac{\partial}{\partial z}v\right)\right]$$

with $b > 0$. The dynamical equations (4.29) becomes:

$$\frac{\partial}{\partial t}\begin{bmatrix}\varepsilon\\ p\end{bmatrix} = \begin{bmatrix}0 & \frac{\partial}{\partial z}\\ \frac{\partial}{\partial z} & 0\end{bmatrix}\begin{bmatrix}T\,\varepsilon\\ \frac{p}{\mu}\end{bmatrix} + \begin{bmatrix}0\\ \frac{\partial}{\partial z}\left[b\left(\frac{\partial}{\partial z}\frac{p}{\mu}\right)\right]\end{bmatrix} \tag{4.45}$$

And again by introducing the pair of port variables (v_R, F_R) and identifying

$$\frac{\partial}{\partial t}\begin{bmatrix}\varepsilon\\ p\end{bmatrix} = \begin{bmatrix}f_1\\ f_2\end{bmatrix} \qquad\qquad \begin{bmatrix}T\,\varepsilon\\ \frac{p}{\mu}\end{bmatrix} = \begin{bmatrix}\delta_\varepsilon H_0\\ \delta_p H_0\end{bmatrix} = \begin{bmatrix}e_1\\ e_2\end{bmatrix}$$

the dynamical equations (4.45) imply the following relation on the power variables (f_1, f_2), (e_1, e_2) and (f_R, e_R) defined by a skew-symmetric operator extending the canonical operator in (4.45):

$$\begin{bmatrix}f_1\\ f_2\\ f_R\end{bmatrix} = \begin{bmatrix}0 & \frac{\partial}{\partial z} & 1\\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial z}\\ 1 & \frac{\partial}{\partial z} & 0\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_R\end{bmatrix} \tag{4.46}$$

completed with the dissipative relation: $e_R = b\,f_R$.

By considering the following extension of the Stokes-Dirac structure (4.24), associated with the differential operator in (4.46) defined according to [211, chap. 6] with respect tot the pairing (4.44):

$$\mathscr{D}_{ext} = \left\{ \left(\begin{bmatrix}f_1\\ f_2\\ f_\delta\\ f_R\end{bmatrix}, \begin{bmatrix}e_1\\ e_2\\ e_\delta\\ e_R\end{bmatrix} \right) \in \mathscr{F}\times C^\infty([a,b])\times\mathscr{E}\times C^\infty([a,b]) \;\middle|\; \begin{bmatrix}f_1\\ f_2\\ f_R\end{bmatrix} = \begin{bmatrix}0 & \frac{\partial}{\partial z} & 1\\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial z}\\ 1 & \frac{\partial}{\partial z} & 0\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_R\end{bmatrix} \text{ and } \begin{bmatrix}f_\delta\\ e_\delta\end{bmatrix} = \begin{bmatrix}0\,1\,0\\ 1\,0\,0\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_R\end{bmatrix}\Bigg|_{a,b} \right\} \tag{4.47}$$

the dynamics of the lossy vibrating string is formulated in terms of a dissipative Hamiltonian system defined with respect to the Dirac structure $\mathscr{D}_{ext}$ and generated by the Hamiltonian (4.28):

$$\left(\begin{bmatrix}\frac{\partial Q}{\partial t}\\ \frac{\partial \phi}{\partial t}\\ f_\delta\\ f_R\end{bmatrix}, \begin{bmatrix}\frac{\delta H_0}{\delta Q}\\ \frac{\delta H_0}{\delta \phi}\\ e_\delta\\ b\,f_R\end{bmatrix} \right) \in \mathscr{D}_{ext}$$

In [112, 211, 212] it is shown how the system may be *reduced* (by eliminating the variable I_R) to a dissipative boundary port Hamiltonian system expressed with

respect to a differential operator which is the sum of a (formally) skew-symmetric and symmetric operator:

$$\begin{bmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & -\frac{\partial}{\partial z} b \frac{\partial}{\partial z} \end{bmatrix}$$

and how one may define port boundary variables associated with this operator.

4.2 Stokes-Dirac structures and distributed port Hamiltonian systems

In the preceding section we have defined infinite-dimensional port Hamiltonian systems which we have also called boundary port Hamiltonian system associated with systems of two conservation laws defined on one-dimensional spatial domains. In this section we shall generalize the definition of Stokes-Dirac structures and boundary port Hamiltonian systems to higher-dimensional spatial domains according to [143, 184]. Therefore we shall consider systems of two conservation laws defined on higher-order spatial domains and formulate them in an coordinate and dimension independent form using so-called differential forms and not the more usual vector calculus formulation. This will allow us to give a dimension-independent definition of Stokes-Dirac structures and boundary port Hamiltonian systems.

4.2.1 Reminder on differential forms

In this paragraph we shall first introduce differential forms on the case of one-dimensional spatial domains and then very briefly recall their definition for a spatial domain of arbitrary dimension. Differential forms are tightly related to integration in $\mathbb{R}^n$ as it appears clearly in their use to express systems of conservation laws [77]. In relation with integration in $\mathbb{R}^n$ the reader is referred to the textbooks [7, 47, 216] or in the context of symplectic geometry to [2, 92, 116].

4.2.1.1 Integration in $\mathbb{R}^n$ and differential forms

Let us now introduce on the 1-dimensional case, the use of differential forms in the formulation of the systems of conservation laws. Until now we have considered the state variables α and the flux variables β as functions on the space-time domain $Z \times I$. However considering the balance equation on some interval $[c, d] \subset Z$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_c^d \alpha \mathrm{d}z = -(\beta(d) - \beta(c))$$

associated with a conservation law where $\alpha$is the conserved quantity (such as the internal energy or the momentum) and β is the flux variable (such as the heat flux or the elastic strain). From the expression of the conservation law, it appears clearly that they are of different nature: the state variables α correspond to a conserved quantity obtained through integration on the spatial domain and the flux variables β correspond to a function evaluated at some point (for instance on the boundary points of the spatial domain). This may be expressed if one represents the different variables as differential forms. Indeed note that for any interval $[c,d] \subset Z$, the conserved quantity

$$\int_c^d \alpha \mathrm{d}z$$

is a real number. In this sense the integration may be seen as a *pairing between 1-dimensional spatial domains and the variable* α and with some defining assumptions, the variable α is then called *differential form of degree* 1. The flux variables β are real functions which are evaluated on points of the spatial domain. This may also be seen as a pairing of points, sets of dimension 0, of the spatial domain Z, with the variable β which is then called *differential form of degree* 0. Now, considering infinitesimal variations of the integration intervals, it may be shown that differential forms of degree 1 may also be defined as elements of the dual space to the vector space of vector fields defined on the spatial domain Z.

In the general case treated in this chapter, the spatial domain is an n-dimensional oriented connected open smooth sub-manifold denoted by $Z \subset \mathbb{R}^n$ of dimension k. In this case one may give analogous definitions of differential forms of degree k both as dual to any open domains of dimension k but also as alternated k-forms on the space of vector fields on Z.

Definition 4.3. For any point $z \in Z$, denote by T_zZ the tangent space of Z at the point z. Denote by A_z^k the vector space of *k-alternated forms* on T_zZ, that is k-linear (i.e. linear with respect to each of its arguments) forms $\omega_z^k : (T_zZ)^k \to \mathbb{R}$ that satisfy for any permutation π in $\{1,..,k\}$ and for any k-tuple $(\xi_1,\ldots,\xi_k) \in (T_zZ)^k$:

$$\omega_z^k\left(\xi_{\pi(1)},\ldots,\xi_{\pi(k)}\right) = \sigma(\pi)\,\omega_z^k(\xi_1,\ldots,\xi_k)$$

where $\sigma(\pi)$ denotes the signature of the permutation π.

Denote by A^k the union: $A^k = \cup_{z\in Z} A_z^k$. A *smooth differential form of degree k*, denoted by ω^k is a smooth section of A^k that is a smooth map: $Z \to A_z^k$. By definition the forms of degree 0 are identified with real functions. The space of forms of degree 1 is the dual $\mathscr{T}(^*M)$ to the space of vector fields $\mathscr{T}(M)$ on Z.

In the sequel we shall smooth differential form of degree k call simply *k-forms* and denote the set of k-forms on Z by $\Omega^k(Z)$. It may also be noted that any k-form defined on a n-dimensional manifold Z is zero for $k > n$ as a consequence that is is an alternated form.

4.2.1.2 Exterior product, exterior derivation and Stokes' theorem

It follows directly from the definition that the set of k-forms $\Omega^k(Z)$ is a real vector space. But it endows far more structure which we briefly recall. Denote by $\Omega = \bigoplus_{k\geq 0}\Omega^k(Z)$ the algebra of differential forms over Z. It is endowed with an exterior product $\wedge$ which endows Ω with a structure of graded algebra and which we shall use later to define a pairing between differential forms in order to extend Stokes-Dirac structures to higher dimensional spatial domains.

Definition 4.4. Consider a k-form $\omega^k \in \Omega^k(Z)$ and a l-form $\omega^l \in \Omega^l(Z)$, their *exterior product* (also called *wedge product* or *Grassman product*) is the $(k+l)$-form $\omega^k \wedge \omega^l \in \Omega^{k+l}(Z)$ such that for any $(k+l)$-tuple $(\xi_1,\ldots,\xi_{k+l}) \in \mathscr{T}(M)^{k+l}$:

$$\left(\omega^k \wedge \omega^l\right)(\xi_1,\ldots,\xi_{k+l}) = \sum_{\pi\in\mathfrak{S}} \sigma(\pi)\,\omega^k\left(\xi_{\pi(1)},\ldots,\xi_{\pi(k)}\right)\omega^l\left(\xi_{\pi(k+1)},\ldots,\xi_{\pi(k+l)}\right)$$

where $\mathfrak{S}$ denotes the sets of shuffle permutation that is the permutation π satisfying $\pi(1) < \ldots < \pi(k)$ and $\pi(k+1) < \ldots < \pi(k+l)$.

The exterior product is an associative and bi-linear product and satisfies Z_2-commutativity, i.e. $\omega^k \wedge \omega^l = (-1)^{kl}\,\omega^l \wedge \omega^k$.

Note 4.3. Using the wedge product, any k-form of an n-dimensional manifold Z may be written as follows using some coordinates. Consider some coordinate system of Z denoted by $\left(z^1,\ldots,z^n\right)$ which may also be seen as coordinate functions. Then the n 1-forms $\left(\mathrm{d}z^1,\ldots,\mathrm{d}z^n\right)$ form a basis of the cotangent bundle $\mathscr{T}Z$, that is of the 1-forms of Z. It may be shown that any k-form $\omega^k \in \Omega^k(Z)$ may be written as a linear combination of monomial (wedge product of k 1-forms), using Einstein's summation rule, as follows:

$$\omega^k = \frac{1}{k!}\,\omega^k_{i_1,\ldots,i_k}\mathrm{d}z^{i_1}\wedge\ldots\wedge\mathrm{d}z^{i_k} = \omega^k_{I_1,\ldots,I_k}\mathrm{d}z^{I_1}\wedge\ldots\wedge\mathrm{d}z^{I_k}$$

where $i_1,\ldots,i_k$ is an indexed set of integers in $\{1,\ldots,n\}$ and the functions $\omega^k_{i_1,\ldots,i_k}$ satisfies the alternation property

$$\omega^k_{i_1,\ldots,i_k}\left(z^1,\ldots,z^n\right) = \sigma(\pi)\;\omega^k_{i_{\pi(1)},\ldots,i_{\pi(k)}}\left(z^1,\ldots,z^n\right)$$

for any permutation in $\{1,\ldots k\}$ whereas $I_1,\ldots,I_k$ is an ordered index set of integers in $\{1,\ldots,n\}$.

As an example consider $Z=\mathbb{R}^3$ with coordinates $\left(z^1,z^2,z^3\right)$: the 0-forms are real functions $f\left(z^1,z^2,z^3\right)$, the 1-forms may be written $\omega^1 = \omega^1_1\,\mathrm{d}z^1 + \omega^1_2\,\mathrm{d}z^2 + \omega^1_3\,\mathrm{d}z^3$, the 2-forms are written: $\omega^2 = \omega^2_{12}\,\mathrm{d}z^1\wedge\mathrm{d}z^2 + \omega^2_{13}\,\mathrm{d}z^1\wedge\mathrm{d}z^3 + \omega^2_{23}\,\mathrm{d}z^2\wedge\mathrm{d}z^3$ and the 3-forms: $\omega^3 = \omega^3_{123}\,\mathrm{d}z^1\wedge\mathrm{d}z^2\wedge\mathrm{d}z^3$.

The algebra of differential forms Ω is also endowed with a derivation which is called exterior derivation, denoted byd and defined as follows.

Definition 4.5. The *exterior derivation* (also called *co-boundary map*) denoted by d is a derivation of degree 1 which maps $\Omega^k(Z)$ into $\Omega^{k+1}(Z)$ and satisfies:

- linearity;
- anti-derivation, i.e. $\mathrm{d}\left(\omega^k \wedge \omega^l\right) = (-1)^k \left(\mathrm{d}\omega^k \wedge \omega^l\right) + \left(\omega^k \wedge \mathrm{d}\omega^l\right)$, with $\omega^k \in \Omega^k(Z)$ and $\omega^l \in \Omega^l(Z)$;
- if ω^0 is a smooth function, $\mathrm{d}\omega^0$ is the differential of the function;
- $\mathrm{d} \circ \mathrm{d} = 0$;
- the derivation is local: for any open set $U \subset Z$, if the restrictions to U of two k-forms coincide then also the restrictions of their exterior derivatives.

In some coordinate system $\left(z^1, \ldots, z^n\right)$ of n-dimensional manifold Z, the exterior derivative of the k-form $\omega^k = \omega^k_{I_1,\ldots,I_k} \mathrm{d}z^{I_1} \wedge \ldots \wedge \mathrm{d}z^{I_k}$ is written:

$$\mathrm{d}\omega^k = \mathrm{d}\omega^k_{I_1,\ldots,I_k} \mathrm{d}z^{I_1} \wedge \ldots \wedge \mathrm{d}z^{I_k}$$

Note 4.4. The exterior derivation has a classical interpretation in terms of vector calculus. Consider that the spatial domain Z is 3-dimensional, then the exterior derivative of a 0-form (or function) corresponds to the gradient operator, the exterior derivative a 1-form to the curl and the exterior derivative of a 2-form corresponds to the divergence operator. Let us recall the coordinate expression of the exterior derivative in the case of the spatial domain $Z = \mathbb{R}^3$ with coordinates $\left(z^1, z^2, z^3\right)$. Then the exterior derivative of any 0-form (or function) $f\left(z^1, z^2, z^3\right)$ is:

$$\mathrm{d}f = \frac{\partial f}{\partial z^1} \mathrm{d}z^1 + \frac{\partial f}{\partial z^2} \mathrm{d}z^2 + \frac{\partial f}{\partial z^3} \mathrm{d}z^3.$$

Then the exterior derivative of any 1-form $\omega^1 = \omega^1_1 \mathrm{d}z^1 + \omega^1_2 \mathrm{d}z^2 + \omega^1_3 \mathrm{d}z^3$ is the 2-form

$$\begin{aligned}\mathrm{d}\omega^1 = &\left(\frac{\partial \omega^1_2}{\partial z^1} - \frac{\partial \omega^1_1}{\partial z^2}\right) \mathrm{d}z^1 \wedge \mathrm{d}z^2 + \left(\frac{\partial \omega^1_3}{\partial z^1} - \frac{\partial \omega^1_3}{\partial z^1}\right) \mathrm{d}z^1 \wedge \mathrm{d}z^3 + \\ &+ \left(\frac{\partial \omega^1_3}{\partial z^2} - \frac{\partial \omega^1_2}{\partial z^3}\right) \mathrm{d}z^2 \wedge \mathrm{d}z^3.\end{aligned}$$

And the exterior derivative of a 2-form $\omega^2 = \omega^2_{12} \mathrm{d}z^1 \wedge \mathrm{d}z^2 + \omega^2_{13} \mathrm{d}z^1 \wedge \mathrm{d}z^3 + \omega^2_{23} \mathrm{d}z^2 \wedge \mathrm{d}z^3$ is the 3-form:

$$\mathrm{d}\omega^2 = \left(\frac{\partial \omega^2_{12}}{\partial z^3} + \frac{\partial \omega^2_{13}}{\partial z^2} + \frac{\partial \omega^2_{23}}{\partial z^1}\right) \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3.$$

Finally we shall recall Stokes' theorem which is fundamental for the definition of the port variables defined on the boundary and which gave the name to the Dirac structure underlying systems of conservation laws with energy flux at the boundary.

Theorem 4.1. *Consider a spatial domain $Z \in \mathbb{R}^n$ being an k-dimensional smooth manifold with smooth $(k-1)$-dimensional boundary ∂Z. Then for any $(k-1)$-form*

ω^{k-1} with compact support in $\mathbb{R}^n$, one has:

$$\int_Z d\omega^{k-1} = \int_{\partial Z} \omega^{k-1} \tag{4.48}$$

4.2.1.3 Duality and variational derivative

Let us first define conjugated spaces on which one may define pairs of conjugated variables composing the bond space with the aim to define Dirac structures. Consider firstly the two (linear) spaces of differential forms defined on the n-dimensional domain Z, $\Omega^k(Z)$ and $\Omega^{n-k}(Z)$. They are conjugated in the sense that their degree is complementary with respect to the dimension n of the spatial domain and that there is a natural pairing between $\Omega^k(Z)$ and $\Omega^{n-k}(Z)$ given by

$$\left\langle \omega^{n-k} \mid \omega^k \right\rangle := \int_Z \omega^{n-k} \wedge \omega^k \quad (\in \mathbb{R}) \tag{4.49}$$

with $\omega^k \in \Omega^k(Z)$, $\omega^{n-k} \in \Omega^{n-k}(Z)$. In fact, the pairing (4.49) is *non-degenerate* in the sense that if $\left\langle \omega^{n-k} \mid \omega^k \right\rangle = 0$ for all $\omega^k \in \Omega^k(Z)$, respectively for all $\omega^{n-k} \in \Omega^{n-k}(Z)$, then $\omega^{n-k} = 0$, respectively $\omega^k = 0$.

Similarly one may define a pairing for conjugated pairs of exterior form defined on the boundary ∂Z of the domain Z as follows:

$$\left\langle \omega^{n-k-1} \mid \omega^k \right\rangle := \int_{\partial Z} \omega^{n-k} \wedge \omega^k \quad (\in \mathbb{R}) \tag{4.50}$$

with $\omega^k \in \Omega^k(\partial Z)$, $\omega^{n-k-1} \in \Omega^{n-k}(\partial Z)$.

Secondly, let us adapt the definition of the *variational derivative* of a functional $H(\alpha)$ with respect to the differential form $\alpha \in \Omega^p(Z)$.

Proposition 4.2. *Consider a density $\mathscr{H} : \Omega^p(Z) \times Z \to \Omega^n(Z)$ where $p \in \{1,..,n\}$ and denote by $H := \int_Z \mathscr{H} \in \mathbb{R}$ the associated functional, then one shows that for any $\alpha, \Delta\alpha \in \Omega^p(Z)$ with compact support strictly included in Z and $\varepsilon \in \mathbb{R}$:*

$$H(\alpha + \varepsilon\Delta\alpha) = \int_Z \mathscr{H}(\alpha + \varepsilon\Delta\alpha) = \int_Z \mathscr{H}(\alpha) + \varepsilon \int_Z \left[\frac{\delta H}{\delta \alpha} \wedge \Delta\alpha \right] + O\left(\varepsilon^2\right)$$

for a certain uniquely defined differential form $\frac{\delta H}{\delta \alpha} \in \Omega^{n-p}(Z)$ which is called the variational derivative of H with respect to $\alpha \in \Omega^p(Z)$.

4.2.1.4 Hodge star operator

The Hodge star operator is associated with an inner product on exterior differential forms and is essential in the definition of the Hamiltonian (or energy) functions of many physical systems which admits “quadratic” energy.

Definition 4.6. Assume that the vector space $\Omega^k(Z)$ of k-forms on Z is endowed with an inner product denoted by $\langle \cdot, \cdot \rangle$, then the Hodge star of the k-form ω^k, denoted by $*\omega^k$ is the defined by:

$$\left\langle \alpha, \omega^k \right\rangle = \int_Z \alpha \wedge \left(*\omega^k \right) \quad \forall \alpha \in \Omega^k(Z)$$

It may be noted that very often the inner products for each space of k-forms are founded on a common structure, a Riemannian structure on Z (i.e. the tangent spaces and the space of vector fields are endowed with an inner product). Then the Hodge star of a k-form is entirely defined by considering the following identity:

$$*\omega_z^k(\xi_{k+1}, \ldots, \xi_n) = \omega_z^k(\xi_1, \ldots, \xi_k)$$

for any oriented orthonormal basis $(\xi_1, \ldots, \xi_n)$ of $T_z Z$.

4.2.2 Conservation laws and balance equations expressed using k-forms

4.2.2.1 The example of heat conduction in a 3-dimensional medium

Let us present the most elementary conservation law on the example of the single energy balance equation in a medium subject to heat conduction as in Sect. 4.1.1 but now considering a 3-dimensional connected bounded spatial domain $Z \subset \mathbb{R}^3$ having a 2-dimensional smooth boundary ∂Z.

One writes the conservation of internal energy, hence the state variable is the internal energy density $u \in \Omega^3(Z)$ and the *conservation law* is written:

$$\frac{\partial u}{\partial t} = -\mathrm{d}J_Q \tag{4.51}$$

where the flux variable is simply the heat flux, a 2-form: $J_Q \in \Omega^2(Z)$.

Let us describe now the thermodynamic properties by Gibbs relation, choosing as *thermodynamical potential* the entropy function which is also an extensive variable hence is described by the 3-form: $s = s(u) \in \Omega^3(Z)$ (and assuming that the system is closed and its transformation are isochore). The conjugated extensive variable is the 0-form (or function) denoted by $\beta \in \Omega^0(Z)$ and defined by:

$$\beta = \frac{\delta s}{\delta u}(u) \tag{4.52}$$

which in terms of the temperature is simply: $\beta = \frac{1}{T}$.

The heat conduction phenomenon is described by a linear relation expressing the heat flux in terms of the *driving force* $F' \in \Omega^1(Z)$:

$$F' = -\mathrm{d}\beta \tag{4.53}$$

which is interpreted in vector calculus as the gradient of the reciprocal of the temperature, and expressing the *law of fluxes*, here Fourier's law:

$$J_Q = -\frac{1}{\beta^2}\,\lambda'(\beta) * F'(\beta)$$

with $\lambda'(\beta)$ being a smooth positive real valued function defining the heat conduction coefficient This decomposition of the energy balance equation for the heat conduction according to the axioms of irreversible thermodynamics again may appear *two conjugated relations*, namely the conservation law (4.51) and the definition of the driving force (4.53):

$$f = \mathrm{d}J_Q \qquad\qquad F' = -\mathrm{d}\beta$$

These relations are canonical in the sense that they do not depend on any parameter and involve uniquely the exterior derivation.

Now let us write the *global entropy balance equation*:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_Z s &= \int_Z \frac{\delta s}{\delta u} \wedge \frac{\partial u}{\partial t} = \int_Z \frac{\delta s}{\delta u} \wedge (-\mathrm{d}J_Q)\\ &= \int_Z \left(\mathrm{d}\frac{\delta s}{\delta u}\right) \wedge J_Q - [\beta \wedge J_Q]_{\partial Z}\\ &= \underbrace{\int_Z \frac{1}{\beta^2}\,\lambda'(\beta)\,F'(\beta) \wedge *F'(\beta)}_{\textit{entropy creation term}} - [\beta \wedge J_Q]_{\partial Z}\end{aligned}$$

The first term of the entropy balance equation correspond to a positive entropy creation term, in accordance with the second principle of Thermodynamics and the second term corresponds to the entropy flux outgoing through the boundary of the spatial domain. This leads to define in a natural way a pair of conjugated variables at the boundary as follows:

$$f_\partial = J_Q\,|_{\partial Z} \qquad\qquad e_\partial = \beta\,|_{\partial Z}$$

4.2.2.2 Systems of balance equations

Sect. 3.4 in Chapter 3 gives several examples of systems of conservations laws which consists in balance equations coupled through the definition of the flux variables. For heat and mass transfer systems, the models consist in balance equations on the mass or number of models of the species, the volume (or space balance) , the entropy or energy. For elasto-dynamic systems they consist in momentum balance and kinematic equations which may also be interpreted as balance equations and for electro-magnetic systems the model consists in coupled electrical charge

and magnetic flux balance equations. In general all or parts of these systems may be coupled in order to describe cation or water transfer in electrolytes or reactive flows, rheological fluids. This leads to the following very general definition.

Definition 4.7. A *system of balance equations (or conservation laws with source terms)* with spatial domain $Z \subset \mathbb{R}^n$, is defined firstly by a set of N *conserved quantities* $\alpha_i \in \Omega^{k_i}(Z)$, $i \in \{1,\ldots,N\}$ where $N \in \mathbb{N}, k_i \in \{0,\ldots,n\}$ composing the state space $\mathscr{X} = \bigotimes_{i=1,..,N} \Omega^{k_i}(Z)$. Secondly it is defined by the associated set of *conservation laws*:

$$\frac{\partial \alpha_i}{\partial t} + \mathrm{d}\beta_i = g_i \tag{4.54}$$

where $\beta_i \in \Omega^{k_i-1}(Z)$ denote the set of *fluxes* and $g_i \in \Omega^{k_i}(Z)$ denote the set of *distributed source forms*. And thirdly it is defined by the *closure equations* defining the set of fluxes:

$$\beta_i = J(\alpha_j, z)_{j=1,..,N} \tag{4.55}$$

Such a system satisfies the following *balance equations* (the integral form of the conservation laws):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_Z \alpha_i + \int_{\partial Z} \beta_i = \int_Z g_i \tag{4.56}$$

Remark 4.2. A usual case is when the conserved quantities are 3-forms, that is the balance equation is evaluated on volumes of the 3-dimensional space. Then, in vector calculus notations, the conserved quantities may be identified with vector fields u_i on Z as well as the fluxes, denoted by q_i and the interaction vector fields g_i. And the system of conservation laws takes the more familiar form:

$$\frac{\partial u_i}{\partial t}(z,t) + \mathrm{div}_z q_i = g_i \qquad i = 1,..,n \tag{4.57}$$

However, as said before, systems of conservation laws applies more generally to differential forms of any dimension. Maxwell's equations give a classical example where the conserved quantities are not differential forms of degree 3 but actually 2 [92]: they are the induction fluxes through some surface.

4.2.3 Systems of two conservation laws

In the sequel, as in the case of 1-dimensional spatial domains, we shall consider a particular class of systems of conservation laws where the fluxes, given by the closure equations, describe the canonical interaction of two physical domains (for instance involving kinetic energy and elastic energy in the case of the vibrating string or electric and magnetic energies for electromagnetic fields).

Definition 4.8. *Systems of two conservation laws with canonical inter-domain coupling* are systems of two conservation laws ($N = 2$) with no distributed interaction

forms, index $i \in \{p,q\}$ where p and q are integers satisfying $p+q=n+1$ and the conserved quantities are $\alpha_p \in \Omega^p(Z)$ and $\alpha_q \in \Omega^q(Z)$. The closure equations are generated by *Hamiltonian density function* $\mathscr{H} : \Omega^p(Z) \times \Omega^q(Z) \times Z \to \Omega^n(Z)$ resulting in the total Hamiltonian

$$H := \int_Z \mathscr{H} \in \mathbb{R}$$

and defined by:

$$\begin{bmatrix} \beta_p \\ \beta_q \end{bmatrix} = \varepsilon \begin{bmatrix} 0 & (-1)^r \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta H}{\delta \alpha_p} \\ \frac{\delta H}{\delta \alpha_q} \end{bmatrix} \tag{4.58}$$

where $r = p\,q+1$, $\varepsilon \in \{-1,+1\}$ depending on sign convention on the physical domain.

This system of two conservation laws may be interpreted in the following way. Each conservation law correspond to a physical domain (e.g. electric and magnetic) and is defined with respect to some conserved quantity (e.g. electrical charge and magnetic flux) represented by the differential forms α_p and α_q. One may notice that the degrees of these differential form are not independent and satisfy the complementarity condition: $p+q=n+1$. This is a consequence of the canonical interconnection relation (4.58). It may also be noticed that this relation is the strict analogue of the canonical coupling defined by the symplectic gyrator in the finite-dimensional bond graph language.

Combining the equations (4.54) and (4.58), the system of two conservation laws may be written as follows:

$$\frac{\partial \alpha}{\partial t} = \varepsilon \begin{bmatrix} 0 & (-1)^r \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta H}{\delta \alpha_p} \\ \frac{\delta H}{\delta \alpha_q} \end{bmatrix} \tag{4.59}$$

This system is an infinite-dimensional Hamiltonian system defined with respect to the *matrix differential operator*:

$$\mathscr{J} = \varepsilon \begin{bmatrix} 0 & (-1)^r \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \tag{4.60}$$

and generated by the Hamiltonian function H [157]. But, in order that the system (4.16) really defines a Hamiltonian system, one has to check that the operator $\mathscr{J}$ is a *Hamiltonian operator*, that is it should be skew-symmetric and satisfy the Jacobi identities. Let us first check the skew-symmetry of the operator. And therefore define the vector of *effort variables* associated with some real-valued functions $H(\alpha_i)$ of a k-form $\alpha_i \in \Omega^i(Z), i \in \{p,q\}$ as the variational derivative: $e_i = \delta_{\alpha_i} H \in \Omega^{n-i}(Z)$.

Let us check under which conditions the bracket:

$$\{H^1,H^2\}_{\mathscr{J}} := \left\langle \begin{bmatrix} e_p^1 \\ e_q^1 \end{bmatrix} \mid \mathscr{J} \begin{bmatrix} e_p^2 \\ e_q^2 \end{bmatrix} \right\rangle = \varepsilon \int_Z \left[e_p^1 \wedge (-1)^r \mathrm{d}e_q^2 + e_q^1 \wedge \mathrm{d}e_p^2 \right] \tag{4.61}$$

is skew-symmetric. Therefore let us compute, using the properties of the exterior derivation and Stokes' theorem:

$$\begin{aligned}\{e^1,e^2\}_{\mathscr{J}}+\{e^2,e^1\}_{\mathscr{J}} &= \varepsilon\int_Z \left[e_p^1\wedge(-1)^r \mathrm{d}e_q^2+e_q^2\wedge \mathrm{d}e_p^1\right]+\\ &\quad+\varepsilon\int_Z \left[e_p^2\wedge(-1)^r \mathrm{d}e_q^1+e_q^1\wedge \mathrm{d}e_p^2\right]\\ &= -\varepsilon\int_Z \mathrm{d}\left(e_p^1\wedge e_q^2+e_q^1\wedge e_p^2\right)\\ &= -\varepsilon\int_{\partial Z}\left(e_p^1\wedge e_q^2+e_q^1\wedge e_p^2\right)\end{aligned} \tag{4.62}$$

Hence the matrix differential operator $\mathscr{J}$ is a skew-symmetric operator *for effort variables with domain strictly included in the spatial domain Z*.

However for effort variables that are not zero at the boundary of the spatial domain, the matrix differential operator is no more skew-symmetric and some boundary terms appear. In terms of physical modelling this implies that there is some flow of energy at the boundary of the domain. The Hamiltonian function is not conserved, but obeys the following *balance equation*:

$$\frac{\mathrm{d}}{\mathrm{d}t}H=\varepsilon\int_{\partial Z}\left(\beta_p^1\wedge\beta_q^2+\beta_q^1\wedge\beta_p^2\right) \tag{4.63}$$

If the Hamiltonian represents the energy of the system, then the right hand side of the equation (4.63) represents the flow of energy per time unit ingoing the system through the boundary of its spatial domain.

This suggests, in the same way as for 1-dimensional domains, to introduce some boundary port variables, defined on the boundary ∂Z of the spatial domain. Define the *flow variables* to be the time variation of the state and denote it by:

$$\begin{bmatrix}f_p\\ f_q\end{bmatrix}=\begin{bmatrix}\frac{\partial\alpha_p}{\partial t}\\ \frac{\partial\alpha_q}{\partial t}\end{bmatrix}\in\Omega^p(Z)\times\Omega^q(Z) \tag{4.64}$$

and define the vector of *effort variables* to be the vector of the variational derivative of the Hamiltonian function with respect to the two state variables (the generating forces for a physical system) and denote it by:

$$\begin{bmatrix}e_p\\ e_q\end{bmatrix}=\begin{bmatrix}\frac{\delta H}{\delta\alpha_p}\\ \frac{\delta H}{\delta\alpha_q}\end{bmatrix}\in\Omega^{n-p}(Z)\times\Omega^{n-q}(Z) \tag{4.65}$$

The flow and effort variables are *power conjugated* as their product is the time variation of the Hamiltonian function:

$$\frac{\mathrm{d}H}{\mathrm{d}t}=\int_Z\left(\frac{\delta H}{\delta\alpha_p}\wedge\frac{\partial\alpha_p}{\partial t}+\frac{\delta H}{\delta\alpha_q}\wedge\frac{\partial\alpha_q}{\partial t}\right)=\int_Z\left(e_p\wedge f_p+e_q\wedge f_q\right) \tag{4.66}$$

Using the closure relations (4.58) and the properties of the exterior derivation and Stokes' theorem in the same way as for the derivation of the skew-symmetry property of the differential operator in (4.62), one may write the variation of the Hamiltonian function:

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &= \int_Z [\varepsilon\beta_q \wedge (-\mathrm{d}\beta_p) + (-1)^r \beta_p \wedge \varepsilon(-\mathrm{d}\beta_q)] \\ &= -\varepsilon \int_{\partial Z} \beta_q \wedge \beta_p\end{aligned} \tag{4.67}$$

Now we may define *flow and effort variables on the boundary* of the system as the restriction, denoted by $|_{\partial Z}$, of the flux variables to the boundary ∂Z of the domain Z:

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} \beta_q|_{\partial Z} \\ \beta_p|_{\partial Z} \end{bmatrix} \tag{4.68}$$

They are also power conjugated variables as their product defined in (4.68) is the time variation of the Hamiltonian functional, i.e. the total energy of the physical system.

On the pairs of power conjugated variables, the differential forms (f_p, e_p) and (f_q, e_q) on the domain Z and the differential forms (f_∂, e_∂) defined on the boundary ∂Z, one may define an *interconnection structure*, underlying the system of two conservation laws with canonical inter-domain coupling of Definition 4.8. This interconnection structure is defined by the equation (4.68) and, combining the conservation laws (4.57) with the closure equation (4.58), by:

$$\begin{bmatrix} f_q \\ f_p \end{bmatrix} = \varepsilon \begin{bmatrix} 0 & (-1)^r\,\mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e_q \\ e_p \end{bmatrix} \tag{4.69}$$

This interconnection satisfies the power continuity in the sense that the the power conjugated variables satisfying (4.68) and (4.69), also satisfy the *power continuity relation*:

$$\int_Z (e_p \wedge f_p + e_q \wedge f_q) + \varepsilon \int_{\partial Z} f_\partial \wedge e_\partial = 0 \tag{4.70}$$

This expression is the straightforward consequence of the two expressions of the variation of the Hamiltonian H in (4.66) (4.67).

In the next section 4.2.4 we shall show how this power continuous interconnection structure defines a Dirac structure.

4.2.4 Stokes-Dirac structures

In this section we shall relate the power continuous interconnection structure that we have derived in Sect. 4.2.3 for systems of two conservation laws with canonical inter-domain coupling with a geometric object called *Dirac structure*.

4.2.4.1 Definition

In this subsection we shall show that the interconnection structure defined by (4.68) and (4.69) for the system of two conservation laws with canonical coupling and non-zero energy flow through the boundary, leads to the definition of a Dirac structure called *Stokes-Dirac structure* [140, 141, 184].

Considering the definition, proposed in Sect. 4.2.3, of the variables defining the interconnection structure equations (4.69) and (4.68), we shall consider the linear space of flow variables

$$\mathscr{F}_{p,q} := \Omega^p(Z) \times \Omega^q(Z) \times \Omega^{n-p}(\partial Z) \ni (f_p, f_q, f_\partial) \tag{4.71}$$

for any pair p, q of positive integers satisfying

$$p + q = n + 1, \tag{4.72}$$

and the space of effort variables

$$\mathscr{E}_{p,q} := \Omega^{n-p}(Z) \times \Omega^{n-q}(Z) \times \Omega^{n-q}(\partial Z) \ni (e_p, e_q, e_\partial) \tag{4.73}$$

Then the pairings (4.49) and (4.50) yields a (non-degenerate) pairing between the space of flow variables $\mathscr{F}_{p,q}$ and the space of effort variables $\mathscr{E}_{p,q}$. Note that by (4.72), $(n-p)+(n-q) = n-1$. In order to define the Dirac structure, we shall use the symmetrization of this pairing with values in $\mathbb{R}$, on the product space $\mathscr{B}_{p,q} = \mathscr{F}_{p,q} \times \mathscr{E}_{p,q}$, also called *bond space*:

$$\begin{aligned} &\ll \left(f_p^1, f_q^1, f_b^1, e_p^1, e_q^1, e_b^1\right), \left(f_p^2, f_q^2, f_b^2, e_p^2, e_q^2, e_b^2\right) \gg := \\ &:= \int_Z \left[e_p^1 \wedge f_p^2 + e_q^1 \wedge f_q^2 + e_p^2 \wedge f_p^1 + e_q^2 \wedge f_q^1\right] + \int_{\partial Z} \left[e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1\right] \end{aligned} \tag{4.74}$$

where for $i = 1, 2$

$$\begin{aligned} f_p^i &\in \Omega^p(Z) & f_q^i &\in \Omega^q(Z) \\ e_p^i &\in \Omega^{n-p}(Z) & e_p^i &\in \Omega^{n-q}(Z) \\ f_b^i &\in \Omega^{n-p}(\partial Z) & e_b^i &\in \Omega^{n-q}(\partial Z) \end{aligned} \tag{4.75}$$

The next theorem says now that the interconnection relations (4.60) and (4.68) define a Dirac structure on the bond space $\mathscr{B}_{p,q}$.

Proposition 4.3. *Consider the space of flow variables $\mathscr{F}_{p,q}$ and the space of effort variables $\mathscr{E}_{p,q}$ given in (4.71) and (4.73) with the integers p and q satisfying (4.72) and the pairing defined in (4.74). Define the following linear subspace $\mathscr{D}$ of the bond space $\mathscr{B}_{p,q} = \mathscr{F}_{p,q} \times \mathscr{E}_{p,q}$:*

$$\mathscr{D} = \left\{ (f_p, f_q, f_b, e_p, e_q, e_b) \in \mathscr{F}_{p,q} \times \mathscr{E}_{p,q} \mid \right.$$
$$\left. \begin{bmatrix} f_p \\ f_q \end{bmatrix} = \varepsilon \begin{bmatrix} 0 & (-1)^r d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \ \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} e_{p|\partial Z} \\ e_{q|\partial Z} \end{bmatrix} \right\} \quad (4.76)$$

where $r := pq+1$. Then $\mathscr{D} = \mathscr{D}^\perp$ that is $\mathscr{D}$ is a Dirac structure.

Proof. Let us first discuss the sign convention and reduce the proof to the case of $\varepsilon = 1$ for the sake of simplifying the notations. Therefore, one may notice that the transformation:

$$(f_p, f_q, f_b, e_p, e_q, e_b) \mapsto (\varepsilon f_p, \varepsilon f_q, \varepsilon f_b, e_p, e_q, e_b) \quad (4.77)$$

leaves invariant the equation:

$$\ll \left(f_p^1, f_q^1, f_b^1, e_p^1, e_q^1, e_b^1\right), \left(f_p^2, f_q^2, f_b^2, e_p^2, e_q^2, e_b^2\right) \gg = 0$$

which characterizes the conditions of isotropy and co-isotropy of the Dirac structures. Hence the proof is then given in two parts, setting $\varepsilon = 1$. Firstly we show the isotropy: $\mathscr{D} \subset \mathscr{D}^\perp$, and secondly the co-isotropy: $\mathscr{D}^\perp \subset \mathscr{D}$.

- $\mathscr{D} \subset \mathscr{D}^\perp$: let $(f_p^1, f_q^1, f_b^1, e_p^1, e_q^1, e_b^1) \in \mathscr{D}$, and consider any $(f_p^2, f_q^2, f_b^2, e_p^2, e_q^2, e_b^2) \in \mathscr{D}$. By substitution of (4.76) into (4.74) the right-hand side of (4.74) becomes

$$\int_Z \left[(-1)^r e_p^1 \wedge \mathrm{d}e_q^2 + e_q^1 \wedge \mathrm{d}e_p^2 + (-1)^r e_p^2 \wedge \mathrm{d}e_q^1 + e_q^2 \wedge \mathrm{d}e_p^1\right] - \\ - (-1)^{n-q} \int_{\partial Z} \left[e_q^1 \wedge e_p^2 + e_q^2 \wedge e_p^1\right] \quad (4.78)$$

By the properties of the exterior derivative (see Sect. 4.2.1.2):

$$\begin{aligned} \mathrm{d}(e_q^2 \wedge e_p^1) &= \mathrm{d}e_q^2 \wedge e_p^1 + (-1)^{n-q} e_q^2 \wedge \mathrm{d}e_p^1 \\ \mathrm{d}(e_q^1 \wedge e_p^2) &= \mathrm{d}e_q^1 \wedge e_p^2 + (-1)^{n-q} e_q^1 \wedge \mathrm{d}e_p^2 \end{aligned} \quad (4.79)$$

and by the properties of the wedge product:

$$\begin{aligned} e_p^1 \wedge \mathrm{d}e_q^2 &= (-1)^{(n-p)(n-q+1)} \mathrm{d}e_q^2 \wedge e_p^1 \\ e_p^2 \wedge \mathrm{d}e_q^1 &= (-1)^{(n-p)(n-q+1)} \mathrm{d}e_q^1 \wedge e_p^2 \end{aligned} \quad (4.80)$$

Hence the first and fourth term in the $\int_Z$ integral in (4.78) can be rewritten as

$$\begin{aligned} (-1)^r e_p^1 \wedge \mathrm{d}e_q^2 + e_q^2 \wedge \mathrm{d}e_p^1 &= (-1)^{r+(n-p)(n-q+1)} \mathrm{d}e_q^2 \wedge e_p^1 + e_q^2 \wedge \mathrm{d}e_p^1 \\ &= (-1)^{n-q} \mathrm{d}e_q^2 \wedge e_p^1 + e_q^2 \wedge \mathrm{d}e_p^1 \\ &= (-1)^{n-q} \mathrm{d}(e_q^2 \wedge e_p^1) \end{aligned} \quad (4.81)$$

since by $p+q=n+1$ and $r=pq+1$, $r+(n-p)(n-q+1)=r+(q-1)p=2pq-p+1$ and $(-1)^{2pq-p+1}=(-1)^{1-p}=(-1)^{n-q}$. Similarly, the second term together with third term can be written as

$$e_q^1\wedge \mathrm{d}e_p^2+(-1)^r e_p^2\wedge \mathrm{d}e_q^1=(-1)^{n-q}\mathrm{d}(e_q^1\wedge e_p^2) \tag{4.82}$$

Substitution of (4.81) (4.82) in the $\int_Z$ integral in (4.78) then yields by Stokes' theorem that this integral is equal to

$$(-1)^{n-q}\int_Z \mathrm{d}(e_q^2\wedge e_p^1)+\mathrm{d}(e_q^1\wedge e_p^2)=(-1)^{n-q}\int_{\partial Z}\left[e_q^1\wedge e_p^2+e_q^2\wedge e_p^1\right], \tag{4.83}$$

showing that (4.78) is zero, and thus $\mathscr{D}\subset\mathscr{D}^\perp$.

- $\mathscr{D}^\perp\subset\mathscr{D}$: let $(f_p^1,f_q^1,f_b^1,e_p^1,e_q^1,e_b^1)\in\mathscr{D}^\perp$, which implies that for all the elements $(f_p^2,f_q^2,f_b^2,e_p^2,e_q^2,e_b^2)\in\mathscr{D}$ the right-hand side of (4.74) is zero, and hence by substitution of (4.76)

$$\begin{aligned}\int_Z\left[(-1)^r e_p^1\wedge \mathrm{d}e_q^2+e_q^1\wedge \mathrm{d}e_p^2+e_p^2\wedge f_p^1+e_q^2\wedge f_q^1\right]+\\ +\int_{\partial Z}\left[e_b^1\wedge e_p^2-(-1)^{n-q}e_q^2\wedge f_b^1\right]=0\end{aligned} \tag{4.84}$$

for all e_p^2 and e_q^2. Now, consider first e_p^2 and e_q^2 which are zero on the boundary ∂Z, implying that

$$\int_Z\left[(-1)^r e_p^1\wedge \mathrm{d}e_q^2+e_q^1\wedge \mathrm{d}e_p^2+e_p^2\wedge f_p^1+e_q^2\wedge f_q^1\right]=0 \tag{4.85}$$

for all e_p^2 and e_q^2 with $e_p^2|_{\partial Z}=e_q^2|_{\partial Z}=0$. By the first line of (4.79) and (4.80)

$$\begin{aligned}(-1)^r e_p^1\wedge \mathrm{d}e_q^2&=(-1)^{r+(n-p)(n-q+1)}\mathrm{d}e_q^2\wedge e_p^1\\ &=(-1)^{n-q}\mathrm{d}e_q^2\wedge e_p^1\\ &=(-1)^{n-q}\mathrm{d}(e_q^2\wedge e_p^1)-e_q^2\wedge \mathrm{d}e_p^1\end{aligned} \tag{4.86}$$

Similarly, by the second line of (4.79), (4.80)

$$\begin{aligned}e_q^1\wedge \mathrm{d}e_p^2&=(-1)^{n-q}\mathrm{d}(e_q^1\wedge e_p^2)-(-1)^{n-q}\mathrm{d}e_q^1\wedge e_p^2\\ e_p^2\wedge f_p^1&=(-1)^{(n-p)p}f_p^1\wedge e_p^2\end{aligned} \tag{4.87}$$

Since $e_p^2|_{\partial Z}=e_q^2|_{\partial Z}=0$, substitution of (4.86), (4.87) into (4.85) then yields by Stokes' theorem

$$\int_Z\left[-e_q^2\wedge \mathrm{d}e_p^1-(-1)^{n-q}\mathrm{d}e_q^1\wedge e_p^2+(-1)^{(n-p)p}f_p^1\wedge e_p^2+e_q^2\wedge f_q^1\right]=0 \tag{4.88}$$

for all e_p^2 and e_q^2 with $e_p^2|_{\partial Z}=e_q^2|_{\partial Z}=0$. Clearly, this implies

$$f_q^1 = \mathrm{d}e_p^1 \qquad (-1)^{(n-p)p} f_p^1 = (-1)^{(n-q)} \mathrm{d}e_q^1 \tag{4.89}$$

where the last equality is easily seen to be equivalent to

$$f_p^1 = (-1)^r \mathrm{d}e_q^1 \tag{4.90}$$

Finally, substitute (4.89), (4.90) into (4.84) to obtain

$$\int^Z \left[(-1)^r e_p^1 \wedge \mathrm{d}e_q^2 + e_q^2 \wedge \mathrm{d}e_p^1 + e_q^1 \wedge \mathrm{d}e_p^2 + (-1)^r e_p^2 \wedge \mathrm{d}e_q^1\right] + \\ + \int_{\partial Z} \left[e_b^1 \wedge e_p^2 - (-1)^{n-q} e_q^2 \wedge f_b^1\right] = 0 \tag{4.91}$$

for all e_p^2, e_q^2. Substituting again (4.86) and the first line of (4.87), noting that $(-1)^{n-q}\mathrm{d}e_q^1 \wedge e_p^2 = (-1)^r e_p^2 \wedge \mathrm{d}e_q^1$, this yields

$$\int_Z \left[(-1)^{n-q}\mathrm{d}(e_q^2 \wedge e_p^1) + (-1)^{n-q}\mathrm{d}(e_q^1 \wedge e_p^2)\right] + \\ + \int_{\partial Z} \left[e_b^1 \wedge e_p^2 - (-1)^{n-q} e_q^2 \wedge f_b^1\right] = 0 \tag{4.92}$$

and hence by Stokes' theorem

$$\int_{\partial Z} \left[(-1)^{n-q} e_q^2 \wedge e_p^1 - (-1)^{n-q} e_q^2 \wedge f_b^1 + (-1)^{n-q} e_q^1 \wedge e_p^2 + e_b^1 \wedge e_p^2\right] = 0 \tag{4.93}$$

for all e_p^2, e_q^2, implying that

$$f_b^1 = e_p^1|_{\partial Z} \qquad e_b^1 = -(-1)^{n-q} e_q^1|_{\partial Z} \tag{4.94}$$

showing that indeed $(f_p^1, f_q^1, f_b^1, e_p^1, e_q^1, e_b^1) \in \mathscr{D}$.

By defining the Stokes-Dirac structure on the bond space $\mathscr{B}_{p,q}$, we have extended the Hamiltonian operator $\mathscr{J}$ in the following sense. The Hamiltonian operator $\mathscr{J}$ defines a Poisson bracket (4.61) and hence a Dirac structure, on the space of efforts with domain being compact and strictly included in the spatial domain of the system Z. By introducing the port variables (4.68) and defining the Stokes-Dirac structure, we have hence extended the definition of Dirac structure to efforts which domain being the complete domain Z, i.e. which may not vanish at its boundary.

Remark 4.3. One may note that the Stokes-Dirac structures may be composed by considering spatial domains with common boundary. Indeed, let Z_1, Z_2 be two n-dimensional manifolds with boundaries $\partial Z_1, \partial Z_2$, such that:

$$\begin{aligned} \partial Z_1 &= \Gamma \cup \Gamma_1, \quad \Gamma \cap \Gamma_1 = \emptyset \\ \partial Z_2 &= \Gamma \cup \Gamma_2, \quad \Gamma \cap \Gamma_2 = \emptyset \end{aligned} \tag{4.95}$$

for certain $(n-1)$-dimensional manifolds Γ, Γ_1 and Γ_2, that is Z_1 and Z_2 have the boundary Γ in common. Then the Stokes-Dirac structures $\mathscr{D}_1$ and $\mathscr{D}_2$ defined with respect to the spatial domains Z_1 and Z_2 according to (4.76) may be composed to the Stokes-Dirac structure associated with the spatial domain $Z_1 \cup Z_2$ using the constraint relations on the port boundary variables $\left(f_b^1, e_b^1\right)$ of $\mathscr{D}_1$ and $\left(f_b^2, e_b^2\right)$ of $\mathscr{D}_2$: $f_1|_\Gamma = -f_b^2|_\Gamma$ and $e_b^1|_\Gamma = e_b^2|_\Gamma$. Note that these relations define themselves also a Dirac structure.

4.2.4.2 Poisson bracket associated with the Stokes-Dirac structure

Although Dirac structures are a generalization of Poisson structure and are not defined by a Poisson bracket, one may also associate a Poisson bracket to any Dirac structure on some set of real valued functions [54, 67]. Therefore in this paragraph, we shall define the Poisson bracket associated with the Stokes-Dirac structure and show how this extends the Poisson bracket associated with the differential operator (4.61).

First, define the space of the *admissible effort variables*:

$$\mathscr{E}_{\text{adm}} = \left\{ e \in \mathscr{E}_{p,q} | \exists f \in \mathscr{F}_{p,q} \text{ such that } (f,e) \in \mathscr{D} \right\} \tag{4.96}$$

On the space of admissible efforts $\mathscr{E}_{\text{adm}}$ one may define the *bi-linear form* such that for any pair $(e_1, e_2) \in \mathscr{E}_{\text{adm}}^2$:

$$[e_1, e_2] := \langle e_1 \mid f_2 \rangle \in \mathbb{R} \tag{4.97}$$

for any $f_2 \in \mathscr{F}_{p,q}$ such that $(f_2, e_2) \in \mathscr{D}$. This bi-linear form is well-defined, since for any other $f_2' \in \mathscr{F}$ such that $(f_2', e_2) \in \mathscr{D}$ we obtain by linearity $(f_2 - f_2', 0) \in \mathscr{D}$, and hence

$$0 = \ll (f_1, e_1), (f_2 - f_2', 0) \gg = \langle e_1 \mid f_2 \rangle - \langle e_1 \mid f_2' \rangle \tag{4.98}$$

Furthermore, the bi-linear product $[,]$ is *skew-symmetric* since if $(f_1, e_1), (f_2, e_2) \in \mathscr{D}$, then

$$0 = \ll (f_1, e_1), (f_2, e_2) \gg = \langle e_1 \mid f_2 \rangle + \langle e_2 \mid f_1 \rangle \tag{4.99}$$

Now, let us define on the space of flow variables $\mathscr{F}_{p,q}$ the set of *admissible mappings*

$$\mathscr{K}_{\text{adm}} = \left\{ K : \mathscr{F}_{p,q} \to \mathbb{R} \mid \delta K \in \mathscr{E}_{\text{adm}} \right\} \tag{4.100}$$

Considering the definition of the Stokes-Dirac structure $\mathscr{D}$ given in Proposition 4.3 and particularly the last line of (4.76), the set of admissible functions K_{adm} may also be written as follows:

$$\mathscr{K}_{\text{adm}} = \left\{ K : \mathscr{F}_{p,q} \to \mathbb{R} \mid \delta_{f_\partial} K = -(-1)^{n-q} \delta_{f_q} K|_{\partial Z} \right\} \tag{4.101}$$

Using the skew-symmetric bracket defined on the admissible effort variables in the equation (4.97), one may define the following bracket on the space of admissible functions $\mathscr{K}_{\text{adm}}$:

$$\{K_1, K_2\}_{\mathscr{D}} := [\delta K_1, \delta K_2] \qquad K_1, K_2 \in K_{\text{adm}} \tag{4.102}$$

By skew-symmetry of $[\,,\,]$, it immediately follows that also $\{\,,\,\}_{\mathscr{D}}$ is *skew-symmetric*. Using the constitutive relations (4.76) of the Stokes-Dirac structure, the bracket on K_{adm} may be calculate as follows (using only the variational derivatives with respect to the two first flow variables f_p and f_q):

$$\begin{aligned}\{k^1, k^2\}_{\mathscr{D}} = &\int_Z \left[\delta_{f_p} K^1 \wedge (-1)^r \mathrm{d}(\delta_{f_q} K^2) + (\delta_{f_q} K^1) \wedge \mathrm{d}(\delta_{f_p} K^2)\right] \\ &- \int_{\partial Z} (-1)^{n-q} (\delta_{f_q} K^1) \wedge (\delta_{f_p} K^2)\end{aligned} \tag{4.103}$$

Furthermore, it is straightforward to check that the bracket $\{\,,\,\}_{\mathscr{D}}$ also satisfies the Jacobi-identity

$$\{\{k^1, k^2\}_{\mathscr{D}}, k^3\}_{\mathscr{D}} + \{\{k^2, k^3\}_{\mathscr{D}}, k^1\}_{\mathscr{D}} + \{\{k^3, k^1\}_{\mathscr{D}}, k^2\}_{\mathscr{D}} = 0 \tag{4.104}$$

for all $k^i \in K_{\text{adm}}$.

In summary, the Dirac structure extends the bracket defined in (4.61) and associated with the differential operator (4.60) to functions defined on the flow variables including the boundary variables f_∂. However it depends only on the variational derivatives with respect to the two first flow variables f_p and f_q and hence it extends the bracket (4.61) by the term of the second line of the equation (4.104) which depends only on the boundary terms. This bracket will come to play a role later in the characterization of invariants of the port Hamiltonian system defined with respect to this bracket.

4.2.5 Port Hamiltonian formulation of systems of two conservation laws with boundary energy flow

4.2.5.1 Definition

In the preceding paragraph, we have shown that the interconnection relations associated with a system of two conservation laws in canonical interaction (see equations (4.60) and (4.68)), define a Dirac structure that we have called Stokes-Dirac structure. Considering the definition of the system of the two conservation laws in canonical coupling of Definition 4.8 and the definition of the Stokes-Dirac structure of Proposition 4.3, it is straightforward to formulate the system as a port Hamiltonian system with boundary port variables.

Definition 4.9. The boundary *port-Hamiltonian system of two conservation laws* with n-dimensional manifold of spatial variables Z, state space $\Omega^p(Z) \times \Omega^q(Z)$ (with $p+q=n+1$ and $r=pq+1$), Stokes-Dirac structure $\mathscr{D}$ given by (4.76) and Hamiltonian functional H, is defined as follows:

$$\begin{bmatrix} -\frac{\partial \alpha_p}{\partial t} \\ -\frac{\partial \alpha_q}{\partial t} \end{bmatrix} = \varepsilon \begin{bmatrix} 0 & (-1)^r \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \delta_p H \\ \delta_q H \end{bmatrix}$$
$$\begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} \delta_p H|_{\partial Z} \\ \delta_q H|_{\partial Z} \end{bmatrix} \tag{4.105}$$

As a consequence of the power-continuity relation satisfied by any element $(f_p, f_q, f_b, e_p, e_q, e_b)$ of the Dirac structure $\mathscr{D}$:

$$\int_Z [e_p \wedge f_p + e_q \wedge f_q] + \int_{\partial Z} e_b \wedge f_b = 0 \tag{4.106}$$

one may deduce an additional balance equation (to the two conservation equations on α_p and α_q), namely that the Hamiltonian function H is also a conserved quantity and satisfies also a balance equation.

Proposition 4.4. *Consider the distributed parameter port-Hamiltonian system (4.105). Then the Hamiltonian function H satisfies the* power balance equation*:*

$$\frac{dH}{dt} = \int_{\partial Z} e_b \wedge f_b \tag{4.107}$$

expressing that the increase in energy on the domain Z is equal to the power supplied to the system through its boundary ∂Z.

This port Hamiltonian system represents the dynamics of a system of two conservation laws in canonical interaction and constitutes thereby a generalization to the infinite dimensional case, of the standard Hamiltonian systems defined with respect to a symplectic bracket (in the linear case, the harmonic oscillator).

4.2.5.2 Lossless vibrating string formulated using k-forms

The first example is again the example of the vibrating string which has already been presented in Sect. 4.1.2.5. Now we shall briefly formulate it in terms of boundary port Hamiltonian systems defined on differential forms of the 1-dimensional spatial domain $Z=[a,b]$. In the sequel z denotes some coordinate of the interval Z.

The *strain* is a subject to a conservation law, hence it is defined be the 1-form

$$\alpha_q(t) = \varepsilon(t,z)\mathrm{d}z \tag{4.108}$$

The associated co-energy variable is the *stress* given by the 0-form

$$\sigma = T * \alpha_q \tag{4.109}$$

with T the elasticity modulus and $*$ the Hodge star operator (see Sect. 4.2.1.4). The *elastic-potential energy* is the quadratic function of the strain defined by:

$$U(\alpha_q) = \frac{1}{2}\int_a^b T * \alpha_q \wedge \alpha_q \tag{4.110}$$

and the stress 0-form is then indeed: $\sigma = \delta_{\alpha_q} U$. The kinetic *momentum* is also a conserved quantity subject to a conservation law, hence it is defined as the 1-form

$$\alpha_p(t) = p(t,z)\mathrm{d}z \tag{4.111}$$

The associated co-energy variable is the *velocity* given by the 0-form

$$v = \frac{1}{\mu} * \alpha_p = \delta_p K \tag{4.112}$$

and the *kinetic energy function* is given by the quadratic function

$$K(\alpha_p) = \frac{1}{2}\int_a^b v \wedge p \tag{4.113}$$

In this case the Dirac structure is the Stokes-Dirac structure for $n = p = q = 1$, with $\varepsilon = -1$, leading to the port Hamiltonian system (with $H := U + K$)

$$\begin{aligned} \begin{bmatrix} -\frac{\partial \alpha_p}{\partial t} \\ -\frac{\partial \alpha_q}{\partial t} \end{bmatrix} &= \begin{bmatrix} 0 & -\mathrm{d} \\ -\mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \delta_p H \\ \delta_q H \end{bmatrix} \\ \begin{bmatrix} f_b \\ e_b \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_p H|_{\partial Z} \\ \delta_q H|_{\partial Z} \end{bmatrix} \end{aligned} \tag{4.114}$$

with boundary variables being the velocity and stress 0-forms at the ends of the string.

4.2.5.3 Maxwell's equations

The formulation of Maxwell's equations in terms of differential forms is very classical [92]. In this section we illustrate how it may be extended to a boundary port-Hamiltonian system defined on the Stokes-Dirac structure with 3-dimensional connected bounded spatial domain $Z \subset \mathbb{R}^3$ having a 2-dimensional smooth boundary ∂Z. In the sequel we shall denote by (z_1, z_2, z_3) some coordinates of the spatial domain.

The electromagnetic field is defined by two energy variables that are the *electric field induction* 2-form $\alpha_p = \mathscr{D} \in \Omega^2(Z)$:

$$\mathscr{D} = \frac{1}{2} D_{ij}(t,z)\mathrm{d}z^i \wedge \mathrm{d}z^j \tag{4.115}$$

and the *magnetic field induction* 2-form $\alpha_q = \mathscr{B} \in \Omega^2(Z)$:

$$\mathscr{B} = \frac{1}{2} B_{ij}(t,z) \mathrm{d}z^i \wedge \mathrm{d}z^j \tag{4.116}$$

The corresponding Stokes-Dirac structure with $n = 3$, $p = 2$ and $q = 2$) is given as (cf. (4.76)):

$$\begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix} \qquad \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_{p|\partial Z} \\ e_{q|\partial Z} \end{bmatrix} \tag{4.117}$$

Usually in this case one does *not* start with the definition of the total energy (Hamiltonian) H, but instead with the co-energy variables $\delta_p H$ and $\delta_q H$, given, respectively, as the electric field intensity $\mathscr{E} \in \Omega^1(Z)$:

$$\mathscr{E} = E_i(t,z) \mathrm{d}z^i \tag{4.118}$$

and the magnetic field intensity $\mathscr{H} \in \Omega^1(Z)$:

$$\mathscr{H} = H_i(t,z) \mathrm{d}z^i \tag{4.119}$$

They are related to the energy variables through the constitutive relations of the medium (or material equations)

$$*\mathscr{D} = \varepsilon \mathscr{E} \qquad *\mathscr{B} = \mu \mathscr{H} \tag{4.120}$$

with the scalar functions $\varepsilon(z)$ and $\mu(z)$ denoting the electric permittivity, respectively magnetic permeability. Then one *defines* the Hamiltonian H as

$$H = \int_Z \frac{1}{2} (\mathscr{E} \wedge \mathscr{D} + \mathscr{H} \wedge \mathscr{B}), \tag{4.121}$$

and one immediately verifies that $\delta_p H = \mathscr{E}$ and $\delta_q H = \mathscr{H}$.

Remark 4.4. There are other cases (corresponding to a *nonlinear* theory of the electromagnetic field, such as the Born-Infeld theory, see e.g. [92]) where the Hamiltonian $H = \int_Z h$ with the energy density $h(\mathscr{D}, \mathscr{B})$ being a more general expression than $\frac{1}{2}(\varepsilon^{-1} * \mathscr{D} \wedge \mathscr{D} + \mu^{-1} * \mathscr{B} \wedge \mathscr{B})$.

Assuming that there is no current in the medium Maxwell's equations can now be written as the two conservation laws in the electrical and the magnetic domains, namely Ampère's and Faraday's law (see [92])

$$\begin{aligned} \frac{\partial \mathscr{D}}{\partial t} &= \mathrm{d}\mathscr{H} \\ \frac{\partial \mathscr{B}}{\partial t} &= -\mathrm{d}\mathscr{E} \end{aligned} \tag{4.122}$$

Explicitly taking into account the behavior at the boundary, Maxwell's equations on a domain $Z \subset \mathbb{R}^3$ are then represented as the boundary port-Hamiltonian system with respect to the Stokes-Dirac structure given by (4.117), as

$$\begin{bmatrix} -\frac{\partial \mathscr{D}}{\partial t} \\ -\frac{\partial \mathscr{B}}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \delta_{\mathscr{D}} H \\ \delta_{\mathscr{B}} H \end{bmatrix} \qquad \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} \delta_{\mathscr{D}} H|_{\partial Z} \\ \delta_{\mathscr{B}} H|_{\partial Z} \end{bmatrix} \tag{4.123}$$

The energy-balance (4.107) in the case of Maxwell's equations takes the form

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_{\partial Z} \delta_{\mathscr{B}} H \wedge \delta_{\mathscr{D}} H = \int_{\partial Z} \mathscr{H} \wedge \mathscr{E} = -\int_{\partial Z} \mathscr{E} \wedge \mathscr{H} \tag{4.124}$$

with $\mathscr{E} \wedge \mathscr{H}$ is the 2-form corresponding to the *Poynting vector* and characterizing the outgoing electro-magnetic energy through the boundary of the domain (see [92]).

4.2.6 Extension to distributed port variables and scattering boundary variables

4.2.6.1 Distributed port variables and dissipation

In order to represent systems of conservation laws with sources (or general balance equations), it is necessary to add to the boundary port variables also distributed port variables. An example of this is provided by Maxwell's equations (cf. Sect. 4.2.5.3), where interaction may also take place via the current density J, which directly affects the electric charge distribution in the domain Z. Another situation where such distributed port variables arise concern multi-scale distributed parameter systems, such as for instance models of the heart of fuel cells. In the sequel we shall treat a little more general case where the port variables are defined on some arbitrary spatial domain denoted by S, not necessarily equal to the domain Z.

In order to incorporate the distributed port variables, the effort and flow spaces $\mathscr{F}_{p,q}$ and $\mathscr{E}_{p,q}$ as defined in (4.71), (4.73) are augmented to:

$$\mathscr{F}^a_{p,q} := \mathscr{F}_{p,q} \times \Omega^d(S) \qquad \mathscr{E}^a_{p,q} := \mathscr{E}_{p,q} \times \Omega^{n-d}(S) \tag{4.125}$$

for some m-dimensional manifold S and some $d \in \{0, 1, \cdots, m\}$, with $e^d \in \Omega^d(S)$ denoting the distributed source effort variable, and $f^d \in \Omega^{n-d}(S)$ the conjugated distributed flow variable, corresponding to an energy exchange

$$\int_S f^d \wedge e^d \tag{4.126}$$

The Stokes-Dirac structure (4.76) is now extended to

$$\begin{bmatrix} f_p \\ f_q \\ f^d \end{bmatrix} = \begin{bmatrix} 0 & (-1)^r \mathrm{d} & G_p \\ \mathrm{d} & 0 & G_e \\ -G_p^* & -G_q^* & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \\ e^d \end{bmatrix}$$
$$\begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} e_{p|\partial Z} \\ e_{q|\partial Z} \end{bmatrix} \tag{4.127}$$

with the compound map G:

$$G = \begin{bmatrix} G_p \\ G_q \end{bmatrix} : \Omega^d(S) \to \Omega^p(Z) \times \Omega^q(Z) \tag{4.128}$$

admitting a dual map:

$$G^* = (G_p^*, G_q^*) : \Omega^{n-p}(Z) \times \Omega^{n-q}(Z) \to \Omega^{n-d}(S) \tag{4.129}$$

satisfying

$$\int_Z [e_p \wedge G_p(f_d) + e_q \wedge G_q(f_d)] = \int_S \left[G_p^*(e_p) + G_q^*(e_q)\right] \wedge f_d \tag{4.130}$$

for all $e_p \in \Omega^{n-p}(Z)$, $e_q \in \Omega^{n-q}(Z)$ and $f_d \in \Omega^d(S)$. The following proposition can be easily checked.

Proposition 4.5. *Equations (4.127) determine a Dirac structure $\mathscr{D}^a \subset \mathscr{F}^a_{p,q} \times \mathscr{E}^a_{p,q}$ with respect to the augmented bi-linear form on the bond space $\mathscr{F}^a_{p,q} \times \mathscr{E}^a_{p,q}$ which is obtained by adding to the bi-linear form (4.74) on $\mathscr{F}_{p,q}, \times \mathscr{E}_{p,q}$ the term*

$$\int_S \left[e_d^1 \wedge f_d^2 + e_d^2 \wedge f_d^1\right] \tag{4.131}$$

By identifying $f_p = \frac{\partial \alpha_p}{\partial t}$, $f_q = \frac{\partial \alpha_q}{\partial t}$, $e_p = \delta_p H$ and $e_q = \delta_q H$ in the definition of $\mathscr{D}^a$ given by (4.127) we obtain a port-Hamiltonian system with external variables (f_b, f_d, e_b, e_d), with f_b, e_b the boundary external variables and f_d, e_d the distributed external variables. Furthermore, the energy balance (4.107) extends to

$$\frac{dH}{dt} = \int_{\partial Z} e_b \wedge f_b + \int_S e_d \wedge f_d \tag{4.132}$$

with the first term on the right-hand side denoting the power flow through the boundary, and the second term denoting the distributed power flow.

Note 4.5. It should be noted that the assumption that the map G admits a dual as stated in (4.129) is a restrictive assumption from the physical modelling perspective. It excludes for instance any map involving the exterior derivative as for instance it was the case in the example of the vibrating string with structural damping. If one would like to take account of such a case then the definition of the port variables of the extended Dirac structure should be modified but this topic exceeds the scope of this chapter. For the case of 1-dimensional spatial domains the reader is referred to [211].

Finally, *irreversible processes* may be incorporated in the framework of distributed parameter port-Hamiltonian systems by *completing* the model with some dissipative relations on the port variables (boundary or distributed port variables). For example, for distributed dissipation, let $R : \Omega^{n-d}(S) \to \Omega^d(S)$ be a map satisfying

$$\int_S e_d \wedge R(e_d) \geq 0, \quad \forall e_d \in \Omega^{n-d}(S) \tag{4.133}$$

Then completing the port Hamiltonian system defined with respect to the Dirac structure $\mathscr{D}^a$, with the dissipative relation:

$$f_d = -R(e_d) \tag{4.134}$$

we obtain a port-Hamiltonian system *with dissipation*, satisfying the energy balance equation and dissipation relation:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_{\partial Z} e_b \wedge f_b - \int_S e_d \wedge R(e_d) \leq \int_{\partial Z} e_b \wedge f_b \tag{4.135}$$

Example 4.1. With reference to the Maxwell's equations of Sect. 4.2.5.3, in the case of a non-zero *current density* $J \in \Omega^2(Z)$, we have to modify the first matrix equation of (4.123) to

$$\begin{bmatrix} -\frac{\partial \mathscr{D}}{\partial t} \\ -\frac{\partial \mathscr{B}}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \delta_{\mathscr{D}} H \\ \delta_{\mathscr{B}} H \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} J \tag{4.136}$$

with I denoting the identity operator from $J \in \Omega^2(Z)$ to $\Omega^2(Z)$. Thus, in the notation of (4.128), $f_d = J$, $S = Z$, and $\Omega^d(S) = \Omega^2(Z)$. Furthermore, we add the equation

$$e_d = -\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \delta_{\mathscr{D}} H \\ \delta_{\mathscr{B}} H \end{bmatrix} = -\mathscr{E}, \tag{4.137}$$

yielding the augmented energy balance

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\int_{\partial Z} \mathscr{E} \wedge \mathscr{H} - \int_Z \mathscr{E} \wedge J \tag{4.138}$$

which is known as *Poynting's theorem*. Finally, in order to incorporate energy dissipation we write $J = J_d + \bar{J}$, and we impose Ohm's law

$$*J_d = \sigma \mathscr{E} \tag{4.139}$$

with $\sigma(t,z)$ the specific conductivity of the medium.

4.2.6.2 An introduction to scattering variables

The description of open Hamiltonian systems, exchanging some energy with their environment through the boundary of their spatial domain has led to the definition

of port variables and port Hamiltonian systems defined with respect to Stokes-Dirac structures. The balance equation on the Hamiltonian (for physical systems, the total energy) is then written according to (4.107) as:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_{\partial Z} e_b \wedge f_b$$

where $f_b \in \Omega^{n-p}(\partial Z)$ and $e_b \in \Omega^{n-q}(\partial Z)$ with $p+q = n+1$. The energy flow through the boundary is expressed as the dual product between the conjugated pair of boundary flow and effort variables. However it might be useful for control purposes or model reduction purposes to consider so-called scattering variables. In this paragraph we shall briefly introduce such boundary variables corresponding to the Stokes-Dirac structure. Basically, the finite dimensional scattering theory discussed in Sect. 2.8 is extended in order to locate the *incoming* and *outgoing* power flows on the boundary of the infinite dimensional system. The first step is the generalization of scattering subspaces to the distributed parameter case.

As discussed in Sect. 2.8.3, the scattering subspaces can be defined in a geometric way once the space of power variables has been equipped with a symmetric bilinear operator (the +pairing) and a metric. According to Sect. 4.2.1.3, the pairing for conjugated pairs of exterior forms defined on the boundary ∂Z of the spatial domain Z is (see (4.50)):

$$\langle e \mid f \rangle := \int_{\partial Z} e \wedge f$$

with $f \in \Omega^{n-p}(\partial Z) \equiv \mathscr{F}$ and $e \in \Omega^{n-q}(\partial Z) \equiv \mathscr{E}$, thus leading to the following definition of +pairing:

$$\ll (f_1,e_1),(f_2,e_2) \gg := \langle e_1 \mid f_2 \rangle + \langle e_2 \mid f_1 \rangle = \int_{\partial Z} e_1 \wedge f_2 + \int_{\partial Z} e_2 \wedge f_1 \quad (4.140)$$

with $(f_i,e_i) \in \mathscr{F} \times \mathscr{E}$, $i = 1,2$. The most critical point is now the identification of a suitable norm on the space of power variables. Since ∂Z is a Riemannian manifold, a symmetric positive definite 2-contra-variant tensor g is well-defined and, based on this tensor, the Hodge star operator can be also defined (see Sect. 4.2.1.4). Then, it is possible to prove that, given with $(f_i,e_i) \in \mathscr{F} \times \mathscr{E}$, $i = 1,2$,

$$\langle (f_1,e_1),(f_2,e_2) \rangle := \int_{\partial Z} e_1 \wedge *e_2 + \int_{\partial Z} f_1 \wedge *f_2 \quad (4.141)$$

in an inner product on $\mathscr{F} \times \mathscr{E}$, and if $(f,e) \in \mathscr{F} \times \mathscr{E}$

$$\|(f,e)\|^2 := \langle (f,e),(f,e) \rangle = \int_{\partial Z} e \wedge *e + \int_{\partial Z} f \wedge *f \quad (4.142)$$

is a norm on on $\mathscr{F} \times \mathscr{E}$. Then, taking into account Sect. 2.8.3 and, in particular, Remark 2.11, it is possible to give the following:

Definition 4.10. Suppose that ∂Z is an $(n-1)$-dimensional Riemannian manifold and assume that $\mathscr{F} = \Omega^{n-p}(\partial Z)$ is the space of flows and that $\mathscr{E} = \Omega^{n-q}(\partial Z)$ is

the space of efforts, with $p+q=n+1$. The scattering subspaces $\Sigma^+, \Sigma^- \subset \mathscr{F} \times \mathscr{E}$ are defined as

$$\Sigma^+ := \left\{ (f,e) \in \mathscr{F} \times \mathscr{E} \mid \ll (f,e),(f,e) \gg = \|(f,e)\|^2 \right\}$$
$$\Sigma^- := \left\{ (f,e) \in \mathscr{F} \times \mathscr{E} \mid \ll (f,e),(f,e) \gg = -\|(f,e)\|^2 \right\}$$

From (4.141) and (4.142), we deduce that the scattering subspaces of Definition 4.10 can be written as:

$$\Sigma^+ = \left\{ (f,e) \in \mathscr{F} \times \mathscr{E} \mid f = *e \Leftrightarrow e = (-1)^{(n-p)(p-1)} * f \right\}$$
$$\Sigma^- = \left\{ (f,e) \in \mathscr{F} \times \mathscr{E} \mid f = -*e \Leftrightarrow e = -(-1)^{(n-p)(p-1)} * f \right\} \tag{4.143}$$

These new expressions for Σ^+ and Σ^- are instrumental for proving that all the main properties of Sect. 2.8.2, valid in finite dimensions, have been extended to the distributed parameter case. More precisely, it is easy to verify that

- $\Sigma^+ \cap \Sigma^- = \{(0,0)\}$;
- $\mathscr{F}$, $\mathscr{E}$, Σ^+ and Σ^- are isomorphic spaces;
- If $\sigma^+ \in \Sigma^+$ and $\sigma^- \in \Sigma^-$, then $\langle \sigma^+, \sigma^- \rangle = 0$, which means that σ^+ and σ^- are orthogonal with respect to the scalar product (4.141) defined on $\mathscr{F} \times \mathscr{E}$.

Immediate consequences are that $\mathscr{F} \times \mathscr{E} = \Sigma^+ \oplus \Sigma^-$, with Σ^+ and Σ^- orthogonal in the sense of the inner product (4.141), and that for every $(f,e) \in \mathscr{F} \times \mathscr{E}$, $\sigma^+ \in \Sigma^+$ and $\sigma^- \in \Sigma^-$ are unequivocally determined in such a way that (see (2.239) for the finite dimensional case):

$$(f,e) = \sigma^+ + \sigma^- \tag{4.144}$$

Moreover, it can be verified that the +pairing (4.140) restricted on Σ^+ and Σ^- gives the inner product (4.141). Finally, the scattering power decomposition (2.241) can be proved to hold also in case of distributed parameter systems. In fact, given $(f,e) \in \mathscr{F} \times \mathscr{E}$, we have that:

$$\langle e \mid f \rangle = \frac{1}{2} \left\| \sigma^+ \right\|^2 - \frac{1}{2} \left\| \sigma^- \right\|^2$$

where σ^+ and σ^- are such that (4.144) holds, and the metric $\|\cdot\|$ is the one defined by (4.142) and restricted on Σ^+ and Σ^-.

It is important to note that, as reported in Definition 4.10, the scattering subspaces are subsets of $\mathscr{F} \times \mathscr{E}$. On the other hand, through the Hodge star operator it is possible to establish a bijection between the scattering subspaces and the space of flows $\mathscr{F}$ or the space of efforts $\mathscr{E}$. This can be easily deduced from (4.143). An interesting consequence is that it makes sense to identify the scattering subspaces, for example, with $\mathscr{E}$. If $\Sigma^+_{\mathscr{E}} \equiv \mathscr{E}$ and $\Sigma^-_{\mathscr{E}} \equiv \mathscr{E}$, then (4.144) can be written as

$$(f,e) = (*\sigma^+_{\mathscr{E}}, \sigma^+_{\mathscr{E}}) + (-*\sigma^-_{\mathscr{E}}, \sigma^-_{\mathscr{E}})$$

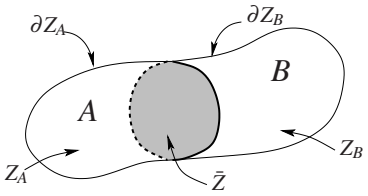

Fig. 4.1 Interconnection of systems A and B over a subset $\bar{Z}$ of their boundary.

with $\sigma_{\mathscr{E}}^+ \in \Sigma_{\mathscr{E}}^+$ and $\sigma_{\mathscr{E}}^- \in \Sigma_{\mathscr{E}}^-$. Consequently, we have that

$$f = *\left(\sigma_{\mathscr{E}}^+ - \sigma_{\mathscr{E}}^-\right) \qquad e = \sigma_{\mathscr{E}}^+ + \sigma_{\mathscr{E}}^- \tag{4.145}$$

and that

$$\sigma_{\mathscr{E}}^+ = \frac{e + (-1)^{(n-1)(n-q)} * f}{2} \qquad \sigma_{\mathscr{E}}^- = \frac{e - (-1)^{(n-1)(n-q)} * f}{2}$$

which are the infinite dimensional counterpart of (2.248). Similar considerations can be drawn by identifying the scattering subspaces with the space of flows $\mathscr{F}$.

As in finite dimensions, scattering variables can be used to describe the power conserving interconnection of distributed parameter systems (see Sect. 2.8.4). Consider two infinite dimensional systems A and B with spatial domain given by two n-dimensional Riemannian manifolds Z_A and Z_B. Then, suppose that their borders ∂Z_A and ∂Z_B are Riemannian manifolds too. On ∂Z_A (resp. on ∂Z_B), assume that the space of flows is given by $\mathscr{F}_A = \Omega^{n-p}(\partial Z_A)$ (resp. by $\mathscr{F}_B = \Omega^{n-p}(\partial Z_B)$), and its dual, the space of efforts, by $\mathrm{e}_A = \Omega^{n-q}(\partial Z_A)$ (resp. by $\mathrm{e}_B = \Omega^{n-q}(\partial Z_B)$). Since a Riemannian metric g^A (resp. g^B) is defined over the border of each spatial domain Z_A and Z_B, it is possible to define the Hodge star operator $*_A$ (resp. $*_B$) and the scattering subspaces Σ_A^+ and Σ_A^- (resp. Σ_B^+ and Σ_B^-) on ∂Z_A (resp. on ∂Z_B), or their equivalent representation in terms of elements of the space of efforts or flows.

If $\bar{Z} = \partial Z_A \cap \partial Z_B \neq \emptyset$, Z_A and Z_B are interconnected through the distributed port $\bar{Z}$, which is supposed to be an $(n-1)$-dimensional Riemannian manifold. This situation is represented in Fig. 4.1. The interconnection is power conserving if $e_A \wedge f_A + e_B \wedge f_B = 0$ on $\bar{Z}$, which leads e.g. to the well-known common-effort and common-flow interconnections.

Assume, for example, a common-effort interconnection between system A and B. Since $\bar{Z}$ belongs to both the boundaries of A and B, we can say that, on this manifold, two Riemannian metrics g^A and g^B are defined. These metrics are strictly related to the *physical properties* of the two systems. Moreover, on $\bar{Z}$ it will be possible to define two Hodge star operators $*_A$ and $*_B$ and, consequently, two distinct scattering decompositions. In other words, the space of power variables of A and B, restricted to $\bar{Z}$, can be described by using scattering variables related to two different scattering decomposition, clearly defined by means of two (different) Hodge star operators. Given $(f_i, e_i) \in \mathscr{F}_i \times \mathscr{E}_i$, with $i = A, B$, from (4.144), in terms of scattering variables it is possible to write that

$$(f_A, e_A) = \sigma_A^+ + \sigma_A^- \qquad (f_B, e_B) = \sigma_B^+ + \sigma_B^-$$

where $\sigma_i^+ \in \Sigma_i^+$ and $\sigma_i^- \in \Sigma_i^-$, with $i = A, B$. In this way, the interconnection can be described in terms of the scattering variables. In case of the common-effort interconnection on $\bar{Z}$, i.e. $f_A = -f_B$ and $e_A = e_B$, if systems A and B share the same metric $g = g^A = g^B$ on their boundary, from (4.145) it is easy to verify that

$$\sigma_A^+ = \sigma_B^- \qquad \sigma_B^+ = \sigma_A^-$$

This means that, if the two systems have the same physical properties, then the outgoing power from A becomes the incoming power in B and, symmetrically, the outgoing power from B becomes the incoming power in A. So, no power reflection is present at the interconnection port $\bar{Z}$, phenomena that is present if the physical properties of the two systems are different, as it can be easily verified. This is exactly what happens also in the finite dimensional case, as reported in Sect. 2.8.4. More details can be found in [129].

4.3 Extension of port-Hamiltonian systems on Stokes-Dirac structures

In Sect. 4.2, we have presented a canonical Dirac structure called Stokes-Dirac structure and associated with systems of two conservation laws subject to the canonical reversible inter-domain coupling relation given in (4.76). And we have shown how one may define a port boundary Hamiltonian system with respect to this Stokes-Dirac structure which accounts for energy flow through the boundary of the spatial domain. However in Sect. 4.1, on the examples of dissipative physical systems, we have considered Dirac structures which may be considered as extensions of the canonical Stokes-Dirac structure. In this section we shall conclude with some extensions of Stokes-Dirac structures appearing in models of reversible physical systems on different paradigmatic examples.

The first example is the formulation of a flexible beam model, the Timoshenko beam, as a boundary port Hamiltonian model. In this case actually there are two physical domains in interaction the kinetic and potential elastic domains like for the vibrating string. However the displacements of a flexible beam are not only translational but also rotational and therefore the Stokes-Dirac structure has to be extended in order to encompasses the kinematic and static relations describing the assumptions on the configuration space. A further extension is necessary in order to formulate the port-Hamiltonian model of a nonlinear flexible link, which is presented as a second example. Finally, the third example concern the 3-dimensional model of an isentropic fluid. In this case also two physical domains, the kinetic and the spatial domains are in canonical interaction but one has also to encompass the convection of momentum. For this purpose we shall recall how the canonical Stokes-Dirac structure may be augmented with a nonlinear term generating the convection.

4.3.1 Timoshenko beam

According to the Timoshenko theory, the motion of a beam can be described by the following system of PDE:

$$\begin{aligned} \rho\frac{\partial^2 w}{\partial t^2} - K\frac{\partial^2 w}{\partial z^2} + K\frac{\partial \phi}{\partial z} &= 0 \\ I_\rho\frac{\partial^2 \phi}{\partial t^2} - EI\frac{\partial^2 \phi}{\partial z^2} + K\left(\phi - \frac{\partial w}{\partial z}\right) &= 0 \end{aligned} \tag{4.146}$$

where t is the time and $z \in [0,L]$ is the spatial coordinate along the beam in its equilibrium position, $w(z,t)$ is the deflection of the beam from the equilibrium configuration and $\phi(z,t)$ is the rotation of the beam's cross section due to bending; the motion takes place in the wz-plane. Denote by $Z := [0,L]$ the spatial domain and by $\partial Z = \{0,L\}$ its boundary. The coefficients ρ, I_ρ, E and I, assumed to be constant, are the mass per unit length, the mass moment of inertia of the cross section, Young's modulus and the moment of inertia of the cross section, respectively. The coefficient K is equal to kGA, where G is the modulus of elasticity in shear, A is the cross sectional area and k is a constant depending on the shape of the cross section.

The mechanical energy is equal to [102]

$$H(t) := \frac{1}{2}\int_0^L \left[\rho\left(\frac{\partial w}{\partial t}\right)^2 + I_\rho\left(\frac{\partial \phi}{\partial t}\right)^2 + K\left(\phi - \frac{\partial w}{\partial z}\right)^2 + EI\left(\frac{\partial \phi}{\partial z}\right)^2\right] \mathrm{d}z \tag{4.147}$$

thus showing the presence of two interactive energy domains, the *kinetic* and the *potential elastic*. The potential elastic energy is a function of the *shear* and of the *bending*, given by the following 1-forms:

$$\varepsilon_t = \mathrm{d}w - *\phi \qquad\qquad \varepsilon_r = \mathrm{d}\phi \tag{4.148}$$

with $w, \phi \in \Omega^0(Z)$. The associated co-energy variables are the 0-forms (functions) *shear force* and the *bending momentum*, given by $\sigma_t(t,z) = K*\varepsilon_t(t,z)$ and $\sigma_r(t,z) = EI*\varepsilon_r(t,z)$. Besides, the kinetic energy is function of the *translational* and *rotational momenta*, i.e. of the following 1-form:

$$p_t = \rho\frac{\partial w}{\partial t} \qquad\qquad p_r = I_\rho\frac{\partial \phi}{\partial t} \tag{4.149}$$

with $\rho, I_\rho \in \Omega^1(Z)$, and the associated co-energy variables are the 0-forms *translational* and *rotational momenta*, which are given by $v_t(t,z) = \frac{1}{\rho}*p_t(t,z)$ and $v_r(t,z) = \frac{1}{I_\rho}*p_r(t,z)$. Consequently, the total energy (4.147) becomes the following (quadratic) functional:

$$H(p_t,p_r,\varepsilon_t,\varepsilon_r) = \frac{1}{2}\int_Z \left(\frac{1}{\rho}*p_t\wedge p_t + \frac{1}{I_\rho}*p_r\wedge p_r + K*\varepsilon_t\wedge\varepsilon_t + EI*\varepsilon_r\wedge\varepsilon_r\right)$$
$$= \int_Z \mathscr{H}(p_t,p_r,\varepsilon_t,\varepsilon_r) \tag{4.150}$$

with $\mathscr{H}:\Omega^1(Z)\times\cdots\times\Omega^1(Z)\times Z\to\Omega^1(Z)$ the energy density. Consider a time function $(p_t(t),p_r(t),\varepsilon_t(t),\varepsilon_r(t))\in\Omega^1(Z)\times\cdots\times\Omega^1(Z)$ with $t\in\mathbb{R}$, and evaluate the energy H along this trajectory. At any time t, the variation of internal energy, that is the power exchanged with the environment, is given by

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_Z \left(\delta_{p_t}H\wedge\frac{\partial p_t}{\partial t} + \delta_{p_r}H\wedge\frac{\partial p_r}{\partial t} + \delta_{\varepsilon_t}H\wedge\frac{\partial\varepsilon_t}{\partial t} + \delta_{\varepsilon_r}H\wedge\frac{\partial\varepsilon_r}{\partial t}\right)$$
$$= \int_Z \left[\left(\frac{1}{\rho}*p_t\right)\wedge\frac{\partial p_t}{\partial t} + \left(\frac{1}{I_\rho}*p_r\right)\wedge\frac{\partial p_r}{\partial t} + (K*\varepsilon_t)\wedge\frac{\partial\varepsilon_t}{\partial t} + (EI*\varepsilon_r)\wedge\frac{\partial\varepsilon_r}{\partial t}\right]$$

The differential forms $\frac{\partial p_t}{\partial t}$, $\frac{\partial p_r}{\partial t}$, $\frac{\partial\varepsilon_t}{\partial t}$ and $\frac{\partial\varepsilon_r}{\partial t}$ are the time derivatives of the energy variables p_t, p_r, ε_t, ε_r and represent the *generalized velocities* (flows), while $\delta_{p_t}H$, $\delta_{p_r}H$, $\delta_{\varepsilon_t}H$, $\delta_{\varepsilon_r}H$ are the variational derivative of the total energy (4.150) They are related to the rate of change of the stored energy and represent the *generalized forces* (efforts).

The distributed port-Hamiltonian formulation of the Timoshenko beam can be obtained either by expressing (4.146) in terms of p_t, p_r, ε_r and ε_t introduced in (4.148) and (4.149), or, in a more rigorous way, by revealing the underlying Stokes-Dirac structure of the model. For this purpose, it is necessary to define the space of power variables. The space of flows is given by

$$\mathscr{F} := \Omega^1(Z)\times\Omega^1(Z)\times\Omega^1(Z)\times\Omega^1(Z)\times\Omega^0(\partial Z)\times\Omega^0(\partial Z) \tag{4.151}$$

while its *dual*, the space of efforts $\mathscr{E}$ can be identified with

$$\mathscr{E} := \Omega^0(Z)\times\Omega^0(Z)\times\Omega^0(Z)\times\Omega^0(Z)\times\Omega^0(\partial Z)\times\Omega^0(\partial Z) \tag{4.152}$$

Then, with the following proposition the Stokes-Dirac structure of the Timoshenko beam is introduced.

Proposition 4.6. *Consider the space of power variables $\mathscr{F}\times\mathscr{E}$ with $\mathscr{F}$ and $\mathscr{E}$ defined in (4.151) and (4.152) and the +pairing $\ll\cdot,\cdot\gg$ given by*

$$\begin{aligned}\ll \left(f^1_{p_t},\ldots,f^{r,1}_b,e^1_{p_t},\ldots,e^{r,1}_b,\right),\left(f^2_{p_t},\ldots,f^{r,2}_b,e^2_{p_t},\ldots,e^{r,2}_b\right)\gg :=\\ := \int_Z \left(f^1_{p_t}\wedge e^2_{p_t} + f^2_{p_t}\wedge e^1_{p_t} + f^1_{p_r}\wedge e^2_{p_r} + f^2_{p_r}\wedge e^1_{p_r}\right) +\\ + \int_Z \left(f^1_{\varepsilon_t}\wedge e^2_{\varepsilon_t} + f^2_{\varepsilon_t}\wedge e^1_{\varepsilon_t} + f^1_{\varepsilon_r}\wedge e^2_{\varepsilon_r} + f^2_{\varepsilon_r}\wedge e^1_{\varepsilon_r}\right) +\\ + \int_{\partial Z}\left(f^{t,1}_b\wedge e^{t,2}_b + f^{t,2}_b\wedge e^{t,1}_b + f^{r,1}_b\wedge e^{r,2}_b + f^{r,2}_b\wedge e^{r,1}_b\right)\end{aligned} \tag{4.153}$$

Define the following linear subspace $\mathscr{D}$ of $\mathscr{F} \times \mathscr{E}$:

$$\mathscr{D} = \left\{ \left(f_{p_t}, f_{p_r}, f_{\varepsilon_t}, f_{\varepsilon_r}, f_b^t, f_b^r, e_{p_t}, e_{p_r}, e_{\varepsilon_t}, e_{\varepsilon_r}, e_b^t, e_b^r\right) \in \mathscr{F} \times \mathscr{E} \mid \right.$$
$$\left. \begin{bmatrix} f_{p_t} \\ f_{p_r} \\ f_{\varepsilon_t} \\ f_{\varepsilon_r} \end{bmatrix} = - \begin{bmatrix} 0 & 0 & d & 0 \\ 0 & 0 & * & d \\ d & -* & 0 & 0 \\ 0 & d & 0 & 0 \end{bmatrix} \begin{bmatrix} e_{p_t} \\ e_{p_r} \\ e_{\varepsilon_t} \\ e_{\varepsilon_r} \end{bmatrix}, \quad \begin{bmatrix} f_b^t \\ f_b^r \\ e_b^t \\ e_b^r \end{bmatrix} = \begin{bmatrix} e_{p_t} \mid_{\partial Z} \\ e_{p_r} \mid_{\partial Z} \\ e_{\varepsilon_t} \mid_{\partial Z} \\ e_{\varepsilon_r} \mid_{\partial Z} \end{bmatrix} \right\} \quad (4.154)$$

Then $\mathscr{D} = \mathscr{D}^{\perp}$, that is $\mathscr{D}$ is a Stokes-Dirac structure.

Proof. The proof is quite similar to the one given for Proposition 4.3. More details can be found in [121].

Assume that the total energy (4.150) is the Hamiltonian of the system.The rate of change of the energy variables p_t, p_r, ε_t and ε_r can be connected to the Stokes-Dirac structure (4.154) by setting

$$f_{p_t} = -\frac{\partial p_t}{\partial t} \qquad f_{\varepsilon_t} = -\frac{\partial \varepsilon_t}{\partial t} \qquad f_{p_r} = -\frac{\partial p_r}{\partial t} \qquad f_{\varepsilon_r} = -\frac{\partial \varepsilon_r}{\partial t} \qquad (4.155)$$

where the minus sign is necessary in order to have a consistent energy flow description. Moreover, the rate of change of the Hamiltonian with respect to the energy variables, that is its variational derivatives, can be related to the Stokes-Dirac structure by setting

$$e_{p_t} = \delta_{p_t} H \qquad e_{\varepsilon_t} = \delta_{\varepsilon_t} H \qquad e_{p_r} = \delta_{p_r} H \qquad e_{\varepsilon_r} = \delta_{\varepsilon_r} H \qquad (4.156)$$

From (4.155) and (4.156), it is possible to obtain the distributed Hamiltonian formulation with boundary energy flow of the Timoshenko beam. We give the following:

Definition 4.11. The port-Hamiltonian model of the Timoshenko beam with Stokes-Dirac structure $\mathscr{D}$ (4.154) and Hamiltonian H (4.150) is given by

$$\begin{bmatrix} \frac{\partial p_t}{\partial t} \\ \frac{\partial p_r}{\partial t} \\ \frac{\partial \varepsilon_t}{\partial t} \\ \frac{\partial \varepsilon_r}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \mathrm{d} & 0 \\ 0 & 0 & * & \mathrm{d} \\ \mathrm{d} & -* & 0 & 0 \\ 0 & \mathrm{d} & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_{p_t} H \\ \delta_{p_r} H \\ \delta_{\varepsilon_t} H \\ \delta_{\varepsilon_r} H \end{bmatrix} \qquad \begin{bmatrix} f_b^t \\ f_b^r \\ e_b^t \\ e_b^r \end{bmatrix} = \begin{bmatrix} \delta_{p_t} H \mid_{\partial Z} \\ \delta_{p_r} H \mid_{\partial Z} \\ \delta_{\varepsilon_t} H \mid_{\partial Z} \\ \delta_{\varepsilon_r} H \mid_{\partial Z} \end{bmatrix} \qquad (4.157)$$

Since the elements of every Stokes-Dirac structure satisfy the power conserving property, we have that, given $(f_{p_t}, \ldots, f_{\varepsilon_r}, e_{p_t}, \ldots, e_{\varepsilon_r}, f_b^t, \ldots, e_b^r) \in \mathscr{D}$, then

$$\int_Z (f_{p_t} \wedge e_{p_t} + f_{p_r} \wedge e_{p_r} + f_{\varepsilon_t} \wedge e_{\varepsilon_t} + f_{\varepsilon_r} \wedge e_{\varepsilon_r}) + \int_{\partial Z} \left(f_b^t \wedge e_b^t + f_b^r \wedge e_b^r\right) = 0$$

and, consequently, the following proposition can be proved.

Proposition 4.7. *Consider the distributed port-Hamiltonian model of the Timoshenko beam (4.157). Then*

$$\frac{dH}{dt}(t) = \int_{\partial Z} \left(e_b^t \wedge f_b^t + e_b^r \wedge f_b^r \right) = \left[e_b^t(t,z) f_b^t(t,z) + e_b^r(t,z) f_b^r(t,z) \right]_{z=0}^{z=L} \tag{4.158}$$

or, in other words, the increase of energy kinetic/potential energy of the beam is equal to the power supplied through the boundary.

Power exchange through the boundaries is not the only way by means of which the system can interact with the environment. The "distributed control" is a well-know control technique that can be fruitfully applied to flexible structures. The actuators are connected along the flexible structure and can act on the system applying forces/couples that are functions of the configuration of the beam. The final result is that vibrations can be damped in a more efficient way than acting only on the boundary of the beam.

In order to introduce a distributed port, the space of power variables $\mathscr{F} \times \mathscr{E}$ defined in (4.151, 4.152) and the Stokes-Dirac structure $\mathscr{D}$ defined in (4.154) have to be modified. The space of power variables becomes $\mathscr{F}_d \times \mathscr{E}_d$, where

$$\mathscr{F}_d := \mathscr{F} \times \underbrace{\Omega^1(Z) \times \Omega^1(Z)}_{\textit{distrib. flow}} \qquad \mathscr{E}_d := \mathscr{E} \times \underbrace{\Omega^1(Z) \times \Omega^1(Z)}_{\textit{distrib. effort}} \tag{4.159}$$

The modified Stokes-Dirac structure that incorporates the distributed port is given by the following:

Proposition 4.8. *Consider the space of power variables $\mathscr{F}_d \times \mathscr{E}_d$ defined in (4.159) and the +pairing operator $\ll \cdot, \cdot \gg$ given by (4.153). Define the following linear subspace $\mathscr{D}_d$ of $\mathscr{F}_d \times \mathscr{E}_d$:*

$$\mathscr{D}_d = \left\{ \left(f_{p_t}, f_{p_r}, f_{\varepsilon_t}, f_{\varepsilon_r}, f_b^t, f_b^r, f_d^t, f_d^r, e_{p_t}, e_{p_r}, e_{\varepsilon_t}, e_{\varepsilon_r}, e_b^t, e_b^r, e_d^t, e_d^r \right) \in \mathscr{F}_d \times \mathscr{E}_d \mid \right.$$
$$\begin{bmatrix} f_{p_t} \\ f_{p_r} \\ f_{\varepsilon_t} \\ f_{\varepsilon_r} \end{bmatrix} = - \begin{bmatrix} 0 & 0 & d & 0 \\ 0 & 0 & * & d \\ d & -* & 0 & 0 \\ 0 & d & 0 & 0 \end{bmatrix} \begin{bmatrix} e_{p_t} \\ e_{p_r} \\ e_{\varepsilon_t} \\ e_{\varepsilon_r} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_d^t \\ f_d^r \end{bmatrix},$$
$$\left. \begin{bmatrix} e_d^t \\ e_d^r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_{p_t} \\ e_{p_r} \\ e_{\varepsilon_t} \\ e_{\varepsilon_r} \end{bmatrix}, \quad \begin{bmatrix} f_b^t \\ f_b^r \\ e_b^t \\ e_b^r \end{bmatrix} = \begin{bmatrix} e_{p_t} \mid_{\partial Z} \\ e_{p_r} \mid_{\partial Z} \\ e_{\varepsilon_t} \mid_{\partial Z} \\ e_{\varepsilon_r} \mid_{\partial Z} \end{bmatrix} \right\} \tag{4.160}$$

Then $\mathscr{D}_d = \mathscr{D}_d{}^{\perp}$, that is $\mathscr{D}_d$ is a Stokes-Dirac structure.

Proof. The proof is very similar to the one given for Proposition 4.3 and Proposition 4.6.

The port-Hamiltonian formulation of the Timoshenko beam with boundary and distributed energy flow can be obtained simply combining the Stokes-Dirac structure $\mathscr{D}_d$ (4.160) with (4.155) and (4.156). The resulting model is given in the following:

Definition 4.12. The port-Hamiltonian model of the Timoshenko beam with Stokes-Dirac structure $\mathscr{D}_d$ (4.160) and Hamiltonian H (4.150) is given by

$$\begin{aligned}
\begin{bmatrix} \frac{\partial p_t}{\partial t} \\ \frac{\partial p_r}{\partial t} \\ \frac{\partial \varepsilon_t}{\partial t} \\ \frac{\partial \varepsilon_r}{\partial t} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & \mathrm{d} & 0 \\ 0 & 0 & * & \mathrm{d} \\ \mathrm{d} & -* & 0 & 0 \\ 0 & \mathrm{d} & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_{p_t} H \\ \delta_{p_r} H \\ \delta_{\varepsilon_t} H \\ \delta_{\varepsilon_r} H \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_d^t \\ f_d^r \end{bmatrix} \\
\begin{bmatrix} e_d^t \\ e_d^r \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_{p_t} H \\ \delta_{p_r} H \\ \delta_{\varepsilon_t} H \\ \delta_{\varepsilon_r} H \end{bmatrix}, \quad \begin{bmatrix} f_b^t \\ f_b^r \\ e_b^t \\ e_b^r \end{bmatrix} = \begin{bmatrix} \delta_{p_t} H \mid_{\partial Z} \\ \delta_{p_r} H \mid_{\partial Z} \\ \delta_{\varepsilon_t} H \mid_{\partial Z} \\ \delta_{\varepsilon_r} H \mid_{\partial Z} \end{bmatrix}
\end{aligned} \tag{4.161}$$

The energy balance equation (4.158) becomes

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_{\partial Z} \left(f_b^t \wedge e_b^t + f_b^r \wedge e_b^r \right) + \int_Z \left(f_d^t \wedge e_d^t + f_d^r \wedge e_d^r \right) \tag{4.162}$$

which expresses the fact that the variation of internal stored energy equals the power supplied to the system through the boundary and the distributed port.

4.3.2 Nonlinear flexible link

In this section, a simple way to model flexible robotic links is presented. Differently from classical approaches and from Euler-Bernoulli [62, 100] or Timoshenko theory [89, 102, 121] (see also Sect. 4.3.1), the proposed model is able to describe large deflections in 3-D space and does not rely on any finite dimensional approximation (e.g. modal approximation). The model has been formulated within the port Hamiltonian formalism because intuitive considerations on the geometric behavior of the elastic link naturally define a Stokes–Dirac structure. Moreover, it should be already clear that port Hamiltonian systems can be easily *interconnected*, thus allowing the description of complex systems as a composition of parts in an object-oriented way. By combining rigid bodies, springs, dampers, joints and, finally, flexible links (i.e. by extending the modelling procedure illustrated in Sect. 3.2, see [124]), it is virtually possible to model and mathematically describe whatever complex mechanical structure formed by beams, without relying on rough finite dimensional approximation of the elastic behavior, usually deduced from a combination of modal analysis and system identification, [46, 61, 62, 155, 207, 215].

Consider a slender flexible beam of length L and with an unstressed configuration which is *not* required to be a straight line. As illustrated in Fig. 4.2 and fol-

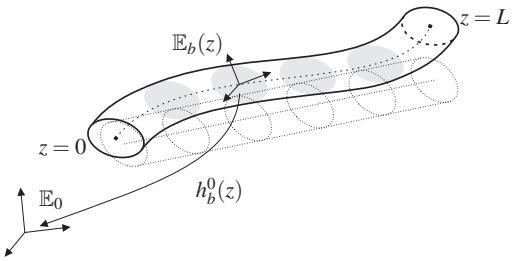

Fig. 4.2 Schematic representation of a flexible link in the deformed and unstressed configurations.

lowing [88, 193], if $z \in [0, L]$ denotes the position along the link in the unstressed configuration, assume that the configuration in the space of the cross section with respect to an inertial reference $\mathbb{E}_0$ is given by $h_b^0(z) \in SE_b^0(3)$, where the subscript b denotes the *body reference* $\mathbb{E}_b(z)$ attached to the cross section. The motion of the cross section is due to a wrench $w_b^{b,0}(z)$ and described by a twist $t_b^{b,0}(z)$. Both quantities are expressed in $\mathbb{E}_b(z)$.

Making use of (3.41), the relative motion between infinitesimally closed cross sections due to elasticity is given by

$$t_\delta^b(z) = \mathrm{Ad}_{h_0^b(z)} \mathrm{d} \left(\mathrm{Ad}_{h_b^0(z)} t_b^{b,0}(z) \right) \tag{4.163}$$

if expressed in body reference. Note that thanks to the use of twists, in case in which a pure rigid body motion is taking place, $t_b^{b,0}(z)$ would be the same for all $z \in [0, L]$ and equal to the rigid body motion.

As shown in Fig. 4.3a, the twist $t_b^{b,0}$ represents the "velocity" of the cross section in space and is defined in each point of the spatial domain $[0, L]$. On the other hand, $t_\delta^b(z)$ provides the relative "velocity" of the cross section in $z + \Delta z$ with respect to the one in z ($\Delta z \to 0$) and can be integrated along the spatial domain for the same instant in time to compute how the link deformation changes in time. Integrating $t_b^{b,0}$ in z does not provide any meaningful quantity. Even if $t_b^{b,0}$ and t_δ^b are both twists, they are different objects. Since $t_b^{b,0}$ is a function in $[0, L]$, it is a $se(3)$-valued zero-form, while t_δ^b is a $se(3)$-valued one form. Roughly speaking, t_δ^b requires an infinitesimal Δz to make sense or, equivalently, can be integrated along the line.

The relative motion $t_\delta^b(z)$ is related to a deformation and, consequently, a wrench $w_b^{b,0}$ acting on the cross section due to the elastic forces is present. As reported in Fig. 4.3b, the difference between the elastic forces acting on the cross sections in $z + \Delta z$ and in z generates a wrench w_δ^b on the link element $[z, z + \Delta z]$ which is responsible of the motion. Making use of (3.44), i.e. dually with respect to (4.163), we have that

$$w_\delta^b(z) = \mathrm{Ad}^*_{h_b^0(z)} \mathrm{d} \left(\mathrm{Ad}^*_{h_0^b(z)} w_b^{b,0}(z) \right) \tag{4.164}$$

where $w_b^{b,0}$ and w_δ^b are respectively $se^*(3)$-valued zero and one-forms. The integral over the spatial domain of w_δ^b provides the total wrench acting on the link.

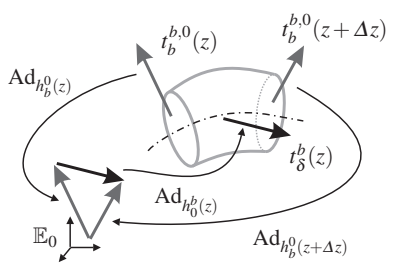

Fig. 4.3a Computation of $t_\delta^b(z)$.

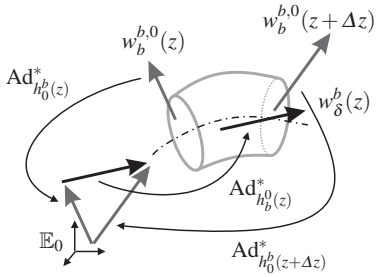

Fig. 4.3b Computation of $w_b^{b,0}(z)$.

The link can interact with the environment through a pair of ports at its extremities. The boundary port variables are the twist and wrench of the cross sections in $z=0$ and $z=L$,

$$\left(t_b^{b,0}(0), w_b^{b,0}(0)\right) \qquad \left(t_b^{b,0}(L), w_b^{b,0}(L)\right) \tag{4.165}$$

Note that these quantities are expressed in body references $\mathbb{E}_b(0)$ and $\mathbb{E}_b(L)$.

Equations (4.163) and (4.164), together with the boundary terms (4.165), are fundamental in the definition of the Stokes–Dirac structure describing the energetic structure of the flexible link. Before providing such a Stokes–Dirac structure, it is necessary to define the space of power variables. If $\mathscr{V}$ is a (finite dimensional) linear space, denote with $\Omega^k_{\mathscr{V}}(Z)$ the space of $\mathscr{V}$-valued k-forms on Z, [77]. Then, as far as the space of flows is concerned, let us assume $\mathscr{F}^d = \Omega^1_{se(3)}(Z) \times \Omega^1_{se^*(3)}(Z)$, while the space of efforts is $\mathscr{E}^d = \Omega^0_{se^*(3)}(Z) \times \Omega^0_{se(3)}(Z)$. This choice will become more clear later. From (4.165), the *natural* space of boundary power variables is given by

$$\mathscr{F}^B \times \mathscr{E}^B = \Omega^0_{se(3)}(\partial Z) \times \Omega^0_{se^*(3)}(\partial Z) = \Big(se(3) \times se^*(3)\Big) \times \Big(se(3) \times se^*(3)\Big)$$

since both the pairs twist/wrench in $z=0$ and $z=L$ have to be taken into account.

Given $(f^q, f^p, f^B, e^q, e^p, e^B) \in \mathscr{F}^d \times \mathscr{F}^B \times \mathscr{E}^d \times \mathscr{E}^B$, the associated power (i.e. the dual product) can be computed as follows:

$$\begin{aligned}\big\langle (e^q, e^p, e^B) \mid (f^q, f^p, f^B) \big\rangle = \\ = \int_Z * \big(\langle e^q \mid *f^q \rangle + \langle *f^p \mid e^p \rangle \big) + \big\langle e^B \mid f^B \big\rangle \Big|_{z=0}^{z=L}\end{aligned} \tag{4.166}$$

where $*$ is the Hodge star operator mapping one-forms to zero-forms and vice-versa (see Sect. 4.2.1.4). The dual product on the right side is the natural pairing between elements of $se(3)$ and $se^*(3)$. Note that (4.166) makes sense since $*f^q$ and $*f^p$ are zero-forms (i.e. functions) on Z and then $*\big(\langle e^q \mid *f^q \rangle + \langle *f^p \mid e^p \rangle \big)$ a one-form that can be integrated on Z.

Proposition 4.9. *The set*

$$\mathscr{D} = \left\{ (f^q, f^p, f^B, e^q, e^p, e^B) \in \mathscr{F}^d \times \mathscr{F}^B \times \mathscr{E}^d \times \mathscr{E}^B \;\middle|\; \begin{bmatrix} f^q \\ f^p \end{bmatrix} = - \begin{bmatrix} \mathrm{Ad}_{h_0^b(z)} d \left(\mathrm{Ad}_{h_b^0(z)} e^p \right) \\ \mathrm{Ad}^*_{h_b^0(z)} d \left(\mathrm{Ad}^*_{h_0^b(z)} e^q \right) \end{bmatrix}, \; \begin{bmatrix} f^B \\ e^B \end{bmatrix} = \begin{bmatrix} e^q \mid_{\partial Z} \\ e^p \mid_{\partial Z} \end{bmatrix} \right\} \tag{4.167}$$

is a Stokes–Dirac structure with respect to the +pairing determined by the dual product (4.166).

Proof. As discussed in [88] in the planar case, it is possible to find a coordinate change that maps (4.167) into the Stokes–Dirac structure of Definition 4.8. An alternative method is based on integration by parts and takes advantage of the skew-adjoint properties of the differential operator that defines the subset (4.167), as in the proof of Proposition 4.3. This method is extensively adopted also in [125–127] in similar proofs involving also higher order differential operators.

The energy (state) variables associated with the flexible link are the infinitesimal "deformation" q, i.e. the strain, and momentum p, expressed in body reference. More in details, the state space is defined as $\mathscr{X} = \Omega^1_{se(3)}(Z) \times \Omega^1_{se^*(3)}(Z)$, with q and p $se(3)$-valued and $se^*(3)$-valued one-forms respectively. From a *physical* point of view, it is necessary that these quantities are 1-forms because they are *densities*.

Once the space of state variables has been defined, the last step is the definition of a Hamiltonian function, given by the sum of two contributions: the kinetic energy and the potential elastic one, due to deformation. As far as the kinetic energy is concerned, denote by I the inertia tensor which defines a quadratic form on $se(3)$. As discussed in Sect. 3.2, this tensor uniquely defines a bijection between elements in $se(3)$ and elements in $se^*(3)$. More precisely, given $t_b^{0,b} \in se(3)$, there is a unique $p \in se^*(3)$ such that $p(\bar{t}_b) = \left\langle t_b^{0,b}, \bar{t}_b \right\rangle_I$, for every $\bar{t}_b \in se(3)$. The quantity p is the momentum of the cross section. Since I is non-singular, it is possible to define the following one-form which represents the kinetic energy density of the link:

$$\mathscr{K}(p) = \frac{1}{2} * \langle *p, *p \rangle_Y \tag{4.168}$$

where $Y = I^{-1}$. Note that $*p$ is a function of $z \in Z$ with values in $se^*(3)$. As far as the elastic energy contribution is concerned, denote by C the compliance tensor, with inverse C^{-1}, which defines a quadratic form on $se(3)$. Then, the elastic energy density is given by the following one-form, [75, 188]:

$$\mathscr{W}(q) = \frac{1}{2} * \langle *q, *q \rangle_{C^{-1}} \tag{4.169}$$

Again, $*q$ is a function of $z \in Z$ with values in $se(3)$. Relation (4.169) holds under the hypothesis of linear elasticity theory since the corresponding energy density is *quadratic* in the strain. If non-linear effects have to be taken into account, (4.169) should be replaced by a map $\mathscr{W} : Z \times \Omega^1_{se(3)}(Z) \to \Omega^1_{\mathbb{R}}(Z)$, in which the dependence

on the spatial variables has been also considered. For simplicity, in the remaining part of the paper, a quadratic elastic energy density in the form (4.169) is assumed. Finally, from (4.168) and (4.169), the total energy (Hamiltonian) function is given by

$$H(p,q) = \frac{1}{2}\int_Z \Big(\mathscr{K}(p) + \mathscr{W}(q)\Big) = \frac{1}{2}\int_Z *\Big(\langle *p, *p\rangle_Y + \langle *q, *q\rangle_{C^{-1}}\Big) \tag{4.170}$$

The port Hamiltonian model of a dynamical system follows from the corresponding Stokes–Dirac structure once the port behavior of the energy storing components (and in case of the resistive part) has been specified. Consequently, the port Hamiltonian model of the link results from (4.167) and from

$$-f^q = \frac{\partial q}{\partial t}\left(= t^b_\delta\right) \qquad -f^p = \frac{\partial p}{\partial t} - p \wedge t^{b,0}_b \left(= w^b_\delta\right) \tag{4.171}$$

and

$$e^q = \delta_q H \left(= w^{b,0}_b\right) \qquad e^p = \delta_p H \left(= t^{b,0}_b\right) \tag{4.172}$$

The second relation in (4.171) is simply the second law of dynamics in body reference, with $p \wedge t^{b,0}_b \equiv \mathrm{ad}^*_{t^{b,0}_b} p$ (cf. Sect. 3.2).

The port Hamiltonian representation of the link follows immediately. Note how the results presented in [88] have been easily extended in order to cope with deformation in 3-dimensional space.

$$\begin{cases} \dfrac{\partial q}{\partial t} = \mathrm{Ad}_{h^b_0(z)}\mathrm{d}\left(\mathrm{Ad}_{h^0_b(z)}\,\delta_p H\right) \\ \dfrac{\partial p}{\partial t} = \mathrm{Ad}^*_{h^0_b(z)}\mathrm{d}\left(\mathrm{Ad}^*_{h^b_0(z)}\,\delta_q H\right) + p \wedge \delta_p H \end{cases} \tag{4.173}$$

Furthermore, the boundary terms are given by

$$\begin{aligned} f^B(0) &= \delta_p H\,|_{z=0} & e^B(0) &= \delta_q H\,|_{z=0} \\ f^B(L) &= \delta_p H\,|_{z=L} & e^B(L) &= \delta_q H\,|_{z=L} \end{aligned} \tag{4.174}$$

From the power conserving properties of a Dirac structure, the following energy balance relation easily follows:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \left\langle e^B(L) \mid f^B(L)\right\rangle - \left\langle e^B(0) \mid f^B(0)\right\rangle \tag{4.175}$$

This relation states an obvious property of this physical system, i.e. the fact that the variation of internal energy equals the total power flow at its boundary. Since no dissipative effect is considered, if the boundary energy flow is set to zero (i.e. in the case of a flexible beam clamped at both its extremities) energy is conserved. At the

end of this section, a way to include dissipative effects, but also distributed actuation or gravity, in terms of port interconnection is illustrated.

Note that the model (4.173) depends on the configuration of the cross section with respect to the inertial reference $h_b^0(z)$, but the state variable of the system is the strain q related to the infinitesimal deformation. However, if $\hat{h}_b^0 : Z \to SE_b^0(3)$ denotes the unstressed configuration, the "twist" $\hat{n} \in \Omega^1_{se(3)}(Z)$ describing how the unstressed configuration evolves in the spatial variable s and expressed in body reference is given by (see also [193]):

$$\hat{n} = \left(\hat{h}_b^0\right)^{-1} \mathrm{d}\hat{h}_b^0 \tag{4.176}$$

Similarly to what is presented in Sect. 3.2.3, for all $t \in se(3)$, relation (4.163) becomes

$$\mathrm{Ad}_{h_0^b}\mathrm{d}\left(\mathrm{Ad}_{h_b^0} t\right) = \mathrm{d}t + \mathrm{ad}_{(q+\hat{n})}t \tag{4.177}$$

and, dually, for every $w \in se^*(3)$, (4.164) can be written as

$$\mathrm{Ad}^*_{h_b^0}\mathrm{d}\left(\mathrm{Ad}^*_{h_0^b} w\right) = \mathrm{d}w - \mathrm{ad}^*_{(q+\hat{n})}w \tag{4.178}$$

By combining (4.173) with (4.177) and (4.178), the dynamics of the flexible link assume the following equivalent expression:

$$\begin{cases} \dfrac{\partial q}{\partial t} = \mathrm{d}\delta_p H + \mathrm{ad}_{(q+\hat{n})}\delta_p H \\ \dfrac{\partial p}{\partial t} = \mathrm{d}\delta_q H - \mathrm{ad}^*_{(q+\hat{n})}\delta_q H + p \wedge \delta_p H \end{cases} \tag{4.179}$$

Note the invariance of the dynamics to group action. The boundary terms remain the same as in (4.174).

In (4.179), it is not immediate to see how the effects of a gravity field can be taken into account since the configuration of the cross section with respect to the inertial reference $h_b^0(z)$ is not available. Moreover, at present stage it is not clear how dissipative effects can be taken into account and if it is possible to include distributed actuation along the spatial domain of the flexible link. As discussed in Sect. 4.2.6.1 for Maxwell's equations, in Sect. 4.3.1 for the Timoshenko beam and in [126] for piezo-electric materials, a possible solution can be to modify the Stokes–Dirac structure (4.167) by including a distributed port along the domain. In the case of flexible link, this distributed port is characterized by a space of power variables $\mathscr{F}^e \times \mathscr{E}^e = \Omega^0_{se(3)}(Z) \times \Omega^1_{se^*(3)}(Z)$, in which flows are twists, defined point-wise, and the efforts are wrench density. If it is supposed that these quantities are expressed in body reference, by taking into account (4.177) and (4.178), the Stokes–Dirac structure (4.167) modifies into

$$\mathscr{D} = \left\{ (f^q, f^p, f^e, f^B, e^q, e^p, e^e, e^B) \in \mathscr{F}^d \times \mathscr{F}^e \times \mathscr{F}^B \times \mathscr{E}^d \times \mathscr{E}^e \times \mathscr{E}^B \;\middle|\; \begin{bmatrix} f^q \\ f^p \end{bmatrix} = - \begin{bmatrix} \mathrm{d}e^p + \mathrm{ad}_{(q+\hat{n})} e^p \\ \mathrm{d}e^q - \mathrm{ad}^*_{(q+\hat{n})} e^q \end{bmatrix} - \begin{bmatrix} 0 \\ e^e \end{bmatrix}, \quad f^e = e^p, \quad \begin{bmatrix} f^B \\ e^B \end{bmatrix} = \begin{bmatrix} e^q \,|_{\partial\mathscr{Z}} \\ e^p \,|_{\partial\mathscr{Z}} \end{bmatrix} \right\} \tag{4.180}$$

Note that the boundary terms (f^B, e^B) defined in (4.167) remain the same. Thanks to this distributed port it is possible to act on the flexible link along its spatial domain by imposing a distributed "force" and, at the same time, to measure the "velocity" of the cross section in space. This distributed port can be *terminated* on a distributed impedance in order to model dissipative effects, such as viscous friction, as in Sect. 4.2.6.1. In the linear case, it is sufficient to introduce a quadratic form D on $se(3)$ describing dissipation in body frame and to impose that $e^e = -*D\, f^e = -*D\, \delta_p H$. Note that the power flow is always less than zero. As far as concerns gravity, a possible solution can be to integrate in time $f^e = \delta_p H \equiv t_b^{0,b}$ so that position and orientation in space of the cross section is known. Then, gravity is a distributed source of effort modulates by orientation information that has to be interconnected to e^e.

Finally, suppose that deformation is small, that is $q \simeq 0$. If the unstressed configuration is assumed to be a straight line along the x-axis of the fixed reference frame, (4.179) simplifies to the superposition of a Timoshenko beam dynamics in the y and z directions and of a transverse wave in the x direction. This can be proved by writing (4.179) in coordinates and by taking into account the port Hamiltonian formulation of the Timoshenko beam equations presented in Sect. 4.3.1. This result has been originally verified in [88] for the planar case by following a different procedure.

4.3.3 Ideal isentropic fluid

The flow of an ideal compressible isentropic fluid in three dimensions consists in a system of two conservation laws, the mass and the momentum balance equations. It is described in Eulerian representation by the Euler equations [2, chap. 5], [8, chap. 1], [157, chap. 2] which we recall below in the vector calculus notation:

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\nabla \cdot (\rho v) \\ \frac{\partial v}{\partial t} &= -v \cdot \nabla v - \frac{1}{\rho} \nabla p \end{aligned} \tag{4.181}$$

with $\rho(z,t) \in \mathbb{R}$ the mass density at the spatial position $z \in \mathbb{R}^3$ at time t, $v(z,t) \in \mathbb{R}^3$ the (Eulerian) velocity of the fluid at spatial position z and time t, and $p(z,t)$ the pressure function, derivable from an internal energy function $U(\rho)$ as

$$p(z,t) = \rho^2(z,t)\frac{\partial U}{\partial \rho}(\rho(z,t)) \tag{4.182}$$

The Hamiltonian formulation of the ideal compressible isentropic fluid is classical and has been discussed with respect to different choices of state variables and Poisson brackets in particular in [8, 134, 135, 148, 149]. In the sequel we shall firstly construct a Dirac structure which extends the Poisson bracket of the Hamiltonian formulation arising from the formulation of a system of two conservation laws in order to encompass also possible exchange of the conserved quantities and energy through the boundary of the spatial domain. Therefore we shall identify the functions and vector fields in (4.181) and (4.182) with different differential forms. Secondly we shall define a boundary port-Hamiltonian system on this Dirac structure which extends the Hamiltonian formulation of (4.181).

Let $D \subset \mathbb{R}^3$ be a given domain, filled with the fluid. We assume the existence of a Riemannian metric $\langle \cdot, \cdot \rangle$ on D; usually the standard Euclidean metric on $\mathbb{R}^3$. Let $Z \subset D$ be any 3-dimensional manifold with boundary ∂Z. First we identify the mass-density ρ with a 3-form on Z (see e.g. [134, 135]), that is, with an element of $\Omega^3(Z)$. Furthermore, we identify the Eulerian vector field v with a 1-form on Z, that is, with an element of $\Omega^1(Z)$. By the existence of the Riemannian metric on Z we can, by "index raising" or "index lowering", identify vector fields with 1-forms and vice versa. The vector space of flow variables $\mathscr{F}_{p,q}$ and of the effort variables $\mathscr{E}_{p,q}$ with $n = 3$, $p = 3$ and $q = 1$ as defined in Sect. 4.2.4 are hence defined as:

$$\mathscr{F}_{p,q} = \Omega^3(Z) \times \Omega^1(Z) \times \Omega^0(\partial Z) \tag{4.183}$$

and

$$\mathscr{E}_{p,q} = \Omega^0(Z) \times \Omega^2(Z) \times \Omega^2(\partial Z) \tag{4.184}$$

In order to account for the convection term in the momentum balance equation, the Hamiltonian formulation of the system (4.181) is defined on a Poisson structure which extends the Poisson bracket associated with the canonical inter-domain coupling, to a Poisson bracket including a Lie-Poisson bracket associated with divergence free vector fields [8, chap. 1]. This Poisson bracket is extended to the following Dirac structure.

Definition 4.13. The vector space, called Stokes-Dirac structure with convection term and denoted by $\mathscr{D}^m$ defined by

$$\mathscr{D}^m := \left\{ \left(f_p, f_v, f_b, e_\rho, e_v, e_b\right) \in \mathscr{F}_{p,q} \times \mathscr{E}_{p,q} \mid \begin{bmatrix} f_\rho \\ f_v \end{bmatrix} = \begin{bmatrix} \mathrm{d}e_v \\ \mathrm{d}e_\rho + \frac{1}{*\rho} *((*\mathrm{d}v) \wedge (*e_v)) \end{bmatrix}, \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} e_{\rho|\partial Z} \\ -e_{v|\partial Z} \end{bmatrix} \right\} \tag{4.185}$$

where $*$ denotes the Hodge star operator (corresponding to the Riemannian metric on Z) is a Dirac structure of the bond space $\mathscr{F}_{p,q} \times \mathscr{E}_{p,q}$ defined in (4.183) and (4.184).

A fundamental difference of the modified Stokes-Dirac structure $\mathscr{D}^m$ with respect to the standard Stokes-Dirac structure is that $\mathscr{D}^m$ is non-linear in the sense that it depends explicitly on the energy variables ρ and v (via the terms $*\rho$ and $\mathrm{d}v$ in the additional term $\frac{1}{*\rho} *((*\mathrm{d}v) \wedge (*e_v))$. Completely similar to the proof of Proposition 4.3, it is shown that $(\mathscr{D}^m(\rho, v))^\perp = \mathscr{D}^m(\rho, v)$ for all ρ, v; the crucial additional observation is that the expression

$$e_v^2 \wedge *((*\mathrm{d}v) \wedge (*e_v^1)) \tag{4.186}$$

is *skew-symmetric* in $e_v^1, e_v^2 \in \Omega^2(Z)$.

Remark 4.5. In the standard Euclidean metric, identifying via the Hodge star operator 2-forms β_i with 1-forms, and representing 1-forms as vectors, we have in vector calculus notation the equality

$$\beta_2 \wedge *(\alpha \wedge *\beta_1) = \alpha \cdot (\beta_1 \times \beta_2) \tag{4.187}$$

for all 2-forms β_1, β_2 and 1-forms α. This shows clearly the skew-symmetry of (4.186).

The Eulerian equations (4.181) for an ideal isentropic fluid are obtained in the port-controlled Hamiltonian representation by considering the Hamiltonian

$$H(\rho, v) := \int_Z \left[\frac{1}{2} \left\langle v^\sharp, v^\sharp \right\rangle \rho + U(*\rho)\rho \right] \tag{4.188}$$

with $v^\sharp$ the vector field corresponding to the 1-form v ("index lowering"), and $U(*\rho)$ the potential energy. Indeed, by making the substitutions (4.64) and (4.65) in $\mathscr{D}^m$, and noting that

$$\delta H = (\delta_\rho H, \delta_v H) = \left(\frac{1}{2} \left\langle v^\sharp, v^\sharp \right\rangle + \frac{\partial}{\partial \tilde{\rho}} \left(\tilde{\rho} U(\tilde{\rho})\right), \; i_{v^\sharp} \rho \right) \tag{4.189}$$

with $\tilde{\rho} := *\rho$, the port-Hamiltonian system takes the form

$$\begin{aligned} -\frac{\partial \rho}{\partial t} &= \mathrm{d}(i_{v^\sharp}\rho) \\ -\frac{\partial v}{\partial t} &= \mathrm{d}\left(\frac{1}{2}\left\langle v^\sharp, v^\sharp\right\rangle + w(*\rho)\right) + \frac{1}{*\rho}\left((*\mathrm{d}v)\wedge(*i_{v^\sharp}\rho)\right) \\ f_b &= \left[\frac{1}{2}\left\langle v^\sharp, v^\sharp\right\rangle + w(*\rho)\right]_{|\partial Z} \\ e_b &= -i_{v^\sharp}\rho_{|\partial Z} \end{aligned} \tag{4.190}$$

with

$$w(\tilde{\rho}) := \frac{\partial}{\partial \tilde{\rho}}(\tilde{\rho}U(\tilde{\rho})) \tag{4.191}$$

the *enthalpy*. The expression $\delta_\rho H = \frac{1}{2}\left\langle v^\sharp, v^\sharp\right\rangle + w(\tilde{\rho})$ is known as the *Bernoulli function*.

The first two equations of (4.190) can be seen to represent the Eulerian equations (4.181). The first equation corresponds to the *mass balance equation*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\varphi_t(V)} \rho = 0 \tag{4.192}$$

where V denotes an arbitrary volume in Z, and φ_t is the flow of the fluid (transforming the material volume V at $t = 0$ to the volume $\varphi_t(V)$ at time t). Indeed, (4.192) for any V is equivalent to

$$\frac{\partial \rho}{\partial t} + L_{v^\sharp}\rho = 0 \tag{4.193}$$

Since by Cartan's magical formula $L_{v^\sharp}\rho = \mathrm{d}(i_{v^\sharp}\rho) + i_{v^\sharp}\mathrm{d}\rho = \mathrm{d}(i_{v^\sharp}\rho)$ (since $\mathrm{d}\rho = 0$) this yields the first line of (4.190).

For the identification of the second equation of (4.190) with the second equation of (4.186) we note the following. Interpret $\nabla\cdot$ in (4.181) as the co-variant derivative corresponding to the assumed Riemannian metric $\langle\cdot,\cdot\rangle$ on Z. For a vector field u on Z, let $u^\flat$ denote the corresponding 1-form $u^\flat := i_u\langle\cdot,\cdot\rangle$ ("index raising"). The co-variant derivative ∇ is related to the Lie derivative by the following formula (see for a proof [8], p. 202):

$$L_u u^\flat = (\nabla_u u)^\flat + \frac{1}{2}\mathrm{d}\langle u, u\rangle \tag{4.194}$$

Since by Cartan's magical formula $L_u u^\flat = i_u \mathrm{d}u^\flat + \mathrm{d}(i_u u^\flat) = i_u \mathrm{d}u^\flat + \mathrm{d}\langle u, u\rangle$, (4.194) can be also written as

$$(\nabla_u u)^\flat = i_u \mathrm{d}u^\flat + \frac{1}{2}\mathrm{d}\langle u, u\rangle \tag{4.195}$$

This is the coordinate-free analog of the well-known vector calculus formula $u\cdot\nabla u = \operatorname{curl} u\times u + \frac{1}{2}\nabla|u|^2$. Furthermore we have the identity

$$i_{v^\sharp}\mathrm{d}v = \frac{1}{*\rho}*\left((*\mathrm{d}v)\wedge(*i_{v^\sharp}\rho)\right) \tag{4.196}$$

Finally, we have the following well-known relation between enthalpy and pressure (obtained from (4.187) and (4.191))

$$\frac{1}{\tilde{\rho}}\mathrm{d}p = \mathrm{d}(w(\tilde{\rho})). \tag{4.197}$$

Hence by (4.195) (with $u = v^\sharp$), (4.108) and (4.197), we may rewrite the second equation of (4.190) as

$$-\frac{\partial v}{\partial t} = \left(\nabla_{v^\sharp} v^\sharp\right)^\flat + \frac{1}{*\rho}\mathrm{d}p \tag{4.198}$$

which is the coordinate-free formulation of the second equation of (4.181).

The boundary variables f_b and e_b given in (4.190) are respectively the *stagnation pressure* at the boundary divided by ρ, and the (incoming) *mass flow* through the boundary. The energy-balance (4.107) can be written out as

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &= \int_{\partial Z} e_b \wedge f_b = -\int_{\partial Z} i_{v^\sharp}\rho \wedge \left[\frac{1}{2}\left\langle v^\sharp, v^\sharp\right\rangle + w(*\rho)\right] \\ &= -\int_{\partial Z} i_{v^\sharp}\left[\frac{1}{2}\left\langle v^\sharp, v^\sharp\right\rangle \rho + w(*\rho)\rho\right] \\ &= -\int_{\partial Z} i_{v^\sharp}\left[\frac{1}{2}\left\langle v^\sharp, v^\sharp\right\rangle \rho + U(*\rho)\rho\right] - \int_{\partial Z} i_{v^\sharp}(*p)\end{aligned} \tag{4.199}$$

where for the last equality we have used the relation (following from (4.182), (4.191))

$$w(*\rho)\rho = U(*\rho)\rho + *p \tag{4.200}$$

The first term in the last line of (4.199) corresponds to the convected energy through the boundary ∂Z, while the second term is (minus) the external work (static pressure times velocity).

Alternatively, emphasizing the interpretation of v as a 1-form, there is an alternative interpretation of the momentum balance equation as *Kelvin's circulation theorem*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\varphi_t(C)} v = 0 \tag{4.201}$$

where C denotes any *closed* contour. Indeed, (4.201) for any closed C is equivalent to the 1-form $\frac{\partial v}{\partial t} + L_{v^\sharp} v$ being *closed*. By (4.194) this is equivalent to requiring

$$\frac{\partial v}{\partial t} + \left(\nabla_{v^\sharp} v^\sharp\right)^\flat \tag{4.202}$$

to be closed, that is

$$\frac{\partial v}{\partial t} + \left(\nabla_{v^\sharp} v^\sharp\right)^\flat = -\mathrm{d}k \tag{4.203}$$

for some (possibly locally defined) $k : Z \to \mathbb{R}$. Now additionally requiring that this function k depends on z through ρ, that is

$$k(z) = w(\rho(z)) \tag{4.204}$$

for some function w, we recover (4.198) with $\frac{1}{*\rho}\mathrm{d}p$ replaced by $\mathrm{d}w$, i.e. the differential of the enthalpy.

Remark 4.6. In the case of a one- or two-dimensional fluid flow the extra term in the Dirac structure $\mathscr{D}^m$ as compared with the standard Stokes-Dirac structure $\mathscr{D}$ vanishes, and so in these cases the fluid dynamics is defined as a boundary port-Hamiltonian system with respect to the canonical Stokes-Dirac structure (with ρ being a 1-form, respectively, a 2-form).

Furthermore, if in the 3-dimensional case the 2-form $\mathrm{d}v(t)$ happens to be zero at a certain time-instant $t = t_0$ (*irrotational flow*), then it *continues* to be zero for all time $t \geq t_0$. Hence also in this case the extra term (4.186) in the modified Stokes-Dirac structure $\mathscr{D}^m$ vanishes, and the port-Hamiltonian system describing the Euler equations reduces to the standard distributed-parameter port-Hamiltonian system given in Proposition 4.3.

Remark 4.7. For the modified Stokes-Dirac structure $\mathscr{D}^m$ given in (4.185), the space of admissible mappings K_{adm} given in (4.100) is the same as for the Stokes-Dirac structure, but the resulting skew-symmetric bracket has an additional term:

$$\begin{aligned}\{k^1,k^2\}_{\mathscr{D}^m} = \int_Z [&(\delta_\rho k^1)\wedge(-1)^r \mathrm{d}(\delta_q k^2) + (\delta_q k^1)\wedge \mathrm{d}(\delta_p k^2) \\ &+ \frac{1}{*\rho}\delta_v k^1 \wedge *((*\mathrm{d}v)\wedge(*\delta_v k^2))] \\ &- \int_{\partial Z}(-1)^{n-q}(\delta_q k^1)\wedge(\delta_p k^2)\end{aligned} \tag{4.205}$$

For the skew-symmetry of the additional term see (4.186) and Remark 4.5.

4.4 Conclusion

In this chapter we have presented the port-Hamiltonian formulation of distributed parameter systems. The core of the formulation is to consider systems of conservation laws and to use the intrinsic variables associated with conservation laws, namely exterior differential forms. We have considered essentially systems of two conservation laws in reversible interaction as they appear in elasto-dynamic and electro-magnetic systems. In this case we have defined a canonical Dirac structure called Stokes-Dirac structure which is canonical in the sense that it is defined solely on two adjoint (exterior) derivation and does not depend on any parameter. On this Dirac structure we have defined port-Hamiltonian systems which are extensions of infinite-dimensional Hamiltonian systems in the sense that they allow to encompass open physical systems which exchange energy and other conserved quantities at the boundary of their spatial domain.

Extensions of this canonical Stokes-Dirac structure and the associated boundary port Hamiltonian system have been presented for the Timoshenko beam, the ideal compressible fluid – where the Stokes-Dirac structure has to be completed in order to account for complex geometric configuration spaces as well as the convection of momentum – and for a nonlinear 3-D model of a flexible link. It should be mentioned that we did not present in this chapter other extensions to Stokes-Dirac structures associated linear matrix differential operator for which the reader is referred to [113, 211].

Finally it should also be mentioned that Stokes-Dirac structures also arise in dissipative systems. In this case they arise both from the formulation of the distributed parameter system as a system of conservation laws and the basic axioms of Irreversible Thermodynamics on the definition of the flux variables as linear functions of the generating force due to thermodynamical non-equilibrium. This has been illustrated on different examples of distributed parameter systems defined on one-dimensional spatial domains in this chapter and the reader is also referred to [211, 212].

Chapter 5
Control of Finite-Dimensional Port-Hamiltonian Systems

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Abstract We discuss in this chapter a number of approaches to exploit the model structure of port-Hamiltonian systems for control purposes. Actually, the formulation of physical control systems as port-Hamiltonian systems may lead in some cases to a *re-thinking* of standard control paradigms. Indeed, it opens up the way to formulate control problems in a way that is different and perhaps broader than usual. For example, formulating physical systems as port-Hamiltonian systems naturally leads to the consideration of 'impedance' control problems, where the *behavior* of the system at the interaction port is sought to be shaped by the addition of a controller system, and it suggests *energy-transfer* strategies, where the energy is sought to be transferred from one part the system to another. Furthermore, it naturally leads to the investigation of a particular type of dynamic controllers, namely those that can be also represented as port-Hamiltonian systems and that are attached to the given plant system in the same way as a physical system is interconnected to another physical system. As an application of this strategy of 'control by interconnection' within the port-Hamiltonian setting we consider the problem of (asymptotic) stabilization of a desired equilibrium by shaping the Hamiltonian into a Lyapunov function for this equilibrium. From a mathematical point of view we will show that the mathematical formalism of port-Hamiltonian systems provides various useful techniques, ranging from Casimir functions, Lyapunov function generation, shaping of the Dirac structure by composition, and the possibility to combine finite-dimensional and infinite-dimensional systems.

5.1 Introduction

As discussed in Chapter 2, network modeling of complex physical systems (with components from different physical domains) leads to a class of nonlinear systems, called *port-Hamiltonian systems* (see also [59, 138, 139, 144, 180, 183, 184]). Port-Hamiltonian systems are defined by a Dirac structure (formalizing the power-conserving interconnection structure of the system), an energy function (the Hamil-

tonian), and a resistive structure. Key property of Dirac structures is that the power-conserving interconnection or composition of Dirac structures again defines a Dirac structure, see Sect. 2.5 and [139, 179]. This implies that any power-conserving interconnection of port-Hamiltonian systems is again a port-Hamiltonian system, with Dirac structure being the composition of the Dirac structures of its constituent parts, Hamiltonian being the sum of the Hamiltonians, and total resistive structure determined by the resistive structures of the components taken together.

In this Chapter we describe how this framework may be exploited for control purposes. More precisely, we review the framework of *control by interconnection* within the port-Hamiltonian setting in a way that is more general than previous expositions. We discuss three sets of control problems which may be naturally addressed within this framework. First we present the problem of transferring the energy of one part of the system towards another part (Energy Control). Secondly, we recall the approach of asymptotic stabilization by Casimir generation for the closed-loop system and how this relies on the composition of Dirac structures. Thirdly, we pose the problem of Port Control, where by the addition of a port-Hamiltonian controller system we seek to shape the behavior of the system at the interaction port. We also focus on the precise role of energy dissipation in control strategies for port-Hamiltonian systems. We show in general that for a function to be a conserved quantity (Casimir) for *one* non-degenerate resistive relation at the resistive port it actually needs to be a Casimir for *all* resistive relations, as anticipated in Sect. 2.6.2.

Finally we present two recent control techniques, the interconnection and damping assignment passivity–based control (IDA–PBC), a technique that regulates the behavior of nonlinear systems assigning a desired port–Hamiltonian structure to the closed–loop, and the power shaping methodology that allow us to formulate the stabilization problem in terms of power (as opposed to energy) shaping, adding "derivative actions" in the control.

An important special case of port-Hamiltonian systems is the class of *input-state-output port-Hamiltonian systems*, introduced in Sect. 2.2.3, where there are no algebraic constraints on the state space variables, and the flow and effort variables at the resistive, control and interaction port are split into conjugated input-output pairs. Such input-state-output port-Hamiltonian systems are of the form expressed in (2.51), here reported for clarity sake:

$$P:\begin{cases} \dot{x} = \big[J(x) - R(x)\big]\dfrac{\partial H}{\partial x}(x) + g(x)u + k(x)d \\ y = g^{\mathrm{T}}(x)\dfrac{\partial H}{\partial x}(x) \qquad\qquad x \in \mathscr{X} \\ z = k^{\mathrm{T}}(x)\dfrac{\partial H}{\partial x}(x) \end{cases} \tag{5.1}$$

In (5.1), u, y are the input-output pairs corresponding to the control port $\mathscr{C}$, while d, z denote the input-output pairs of the interaction port $\mathscr{I}$ (see Fig. 2.2). Here the matrix $J(x)$ is skew-symmetric, that is $J(x) = -J^{\mathrm{T}}(x)$. The matrix $R(x) = R^{\mathrm{T}}(x) \geq 0$ specifies the resistive structure. Note that $y^{\mathrm{T}}u$ and $z^{\mathrm{T}}d$ still denote the power corre-

sponding to the control, respectively, interaction port. For more details, in particular extensions of (5.1) to feed-through terms, we refer to Sect. 2.2.4 and [74, 180].

5.2 Energy–balancing control

In this chapter we are interested in lumped–parameter systems interconnected to the external environment through some *port power variables* $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$, which are conjugated in the sense that their product has units of power (e.g., currents and voltages in electrical circuits, or forces and velocities in mechanical systems). We assume the system satisfies the *energy–balance equation*

$$\underbrace{H[x(t)] - H[x(0)]}_{stored\ \ energy} = \underbrace{\int_0^t u^{\mathrm{T}}(s)y(s)\mathrm{d}s}_{supplied} - \underbrace{d(t)}_{dissipated} \tag{5.2}$$

where $x \in \mathbb{R}^n$ is the state vector, $H(x)$ is the *total energy* function, and $d(t)$ is a non-negative function that captures the dissipation effects (e.g., due to resistances and frictions). Energy balancing is, of course, a universal property of physical systems; therefore, our class, which is nothing other than the well–known *passive* systems, captures a very broad range of applications that include nonlinear and time–varying dynamics.

The control objective is to regulate the static behavior, that is, the equilibrium, which is determined by the shape of the energy function. It is therefore natural to recast our control problem in terms of finding a dynamical system and an interconnection pattern such that the overall energy function takes the desired form. There are at least two important advantages of adopting such an "energy shaping" perspective of control:

1. The energy function determines not just the static behavior, but also, via the energy transfer between subsystems (through the ports), its transient behavior. Focusing our attention on the systems energy, we can then aim, not just at stabilization, but also at *performance* objectives that can, in principle, be expressed in terms of "optimal" energy transfer. Performance and not stability is, of course, the main concern in applications.
2. Practitioners are familiar with energy concepts, which can serve as a *lingua franca* to facilitate communication with control theorists, incorporating prior knowledge and providing physical interpretations of the control action.

Passivity–based control techniques achieve stabilization of nonlinear *feedback passive* systems assigning a storage function with a minimum at the desired equilibrium. For physical systems a natural candidate storage function is the difference between the stored and the supplied energies—leading to the so-called *energy–balancing control*, whose underlying stabilization mechanism is particularly appealing. Two important corollaries follow from (5.2)

- The energy of the uncontrolled system (i.e., with $u \equiv 0$) is non-increasing (that is, $H[x(t)] \leq H[x(0)]$), and it will actually decrease in the presence of dissipation. If the energy function is bounded from below, the system will eventually stop at a point of minimum energy. Also, as expected, the rate of convergence of the energy function is increased if we extract energy from the system, for instance, setting $u = -K_{di}y$, with $K_{di} = K_{di}^{\mathrm{T}} > 0$ a so–called damping injection gain.
- Given that
$$-\int_0^t u^{\mathrm{T}}(s)y(s)\mathrm{d}s \leq H[x(0)] < \infty$$
the total amount of energy that can be extracted from a passive system is bounded[1].

Usually, the point where the open–loop energy is minimal (which typically coincides with the zero state) is not the one of interest, and control is introduced to operate the system around some nonzero equilibrium point, say $x_\star$. Hence, the control problem consists in finding a control input $u = \beta(x) + v$ such that the energy supplied by the controller can be expressed as a function of the state. Indeed, from (5.2) we see if we can find a function $\beta(x)$ satisfying

$$-\int_0^t \beta^{\mathrm{T}}[x(s)]y(s)\mathrm{d}s = H_a[x(t)] + \kappa$$

for some function $H_a(x)$, then the control $u = \beta(x) + v$ will ensure that the map $v \to y$ is passive with new energy function

$$H_d(x) = H(x) + H_a(x). \tag{5.3}$$

For port–Hamiltonian systems, the following proposition characterizes the class of functions $\beta(x)$ and $H_a(x)$ such that the closed–loop system satisfies the new energy–balancing equation

$$H_d[x(t)] - H_d[x(0)] = \int_0^t v^{\mathrm{T}}(s)y(s)\mathrm{d}s - d_d(t)$$

with the dissipation term $d_d(t) \geq 0$ to increase the convergence rate.

Proposition 5.1 ([180]). *Consider the port–Hamiltonian system (5.1) without interaction ports, if we can find a function $\beta(x)$ and a vector function $K(x)$ satisfying*

$$\big[J(x) - R(x)\big]K(x) = g(x)\beta(x)$$

such that

[1] This property, which (somehow misleadingly) is often stated with the inequality inverted, will be instrumental in identifying the class of systems that are stabilizable with energy balancing PBC.

$$
\begin{aligned}
&i) \quad \frac{\partial K}{\partial x}(x) = \frac{\partial K}{\partial^T x}(x) \\
&ii) \quad K(x_\star) = -\frac{\partial H}{\partial x}(x_\star) \\
&iii) \quad \frac{\partial K}{\partial x}(x_\star) > -\frac{\partial^2 H}{\partial x^2}(x_\star)
\end{aligned}
$$

Then the closed–loop system is a port-Hamiltonian system of the form

$$\dot{x} = \big[J(x) - R(x)\big]\frac{\partial H_d}{\partial x}(x) \tag{5.4}$$

with H_d given by (5.3) and H_a satisfying $K = \frac{\partial H_a}{\partial x}(x)$. Furthermore, $x_\star$ is an stable equilibrium point of (5.4).

5.2.1 Dissipation obstacle

Unfortunately, energy–balancing stabilization is stymied by the existence of pervasive dissipation, that appears in many engineering applications. Indeed, it has been shown in [161] that a necessary condition to satisfy Proposition 5.1 is that the natural damping of the port–Hamiltonian system satisfies

$$R(x)K(x) = 0.$$

Since $R(x)$ is usually diagonal, this condition requires that no damping is present in the coordinates that need to be shaped, that is, the coordinate where the function $H(x)$ has to be modified. To characterize the dissipation obstacle it is convenient to adopt a *control–by–interconnection* viewpoint, which clearly reveals the limitations of energy–balancing control, as we will see in the following sections. Some control methodologies as the interconnection and damping assignment passivity–based control and power shaping, has been proposed in [159] and [161] respectively to overcome the dissipation obstacle. See also Sect. 2.6.2.1 and Sect. 5.4.6 for further details.

5.3 Control by port–interconnection

Control by port-interconnection is based on designing a controller system which is *interconnected* to the control port with port-variables (f_C, e_C). *In principle* this implies that we only consider *collocated* control, where the controller will only use the information about the plant port-Hamiltonian system that is contained in the conjugated pairs (f_C, e_C) of port variables of the control port, without using additional information about the plant (e.g. corresponding to observation on other parts of the plant system). In the second place, we will restrict attention to controller systems

which are themselves *also* port-Hamiltonian systems. There are two main reasons for this. One is that by doing so the closed-loop system is *again* a port-Hamiltonian system, allowing to easily ensure some desired properties. Furthermore, it will turn out that the port-Hamiltonian framework suggests useful ways to construct port-Hamiltonian controller systems. Second reason is that port-Hamiltonian controller systems allow in principle for a physical system realization (thus linking to passive control and systems design) and physical *interpretation* of the controller action. Of course, this raises the *realization problem* which (controller) systems can be represented as port-Hamiltonian systems, for which some partial answers are available [180].

Since we do not know the environment (or only have very limited information about it), but on the other hand, the system *will* interact with this unknown environment, the task of the controller is often two-fold:

1. to achieve a desired control goal (e.g. set-point regulation or tracking) if the interaction with the environment is marginal or can be compensated;
2. to make sure that the controlled system has a desired interaction behavior with its environment. It is fair to say that up to now the development of the theory of control of port-Hamiltonian systems has mostly concentrated on the second aspect (which at the same time, is often underdeveloped in other control theories).

Most successful approaches to deal with the second aspect of the control goal are those based on the concept of "*passivity*" (see also Sect. 2.6.1), such as *dissipativity theory*, *impedance control* and *Intrinsically Passive Control* (IPC). In fact, the port-Hamiltonian control theory can be regarded as an enhancement to the theory of passivity, making a much closer link with complex physical systems modeling at one hand and with the theory of dynamical systems (in particular, Hamiltonian dynamics) at the other hand.

As said above, in this section we will throughout consider controller systems which are again port-Hamiltonian systems, in the same way as the plant system is a port-Hamiltonian system. We will use the same symbols as above for the internal and external ports and port-variables of the controller port-Hamiltonian system, with an added over-bar $\bar{\cdot}$ or a superscript $\cdot^c$ in order to distinguish it from the plant system. (The interaction port of the controller system may be thought of as an extra possibility for additional controller action (outer-loop control).) In order to further distinguish the plant system and the controller we denote the state space of the plant system by $\mathscr{X}_p$ with coordinates x_p, the Dirac structure by $\mathscr{D}_p$ and its Hamiltonian by H_p, while we will denote the state space manifold of the controller system by $\mathscr{X}_c$ with coordinates x_c, its Dirac structure by $\mathscr{D}_c$ and its Hamiltonian by $H_c : \mathscr{X}_c \to \mathbb{R}$. The interconnection of the plant port-Hamiltonian system with the controller port-Hamiltonian system is obtained by equalizing the port variables at the control port by:

$$f_C = -\bar{f}_C \qquad\qquad e_C = \bar{e}_C \tag{5.5}$$

where $\bar{f}_C$ and $\bar{e}_C$ denote the control port variables of the controller system. Here, the minus sign is inserted to have a uniform notion of direction of power flow. Clearly,

this 'synchronizing' interconnection is power-conserving, that is

$$e_C^{\mathrm{T}} f_C + \bar{e}_C^{\mathrm{T}} \bar{f}_C = 0$$

Remark 5.1. A sometimes useful alternative for the power-conserving interconnection (5.5) is the *gyrating* power-conserving interconnection (see Sect. 2.5.1)

$$f_C = -\bar{e}_C \qquad\qquad e_C = \bar{f}_C \tag{5.6}$$

In fact, the standard feedback interconnection can be regarded to be of this type. Consider a plant input-state-output port-Hamiltonian system as in (5.1)

$$P : \begin{cases} \dot{x}_p = \big[J_p(x_p) - R_p(x_p)\big] \dfrac{\partial H_p}{\partial x_p}(x_p) + g_p(x_p)u + k_p(x_p)d \\ y = g_p^{\mathrm{T}}(x_p) \dfrac{\partial H_p}{\partial x_p}(x_p) \qquad\qquad x_p \in \mathscr{X}_p \\ z = k_p^{\mathrm{T}}(x_p) \dfrac{\partial H_p}{\partial x_p}(x_p) \end{cases}$$

together with a controller input-state-output port-Hamiltonian system

$$C : \begin{cases} \dot{x}_c = \big[J_c(x_c) - R_c(x_c)\big] \dfrac{\partial H_c}{\partial x_c}(x_c) + g_c(x_c)\bar{u} + k_c(x_c)\bar{d} \\ \bar{y} = g_c^{\mathrm{T}}(x_c) \dfrac{\partial H_c}{\partial x_c}(x_c) \qquad\qquad x_c \in \mathscr{X}_c \\ \bar{z} = k_c^{\mathrm{T}}(x_c) \dfrac{\partial H_c}{\partial x_c}(x_c) \end{cases}$$

Then the standard feedback interconnection

$$u = -\bar{y} \qquad\qquad y = \bar{u}$$

can be seen to be equal to the gyrating interconnection (5.6).

For both interconnection constraints it directly follows from the theory of composition of Dirac structures (see Sect. 2.5) that the interconnected (closed-loop) system is again a port-Hamiltonian system with Dirac structure determined by the Dirac structures of the plant port-Hamiltonian system and the controller port-Hamiltonian system. The resulting interconnected port-Hamiltonian system has state space $\mathscr{X}_p \times \mathscr{X}_c$, Hamiltonian $H_p + H_c$, resistive ports $(f_R, e_R, \bar{f}_R, \bar{e}_R)$ and interaction ports $(f_I, e_I, \bar{f}_I, \bar{e}_I)$, satisfying the power-balance

$$\frac{\mathrm{d}}{\mathrm{d}t}(H_p + H_c) = e_R^{\mathrm{T}} f_R + \bar{e}_R^{\mathrm{T}} \bar{f}_R + e_I^{\mathrm{T}} f_I + \bar{e}_I^{\mathrm{T}} \bar{f}_I \leq e_I^{\mathrm{T}} f_I + \bar{e}_I^{\mathrm{T}} \bar{f}_I \tag{5.7}$$

since both $e_R^{\mathrm{T}} f_R \leq 0$ and $\bar{e}_R^{\mathrm{T}} \bar{f}_R \leq 0$. Hence we immediately recover the state space formulation of the passivity theorem, see e.g. [180], if H_p and H_c are both non-

negative, implying that the plant and the controller system are passive (with respect to their controller and interaction ports and storage functions H_p and H_c), then also the closed-loop system is passive (with respect to the interaction ports and storage function $H_p + H_c$.)

Nevertheless, we will show in the next sections that, due to the Hamiltonian structure, we can go beyond the passivity theorem, and that we can derive conditions which ensure that we can passify and/or stabilize plant port-Hamiltonian systems for which the Hamiltonian H_p is *not* non-negative (or bounded from below).

5.3.1 Energy control

Consider two port-Hamiltonian systems Σ_i (without internal dissipation) in input-state-output form

$$\Sigma_i : \begin{cases} \dot{x}_i = J_i(x_i)\dfrac{\partial H_i}{\partial x_i} + g_i(x_i)u_i \\ y_i = g_i^{\mathrm{T}}(x_i)\dfrac{\partial H_i}{\partial x_i} \end{cases} \qquad i = 1, 2$$

both satisfying the power-balance

$$\frac{\mathrm{d}}{\mathrm{d}t}H_i = y_i^{\mathrm{T}} u_i$$

Suppose now that we want to transfer the energy from the port-Hamiltonian system Σ_1 to the port-Hamiltonian system Σ_2, while keeping the total energy $H_1 + H_2$ constant. This can be done by using the following output feedback

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & -y_1 y_2^{\mathrm{T}} \\ y_2 y_1^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{5.8}$$

Since the matrix in (5.8) is skew-symmetric it immediately follows that the closed-loop system composed of systems Σ_1 and Σ_2 linked by the power-conserving feedback is energy-preserving, that is $\frac{\mathrm{d}}{\mathrm{d}t}(H_1 + H_2) = 0$. However, if we consider the individual energies then we notice that

$$\frac{\mathrm{d}}{\mathrm{d}t}H_1 = -y_1^{\mathrm{T}} y_1 y_2^{\mathrm{T}} y_2 = -||y_1||^2 ||y_2||^2 \leq 0$$

implying that H_1 is decreasing as long as $||y_1||$ and $||y_2||$ are different from 0. Conversely, as expected since the total energy is constant,

$$\frac{\mathrm{d}}{\mathrm{d}t}H_2 = y_2^{\mathrm{T}} y_2 y_1^{\mathrm{T}} y_1 = ||y_2||^2 ||y_1||^2 \geq 0$$

implying that H_2 is increasing at the same rate. In particular, if H_1 has a minimum at the zero equilibrium, and Σ_1 is zero-state observable, then all the energy H_1 of Σ_1 will be transferred to Σ_2, provided that $||y_2||$ is not identically zero, which again can be guaranteed by assuming that H_2 has a minimum at the zero equilibrium, and that Σ_2 is zero-state observable.

If there is internal energy dissipation, then this energy transfer mechanism still works. However, the fact that H_2 grows or not will depend on the balance between the energy delivered by Σ_1 to Σ_2 and the internal loss of energy in Σ_2 due to dissipation.

We conclude that this particular scheme of power-conserving energy transfer is accomplished by a skew-symmetric output feedback, which is *modulated* by the values of the output vectors of both systems. Of course this raises, among others, the question of the efficiency of the proposed energy-transfer scheme, and the need for a systematic quest of similar power-conserving energy-transfer schemes. We refer to [69] for a similar but different energy-transfer scheme directly motivated by the structure of the example (control of a snake-board).

5.3.2 Stabilization by Casimir generation

We know that the interconnection of a plant port-Hamiltonian system with a controller port-Hamiltonian system leads to a closed-loop port-Hamiltonian system, with closed-loop Dirac structure being the composition of the plant and the controller Dirac structure. Furthermore, we immediately obtain the power-balance (5.7)

$$\frac{\mathrm{d}}{\mathrm{d}t}(H_p + H_c) = e_R^{\mathrm{T}} f_R + \bar{e}_R^{\mathrm{T}} \bar{f}_R + e_I^{\mathrm{T}} f_I + \bar{e}_I^{\mathrm{T}} \bar{f}_I \leq e_I^{\mathrm{T}} f_I + \bar{e}_I^{\mathrm{T}} \bar{f}_I$$

What does this mean about the stability properties of the closed-loop system, and how can we design the controller port-Hamiltonian system in such a way that the closed-loop system has desired stability properties? Let us therefore first consider the stability of an arbitrary port-Hamiltonian system $\Sigma = (\mathscr{X}, H, \mathscr{R}, \mathscr{C}, \mathscr{I}, \mathscr{D})$ *without* control or interaction ports, that is, an autonomous port-Hamiltonian system $\Sigma = (\mathscr{X}, H, \mathscr{R}, \mathscr{D})$. Clearly, the power-balance (5.7) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t} H = e_R^{\mathrm{T}} f_R \leq 0 \tag{5.9}$$

Hence we immediately infer by standard Lyapunov theory that if $x_\star$ is a minimum of the Hamiltonian H then it will be a *stable* equilibrium of the autonomous port-Hamiltonian system $\Sigma = (\mathscr{X}, H, \mathscr{R}, \mathscr{D})$, which is actually *asymptotically stable* if the dissipation term $e_R^{\mathrm{T}} f_R$ is negative *definite* outside $x_\star$, or alternatively if some sort of detectability condition is satisfied, guaranteeing asymptotic stability by the use of LaSalle's Invariance principle. However, what can we say if $x_\star$ is an equilibrium that is *not* a minimum of H, and thus we cannot directly use H as a Lyapunov function?

A well-known method in Hamiltonian systems, sometimes called the Energy-Casimir method, is to use in the Lyapunov analysis next to the Hamiltonian *other* conserved quantities (dynamical invariants) which may be present in the system. Indeed, if we may find other conserved quantities then candidate Lyapunov functions can be sought within the class of *combinations* of the Hamiltonian H and those conserved quantities. In particular, if we can find a conserved quantity $C : \mathscr{X} \to \mathbb{R}$ such that $V := H + C$ has a minimum at the equilibrium $x_\star$ then we can still infer stability or asymptotic stability by replacing (5.9) by

$$\frac{\mathrm{d}}{\mathrm{d}t} V = e_R^{\mathrm{T}} f_R \leq 0$$

and thus using V as a Lyapunov function. Functions that are conserved quantities of the system for *every* Hamiltonian are called Casimir functions. Casimirs are completely characterized by the Dirac structure of the port-Hamiltonian system, as discussed in Sect. 2.6.2.

Casimir function can play an important role in the design of a controller port-Hamiltonian system such that the closed-loop system has desired stability properties. Suppose we want to stabilize the plant with port-Hamiltonian formulation $(\mathscr{X}_p, H_p, \mathscr{R}, \mathscr{C}, \mathscr{D}_p)$ around a desired equilibrium $x_{p_\star}$. We know that for every controller port-Hamiltonian system the closed-loop system satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}(H_p + H_c) = e_R^{\mathrm{T}} f_R + \bar{e}_R^{\mathrm{T}} \bar{f}_R \leq 0$$

What if $x_\star$ is not a minimum for H_p? A possible strategy is to *generate* Casimir functions $C(x_p, x_c)$ for the closed-loop system by choosing the controller port-Hamiltonian system in an appropriate way. Thereby we generate candidate Lyapunov functions for the closed-loop system of the form

$$V(x_p, x_c) := H_p(x_p) + H_c(x_c) + C(x_p, x_c)$$

where the controller Hamiltonian function $H_c : \mathscr{X}_c \to \mathbb{R}$ still has to be designed. The goal is thus to construct a function V as above in such a way that V has a minimum at $(x_{p_\star}, x_{c_\star})$ where $x_{c_\star}$ still remains to be chosen. This strategy is based on finding all the achievable Casimirs of the closed-loop system. Furthermore, since the closed-loop Casimirs are based on the closed-loop Dirac structures, this reduces to finding all the achievable closed-loop Dirac structures.

A comprehensive analysis of the finite dimensional case has been presented in [45]. A brief recap of the main results is provided in Sect. 5.4. Moreover, in Sect. 6.4 (but cf. also [122, 164]), we show how to characterize the set of achievable Dirac structures and achievable Casimirs in cases which are of a *mixed* finite-dimensional and infinite-dimensional nature. In particular we study the case of an ideal transmission line connected at both ends to a finite-dimensional port-Hamiltonian system.

5.3.3 Port control

In broad terms, the *Port Control* problem is to design, given the plant port-Hamiltonian system, a controller port-Hamiltonian system such that the *behavior* at the interaction port of the plant port-Hamiltonian system is a desired one, or close to a desired one. This means that by adding the controller system we seek to shape the external behavior at the interaction port of the plant system. If the desired external behavior at this interaction port is given in input-output form as a desired (dynamic) impedance, then this amounts to the Impedance Control problem as introduced and studied by Hogan and co-workers [90]; see also [195] for subsequent developments.

The Port Control problem, as stated in this generality, immediately leads to two fundamental questions:

1. Given the plant port-Hamiltonian system, and the controller port-Hamiltonian system to be arbitrarily designed, what are the achievable behaviors of the closed-loop system at the interaction port of the plant?
2. If the desired behavior at the interaction port of the plant is not achievable, then what is the closest achievable behavior?

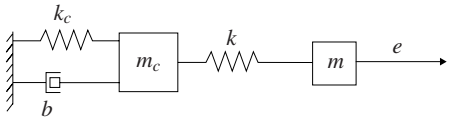
Of course, the second question leaves much room for interpretation, since there is no obvious interpretation of what we mean by 'closest behavior'. Also the first question in its full generality is not easy to answer, and we shall only address an important sub-problem.

An obvious observation is that the desired behavior, in order to be achievable, needs to be the port behavior of a port-Hamiltonian system. This leads already to the problem of characterizing those external behaviors which are port behaviors of port-Hamiltonian systems. Secondly, the Port Control problem can be split into a number of sub-problems. Indeed, we know that the closed-loop system arising from interconnection of the plant port-Hamiltonian system with the controller port-Hamiltonian system is specified by a Hamiltonian which is just the sum of the plant Hamiltonian and the controller Hamiltonian, and a resistive structure which is the "product" of the resistive structure of the plant and of the controller system, together with a Dirac structure which is the *composition* of the plant Dirac structure and the controller Dirac structure. Therefore an important sub-problem is again to characterize the *achievable closed-loop Dirac structures*. A fundamental problem in addressing the Port Control problem in general theoretical terms is the lack of a systematic way to specify 'desired behavior'.

Example 5.1. Consider the plant system (in input-state-output port-Hamiltonian form)

$$\begin{cases} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} \end{cases}$$

Fig. 5.1 Controlled mass.



with q the position and p being the momentum of the mass m, in feedback interconnection $u = -\bar{y} + f_I$, $\bar{u} = y = e_I$, with the controller system (see Fig. 5.1):

$$\begin{cases} \begin{bmatrix} \dot{\Delta q}_c \\ \dot{p}_c \\ \dot{\Delta q} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -b & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_c}{\partial \Delta q_c} \\ \frac{\partial H_c}{\partial p_c} \\ \frac{\partial H_c}{\partial \Delta q_c} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bar{u} \\ \bar{y} = \dfrac{\partial H_c}{\partial \Delta q_c} \end{cases} \tag{5.10}$$

where Δq_c is the displacement of the spring k_c, Δq is the displacement of the spring k, and p_c is the momentum of the mass m_c. The plant Hamiltonian is $H(p) = \frac{1}{2m}p^2$, and the controller Hamiltonian is given as

$$H_c(\Delta q_c, p_c, \Delta q) = \frac{1}{2}\left[\frac{p_c^2}{m_c} + k(\Delta q)^2 + k_c(\Delta q_c)^2\right]$$

The variable $b > 0$ is the damping constant, and f_I is an external force, with e_I denoting the corresponding velocity. The closed-loop system possesses the Casimir function

$$C(q, \Delta q_c, \Delta q) = \Delta q - (q - \Delta q_c),$$

implying that along the solutions of the closed-loop system

$$\Delta q = q - \Delta q_c + c \tag{5.11}$$

with c a constant depending on the initial conditions. With the help of LaSalle's Invariance principle it can be shown that *restricted* to the invariant manifolds (5.11) the system is asymptotically stable for the equilibrium $q = \Delta q_c = p = p_c = 0$.

The problem of Port Control is to determine the controller system given by (5.10), or by a more general expression, in such a way that the port behavior in the port variables f_I, e_I is a desired one. In this particular (simple and linear) example the desired behavior can be quantified e.g. in terms of a desired stiffness and damping of the closed-loop system, which is easily expressed in terms of the closed-loop transfer function from f_I to e_I. Of course, on top of the requirements on the closed-loop transfer function we would also require internal stability of the closed-loop system. For an appealing example of port control of port-Hamiltonian systems within a context of hydraulic systems we refer to [105].

5.4 Achievable Dirac structures with dissipation

The problem of control by interconnection of a plant port-Hamiltonian system P is to find a controller port-Hamiltonian system C such that the closed-loop system has the desired properties. The closed-loop system is again a port-Hamiltonian system with Hamiltonian equal to the sum of the Hamiltonians of the plant and the controller system, a total resistive structure depending on the resistive structures of the plant and controller systems and the Dirac structure being the composition of the Dirac structure of the plant and controller port-Hamiltonian systems. Desired properties of the closed-loop may include for example internal stability of the system and behavior at the interaction port.

Within the framework of control by interconnection of port-Hamiltonian systems, which relies on the existence of Casimirs for the closed-loop system, the problem is restricted to finding achievable Dirac structure of the closed-loop system, that is given a $\mathscr{D}_p$ and a (to be designed) $\mathscr{D}_c$, what are the achievable $\mathscr{D}_p \parallel \mathscr{D}_c$. Consider here the case where $\mathscr{D}_p$ is given a Dirac structure (finite-dimensional) with dissipation, and $\mathscr{D}_c$ a to be designed controller Dirac structure with dissipation. We investigate what are the achievable $\mathscr{D}_p \parallel \mathscr{D}_c$, the closed-loop structures.

5.4.1 Achievable Dirac structures

Theorem 5.1. *Given a plant Dirac structure $\mathscr{D}_P$ with port variables (f_1, e_1, f, e), and a desired Dirac structure $\mathscr{D}$ with port variables (f_1, e_1, f_2, e_2). Then there exists a controller Dirac structure $\mathscr{D}_C$ such that $\mathscr{D} = \mathscr{D}_P \parallel \mathscr{D}_C$ if and only if the following two conditions are satisfied*

$$\mathscr{D}_P^0 \subset \mathscr{D}^0 \tag{5.12}$$

$$\mathscr{D}^\pi \subset \mathscr{D}_P^\pi \tag{5.13}$$

where

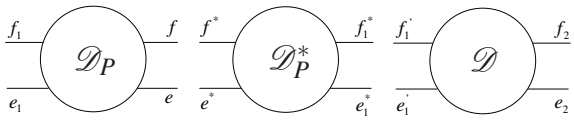
$$\begin{cases} \mathscr{D}_P^0 := \{(f_1, e_1) \mid (f_1, e_1, 0, 0) \in \mathscr{D}_P\} \\ \mathscr{D}_P^\pi := \{(f_1, e_1) \mid \exists (f, e) : (f_1, e_1, f, e) \in \mathscr{D}_P\} \\ \mathscr{D}^0 := \{(f_1, e_1) \mid (f_1, e_1, 0, 0) \in \mathscr{D}\} \\ \mathscr{D}^\pi := \{(f_1, e_1) \mid \exists (f_2, e_2) : (f_1, e_1, f_2, e_2) \in \mathscr{D}\} \end{cases} \tag{5.14}$$

The following proof of Theorem 5.1 is based on the following 'copy' (or 'internal model') $\mathscr{D}_P^*$ of the plant Dirac structure $\mathscr{D}_P$:

$$\mathscr{D}_P^* := \left\{(f_1, e_1, f_2, e_2) \mid (-f_1, e_1, -f_2, e_2) \in \mathscr{D}_P\right\} \tag{5.15}$$

It is easily seen that $\mathscr{D}_P^*$ is a Dirac structure if and only if $\mathscr{D}_P$ is a Dirac structure.

Fig. 5.2 $\mathscr{D}_P \parallel \mathscr{D}_P^* \parallel \mathscr{D}$.



Proof. Necessity of (5.12) and (5.13) is obvious. Sufficiency is shown using the controller Dirac structure

$$\mathscr{D}_C := \mathscr{D}_P^* \parallel \mathscr{D}$$

(see Fig. 5.2). To check that $\mathscr{D} \subset \mathscr{D}_P \parallel \mathscr{D}_C$, consider $(f_1, e_1, f_3, e_3) \in \mathscr{D}$. Because $(f_1, e_1) \in \mathscr{D}^\pi$, applying (5.13) yields that $\exists (f_2, e_2)$ such that $(f_1, e_1, f_2, e_2) \in \mathscr{D}_P$. It follows that $(-f_1, e_1, -f_2, e_2) \in \mathscr{D}_P^*$. Recall now the following interconnection constraints in Fig. 5.2:

$$f_2 = -f_2^* \qquad e_2 = e_2^* \qquad f_1^* = -f_1' \qquad e_1^* = e_1'$$

By taking $(f_1', e_1') = (f_1, e_1)$ in Fig. 5.2, it follows that $(f_1, e_1, f_2, e_2) \in \mathscr{D}_P \parallel \mathscr{D}_C$. Therefore, $\mathscr{D} \subset \mathscr{D}_P \parallel \mathscr{D}_C$. To check that $\mathscr{D}_P \parallel \mathscr{D}_C \subset \mathscr{D}$, consider $(f_1, e_1, f_2, e_2) \in \mathscr{D}_P \parallel \mathscr{D}_C$. Then there exist $f = -f^*$, $e = e^*$, $f_1^* = -f_1'$ and $e_1^* = e_1'$ such that

$$(f_1, e_1, f, e) \in \mathscr{D}_P \tag{5.16}$$

$$(f_1^*, e_1^*, f^*, e^*) \in \mathscr{D}_P^* \Longleftrightarrow (f_1', e_1', f, e) \in \mathscr{D}_P \tag{5.17}$$

$$(f_1', e_1', f_2, e_2) \in \mathscr{D} \tag{5.18}$$

Subtracting (5.17) from (5.16), making use of the the linearity of $\mathscr{D}_P$, we get

$$(f_1 - f_1', e_1 - e_1', 0, 0) \in \mathscr{D}_P \Longleftrightarrow (f_1 - f_1', e_1 - e_1') \in \mathscr{D}_P^0 \tag{5.19}$$

Using (5.19) and (5.12) we get

$$(f_1 - f_1', e_1 - e_1', 0, 0) \in \mathscr{D} \tag{5.20}$$

Finally, adding (5.18) and (5.20), we obtain $(f_1, e_1, f_2, e_2) \in \mathscr{D}$, and so $\mathscr{D}_P \parallel \mathscr{D}_C \subset \mathscr{D}$. Finally we show that conditions (5.12) and (5.13) are equivalent. In fact we prove that $(\mathscr{D}^0)^\perp = \mathscr{D}^\pi$ and the same for $\mathscr{D}_P$. Here, $^\perp$ denotes the orthogonal complement with respect to the canonical bi-linear form on $\mathscr{F}_1 \times \mathscr{F}_1^*$ defined as

$$\ll (f_1^a, e_1^a), (f_1^b, e_1^b) \gg := \left\langle e^a \mid f^b \right\rangle + \left\langle e^b \mid f^a \right\rangle$$

for $(f_1^a, e_1^a), (f_1^b, e_1^b) \in \mathscr{F}_1 \times \mathscr{F}_1^*$. Then, since $\mathscr{D}_P^0 \subset \mathscr{D}^0$ implies $(\mathscr{D}^0)^\perp \subset (\mathscr{D}_P^0)^\perp$, the equivalence between (5.12) and (5.13) is immediate. In order to show $(\mathscr{D}^0)^\perp = \mathscr{D}^\pi$ first take $(f_1, e_1) \in (\mathscr{D}^\pi)^\perp$, implying that

$$\ll (f_1, e_1), (\tilde{f}_1, \tilde{e}_1 \gg = \left\langle e_1 \mid \tilde{f}_1 \right\rangle + \langle \tilde{e}_1 \mid f_1 \rangle = 0$$

for all $(\tilde{f}_1, \tilde{e}_1)$ for which there exists $(\tilde{f}_2, \tilde{e}_2)$ such that $(\tilde{f}_1, \tilde{e}_1, \tilde{f}_2, \tilde{e}_2) \in \mathscr{D}$. This implies that $(f_1, e_1, 0, 0) \in \mathscr{D}^\perp = \mathscr{D}$ and thus that $(f_1, e_1) \in \mathscr{D}^0$. Hence, $(\mathscr{D}^\pi)^\perp \subset \mathscr{D}^0$ and thus $(\mathscr{D}^0)^\perp = \mathscr{D}^\pi$, implying that there exists (f_2, e_2) such that $(f_1, e_1, f_2, e_2) \in \mathscr{D} = \mathscr{D}^\perp$. Hence

$$\langle e_1 \mid \tilde{f}_1 \rangle + \langle \tilde{e}_1 \mid f_1 \rangle + \langle e_2 \mid \tilde{f}_2 \rangle + \langle \tilde{e}_2 \mid f_2 \rangle = 0$$

for all $(\tilde{f}_1, \tilde{e}_1, \tilde{f}_3, \tilde{e}_3) \in \mathscr{D}$, implying that $\langle e_1 \mid \tilde{f}_1 \rangle + \langle \tilde{e}_1 \mid f_1 \rangle = 0$ for all $(\tilde{f}_1, \tilde{e}_1, 0, 0) \in \mathscr{D}$ and thus $(f_1, e_1) \in (\mathscr{D}^0)^\perp$.

5.4.2 Achievable Resistive structures

Similar analysis could also be done for Resistive structures in which case we formulate the problem as follows. We are given a $\mathscr{R}_1$ and the to-be-designed $\mathscr{R}_2$, then what are the achievable Resistive structures $\mathscr{R}_1 \parallel \mathscr{R}_2$?

Theorem 5.2. *Given a Resistive structure $\mathscr{R}_1$ with port variables $(f_{R1}, e_{R1}, f_2, e_2)$ and a desired Resistive structure $\mathscr{R}$ with port variables $(f_{R1}, e_{R1}, f_{R3}, e_{R3})$. Then there exist an $\mathscr{R}_2$ such that $\mathscr{R} = \mathscr{R}_1 \parallel \mathscr{R}_2$ if and only if the following two conditions (these are no more equivalent) are satisfied*

$$\mathscr{R}_1^0 \subset \mathscr{R}^0 \tag{5.21}$$

$$\mathscr{R}^\pi \subset \mathscr{R}_1^\pi \tag{5.22}$$

where

$$\begin{aligned}
\mathscr{R}_1^0 &:= \{(f_{R1}, e_{R1}) \mid (f_{R1}, e_{R1}, 0, 0) \in \mathscr{R}_1\} \\
\mathscr{R}_1^\pi &:= \{(f_{R1}, e_{R1}) \mid \exists (f_2, e_2) \; s.t. \; (f_{R1}, e_{R1}, f_2, e_2) \in \mathscr{R}_1\} \\
\mathscr{R}^0 &:= \{(f_{R1}, e_{R1}) \mid (f_{R1}, e_{R1}, 0, 0) \in \mathscr{R}\} \\
\mathscr{R}^\pi &:= \{(f_{R1}, e_{R1}) \mid \exists (f_{R3}, e_{R3}) \; s.t \; ((f_{R1}, e_{R1}, f_{R3}, e_{R3}) \in \mathscr{R})\}
\end{aligned}$$

Proof. We again follow the same proof as that of achievable Dirac structures, we now define the "copy" $\mathscr{R}_1^*$ of $\mathscr{R}_1$ as

$$\mathscr{R}_1^* := \left\{ (f_{R1}, e_{R1}, f_2, e_2) \mid (-f_{R1}, e_{R1}, -f_2, e_2) \right\} \in \mathscr{R}_1$$

again its clear that $\mathscr{R}_1^*$ is a resistive structure if and only if $\mathscr{R}_1$ is a Resistive structure, and by defining $\mathscr{R}_2 := \mathscr{R}_1^* \parallel \mathscr{R}$. Rest of the proof follows the same procedure as in Theorem 5.1 and hence we omit the details here.

Remark 5.2. It should be noted here that the conditions (5.21) and (5.22) are no longer equivalent as in the case of Dirac structures. This is due to the property of the Resistive structure that $\mathscr{R}^\perp = (-\mathscr{R})$, where again $-\mathscr{R}$ is a pseudo resistive structure corresponding to negative resistance.

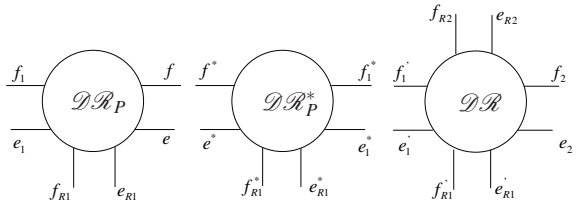


Fig. 5.3 $\mathscr{D}\mathscr{R} = \mathscr{D}\mathscr{R}_P \parallel \mathscr{D}\mathscr{R}_P^* \parallel \mathscr{D}\mathscr{R}$.

5.4.3 Achievable Dirac structures with dissipation

We now use the results in the previous two subsections to study the problem of the achievable Dirac structures with dissipation for the closed loop system. We formulate the problem as follows. Given a $\mathscr{D}_P$ with a $\mathscr{R}_P$ (i.e a plant system with dissipation) and a (to be designed) $\mathscr{D}_C$ with $\mathscr{R}_C$ (a controller system with dissipation), what are the achievable $(\mathscr{D}_P \parallel \mathscr{R}_P) \parallel (\mathscr{D}_C \parallel \mathscr{R}_C)$. For ease of notation we henceforth use $\mathscr{D}\mathscr{R}_P$ for $(\mathscr{D}_P \parallel \mathscr{R}_P)$ and $\mathscr{D}\mathscr{R}_C$ for $(\mathscr{D}_C \parallel \mathscr{R}_C)$.Consider here the case where $\mathscr{D}\mathscr{R}_P$ is given a Dirac structure with dissipation (finite-dimensional), and $\mathscr{D}\mathscr{R}_C$ a to be designed controller Dirac structure with dissipation. We investigate what are the achievable $\mathscr{D}\mathscr{R}_P \parallel \mathscr{D}\mathscr{R}_C$, the closed-loop structures.

Theorem 5.3. *Given a plant Dirac structure with dissipation $\mathscr{D}\mathscr{R}_P$ with port variables $f_1, e_1, f_{R1}, e_{R1}, f, e$ and a desired Dirac structure with Dissipation $\mathscr{D}\mathscr{R}$ with port-variables f_1, e_1, f_{R1}, e_{R1}, f_2, e_2, f_{R2} and e_{R2}. Here (f_1, e_1), (f_{R1}, e_{R1}) respectively denote the flow and effort variables corresponding to the energy storing elements and the energy dissipating elements of the plant system. and similarly with the controller system. Then there exists a controller system $\mathscr{D}\mathscr{R}_C$ such that $\mathscr{D}\mathscr{R} = \mathscr{D}\mathscr{R}_P \parallel \mathscr{D}\mathscr{R}_C$ if and only if the following two conditions are satisfied*

$$\mathscr{D}\mathscr{R}_P^0 \subset \mathscr{D}\mathscr{R}^0 \tag{5.23}$$

$$\mathscr{D}\mathscr{R}^\pi \subset \mathscr{D}\mathscr{R}_P^\pi \tag{5.24}$$

where

$$\begin{aligned}
\mathscr{D}\mathscr{R}_P^0 &:= \left\{(f_1, e_1, f_{R1}, e_{R1}) \mid (f_1, e_1, f_{R1}, e_{R1}, 0, 0) \in \mathscr{D}\mathscr{R}_P\right\} \\
\mathscr{D}\mathscr{R}_P^\pi &:= \left\{(f_1, e_1, f_{R1}, e_{R1}) \mid \exists (f, e) \; s.t. \; (f_1, e_1, f_{R1}, e_{R1}, f, e) \in \mathscr{D}\mathscr{R}_P\right\} \\
\mathscr{D}\mathscr{R}^0 &:= \left\{(f_1, e_1, f_{R1}, e_{R1}) \mid (f_1, e_1, f_{R1}, e_{R1}, 0, 0, 0, 0) \in \mathscr{D}\mathscr{R}\right\} \\
\mathscr{D}\mathscr{R}^\pi &:= \left\{(f_1, e_1, f_{R1}, e_{R1}) \mid \exists (f_2, e_2, f_{R2}, e_{R2}) \right. \\
&\qquad \left. s.t \; ((f_1, e_1, f_{R1}, e_{R1}, f_2, e_2, e_{R2}, e_{R2}) \in \mathscr{D}\mathscr{R})\right\}
\end{aligned} \tag{5.25}$$

Proof. The proof is again based on the copy $\mathscr{D}\mathscr{R}_P^*$ of the plant system defined as

$$\mathscr{D}\mathscr{R}_P^* := \left\{(f_1, e_1, f_{R1}, e_{R1}, f, e) \mid (-f_1, e_1, -f_{R1}, e_{R1}, -f, e) \in \mathscr{D}\mathscr{R}_P\right\}$$

and defining a controller system $\mathscr{DR}_C := \mathscr{DR}_P^* \parallel \mathscr{DR}$. We follow the same procedure for the proof as in the case of achievable Dirac structures, Theorem(5.1). Necessity of conditions (5.23) and (5.24) is obvious. Sufficiency is shown by using the controller Dirac structure with dissipation

$$\mathscr{DR}_C := \mathscr{DR}_P^* \parallel \mathscr{DR}$$

To check that $\mathscr{DR} \subset \mathscr{DR}_p \parallel \mathscr{DR}_c$, consider $(f_1, e_1, f_{R1}, e_{R1}, f_2, e_2, e_{R2}, e_{R2}) \in \mathscr{DR}$. Because $(f_1, e_1, f_{R1}, e_{R1}) \in \mathscr{DR}^{\pi}$, applying (5.24) yields that $\exists (f, e)$ such that $(f_1, e_1, f_{R1}, e_{R1}, f, e) \in \mathscr{DR}_1$. This implies that $(-f_1, e_1, -f_{R1}, e_{R1}, -f, e) \in \mathscr{DR}_P^*$. With the interconnection constraints, see Fig. 5.3

$$f = -f^* \qquad e = e^* \qquad f_1^* = -f_1' \qquad e_1^* = e_1'$$

By taking $(f_1', e_1', f_{R_1}', e_{R_1}') = (f_1, e_1, f_{R_1,} e_{R_1})$ in Fig. 5.3, it follows that

$$(f_1, e_1, f_{R1}, e_{R1}, f_2, e_2, e_{R2}, e_{R2}) \in \mathscr{DR}_p \parallel \mathscr{DR}_c$$

and hence $\mathscr{DR} \subset \mathscr{DR}_p \parallel \mathscr{DR}_c$. To check that $\mathscr{DR}_p \parallel \mathscr{DR}_c \subset \mathscr{DR}$, consider $(f_1, e_1, f_{R1}, e_{R1}, f_2, e_2, e_{R2}, e_{R2}) \in \mathscr{DR}_p \parallel \mathscr{DR}_c$. Then, there exists $f = -f^*$, $e = e^*$, $f_1^* = -f_1'$ and $e_1^* = e_1'$ such that

$$(f_1, e_1, f_{R1}, e_{R1}, f, e) \in \mathscr{DR}_1 \tag{5.26}$$

$$(f_1^*, e_1^*, f_{R1}^*, e_{R1}^*, f^*, e^*) \in \mathscr{DR}_1^* \Leftrightarrow$$

$$\Leftrightarrow (f_1', e_1', f_{R_1}', e_{R_1}', f, e) \in \mathscr{DR}_p \tag{5.27}$$

$$(f_1', e_1', f_{R1}', e_{R1}', f_2, e_2, e_{R2}, e_{R2}) \in \mathscr{DR} \tag{5.28}$$

subtracting (5.27) from (5.23) and also by making use of the linearity on $\mathscr{DR}_p$ we get

$$(f_1 - f_1', e_1 - e_1', f_{R1} - f_{R1}', e_{R1} - e_{R1}', 0, 0) \in \mathscr{DR}_p \Longleftrightarrow$$

$$\Longleftrightarrow (f_1 - f_1', e_1 - e_1', f_{R1} - f_{R1}', e_{R1} - e_{R1}') \in \mathscr{DR}_p^0 \tag{5.29}$$

Using (5.29) and (5.23) we get

$$(f_1 - f_1', e_1 - e_1', f_{R1} - f_{R1}', e_{R1} - e_{R1}', 0, 0, 0, 0) \in \mathscr{DR} \tag{5.30}$$

Finally, adding (5.30) and (5.28) we get $(f_1, e_1, f_{R1}, e_{R1}, f_2, e_2, e_{R2}, e_{R2}) \in \mathscr{DR}$ and hence $\mathscr{DR}_p \parallel \mathscr{DR}_c \subset \mathscr{DR}$.

Remark 5.3. In this case also it can easily be checked that the conditions (5.23) and (5.24) are no more equivalent as in the case of systems without dissipation.This is again due to the compositional property of a Dirac structure with a resistive structure given by as stated in Remark 5.2.

5.4.4 Achievable Casimirs and constraints

An important application of Theorem 5.1 concerns the characterization of the *Casimir functions* which can be achieved for the closed-loop system by interconnecting a given plant port-Hamiltonian system with associated Dirac structure $\mathscr{D}_P$ with a controller port-Hamiltonian system with associated Dirac structure $\mathscr{D}_C$. This constitutes a cornerstone for passivity-based control of port-Hamiltonian systems as developed e.g. in [160, 161]. Dually, we may characterize the achievable *algebraic constraints* for the closed-loop system.

Recall that a *Casimir function* $C : \mathscr{X} \to \mathbb{R}$ of the port-Hamiltonian system is defined to be a function which is constant along all trajectories of the port-Hamiltonian system, irrespective of the Hamiltonian H and the resistive structure (see Sect. 2.6.2). It follows from the above consideration of the admissible flows that the Casimirs are determined by the subspace $G_1 = \operatorname{Im} E_s^{\mathrm{T}}$, where E_s and F_s are the matrices providing the kernel/image representation (see Sect. 2.4.1) of the Dirac structure $\mathscr{D}$ associated with the port-Hamiltonian system. Indeed, necessarily $f_s = -\dot{x}(t) \in G_1$, and thus

$$\dot{x}(t) \in \operatorname{Im} E_s^{\mathrm{T}} \qquad t \in \mathbb{R}.$$

Therefore $C : \mathscr{X} \to \mathbb{R}$ is a Casimir function if $\frac{\mathrm{d}C}{\mathrm{d}t}(x(t)) = \frac{\partial^{\mathrm{T}} C}{\partial x}(x(t))\dot{x}(t) = 0$ for all $\dot{x}(t) \in \operatorname{Im} E_s^{\mathrm{T}}$. Hence $C : \mathscr{X} \to \mathbb{R}$ is a Casimir of the port-Hamiltonian system if it satisfies the set of partial differential equations

$$\frac{\partial C}{\partial x}(x) \in \ker E_s$$

Geometrically, this can be formulated by defining the following subspace of the dual space of efforts

$$P_0 = \left\{ e_s \in \mathscr{X}^* \mid (0, e_s) \in \mathscr{D} \right\}$$

Indeed, it can be easily seen that $G_1 = P_0^{\perp}$ where $\perp$ denotes orthogonal complement with respect to the dual product $\langle \cdot \mid \cdot \rangle$. Hence C is a Casimir function iff

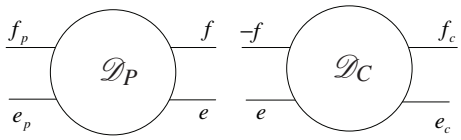
$$\frac{\partial C}{\partial x}(x) \in P_0$$

Remark 5.4. In the case of a non-constant Dirac structure the matrix E_s will depend on x, and $\ker E_s$ will define a co-distribution on the manifold $\mathscr{X}$. Then the issue arises of *integrability* of this co-distribution, see Sect. 2.7 and [59].

Dually, the algebraic constraints for the port-Hamiltonian system are determined by the space P_1, since necessarily $\frac{\partial^{\mathrm{T}} H}{\partial x}(x) \in P_1$, which will induce constraints on the state variables x.

Let us now consider the question of characterizing the set of achievable Casimirs for the closed-loop system $\mathscr{D}_P \parallel \mathscr{D}_C$, where $\mathscr{D}_P$ is the given Dirac structure of the

Fig. 5.4 $\mathscr{D}_P \parallel \mathscr{D}_C$.



plant port-Hamiltonian system with Hamiltonian H, and $\mathscr{D}_C$ is the (to-be-designed) controller Dirac structure. In this case, the Casimirs will depend on the plant state x as well as on the controller state x_c. Since the controller Hamiltonian $H_C(x_c)$ is at our own disposal we will be primarily interested in the dependency of the Casimirs on the plant state x. Since we want to use the Casimirs for shaping the total Hamiltonian $H + H_C$ to a Lyapunov function as anticipated in Sect. 5.3.2 and discussed in [160, 161].

Consider the notation given in Fig. 5.4, and assume the ports in (f_p, e_p) are connected to the (given) energy storing elements of the plant port-Hamiltonian system (that is, $f_p = -\dot{x}$, $e_p = \frac{\partial H}{\partial x}$), while (f_c, e_c) are connected to the (to-be-designed) energy storing elements of a controller port-Hamiltonian system (that is, $f_c = -\dot{x_c}$, $e_c = \frac{\partial H_C}{\partial x_c}$). Note that the number of ports (f_c, e_c) can be freely chosen. In this situation the achievable Casimir functions are functions $K(x, x_c)$ such that $\frac{\partial K}{\partial x}(x, x_c)$ belongs to the space

$$P_{Cas} = \left\{ e_p \mid \exists\, \mathscr{D}_C \text{ s.t. } \exists e_c : (0, e_p, 0, e_3) \in \mathscr{D}_P \parallel \mathscr{D}_C \right\} \tag{5.31}$$

Thus the question of characterizing the achievable Casimirs of the closed-loop system, regarded as functions of the plant state x, is translated to finding a characterization of the space P_{Cas}. This is answered by the following theorem.

Theorem 5.4. *The space P_{Cas} defined in (5.31) is equal to the linear space*

$$\tilde{P} = \left\{ e_p \mid \exists (f, e) : (0, e_p, f, e) \in \mathscr{D}_P \right\}$$

Proof. $P_{Cas} \subset \tilde{P}$ trivially. By using the controller Dirac structure $\mathscr{D}_C = \mathscr{D}_P^*$, we immediately obtain $\tilde{P} \subset P_{Cas}$.

Remark 5.5. For a non-constant Dirac structure on a manifold $\mathscr{X}$ P_{Cas} defines a co-distribution on $\mathscr{X}$.

In a completely dual way we may consider the *achievable constraints* of the closed-loop system, characterized by the space

$$G_{Alg} = \left\{ f_p \mid \exists\, \mathscr{D}_C \text{ s.t. } \exists f_c : (f_p, 0, f_c, 0) \in \mathscr{D}_P \parallel \mathscr{D}_C \right\} \tag{5.32}$$

Theorem 5.5. *The space G_{Alg} defined in (5.32) is equal to the linear space*

$$\left\{ f_p \mid \exists (f, e) : (f_p, 0, f, e) \in \mathscr{D}_P \right\}$$

Remark 5.6. For a non-constant Dirac structure G_{Alg} defines a *distribution* on the manifold $\mathscr{X}$.

Example 5.2. Consider the input-state-output port-Hamiltonian plant system with inputs f and outputs e

$$\begin{cases} \dot{x} = J(x)\dfrac{\partial H}{\partial x}(x) + g(x)f \\ e = g^{\mathrm{T}}(x)\dfrac{\partial H}{\partial x}(x) \end{cases} \qquad x \in \mathscr{X},\ f, e \in \mathbb{R}^m$$

where $J(x)$ is a *skew-symmetric* $n \times n$ matrix. The corresponding Dirac structure is given as the graph of the map

$$\begin{bmatrix} f_p \\ e \end{bmatrix} = \begin{bmatrix} -J(x) & -g(x) \\ g^{\mathrm{T}}(x) & 0 \end{bmatrix} \begin{bmatrix} e_p \\ f \end{bmatrix}$$

It is easily seen that

$$P_{Cas} = \left\{ e_p \mid \exists f \text{ such that } 0 = J(x)e_p + g(x)f \right\}$$

implying that the achievable Casimirs $K(x,x_c)$ are such that $e_p = \frac{\partial K}{\partial x}(x)$ satisfies $J(x)\frac{\partial K}{\partial x}(x) \in \operatorname{Im} g$, that is, K as a function of x is a Hamiltonian function corresponding to a Hamiltonian vector field contained in the distribution spanned by the input vector fields given by the columns of $g(x)$. Similarly, it is easily seen that

$$G_{Alg} = \left\{ f_p \mid \exists f \text{ s.t. } f_p = -g(x)f \right\} = \operatorname{Im} g(x)$$

which implies that the achievable algebraic constraints are of the form $\frac{\partial^{\mathrm{T}} H}{\partial x}(x)g(x) = k(x_c)$. This simply means that the outputs $e = g^{\mathrm{T}}(x)\frac{\partial H}{\partial x}(x)$ can be constrained in any way by interconnecting the system with a suitable controller port-Hamiltonian system.

From a 'control by interconnection' (or 'impedance control') point of view the characterization of the achievable Dirac structures is only a first step towards characterizing the achievable port-Hamiltonian closed-loop behaviors. Also, the translation of specifications on desired closed-loop behavior in terms of the entities defining the closed-loop port-Hamiltonian system is an important open issue for research.

5.4.5 The role of energy dissipation

In the modeling process of physical systems the precise specification of the resistive relations is one of the most difficult parts. Furthermore, precisely these resistive

relations are often subject to time-variation. Therefore, from a robust control perspective it may be desirable to base the construction of the controller in a way that is independent of the precise resistive relations, and to make sure that the controller behavior meets the required specifications for a sufficiently large range of resistive effects. This raises a number of questions, some of which are addressed below.

5.4.6 Casimirs and the dissipation obstacle

Recall that we define a *Casimir function* for a port-Hamiltonian system with dissipation $\Sigma = (\mathscr{X}, H, \mathscr{R}, \mathscr{D})$ to be any function $C : \mathscr{X} \to \mathbb{R}$ such that $e = \frac{\partial C}{\partial x}(x)$ satisfies $(0, e, 0, 0) \in \mathscr{D}$. Indeed, as derived above, this will imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}C = \frac{\partial^{\mathrm{T}} C}{\partial x}(x(t))\dot{x}(t) = \frac{\partial^{\mathrm{T}} C}{\partial x}(x(t))f_S = e^{\mathrm{T}} f_S = 0 \tag{5.33}$$

for every port-Hamiltonian system $(\mathscr{X}, H, \mathscr{R}, \mathscr{D})$ with the same Dirac structure $\mathscr{D}$. As shown in Sect. 2.6.2.1, the definition of Casimir function cannot *relaxed* by requiring that (5.33) only holds for a *specific* resistive relation $R_f f_R + R_e e_R = 0$, where the square matrices R_f and R_e satisfy the symmetry and semi-positive definiteness condition (2.25) together with the dimensionality condition (2.26). In fact, a Casimir for one resistive relation is actually a Casimir for *all* resistive relations, which is closely related to the so-called *dissipation obstacle* in the case of input-state-output port-Hamiltonian systems.

An obvious way to overcome the dissipation obstacle in particular cases is to apply a preliminary feedback (for simplicity we restrict ourselves to input-state-output port-Hamiltonian systems)

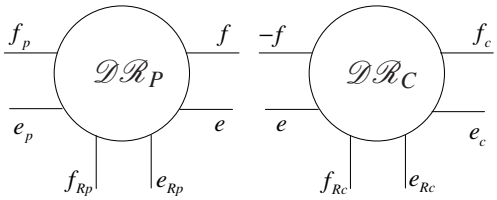
$$u = K(x)y + v, \qquad K(x) = K^{\mathrm{T}}(x) \geq 0$$

(negative damping injection), which leads to the closed-loop system

$$\dot{x} = \big[J(x) - R(x) + K(x)\big]\frac{\partial H}{\partial x} + g(x)v$$

Hence if we are able to design $K(x)$ in such a way that $R(x) - K(x)$ becomes zero, or at least of rank less than $\operatorname{rank} R(x)$ (exact compensation of energy dissipation) then the dissipation obstacle disappears, respectively is mitigated. An obvious drawback of this strategy is that it relies on exact compensation, and hence may be highly sensitive with respect to parameter uncertainty in the resistive relations. Another strategy to overcome the dissipation obstacle in some cases is discussed in [136].

Fig. 5.5 $\mathscr{D}\mathscr{R}_P \parallel \mathscr{D}\mathscr{R}_C$.



5.4.7 Casimirs for any resistive relation

We now consider the question of characterizing the set of achievable Casimirs for the closed-loop system $\mathscr{D}\mathscr{R}_P \parallel \mathscr{D}\mathscr{R}_C$ with $\mathscr{D}\mathscr{R}_P$ the Dirac structure of the plant port-Hamiltonian system with dissipation with Hamiltonian H, and $\mathscr{D}\mathscr{R}_C$ is the controller Dirac structure. For all resistive relations and every port behavior. Then the Casimirs depend on the plant state x and also on the controller state x_c, with the controller Hamiltonian $H_c(x_c)$ at our own disposal.

Consider $\mathscr{D}\mathscr{R}_P$ as in Fig. 5.5, with $(f_p, f_R, f, e_p, e_R, e) \in \mathscr{D}\mathscr{R}_P$ and a controller Dirac structure $\mathscr{D}\mathscr{R}_C$ with $(f_c, f_{R_c}, -f, e_c, e_{R_c}, e) \in \mathscr{D}\mathscr{R}_C$, with (f_p, e_p) are connected to the (given) energy storing elements of the plant port-Hamiltonian system with dissipation, (f_R, e_R) are connected to the energy dissipation elements of the plant system, while (f_c, e_c) connected to the (to be designed) energy storing elements of the controller port-Hamiltonian system with dissipation (that is $f_c = -\dot{x_c}$ and $e_c = \frac{\partial H_c}{\partial x_c}$) while (f_{R_c}, e_{R_c}) are connected to the energy dissipation elements of the controller system and (f, e) being the space of shared flows and efforts between the plant and the controller Dirac structures. In this situation the achievable Casimirs are functions $C(x, x_c)$ such that $\frac{\partial^{\mathrm{T}} C}{\partial x}(x, x_c)$ belongs to the space

$$P_{Cas} = \left\{ e_p \mid \exists \mathscr{D}\mathscr{R}_C \text{ s.t } \exists e_c : (0, e_p, 0, 0, 0, e_c, 0, 0) \in \mathscr{D}\mathscr{R}_P \parallel \mathscr{D}\mathscr{R}_C \right\}$$

The following theorem then addresses the question of characterizing the achievable Casimirs of the closed-loop system, regarded as functions of the plant state x by characterization of the space P_{Cas}.

Proposition 5.2. *The space P_{Cas} defined above is equal to the space*

$$\tilde{P} = \left\{ e_1 \mid \exists (f, e \; s.t \; (0, e_1, 0, 0, f, e) \in \mathscr{D}_p \right\}$$

Proof. We see that $P_{Cas} \subset \tilde{P}$ trivially and by using the controller Dirac structure $\mathscr{D}\mathscr{R}_C = \mathscr{D}\mathscr{R}_P^*$ we obtain $\tilde{P} \subset P_{Cas}$.

5.4.8 Casimirs for a given resistive relation

In case of a specific resistive relation $\mathscr{R}$ given by $R_f f_R + R_e e_R = 0$, where the square matrices R_f and R_e satisfy the symmetry and semi-positive definiteness condition (2.25) together with the dimensionality condition (2.26), if $\mathscr{C} : \mathscr{X} \to \mathbb{R}$ is a Casimir function, then this means that $e = \frac{\partial C}{\partial x}(x)$ satisfies

$$\frac{\partial^{\mathrm{T}} C}{\partial x}(x) f_p = 0$$

for all f_p such that exists e_s, f_R and e_R such that $(f_p, e_p, f_{Rp}, e_{Rp}) \in \mathscr{D}\mathscr{R}$ and $R_f f_R + R_e e_R = 0$. This means that $(0,e) \in (\mathscr{D} \parallel \mathscr{R})^{\perp}$. Since we know by (2.214) in Proposition 2.3 that $(\mathscr{D} \parallel \mathscr{R})^{\perp} = \mathscr{D} \parallel -\mathscr{R}$, and thus the Casimirs are determined by the following subspace

$$(0, e_p, -f_{Rp}, e_{Rp}) \in \mathscr{D}\mathscr{R}$$

We now consider the question of finding all the achievable Casimirs for closed-loop system $\mathscr{D}\mathscr{R}_P \parallel \mathscr{D}\mathscr{R}_C$, with $\mathscr{D}\mathscr{R}_P$ the Dirac structure of the plant port-Hamiltonian system with dissipation with Hamiltonian H, and $\mathscr{D}\mathscr{R}_C$ is the controller Dirac structure; for a given resistive relations and every port behavior. Consider $\mathscr{D}\mathscr{R}_P$ and $\mathscr{D}\mathscr{R}_C$ as above, and in this case the achievable Casimirs are functions $C(x, x_c)$ such that $\frac{\partial^{\mathrm{T}} C}{\partial x}(x, x_c)$ belongs to the space

$$P_{Cas} = \Big\{ e_p \mid \exists\, \mathscr{D}\mathscr{R}_C \text{ s.t.} \\ \exists e_c : (0, e_p, -f_{Rp}, e_{Rp}, 0, e_c, -f_{Rc}, e_{Rc}) \in \mathscr{D}\mathscr{R}_P \parallel \mathscr{D}\mathscr{R}_C \Big\} \quad (5.34)$$

Proposition 5.3. *The space P_{Cas} defined above is equal to the linear space*

$$\tilde{P} = \Big\{ e_1 \mid \exists (f,e) \text{ s.t. } (0, e_1, -f_{Rp}, e_{Rp}, f, e) \in \mathscr{D}_p \Big\}$$

Proof. The proof follows the same procedure as in Proposition 5.2.

5.4.9 Application to control

The characterization of the set of achievable Casimirs in terms of the plant state is useful in the sense that given a plant Dirac structure we can, without defining a controller, determine whether or not there exist Casimir functions for the closed-loop system. This is in addition to the fact that we also have knowledge of the Casimir functions for all R and R_c, with $R \geq 0$ and $R_c \geq 0$.

Consider a port-Hamiltonian system (in the input-output form), with (f_p, e_p) respectively the flows and efforts corresponding to the energy storing elements,

(f_R, e_R) the flows and efforts corresponding to the energy dissipating elements and (f, e) the port variables corresponding to the inputs and outputs. The corresponding Dirac structure is given by

$$\begin{bmatrix} f_p \\ e_R \\ e \end{bmatrix} = \begin{bmatrix} -J(x) & -g_R(x) & -g(x) \\ g_R^{\mathrm{T}} & 0 & 0 \\ g^{\mathrm{T}} & 0 & 0 \end{bmatrix} \begin{bmatrix} e_p \\ f_R \\ f \end{bmatrix} \tag{5.35}$$

The characterization of the space P_{Cas} is given by

$$P_{Cas} = \left\{ e_p \mid \exists f_p \text{ s.t. } 0 = -J(x)e_p - g(x)f \text{ and } 0 = g_R^{\mathrm{T}}(x)e_p \right\}$$

These are the conditions for existence of Casimirs for the closed-loop system in terms of the plant state. The expression implies that the achievable Casimirs do not depend on the coordinate where dissipation enters into the system and is well known in literature as the "dissipation obstacle", cf. Sect. 2.6.2.1. In addition to that they are also the Hamiltonian functions corresponding to the input vector fields given by the columns of $g(x)$.

5.5 Casimirs and stabilization in finite dimensions

5.5.1 Specific Casimirs

Another way to interpret the generation of Casimirs for the closed-loop system is to look at the level sets of the Casimirs as *invariant sub-manifolds* of the combined plant and controller state space $\mathscr{X}_p \times \mathscr{X}_c$. Restricted to every such invariant sub-manifold (part of) the controller state can be expressed as a function of the plant state, whence the closed-loop Hamiltonian restricted to such an invariant manifold can be seen as a *shaped* version of the plant Hamiltonian. To be explicit suppose that we have found Casimirs of the form

$$x_{ci} - F_i(x_p) \qquad\qquad i = 1, \cdots, n_p$$

where n_p is the dimension of the controller state space, then on every invariant manifold $x_{ci} - F_i(x_p) = \alpha_i$, $i = 1, \cdots, n_p$, where $\alpha = (\alpha_1, \cdots, \alpha_{n_p})$ is a vector of constants depending on the initial plant and controller state, the closed-loop Hamiltonian can be written as

$$H_s(x_p) := H_p(x_p) + H_c(F(x_p) + \alpha)$$

where, as before, the controller Hamiltonian H_c still can be assigned. This can be regarded as *shaping* the original plant Hamiltonian H_p to a new Hamiltonian H_s.

Example 5.3. Consider the equations of a normalized pendulum

$$\ddot{d} + \sin q + d\dot{q} = u$$

with d a positive damping constant. The total energy is given by $H(q,p) = \frac{1}{2}p^2 + (1-\cos q)$. The corresponding Dirac structure is given as

$$\begin{bmatrix} -\dot{q} \\ -\dot{p} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sin q \\ p \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ d\,p \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$e_R = \begin{bmatrix} 0 & p \end{bmatrix}$$
$$e = p$$

and comparing with (5.35), we have

$$(f_p, e_p, f_R, e_R, f, e) = \left(-\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix}, \begin{bmatrix} \sin q \\ p \end{bmatrix}, \begin{bmatrix} 0 \\ d\,p \end{bmatrix}, \begin{bmatrix} 0 \\ p \end{bmatrix}, u, e \right)$$

and

$$g_R(x) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

In terms of the plant state (q,p), the achievable Casimirs are such that

$$\frac{\partial C}{\partial p} = 0 \tag{5.36}$$

The above expression implies that any Casimir function for this system does not depend on the p term, which is precisely where dissipation enters into the system. However we can find Casimirs depending on q and can use it for stability analysis as shown below.

Let $q_\star$ be a desired equilibrium position of the pendulum. The objective is to shape the potential energy $P(q) = 1 - \cos q$ in such a way that it has a minimum at $q = q_\star$. Consider a first order controller written in the input-output form as

$$\begin{cases} \dot{x}_c = u_c \\ y_c = \dfrac{\partial H_c}{\partial x_c} \end{cases}$$

with the corresponding elements of the Dirac structure being

$$\left(f_c, e_c, f_{Rc}, e_{Rc}, f', e'\right) = \left(\dot{x}_c, \frac{\partial H_c}{\partial x_c}, 0, 0, u_c, y_c \right)$$

The interconnection constraints between the plant and the controller given as

$$u = y_c \qquad\qquad u_c = -y_p$$

Since we are looking for Casimirs of the form $C = x_c - F(q)$, the solution to (5.36) are functions of the form $F(q) = q$. Choosing

$$P_c(x_c) = \cos x_c + \frac{1}{2}(x_c - q_\star)^2$$

and substituting $x_c = F(q) + c = q + c$ we get the shaped potential energy as

$$\begin{aligned} P_d(q) &= P(q) + P_c(F(q) + c) \\ &= \cos(q + c) + (1 - \cos q) + \frac{1}{2}(q + c - q_\star) \end{aligned}$$

However, in order to obtain a minimum at $q = q_*$ the controller needs to be initialized in such a way that $c = 0$.

5.5.2 Casimirs in extended state–space

In this section a modification of the control by interconnection method to overcome the problem of controller initialization mentioned above is proposed. The key idea is to analyze the closed-loop system in the extended state space $\mathscr{X} \times \mathscr{X}_c$, with the control objective of stabilization of a desired equilibrium (x, x_c), for some x_c satisfying equilibrium conditions of the closed-loop system. To this end, we consider general Casimir functions $C : \mathscr{X} \times \mathscr{X}_c \to \mathbb{R}$. These can be characterized by the space as in (5.34). In this case we see that not only $C(x, x_c)$ is a Casimir function but all functions of the form $\Psi(C(x, x_c))$ are Casimirs of the closed-loop system. Thus we have a whole family of Casimirs to choose from, instead of specific Casimirs. On the basis of the Hamiltonian of the plant, the Hamiltonian of the controller and the corresponding Casimir function a Lyapunov function candidate is built as the sum of the plant and controller Hamiltonians and the Casimir function as

$$V(x, x_c) = H(x) + H_c(x_c) + \Psi(C(x, x_c)) \tag{5.37}$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x, x_c) = -\frac{\partial^{\mathrm{T}} H}{\partial x}(x) R(x) \frac{\partial^{\mathrm{T}} H}{\partial x}(x) - \frac{\partial^{\mathrm{T}} H_c}{\partial x_c}(x_c) R_c(x_c) \frac{\partial^{\mathrm{T}} H_c}{\partial x_c}(x_c) \leq 0$$

and hence $V(x, x_c)$ qualifies as a Lyapunov function for the closed-loop dynamics.

The next step would be to shape the closed-loop energy in the extended state space (x, x_c) in such a way that it has a minimum at $(x_\star, x_{c\star})$, therefore we require that the gradient of (5.37) has an extremum at $(x_\star, x_{c\star})$ and that the Hessian at $(x_\star, x_{c*})$ is positive definite, that is

$$\left[\begin{array}{c} \frac{\partial}{\partial x}[H(x)+\Psi(C(x,x_c))] \\ \frac{\partial}{\partial x_c}[H_c(x_c)+\Psi(C(x,x_c))] \end{array}\right]\Bigg|_{(x_\star,x_{c\star})} = 0 \tag{5.38}$$

and

$$\left[\begin{array}{cc} \frac{\partial^2}{\partial x^2}[H(x)+\Psi(C(x,x_c))] & \frac{\partial^2}{\partial x_c \partial x}\Psi(C(x,x_c))] \\ \frac{\partial^2}{\partial x \partial x_c}\Psi(C(x,x_c))] & \frac{\partial^2}{\partial x_c^2}[H_c(x_c)+\Psi(C(x,x_c))] \end{array}\right]\Bigg|_{(x_\star,x_{c\star})} \geq 0 \tag{5.39}$$

Suppose that $V(x,x_c)$ has a strict local minimum at $(x_\star,x_{c\star})$. Furthermore assume that the largest invariant set under the closed–loop dynamics contained in

$$\left\{(x,x_c)\in\mathscr{X}\times\mathscr{X}_c \mid \frac{\partial^{\mathrm{T}}H}{\partial x}(x)R(x)\frac{\partial^{\mathrm{T}}H}{\partial x}(x)=0, \quad \frac{\partial^{\mathrm{T}}H_c}{\partial x_c}(x_c)R_c(x_c)\frac{\partial^{\mathrm{T}}H_c}{\partial x_c}(x_c)=0\right\}$$

equals $(x_\star,x_{c\star})$. Then $(x_\star,x_{c\star})$ is a locally asymptotically stable equilibrium of the closed–loop system.

Example 5.4. We again consider the case of the normalized pendulum as in the above example. With first order controllers, we need to solve (5.36) for Casimirs of the closed loop system. In other words we are looking for the solution of the PDE

$$\frac{\partial C}{\partial x}(x,x_c)=\frac{\partial C}{\partial x_c}(x,x_c)$$

The solution of this PDE should be of the form $C(x,x_c)=q-x_c$, hence any function of the form $\Psi(q-x_c)$ is a Casimir for the closed-loop system. As in the above example, the objective is to stabilize the system at a desired equilibrium in the extended state space $(q_\star,x_{c\star})$. We shape the potential energy in such a way that it has a minimum at $q=q_\star$, $x_c=x_{c\star}$. This can be achieved by choosing a controller of the form

$$H_c(x_c)=\frac{1}{2}\beta\left(x_c-x_{c\star}-\frac{1}{\beta}\sin q_\star\right)^2$$

and the function $\Psi(C(q,x_c))=\Psi(q-x_c)$ as

$$\Psi(q-x_c)=\frac{1}{2}k\left[q-q_\star-(x_c-x_{c\star})-\frac{1}{k}\sin q_\star\right]^2$$

where β and k are chosen to satisfy (5.38) and (5.39). Simple computations show that β and k should be chosen such that

$$\cos q_\star + k > 0 \qquad\qquad \beta\cos q_\star + k\cos q_\star + k\beta > 0$$

The resulting passivity based input u is then given by

$$u = -\frac{\partial H_c}{\partial x_c}(x_c) = -\beta\left(x_c - x_{c\star} - \frac{1}{\beta}\sin q_\star\right)$$

Remark 5.7. In the same way we can also stabilize a system of n "fully actuated" pendulums, in which case we have to solve n different PDE's for each of the subsystem, in order to find the corresponding Casimir functions.

Example 5.5. The model of a permanent magnet synchronous machine [166], in the case of an isotropic rotor, in the (d,q) frame can be written as a pH system in the input state output form, with the state vector $x = [x_1, x_2, x_3]^{\mathrm{T}}$ and

$$J(x) = \begin{bmatrix} 0 & \frac{LP}{J}x_3 & 0 \\ -\frac{LP}{J}x_3 & 0 & -\Phi \\ 0 & \Phi & 0 \end{bmatrix} \qquad R = \begin{bmatrix} R_s & 0 & 0 \\ 0 & R_s & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

where x_1, x_2 are the stator currents, x_3 is the angular velocity, P is the number of pole pairs, L is the stator inductance, R_s is the stator winding resistance, and Φ and J are the dq back emf constant and the moment of inertia both normalized with P. The inputs are the stator voltages $[v_d,\ v_q]^{\mathrm{T}}$. The energy function of the system is given by

$$H(x) = \frac{1}{2}\left(Lx_1^2 + Lx_2^2 + \frac{J}{P}x_3^2\right)$$

The desired equilibrium to be stabilized is usually selected based on the so–called "maximum torque per ampere" principle as

$$x_\star = \left[0,\ \frac{L\tau_l}{P\Phi},\ \frac{J}{P}x_{3\star}\right]^{\mathrm{T}}$$

where τ_l is the constant load torque[2]. Interconnecting the plant system with a port-Hamiltonian control

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} u_{c1} \\ u_{c2} \end{bmatrix} \qquad \begin{bmatrix} y_{c1} \\ y_{c2} \end{bmatrix} = \begin{bmatrix} \frac{\partial H_c}{\partial \xi_1}(\xi_1, \xi_2) \\ \frac{\partial H_c}{\partial \xi_2}(\xi_1, \xi_2) \end{bmatrix}$$

via the power preserving interconnection

$$v_d = -y_{c1} \qquad u_{c1} = \frac{\partial H}{\partial x_1}(x) \qquad v_q = -y_{c2} \qquad u_{c2} = \frac{\partial H}{\partial x_2}(x)$$

yields the closed–loop system

[2] In the port-Hamiltonian model of the permanent magnet synchronous machine, τ_l acts as a perturbation to the system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ x_3 \\ \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} -R_s & \frac{LP}{J}x_3 & 0 & -1 & 0 \\ -\frac{LP}{J}x_3 & -R_s & -\Phi & 0 & -1 \\ 0 & \Phi & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \\ \frac{\partial H}{\partial x_3} \\ \frac{\partial H_c}{\partial \xi_1} \\ \frac{\partial H_c}{\partial \xi_2} \end{bmatrix}$$

Using Proposition 5.2, we get that the Casimir function is given by $C = \frac{1}{\Phi}x_3 - \xi_2$. Thus, the resulting Lyapunov function would be of the form (5.37)

$$V(x,\xi) = \frac{1}{2}\left(Lx_1^2 + Lx_2^2 + \frac{J}{P}x_3^2\right) + H_c(\xi) + \Psi\left(\frac{1}{\Phi}x_3 - \xi_2\right).$$

However, we can see that the equilibrium assignment condition (5.38) cannot be satisfied, because we need to shape both x_2 and x_3 to assign $x_\star$, and the Casimir depends only on x_3. To overcome this problem, the interconnection matrix $J(x)$ should be modified, but this is not possible with the *Control by Interconnection* technique.

In general, it is not possible to apply the Control by Interconnection to the family of electromechanical systems described in [170]. Firstly, in most cases, the closed–loop matrix $J_{cl}(x,\xi) - R_{cl}(x,\xi)$ is full–rank, leading to Casimir functions of the form $C(x,\xi) = c$, with c a constant vector, which obviously cannot be used to shaped the energy of the system. Secondly, even if we can determine the Casimirs – as in the case of the permanent magnet synchronous machine – these functions do not depend on the coordinates we need to shape. The source of the problem is the lack of interconnection between the electrical and mechanical subsystems, which can be solved modifying the interconnection matrix $J(x)$, [160, 161].

In the case of electromechanical systems, using a control input $u = -\frac{\partial H_c}{\partial \xi} + \bar{v}$, with $\bar{v}$ a constant input, leads to a forced Hamiltonian system with dissipation. The analysis of [136] also allows to modify the interconnection structure to generate Lyapunov function for nonzero equilibria. However, even if Casimirs can be obtained (namely, microelectronics actuators, magnetic levitation system, etc), the stability analysis reveals that the minimum cannot be assigned. This can be viewed as a limitation of this procedure though further investigation remains open.

5.6 Interconnection and damping assignment passivity based-control (IDA–PBC)

IDA–PBC was introduced in [161] as a procedure to control physical systems described by port-Hamiltonian models in input–output form as

$$\begin{cases} \dot{x} = \big[J(x) - R(x)\big] \dfrac{\partial H}{\partial x}(x) + g(x)u \\ y = g^{\mathrm{T}}(x) \dfrac{\partial H}{\partial x}(x) \end{cases} \qquad x \in \mathbb{R}^n,\ u, y \in \mathbb{R}^m \tag{5.40}$$

In the IDA–PBC procedure we select the structure of the closed–loop system as another port-Hamiltonian system and then we characterize all assignable energy functions compatible with this structure. This characterization is given in terms of the solution of a partial differential equation (PDE) which is parametrized by three (designer chosen) matrices that are related with the interconnection between the subsystems, the damping and the kernel of the systems input matrix, respectively. Several interpretations can be given to the role played by these matrices. At the most basic computational level they can be simply viewed as degrees-of-freedom to simplify the solution of the PDE. In the case of physical systems the interconnection and the damping matrices determine the energy exchange and the dissipation of the system, respectively, consequently they can often be judiciously chosen invoking this kind of physical considerations. See [158] for an extensive list of references and applications of this methodology.

The main proposition of IDA–PBC for port–Hamiltonian systems is stated as follows [161]:

Proposition 5.4. *Consider the system (5.40), assume there are matrices* $g^{\perp}(x)$, $J_d(x) = -J_d^T(x)$, $R_d(x) = R_d^T(x) \geq 0$ *and a function* $H_d : \mathbb{R}^n \to \mathbb{R}$ *that verify the PDE*

$$g^{\perp}(x)\big[J(x) - R(x)\big] \frac{\partial H}{\partial x} = g^{\perp}(x)\big[J_d(x) - R_d(x)\big] \frac{\partial H_d}{\partial x} \tag{5.41}$$

where $g^{\perp}(x)$ *is a full-rank left annihilator of* $g(x)$, *i.e.,* $g^{\perp}(x)g(x) = 0$, *and* $H_d(x)$ *is such that*

$$x_\star = \arg\min H_d(x) \tag{5.42}$$

with $x_\star \in \mathbb{R}^n$ *the equilibrium to be stabilized. Then, the closed–loop system (5.40) with* $u = \beta(x)$, *where*

$$\beta(x) = \big[g^T(x)g(x)\big]^{-1} g^T(x) \left\{ \big[J_d(x) - R_d(x)\big] \frac{\partial H_d}{\partial x} - \big[J(x) - R(x)\big] \frac{\partial H}{\partial x} \right\}$$

takes the port-Hamiltonian form

$$\dot{x} = \big[J_d(x) - R_d(x)\big] \frac{\partial H_d}{\partial x} \tag{5.43}$$

with $x_\star$ *a (locally)* stable *equilibrium. It will be* asymptotically *stable if, in addition,* $x_\star$ *is an isolated minimum of* $H_d(x)$ *and the largest invariant set under the closed–loop dynamics (5.43) contained in*

$$\left\{ x \in \mathbb{R}^n \mid \frac{\partial^T H_d}{\partial x} R_d(x) \frac{\partial H_d}{\partial x} = 0 \right\}$$

equals $\{x_\star\}$. *An estimate of its* domain of attraction *is given by the largest* bounded *level set* $\{x \in \mathbb{R}^n \mid H_d(x) \leq c\}$.

Following the ideas of Chapter 2, the IDA–PBC methodology can be expressed in the Dirac framework as follows. Consider the port-Hamiltonian system with state space $\mathscr{X}$, Hamiltonian H corresponding to the energy storage port $\mathscr{S}$, resistive port $\mathscr{R}$ and control port $\mathscr{C}$, given in input-state-output form in (5.40). As in Sect. 2.4.1, if the Dirac structure $\mathscr{D}$ is given in matrix kernel representation as

$$\mathscr{D} = \Big\{ (f_S, e_S, f_R, e_R, f_c, e_c) \in \mathscr{F}_S \times \mathscr{F}_S^* \times \mathscr{F}_R \times \mathscr{F}_R^* \times \mathscr{F}_c \times \mathscr{F}_c^* \mid F_S f_S + E_S e_S + F_R f_R + E_R e_R + F_c f_c + E_c e_c = 0 \Big\}$$

with

$$\begin{aligned} &(i)\ \ E_S F_S^{\mathrm{T}} + F_S E_S^{\mathrm{T}} + E_R F_R^{\mathrm{T}} + F_R E_R^{\mathrm{T}} + E_c F_c^{\mathrm{T}} + F_c E_c^{\mathrm{T}} = 0 \\ &(ii)\ \ \operatorname{rank}\big[F_S \mid E_S \mid F_R \mid E_R \mid F_c \mid E_c\big] = \dim(\mathscr{F}_R \times \mathscr{F}_R \times \mathscr{F}_c) \end{aligned}$$

then, the port–Hamiltonian system (5.40) is given by the set of equations:

$$-F_S \dot{x}(t) + E_S \frac{\partial H}{\partial x}(x(t)) + F_R f_R(t) + E_R e_R(t) + F_c f(t) + E_c e_c(t) = 0 \tag{5.44}$$

where we have set the flows of the energy storing elements $f_S = -\dot{x}$ (the negative sign is included to have a consistent energy flow direction) and the efforts corresponding to the energy storing elements $e_S = \frac{\partial H}{\partial x}$.

Restricting to linear resistive elements, the flow and effort variables connected to the resistive elements are related as $f_R = -\tilde{R} e_R$, with $\tilde{R} = \tilde{R}^{\mathrm{T}} \geq 0$. Substituting these into (5.44) leads to the description of the physical system (5.40) by the set of DAE's

$$-F_S \dot{x}(t) + E_S \frac{\partial H}{\partial x}(x(t)) - F_R \tilde{R} e_R + E_R e_R + F_c f_c(t) + E_c e_c(t) = 0 \tag{5.45}$$

where we can see that (5.40) is a special case of (5.45) by letting

$$F_s = \begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix} \qquad E_S = \begin{bmatrix} J(x) \\ -g_R^{\mathrm{T}}(x) \\ -g^{\mathrm{T}}(x) \end{bmatrix} \qquad F_R = \begin{bmatrix} g_R(x) \\ 0 \\ 0 \end{bmatrix} \qquad E_R = \begin{bmatrix} 0 \\ I_r \\ 0 \end{bmatrix}$$
$$F_c = \begin{bmatrix} g(x) \\ 0 \\ 0 \end{bmatrix} \qquad E_c = \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix}$$

with $r = \dim F_R$, and setting $u = f_c$, $y = e_c$ and $R(x) = g_R^{\mathrm{T}}(x) \tilde{R} g_R(x)$ with g_R representing the input matrix corresponding to the resistive port. As above the objective of IDA–PBC is to find a control input $u = \beta(x)$ such that the closed-loop system (5.43) in implicit form is given by

$$-F_S \dot{x}(t) + E_{S_d} \frac{\partial H_d}{\partial x}(x(t)) - F_{R_d} \tilde{R}_d e_{R_d} + E_{R_d} e_{R_d} = 0 \tag{5.46}$$

with

$$F_S = \begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix} \qquad E_{S_d} = \begin{bmatrix} J_d(x) \\ -g_{R_d}^{\mathrm{T}}(x) \\ 0 \end{bmatrix} \qquad F_{R_d} = \begin{bmatrix} g_{R_d}(x) \\ 0 \\ 0 \end{bmatrix} \qquad E_{R_d} = \begin{bmatrix} 0 \\ I_{r_d} \\ 0 \end{bmatrix}$$

where $J_d(x) = -J_d^{\mathrm{T}}(x)$, $R_d(x) = g_{R_d}^{\mathrm{T}}(x)\tilde{R}_d g_{R_d}(x) = R_d^{\mathrm{T}}(x) \geq 0$, with g_{R_d} representing the input matrix corresponding to the desired resistive port and $r_d = \dim F_{R_d}$.

Multiplying both sides of (5.45) and (5.46) by $F_c^{\perp}$ – a full-rank left annihilator of F_c, i.e. $F_c^{\perp}F_c = 0$ – and eliminating $\dot{x}$, we get

$$F_c^{\perp}\left[E_S\frac{\partial H}{\partial x}(x(t)) - F_R\tilde{R}e_R + E_R e_R + E_c e_c(t)\right] = \\ = F_c^{\perp}\left[E_{S_d}\frac{\partial H_d}{\partial x}(x(t)) - F_{R_d}\tilde{R}_d e_{R_d} + E_{R_d}e_{R_d}\right]$$

assigning $F_c^{\perp} = [g^{\perp}(x) \quad 0 \quad 0]$, with $g^{\perp}(x)$ a full–rank left annihilator of $g(x)$, that is, $g^{\perp}(x)g(x) = 0$, the above equation becomes

$$F_c^{\perp}\left[E_S\frac{\partial H}{\partial x}(x(t)) - F_R\tilde{R}e_R\right] = F_c^{\perp}\left[E_{S_d}\frac{\partial H_d}{\partial x}(x(t)) - F_{R_d}\tilde{R}_d e_{R_d}\right]$$

which is an equivalent representation of the *matching condition* (5.41).

5.6.1 Solving the matching equation

A key step in the IDA–PBC methodology is the solution of (5.41). We underscore the fact that in this equation:

1. $J_d(x)$ and $R_d(x)$ are free-up to the constraint of skew-symmetry and positive semi-definiteness, respectively;
2. $H_d(x)$ may be totally, or partially, fixed provided we can ensure (5.42) – and probably a properness condition;
3. there is an additional degree-of-freedom in $g^{\perp}(x)$ which is not uniquely defined by $g(x)$. As reported in [3], this degree of freedom can be used to linearize a nonlinear PDE that appears in mechanical systems.

Therefore, to solve this equation there are, at least, three ways to proceed:

A. *(Non–Parametrized IDA)* In one extreme case, which was the original one adopted in [161], we *fix* the desired interconnection and dissipation matrices $J_d(x)$ and $R_d(x)$ – hence the name IDA – as well as $g^{\perp}(x)$. This yields a PDE whose solutions define the admissible energy functions $H_d(x)$ for the given interconnection and damping matrices. Among the family of solutions we select one that satisfies (5.42).

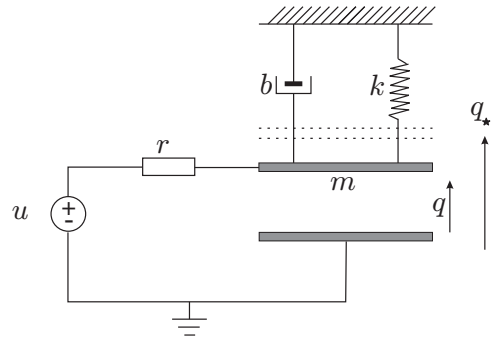


Fig. 5.6 Model of an electrostatic micro-actuator.

1. *(Algebraic IDA)* At the other extreme, originally proposed by [79], one *fixes* the desired energy function, then (5.41) becomes an algebraic equation in $J_d(x)$, $R_d(x)$ and $g^\perp(x)$.
2. *(Parametrized IDA)* For some physical systems it is desirable to restrict the desired energy function to a certain class, for instance, for mechanical systems the sum of a potential energy term, that depends only on the generalized positions, and the kinetic energy that is quadratic in the generalized momenta [162]. Fixing the *structure* of the energy function yields a new PDE for its unknown terms and, at the same time, imposes some constraints on the interconnection and damping matrices.

Example 5.6. Consider the problem of position regulation of the micro–electromechanical system depicted in figure(5.6). The dynamical equations of motion [130, 183], can be represented as a port–Hamiltonian system of the form (5.40) where the state of the system is the air gap q (with $q_\star$ the equilibrium to be stabilized), the momentum p and the charge of the device Q. The plate area, the mass of the plate and the permittivity in the gap are represented by A, m, and ε, respectively. The spring and friction coefficients are given respectively by the positive constants k and b. The input resistance is r and u represents the input voltage which is the control action. As pointed out in [130], p is usually not available for measurement. The interconnection structure is given by

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \frac{1}{r} \end{bmatrix} \qquad g = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{r} \end{bmatrix}$$

with the energy function

$$H(q,p,Q) = \frac{1}{2}k(q-q_\star)^2 + \frac{1}{2m}p^2 + \frac{q}{2A\varepsilon}Q^2$$

Let us start with the Algebraic IDA and assume we want to assign a quadratic energy function, that is,

$$H_d(q,p,Q) = \frac{\gamma_1}{2}(q-q_\star)^2 + \frac{1}{2m}p^2 + \frac{\gamma_2}{2}Q^2$$

where γ_1, γ_2 are positive constants. Denote the desired interconnection and damping matrices, which are to be determined, as

$$J_d = \begin{bmatrix} 0 & J_{12} & J_{13} \\ -J_{12} & 0 & J_{23} \\ -J_{13} & -J_{23} & 0 \end{bmatrix} \qquad R_d = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \tag{5.47}$$

After some simple calculations we get that equation (5.41) becomes an algebraic equation, with one possible solution, the following matrices

$$J_d(Q) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -\frac{Q}{2A\varepsilon\gamma_2} \\ 0 & \frac{Q}{2A\varepsilon\gamma_2} & 0 \end{bmatrix} \qquad R_d = \begin{bmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & r_3 \end{bmatrix}$$

with $\gamma_1 = k$ and $\gamma_2, r_3 > 0$ free parameters. The control law is obtained as

$$\beta(q,p,Q) = \left(\frac{1}{2A\varepsilon\gamma_2}\frac{p}{m} - r_3\gamma_2\right) rQ + \frac{1}{A\varepsilon}qQ$$

Notice that the control depends on the unmeasurable state p. To apply the Non–Parametrized IDA, we use the same structure of $(J_d - R_d)$ in (5.47), this yields the PDEs

$$\begin{aligned} r_1\frac{\partial H_d}{\partial q} + J_{12}\frac{\partial H_d}{\partial p} + J_{13}\frac{\partial H_d}{\partial Q} &= \frac{p}{m} \\ -J_{12}\frac{\partial H_d}{\partial q} - r_2\frac{\partial H_d}{\partial p} + J_{23}\frac{\partial H_d}{\partial Q} &= -k(q-q_\star) - \frac{Q^2}{2A\varepsilon} - \frac{b}{m}p \end{aligned}$$

As there are no clear indications on how to choose the elements of $J_d - R_d$, we fix them as constants. Based on physical considerations we can select the matrix $J_d - R_d$ as

$$J_d - R_d = \begin{bmatrix} 0 & J_{12} & 0 \\ -J_{12} & -r_2 & J_{23} \\ 0 & -J_{23} & -r_3 \end{bmatrix}$$

The solution of the PDEs yields a cumbersome expression and an even more complicated controller that still depends on p. That is, setting $r_2 = J_{12}b$ we get for the desired energy function

$$\begin{aligned} H_d(q,p,Q) = \frac{1}{2J_{12}}\left(\frac{1}{m}p^2 + \frac{k}{2A\varepsilon}(q-q_\star)^2 + \frac{J_{23}^2}{3J_{12}^2A\varepsilon}q^3 + \frac{J_{23}}{J_{12}A\varepsilon}q^2Q + \frac{1}{A\varepsilon}qQ^2\right) \\ + \psi\left(\frac{J_{23}}{J_{12}}q + Q\right) \end{aligned}$$

where $\psi(\cdot)$ is a free function to be selected to guarantee the equilibrium assignment condition (5.42). The expression for the controller is given by

$$\beta(q,p,Q) = -\frac{J_{23}Rr}{J_{12}m}p - r_3 r\left(\frac{J_{23}}{2J_{12}^2 A\varepsilon}q^2 + \frac{1}{4J_{12}A\varepsilon}q + \psi_Q'\right) + \frac{1}{A\varepsilon}qQ$$

Finally, we observe that the mechanical part of the system suggests to consider an energy function consisting of the sum of the open-loop kinetic energy and a function to be defined, that is

$$H_d(q,p,Q) = \frac{1}{2m}p^2 + \varphi(q,Q)$$

This parametrization fixes $(J_d - R_d)$ as

$$J_d - R_d = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -b & 0 \\ 0 & 0 & -r_3 \end{bmatrix}$$

The solution of the PDEs (5.41) yields

$$\varphi(q,Q) = \frac{1}{2}k(q-q_\star)^2 + \frac{1}{2A\varepsilon}qQ^2 + \psi(Q)$$

with $\psi(Q)$ again free for the equilibrium assignment. After some simple calculations we obtain the nice output–feedback control

$$\beta(q,Q) = -(r_3 r - 1)\frac{1}{A\varepsilon}qQ - r_3 r\psi'$$

which does not depend on the unmeasurable state p. This controller contains, as a particular case with $r_3 = \frac{1}{r}$ and $\psi(Q)$ quadratic, the linear charge feedback controller studied in [130], but it also allows to use nonlinear functions to guarantee for instance, saturation levels. We should also mention that the control scheme of [130] has been obtained using a damping control plus a control by interconnection approach.

5.6.2 Energy–balancing of IDA–PBC

The stabilization mechanism of IDA–PBC is particularly clear when applied to port-Hamiltonian systems (5.40) with some "suitable" damping properties. Indeed, it has been shown in [161] that if the natural damping of the port-Hamiltonian system satisfies[3]

$$R(x)\left(\frac{\partial H_d}{\partial x} - \frac{\partial H}{\partial x}\right) = 0 \tag{5.48}$$

[3] Since $R(x)$ is usually diagonal, this condition requires that no damping is present in the coordinates that need to be shaped, that is, the coordinates where the function $H(x)$ has to be modified.

and no additional damping is injected, that is, $R_d(x) = R(x)$, then (along the trajectories of the closed-loop system) the desired energy function may be expressed as

$$H_d(x(t)) = H(x(t)) - \int_0^t u^{\mathrm{T}}(s)y(s)\mathrm{d}s. \tag{5.49}$$

As this expression reveals, in this case, the IDA–PBC assigns as energy function the difference between the energy stored in the system and the energy supplied to it from the environment, hence we say that the controller is *energy–balancing*. For the case of mechanical systems, it can be shown that IDA–PBC is energy–balancing if we modify only the potential energy of the system.

Even when (5.48) is not satisfied the control action of IDA–PBC admits an energy–balancing–like interpretation. In [95] is shown that if the interconnection and the damping are not modified, i.e., $J_d(x) = J(x)$, $R_d(x) = R(x)$, and the matrix $J(x) - R(x)$ is full rank, then $H_d(x(t))$ still satisfies (5.49) with y replaced by the "new output"

$$\tilde{y} = -g^{\mathrm{T}}(x)\left[J(x) - R(x)\right]^{-\mathrm{T}}\left\{\left[J(x) - R(x)\right]\frac{\partial H}{\partial x} + g(x)u\right\}$$

As a partial converse of this result we also have that for single input systems that verify condition (5.48) the new output $\tilde{y}$ exactly coincides with the natural one y. Although $u^{\mathrm{T}}\tilde{y}$ does not have (in general) units of power anymore, it is also shown in [95] that for electromechanical systems the new port variables are obtained from a classical Thevenin-Norton transformation of the voltage-current port variables (u, y).

5.7 Power–shaping stabilization

Energy–balancing control is restricted to systems that do not have "pervasive dissipation", term which refers to the existence of resistive elements whose power does not vanish at the desired equilibrium point. As indicated in Sect. 5.2, systems having pervasive dissipation cannot be stabilized by energy balancing, since they extract an infinite amount of energy from the controller at the desired equilibrium point. Another practical drawback of energy–shaping control is the limited ability to "speed up" the transient response leading to somehow sluggish transients and below par overall performance levels.

To overcome these obstacles, an alternative method was introduced in [159] for nonlinear RLC circuits described by the Brayton–Moser equations (cf. Sect. 2.3.2.1 and [27, 28]). In this new method called *Power–Shaping* the storage function used to identify the passive maps is not the total energy but a function directly related with the power in the circuit. Furthermore, in contrast with the well known passivity property of the (conjugated variables) voltage and current, passivity is established

now with respect to voltage and derivative of the current (or current and derivative of the voltage).

5.7.1 From port-Hamiltonian systems to the Brayton–Moser equations

Differently from what has been presented at the end of Sect. 2.3.2.1, let us now consider a port-Hamiltonian system with dissipation *and* control ports, which can be represented by the set of differential equations

$$\begin{cases} \dot{x} = J(x)\dfrac{\partial H}{\partial x} + g_R(x)f_R + g(x)f_c \\ e_R = g_R^{\mathrm{T}}(x)\dfrac{\partial H}{\partial x} \\ e_c = g^{\mathrm{T}}(x)\dfrac{\partial H}{\partial x} \end{cases} \tag{5.50}$$

where f_c, e_c represent the flows and efforts corresponding to the control port and the resistive relation is specified by an (effort-controlled) Rayleigh dissipation function

$$f_R = -\frac{\partial \tilde{R}}{\partial e_R}(e_R)$$

As in Sect. 2.3.2.1, suppose that the mapping from the energy variables x to the co–energy variables $e = \frac{\partial H}{\partial x}$ is invertible, that is, the inverse transformation from the co–energy variables to the energy variables is given by

$$x = \frac{\partial H^*}{\partial e}(e) \tag{5.51}$$

where H^* is the co–energy or Legendre transformation of H given as $H^* = e^{\mathrm{T}}x - H(x)$. From (5.51) it follows that the dynamics of the port-Hamiltonian system (5.50) can be expressed in the co-energy variables as

$$\begin{cases} \dfrac{\partial^2 H^*}{\partial e^2}\dot{e} = J(x)e - g_R(x)\dfrac{\partial \tilde{R}}{\partial e_R}(e_R) + g(x)f_c \\ e_R = g_R^{\mathrm{T}}(x)e \\ e_c = g^{\mathrm{T}}(x)e \end{cases} \tag{5.52}$$

where we may substitute (5.51) to obtain a differential equation solely in the co–energy variables e. Assume that we may find coordinates $x = (x_q, x_p)$, $\dim x_q = k$, $\dim x_p = n - k$, such that in these coordinates the matrix $J(x)$ takes the form

$$J(x) = \begin{bmatrix} 0 & -B(x) \\ B^{\mathrm{T}}(x) & 0 \end{bmatrix}$$

with $B(x)$ a $k \times (n-k)$ matrix, and moreover the Hamiltonian H splits as $H(x_q, x_p) = H_q(x_q) + H_p(x_p)$. Accordingly to the splitting $x = (x_q, x_p)$ the co–energy variables $e = (e_q, e_p)$ are given by

$$e_q = \frac{\partial H_q}{\partial x_q} \qquad\qquad e_p = \frac{\partial H_p}{\partial x_p}$$

yielding the Legendre transform $H^*(e)$ of $H(x)$ as $H^*(e_q, e_p) = H_q^*(e_q) + H_p^*(e_p)$. For simplicity assume that $g_R(x)$ and $g(x)$ take the form

$$g_R(x_q, x_p) = -\begin{bmatrix} I_k & 0 \\ 0 & I_{n-k} \end{bmatrix} \qquad g(x_q, x_p) = \begin{bmatrix} g_q(x_q) & 0 \\ 0 & g_p(x_p) \end{bmatrix}$$

where $g_q(x)$ and $g_p(x)$ are $k \times k$ and $(n-k) \times (n-k)$ matrices respectively. Then (5.52) becomes

$$\begin{aligned} \begin{bmatrix} \frac{\partial^2 H_q^*}{\partial e_q^2} & 0 \\ 0 & \frac{\partial^2 H_p^*}{\partial e_p^2} \end{bmatrix} \begin{bmatrix} \dot{e}_q \\ \dot{e}_p \end{bmatrix} &= \begin{bmatrix} 0 & -B(x) \\ B^{\mathrm{T}}(x) & 0 \end{bmatrix} \begin{bmatrix} e_q \\ e_p \end{bmatrix} + \begin{bmatrix} \frac{\partial \tilde{R}}{\partial e_q}(e_q, e_p) \\ \frac{\partial \tilde{R}}{\partial e_p}(e_q, e_p) \end{bmatrix} \\ &\quad + \begin{bmatrix} g_q(x_q) & 0 \\ 0 & g_p(x_p) \end{bmatrix} f_c \\ e_R &= -\begin{bmatrix} e_q \\ e_p \end{bmatrix} \\ e_c &= \begin{bmatrix} g_q^{\mathrm{T}}(x_q) & 0 \\ 0 & g_p^{\mathrm{T}}(x_p) \end{bmatrix} \begin{bmatrix} e_q \\ e_p \end{bmatrix} \end{aligned} \tag{5.53}$$

Note the similarities between (5.53) and (2.106). Now define the function

$$P(e_q, e_p, x) = e_q^{\mathrm{T}} B(x) e_p + \tilde{R}(e_q, e_p) \tag{5.54}$$

It follows that the first equation of (5.53) can be alternatively written as

$$\begin{bmatrix} \frac{\partial^2 H_q^*}{\partial e_q^2} & 0 \\ 0 & -\frac{\partial^2 H_p^*}{\partial e_p^2} \end{bmatrix} \begin{bmatrix} \dot{e}_q \\ \dot{e}_p \end{bmatrix} = -\begin{bmatrix} \frac{\partial P}{\partial e_q} \\ \frac{\partial P}{\partial e_p} \end{bmatrix} + \begin{bmatrix} g_q(x_q) & 0 \\ 0 & -g_p(x_p) \end{bmatrix} f_c \tag{5.55}$$

where we may substitute

$$x_q = \frac{\partial H_q^*}{\partial e_q} \qquad\qquad x_p = \frac{\partial H_p^*}{\partial e_p}$$

in order to obtain a differential equation solely in the co–energy variables (e_q, e_p). Note however that if the matrix $B(x)$ and therefore the function P non–trivially depend on x, then (5.55) is not valid if we substitute $x = \frac{\partial H^*}{\partial e}$ in the definition of $P(e_q, e_p, x)$ before taking the partial derivatives of P with respect to e_q and e_p.

Relation (5.55) together with the *mixed potential function* (5.54) correspond to the type of equations that were obtained in [27, 28] for RLC circuits, commonly called the *Brayton–Moser equations* (see example below, generalization of Example 2.8).

Example 5.7. Consider a general nonlinear RLC circuit where the capacitors are described by vector functions representing the capacitor charges $q_C = \hat{q}_C(v_C) : \mathbb{R}^{n_C} \to \mathbb{R}^{n_C}$ and capacitor currents $i_C = \dot{q}_C = C(v_C)\frac{\mathrm{d}}{\mathrm{d}t}v_C$, with capacitance matrix $C(v_C) = \frac{\partial \hat{q}_C}{\partial v_C} \in \mathbb{R}^{n_C} \times \mathbb{R}^{n_C}$. Analogously, the inductors are described by the inductor flux–linkages $\phi_L = \hat{\phi}_L(i_L) : \mathbb{R}^{n_L} \to \mathbb{R}^{n_L}$ and Faraday's law $v_L = \dot{\phi}_L = L(i_L)\frac{\mathrm{d}}{\mathrm{d}t}i_L$, with v_L denoting the inductor voltage. The inductance matrix is given by $L(i_L) = \frac{\partial \hat{\phi}_L}{\partial i_L} \in \mathbb{R}^{n_L} \times \mathbb{R}^{n_L}$. If the network is complete, the state equations in terms of the capacitor charges q_C and inductor flux-linkages ϕ_L is given by (5.50) with the energy variables $x = (q_C, \phi_L)$ and

$$J(x) = \begin{bmatrix} 0 & -B \\ B^{\mathrm{T}} & 0 \end{bmatrix}$$

for some constant matrix B consisting of 0, 1, and -1 elements, fully determined by the topology of the network. The Hamiltonian (total energy) splits as the sum of the electrical and magnetic energy as

$$H(q_C, \phi_L) = H_C(q_C) + H_L(\phi_L)$$

then the co–energy variables are given by

$$e = \begin{bmatrix} e_q \\ e_p \end{bmatrix} = \begin{bmatrix} \frac{\partial H_C}{\partial q_C} \\ \frac{\partial H_L}{\partial \phi_L} \end{bmatrix} = \begin{bmatrix} v_C \\ i_L \end{bmatrix}. \tag{5.56}$$

We have that the inverse transformation from the co–energy variables to the energy variables is given by $x = \frac{\partial H^*}{\partial e}$, with the co–energy function H^* defined as

$$H^* = H_C^*(v_C) + H_L^*(i_L) = \int_0^{v_C} \hat{q}_C(v_C')\mathrm{d}v_C' + \int_0^{i_L} \hat{\phi}_L(i_L')\mathrm{d}i_L' \tag{5.57}$$

The resistive relation is defined by

$$\tilde{R}(e_q, e_p) = \tilde{R}(v_C, i_L) = -G(v_C) + F(i_L)$$

where $F(i_L)$ is the total current potential (content) of all current-controlled resistors and $G(v_C)$ is the total voltage potential (co-content) of all voltage-controlled resistors. Then the mixed–potential function (5.54) becomes

$$P(v_C, i_L) = v_C^{\mathrm{T}} B i_L - G(v_C) + F(i_L). \tag{5.58}$$

Substituting (5.56), (5.57) and (5.58) in (5.55) we get

$$\begin{bmatrix} C(v_C) & 0 \\ 0 & -L(i_L) \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}v_C}{\mathrm{d}t} \\ \frac{\mathrm{d}i_L}{\mathrm{d}t} \end{bmatrix} = - \begin{bmatrix} \frac{\partial P}{\partial v_C} \\ \frac{\partial P}{\partial i_L} \end{bmatrix} + \begin{bmatrix} g_q & 0 \\ 0 & -g_p \end{bmatrix} \begin{bmatrix} i_S \\ v_S \end{bmatrix}$$

where we have set $f_c = (i_S, v_S)$ with $i_S \in \mathbb{R}^{n_{i_S}}$ the controlled current sources, $v_S \in \mathbb{R}^{n_{v_S}}$ the controlled voltage sources, $g_q \in \{-1,0,1\}^{n_C \times n_{i_S}}$ and $g_q \in \{-1,0,1\}^{n_L \times n_{v_S}}$.

5.7.2 Geometry of Brayton–Moser's equation

From a geometric viewpoint, the Brayton–Moser equations (5.55) can be interpreted as *gradient equations* with respect to the mixed-potential function P (5.54) and the indefinite inner product or pseudo-*Riemannian metric* defined by the symmetric matrix

$$\begin{bmatrix} \frac{\partial^2 H_q^*}{\partial e_q^2} & 0 \\ 0 & -\frac{\partial^2 H_p^*}{\partial e_p^2} \end{bmatrix}$$

Notice however that if the Hamiltonian H does not split as $H(x_q, x_p) = H_q(x_q) + H_p(x_p)$, we obtain instead of (5.55) the more general equation

$$\begin{bmatrix} \frac{\partial^2 H^*}{\partial e_q^2} & \frac{\partial^2 H^*}{\partial e_q \partial e_p} \\ -\frac{\partial^2 H^*}{\partial e_p \partial e_q} & -\frac{\partial^2 H^*}{\partial e_p^2} \end{bmatrix} \begin{bmatrix} \dot{e}_q \\ \dot{e}_p \end{bmatrix} = - \begin{bmatrix} \frac{\partial P}{\partial e_q} \\ \frac{\partial P}{\partial e_p} \end{bmatrix} + \begin{bmatrix} g_q(x_q) & 0 \\ 0 & -g_p(x_p) \end{bmatrix} f_c \tag{5.59}$$

Hence in this case the most left matrix appearing in (5.59) is not symmetric anymore, and therefore does not define a pseudo–Riemannian metric. In the following we will consider the Brayton–Moser equations defined in (5.55).

Like Port–Hamiltonian systems, the Brayton–Moser equations (5.55) can be also written in the formalism of Dirac structures, using a non–canonical Dirac structure representation [22]. For instance, consider the following non–canonical Dirac structure

$$\mathscr{D}(e_q,e_p)=\left\{(f_S,e_S,f_c,e_c)\in\mathscr{F}_S\times\mathscr{F}_S^*\times\mathscr{F}_c\times\mathscr{F}_c^*\mid \right.$$
$$-\begin{bmatrix}\frac{\partial^2 H_q^*}{\partial e_q^2} & 0\\ 0 & -\frac{\partial^2 H_p^*}{\partial e_p^2}\end{bmatrix} f_S=e_S+\begin{bmatrix}-g_q & 0\\ 0 & g_p\end{bmatrix} f_c,$$
$$\left. e_c=\begin{bmatrix}-g_q^{\mathrm{T}} & 0\\ 0 & g_p^{\mathrm{T}}\end{bmatrix} f_S\right\} \quad (5.60)$$

defined with respect to the bi-linear form

$$\ll (f_{S_1},e_{S_1},f_{c_1},e_{c_1}),(f_{S_2},e_{S_2},f_{c_2},e_{c_2})\gg_{(e_q,e_p)}=$$
$$=e_{S_1}^{\mathrm{T}}f_{S_2}+e_{S_2}^{\mathrm{T}}f_{S_1}+e_{c_1}^{\mathrm{T}}f_{c_2}+e_{c_2}^{\mathrm{T}}f_{c_1}+f_{S_1}^{\mathrm{T}}\left(\mathscr{A}(e_q,e_p)+\mathscr{A}^{\mathrm{T}}(e_q,e_p)\right)f_{S_2}$$

where

$$\mathscr{A}(e_q,e_p)=\begin{bmatrix}\frac{\partial^2 H_q^*}{\partial e_q^2} & 0\\ 0 & -\frac{\partial^2 H_p^*}{\partial e_p^2}\end{bmatrix}$$

Then, the Brayton–Moser equations can be described as a dynamical system with respect to the non–canonical Dirac structure (5.60) by setting the flow variables as the rate of change of the power variables (e_q,e_p), i.e. $f_S=-(\dot{e}_q,\dot{e}_p)$, the input port variables $u=f_c=(u_q,u_p)$, the effort variables as the co-power variables $e_S=(\frac{\partial P}{\partial e_q}\frac{\partial P}{\partial e_p})$. Notice that the flow and effort variables are conjugated variables in the sense that

$$\dot{P}=\frac{\partial^{\mathrm{T}}P}{\partial e_q}\dot{e}_q+\frac{\partial^{\mathrm{T}}P}{\partial e_p}\dot{e}_p.$$

Recall from the last equation of (5.53) that the natural output variables for the Brayton–Moser equations (5.55) are

$$e_c=\begin{bmatrix}g_q^{\mathrm{T}} & 0\\ 0 & g_p^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}e_q\\ e_p\end{bmatrix}$$

since they are power-conjugated to the inputs f_c, i.e. $e_c^{\mathrm{T}}f_c$ has units of power, it turns out that they are not conjugated with respect to $\dot{P}$ which has units of power per second. Hence, redefining the output port variables as

$$y=\begin{bmatrix}-g_q^{\mathrm{T}} & 0\\ 0 & g_p^{\mathrm{T}}\end{bmatrix}f_S=\begin{bmatrix}g_q^{\mathrm{T}}\dot{e}_q\\ -g_p^{\mathrm{T}}\dot{e}_p\end{bmatrix}$$

the Brayton-Moser equations (5.55) can equivalently be written as the set of differential equations

$$\left(-\dot{e}_q, -\dot{e}_p, \frac{\partial P}{\partial e_q}, \frac{\partial P}{\partial e_p}, u_q, u_p, g_q^{\mathrm{T}}\dot{e}_q, -g_p^{\mathrm{T}}\dot{e}_p\right)(t) \in \mathscr{D}(e_q, e_p), \quad t \in \mathbb{R}.$$

The above Dirac structure satisfies the "power balance equation"

$$0 = -\frac{\partial^{\mathrm{T}} P}{\partial e_q}\dot{e}_q - \frac{\partial^{\mathrm{T}} P}{\partial e_p}\dot{e}_p + \dot{e}_q^{\mathrm{T}} g_q u_q - \dot{e}_p^{\mathrm{T}} g_p u_p + \\ + \left[\dot{e}_q^{\mathrm{T}}\ \dot{e}_p^{\mathrm{T}}\right]\left(\mathscr{A}(e_q, e_p) + \mathscr{A}^{\mathrm{T}}(e_q, e_p)\right)\begin{bmatrix}\dot{e}_q \\ \dot{e}_p\end{bmatrix}$$

i.e.

$$\dot{P} = y^{\mathrm{T}} u + \dot{e}^{\mathrm{T}}\Big(\mathscr{A}(e) + \mathscr{A}^{\mathrm{T}}(e)\Big)\dot{e}. \tag{5.61}$$

where $e = (e_q, e_p)$. Relation (5.61) shows that P is not conserved, not even when $u \equiv 0$ (i.e., when the power supplied to the network through its ports is zero).

5.7.3 Stabilization by power–shaping

We have seen that the Brayton–Moser equations (5.55) satisfy the power balancing equation (5.61). However, we cannot establish a power balancing inequality since $\mathscr{A}(e_q, e_p)$ is sign–indefinite. Furthermore, to obtain the passivity property an additional difficulty stems from the fact that the mixed potential $P(e_q, e_p)$ is also not sign definite. To overcome these difficulties, in [27, 28, 159], sufficient conditions have been given under which the equations (5.55) can be equivalently written as

$$\tilde{\mathscr{A}}(e)\dot{e} = \frac{\partial \tilde{P}}{\partial e}(e) + g(e)u, \tag{5.62}$$

with $e = (e_q, e_p)$, $u = (u_q, u_p)$ and

$$g(e) = \begin{bmatrix} -g_q(e) & 0 \\ 0 & g_p(e) \end{bmatrix}$$

for some new "admissible pair" $(\tilde{\mathscr{A}}, \tilde{P})$, satisfying

$$\tilde{\mathscr{A}}(e) + \tilde{\mathscr{A}}^{\mathrm{T}}(e) \le 0 \qquad\qquad \tilde{P} \ge 0 \tag{5.63}$$

If (5.63) holds, it is clear that $\dot{\tilde{P}} \le u^{\mathrm{T}} y$, i.e, the circuit defines a passive system with port variables (u,y) and storage function $\tilde{P}$, with the output variables given by $y = (g_q^{\mathrm{T}}\dot{e}_q,\ -g_p^{\mathrm{T}}\dot{e}_p)$. From the stability properties of a passive system, we know that if $(\bar{e}_q, \bar{e}_p)$ is a strict local minimum of $\tilde{P}$, then it is a stable equilibrium point of the system (5.62) when $u \equiv 0$. However we would rather like to stabilize another equilibrium point, denoted by $(e_q^\star, e_p^\star)$ corresponding to a possibly non–zero input value $u^\star$.

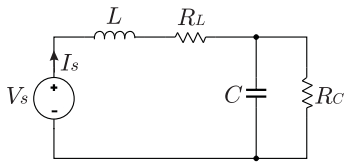


Fig. 5.7 Example of one–port RLC network.

The objective of Power shaping is to find a state feedback law $u = \beta(e)$ such that the dynamics (5.62) becomes

$$\tilde{A}(e)\dot{e} = \frac{\partial \tilde{P}_d}{\partial e}(e) \tag{5.64}$$

with

$$e^\star \text{ being a strict local minimum of } \tilde{P}_d. \tag{5.65}$$

Comparing (5.62) to (5.64) it follows than a power shaping controller $u = \beta(e)$ exists if and only if there exists a solution $\tilde{P}_a$ of the partial differential equation

$$\frac{\partial \tilde{P}_a}{\partial e}(e) = g(e)\beta(e) \tag{5.66}$$

such that (5.65) holds for $\tilde{P}_d = \tilde{P} + \tilde{P}_a$. Notice that (5.66) can be written as

$$g^\perp(e)\frac{\partial \tilde{P}_a}{\partial e}(e) = 0 \tag{5.67}$$

where $g^\perp(e)$ is a full–rank left annihilator of $g(e)$, i.e. $g^\perp(e)g(e) = 0$. If the *matching equation* (5.67) and the minimum condition (5.65) are satisfied, then a controller solving (5.66) is given by

$$u = \beta(e) = \left(g^{\mathrm{T}}(e)g(e)\right)^{-1} g^{\mathrm{T}}(e)\frac{\partial \tilde{P}_a}{\partial e}(e)$$

Example 5.8. Consider the circuit of Fig. 5.7. According to (5.55), the dynamics of the circuit can be written as

$$A\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}v_C \\ \frac{\mathrm{d}}{\mathrm{d}t}i_L \end{bmatrix} = -\begin{bmatrix} \frac{\partial P}{\partial v_C} \\ \frac{\partial P}{\partial i_L} \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} v_S \tag{5.68}$$

with $A = \mathrm{diag}\{C, -L\}$ and the mixed potential function P given by

$$P = \frac{R_C}{2}\left(\frac{v_C}{R_C} - i_L\right)^2 - \frac{1}{2}\left(R_L + R_C\right) i_L^2.$$

The equilibrium points $(i_L^\star, v_C^\star)$ of (5.68) are given by

$$v_C^\star = R_C i_L^\star \qquad\qquad i_L^\star = \frac{v_S^\star}{R_C + R_L}$$

from which it is easy to see that for all (non-zero) equilibrium states, the power extracted from the controller is nonzero. Consequently, it is not possible to stabilize the circuit via energy-balancing. We follow now the power-shaping procedure proposed in [159] to derive an alternative representation of the circuit that reveals the new passivity property. This is given by

$$\tilde{A}\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} v_C \\ \frac{\mathrm{d}}{\mathrm{d}t} i_L \end{bmatrix} = \begin{bmatrix} \frac{\partial \tilde{P}}{\partial v_C} \\ \frac{\partial \tilde{P}}{\partial i_L} \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} v_S$$

with the new admissible pair

$$\tilde{A} = \begin{bmatrix} -C & 0 \\ 2R_C C & -L \end{bmatrix} \qquad \tilde{P} = \frac{R_C}{2}\left(\frac{v_C}{R_C} - i_L\right)^2 + \frac{1}{2}(R_L + R_C) i_L^2$$

Notice the positive sign in the second right hand term of $\tilde{P}$ and that the symmetric part of the matrix $\tilde{A}$ is negative definite under the condition $L > R_C^2 C$. We thus obtain the desired dissipation inequality

$$\dot{\tilde{P}} = \begin{bmatrix} \frac{\mathrm{d}i_L}{\mathrm{d}t} & \dot{v}_C \end{bmatrix} \tilde{A} \begin{bmatrix} \frac{\mathrm{d}i_L}{\mathrm{d}t} \\ \dot{v}_C \end{bmatrix} + \frac{\mathrm{d}i_S}{\mathrm{d}t} v_S \leq \frac{\mathrm{d}i_S}{\mathrm{d}t} v_S$$

Denoting with $(i_L^\star, v_C^\star)$ the desired equilibrium to be stabilized, we will shape the function $\tilde{P}$ to assign a minimum at this point. To this end, we propose to find functions $\tilde{P}_a(i_L), \hat{v}_S(i_L)$ such that

$$\dot{\tilde{P}}_a(i_L) = -\frac{\mathrm{d}i_S}{\mathrm{d}t} \hat{v}_S(i_L), \tag{5.69}$$

yielding a new dissipation inequality for the desired potential function $\tilde{P}_d = \tilde{P} + \tilde{P}_a$. Since $i_L = i_S$, it is clear that, for any arbitrary (differentiable) function $\tilde{P}_a(i_L)$, the function $\hat{v}_s(i_L) = -\frac{\partial \tilde{P}_a}{\partial i_L}(i_L)$ solves (5.69). We propose for simplicity to complete the squares and add a quadratic term in the current errors with

$$\tilde{P}_a = -(R_L + R_C) i_L^\star i_L + \frac{1}{2} K (i_L - i_L^\star)^2$$

with $K \geq 0$ a tuning parameter. This results in the controlled voltage

$$v_S = -K(i_L - i_L^\star) + (R_L + R_C) i_L^\star$$

which globally stabilizes the system with Lyapunov function

$$\tilde{P}_d = \frac{R_C}{2}\left(\frac{v_C}{R_C} - i_L\right)^2 + \frac{1}{2}(R_L + R_C + K)(i_L - i_L^\star)^2$$

5.7.4 Stabilization by Casimir generation

In this Subsection we consider the system (5.62), together with their outputs $y = -g^{\mathrm{T}}(e)\dot{e}$, which can be written in a more general form as

$$\begin{aligned} \tilde{\mathscr{A}}(e)\dot{e} &= \frac{\partial \tilde{P}}{\partial e}(e) + g(e)u \\ y &= -g^{\mathrm{T}}(e)\dot{e}, \end{aligned} \tag{5.70}$$

Following the methodology of Control by Port–Interconnection explained in Sect. 5.3, we can interconnect (5.70) with the controller system

$$\begin{aligned} \dot{x}_c &= u_c \\ y_c &= \frac{\partial H_c}{\partial x_c}(x_c) \end{aligned}$$

with $x_c, u_c, y_c \in \mathbb{R}$ and $H_c(x_c) : \mathbb{R}^m \to \mathbb{R}$, by means of the standard negative feedback

$$\begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix} \begin{bmatrix} y \\ y_c \end{bmatrix}.$$

yielding the closed–loop system

$$\begin{bmatrix} \dot{e} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} \tilde{A}^{-1} & -\tilde{A}^{-1}g \\ -g^{\mathrm{T}}\tilde{A}^{-1} & g^{\mathrm{T}}\tilde{A}^{-1}g \end{bmatrix} \begin{bmatrix} \frac{\partial \tilde{P}}{\partial e} \\ \frac{\partial H_c}{\partial x_c} \end{bmatrix} \tag{5.71}$$

In order to shape the total power function $\tilde{P}(e) + H_c(x_c)$, we are interested in finding m Casimir functions $\mathscr{C}_1(e,x_c), \ldots, \mathscr{C}_m(e,x_c)$ relating e and x_c which are conserved quantities for the closed–loop system (5.71). Consider functions of the form $\mathscr{C}_j(e,x_c) = \tilde{\mathscr{C}}_j(e) - x_{cj},\ j = 1,\ldots,m$, then it can be shown that Casimir functions should satisfy the PDE

$$\frac{\partial \tilde{\mathscr{C}}_j}{\partial e}(e) = -\tilde{A}g_j(e) \equiv \Phi_j(e),$$

where g_j denotes the j-th column of g, $j = 1,\ldots,m$. The above equation has a solution $\tilde{\mathscr{C}}_j$ if and only if the vector Φ_j satisfies the *integrability conditions*

$$\frac{\partial (\Phi_j)_k}{\partial e_l}(e) = \frac{\partial (\Phi_j)_l}{\partial e_k}(e) \qquad k,l = 1,\ldots,n \tag{5.72}$$

where $(\Phi_j)_s$ denotes the s–th element of the vector Φ_j. Assume the $\Phi_j(e)$ satisfies the integrability conditions for every $j = 1,\ldots,m$, then the stability analysis for (5.71) can be carried out with the candidate Lyapunov function (see Sect. 5.3.2 for further details)

$$V(e,x_c) = \tilde{P}(e) + H_c(x_c) + \Psi(\mathscr{C}(e,x_c))$$

where the controller Hamiltonian function $H_c(x_c)$ and the function $\Psi(\mathscr{C}(e,x_c))$ have to be designed, in such a way that $V(e,x_c)$ has a minimum at $(e^\star, x_c^\star)$. Taking the time derivative of $V(e,x_c)$ we can easily check that

$$\dot{V}(e,x_c) = -\frac{\partial^{\mathrm{T}}\tilde{P}}{\partial e} R(e) \frac{\partial \tilde{P}}{\partial e} = \frac{1}{2} \frac{\partial^{\mathrm{T}}\tilde{P}}{\partial e} \left(\tilde{\mathscr{A}}^{-1} + \tilde{\mathscr{A}}^{-\mathrm{T}} \right) \frac{\partial \tilde{P}}{\partial e}$$

Since $\tilde{\mathscr{A}}(e) + \tilde{\mathscr{A}}^{\mathrm{T}}(e) \leq 0$, thus $V(e,x_c)$ qualifies as a Lyapunov function for the closed–loop dynamics.

5.7.5 Remarks

Instrumental for the application of the *power shaping* methodology is the identification of the systems that enjoy new passivity properties, that is, systems for which it is possible to "add differentiation" to the port terminals preserving passivity. In the case of electrical circuits, a complete characterization of the linear RLC circuits that enjoy these new property is given in [81]. For the nonlinear case we refer the reader to [80, 159]. Current research is under way in two directions: extending the technique to other physical systems and assessing the effective advantages of the alternative framework with respect to energy–shaping based schemes like IDA–PBC. See [22] for the geometric formalization of the ideas in [159] and [23].

Chapter 6
Analysis and Control of Infinite-Dimensional Systems

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Abstract Infinite dimensional port Hamiltonian systems have been introduced in Chapter 4 as a novel framework for modeling and control distributed parameter systems. In this chapter, some results regarding control applications are presented. In some sense, it is more correct to speak about *preliminary* results in control of distributed port Hamiltonian systems, since a general theory, as the one discussed in Chapter 5 for the finite dimensional port Hamiltonian systems, has not been completely developed, yet. We start with a short overview on the stability problem for distributed parameter systems in Sect. 6.2, together with some simple but useful stability theorems. Then, in Sect. 6.3, the control by damping injection is generalized to the infinite dimensional case and an application to the boundary and distributed control of the Timoshenko beam is presented. In Sect. 6.4, a simple generalization of the control by interconnection and energy shaping to the infinite dimensional framework is discussed. In particular, the control scheme is developed in order to cope with a simple *mixed finite and infinite dimensional* port Hamiltonian system. Then, an application to the dynamical control of a Timoshenko beam is discussed in Sect. 6.5.

6.1 Introduction

In the last years, stimulated by the applications arising from space exploration, automated manufacturing and other areas of technological development, the control of distributed parameter systems has been an active field of research for control system people. The problem is quite complex since the systems to be controlled are described by a set of partial differential equations, the study of which is not an easy task. It is well-known that the *semi-group theory* provides a large number of results on the analysis of systems of PDEs, and, in particular, on the exponential stability of feedback laws for beam, wave and thermoelastic equations. Classical results can be found in [56, 157], while in [117] some new contributions concerning the stability and feedback stabilization of infinite dimensional systems are reported. In partic-

ular, second order PDEs, such as the Euler-Bernoulli beam equation which arises from control of several mechanical structures (e.g. flexible robots arms and large space structures), are discussed.

As pointed out in Sect. 6.2, when dealing with infinite dimensional systems, the main problem concerns about the intrinsic difficulties related to the proof of stability of an equilibrium point. Moreover, it is important to underscore that this *limitation* does not depend on the particular approach adopted to study the problem. Even if a distributed parameter systems is described within the port Hamiltonian framework, the stability proof of a certain control scheme will always be a difficult task. So the distributed port-Hamiltonian approach does not simplify the control task. Then, we can ask ourself: from the control point of view, what is the advantage related to a port-Hamiltonian description of a distributed parameter system? It is author's opinion that two are the main advantages in adopting the distributed port-Hamiltonian framework. At first, the development of control schemes for infinite dimensional systems is usually based on energy considerations or, equivalently, the stability proof often relies on the properties of an energy-like functional, a generalization of the Lyapunov function to the distributed parameters case. Some examples, related to the stabilization of flexible beams, are in [102, 202]. The Hamiltonian description of a distributed parameter system is given in terms of time evolution of energy variables depending on the variation of the total energy of the system. In this way, the energy of the system, which is generally a *good* Lyapunov function, appears explicitly in the mathematical model of the system itself and, consequently, both the design of the control law and the proof of its stability can be deduced and presented in a more intuitive (in some sense *physical*) and elegant way.

Secondly, the port-Hamiltonian formulation of distributed parameter systems originates from the idea that a system is the result of a network of atomic element, each of them characterized by a particular energetic behavior, as in the finite dimensional case. So, the *mathematical models* originates from the same set of assumptions. This fact is important and allows us to go further: in particular, it is of great interest to understand if also the control schemes developed of finite dimensional port Hamiltonian systems could be generalized in order to deal with distributed parameter ones. For example, suppose that the total energy (Hamiltonian) of the system is characterized by a minimum at the desired equilibrium configuration, [121]. This happens, for example, in the case of flexible beams, for which the zero-energy configuration corresponds to the undeformed beam. In this situation, the controller can be developed in order to behave as a dissipative element to be connected to the system at the boundary or along the distributed port. The amount of dissipated power can be increased in order to reach quickly the configuration with minimum energy. As in the finite dimensional case, it can happen that the minimum of the energy does not correspond to a desired configuration. Then, it is necessary to shape the energy function so that a new minimum is introduced. In other words, it is interesting to investigate if the control by interconnection and energy shaping discussed in Sect. 5.3 can be generalized to the infinite dimensional case. More details in [118, 119, 121, 171].

6.2 Stability for infinite dimensional systems

6.2.1 Arnold's first stability theorem approach

The idea behind the stability of distributed parameter systems remains the same of the finite dimensional case: in order to have (local) asymptotic stability, the equilibrium solutions should be a (local) strict extremum of a proper Lyapunov *functional*, that is the Hamiltonian in the case of distributed port Hamiltonian systems.

In finite dimensions, the positive definiteness of the second differential of the closed-loop Hamiltonian function calculated at the equilibrium configuration is sufficient to show that the steady state solution corresponds to a strict extremum of the Hamiltonian, thus implying the asymptotic stability of the configuration itself. On the other hand, as pointed out in [200], in infinite dimensions, the same condition on the second variation of the Hamiltonian evaluated at the equilibrium is not, in general, sufficient to guarantee asymptotic stability. This is due to the fact that, when dealing with distributed parameter systems, it is necessary to specify the norm associated with the stability argument, because stability with respect to a one norm does not necessarily imply stability with respect to another norm. This is a consequence of the fact that, unlike finite dimensional vector spaces, all norms are not equivalent in infinite dimensions. In particular, in infinite dimensions, not every convergent sequence on the unit ball converges to a point on the unit ball, that is infinite dimensional vector spaces are not compact.

Denote by $\mathscr{X}_\infty$ the configuration space of a distributed parameter system and by $\mathscr{H}_\infty : \mathscr{X}_\infty \to \mathbb{R}$ the corresponding Hamiltonian. Furthermore, denote by $\|\cdot\|$ a norm on $\mathscr{X}_\infty$. The definition of stability for infinite dimensional system can be given as follows:

Definition 6.1. Denote by $\chi^* \in \mathscr{X}_\infty$ an equilibrium configuration for a distributed parameter system. Then, χ^* is said to be stable in the sense of Lyapunov with respect to the norm $\|\cdot\|$ if, for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$\|\chi(0) - \chi^*\| < \delta_\varepsilon \quad \Rightarrow \quad \|\chi(t) - \chi^*\| < \varepsilon$$

for all $t > 0$, where $\chi(0) \in \mathscr{X}_\infty$ is the initial configuration of the system.

In 1965, Arnold proved a set of stability theorems. These theorems are known as Arnold's first and second stability theorems for linear and nonlinear infinite dimensional systems. For a complete overview, refer to [200] where these results are presented within the framework of Hamiltonian fluid dynamics. In this chapter, the stability result we use in some of the proof is known as *Arnold's first nonlinear stability theorem*. Instead of reporting the statement of the theorem, it is more useful to report the underlying mathematical procedure for its proof. This procedure, illustrated in [121, 122, 171, 200], can be treated as a general method for verifying the stability of an equilibrium configuration of a generic non-linear infinite dimensional system. The procedure for proving the stability of the equilibrium χ^* consists of the following steps:

1. Denote by $\mathscr{H}_\infty$ a candidate Lyapunov function which, in the case of distributed port-Hamiltonian systems, is the Hamiltonian function;
2. Show that the equilibrium point χ^* satisfies the first order necessary condition for an extremum of the candidate Lyapunov function, that is verify that

$$\nabla H_\infty(\chi^*) = 0 \tag{6.1}$$

   Furthermore, it is necessary to verify that, at the equilibrium point, the interconnection constraints (boundary conditions) are compatible with the first order condition (6.1).
3. Introduce the nonlinear functional

$$\mathscr{N}(\Delta\chi) := H_\infty(\chi^* + \Delta\chi) - H_\infty(\chi^*) \tag{6.2}$$

   which is proportional to the second variation of H_∞ evaluated in χ^*. This means that its Taylor expansion about $\Delta\chi$ is

$$\mathscr{N}(\Delta\chi) \approx \frac{1}{2}\nabla^2 H(\chi^*)$$

4. Verify if the functional (6.2) satisfies the following convexity condition with respect to a suitable norm on $\mathscr{X}_\infty$, in order to assure its positive definiteness:

$$\gamma_1 \|\Delta\chi\| \leq \mathscr{N}(\Delta\chi) \leq \gamma_2 \|\Delta\chi\|^\alpha \tag{6.3}$$

   with $\alpha, \gamma_1, \gamma_2 > 0$.

A couple of applications of this result will be presented in Sect. 6.4.5.6, in the case of a simple transmission line, and in Sect. 6.5, in order to prove the stabilization property of a dynamical controller for the Timoshenko beam.

6.2.2 La Salle's theorem approach

La Salle's theorem is well-known result for the stability analysis of finite dimensional nonlinear systems. If in a domain about the equilibrium point we can find a Lyapunov function $V(x)$ whose derivative along the trajectories of the system is negative semidefinite, and if we can establish that no trajectory can stay identically at point where $\dot{V}(x) = 0$ except at the equilibrium, then this configuration is asymptotically stable, [101]. This idea is also referred as La Salle's invariance principle.

This result can be generalized in order to cope with distributed parameter systems. First of all, consider a distributed parameter system, e.g. the system (4.105) if the port Hamiltonian formalism is adopted, and denote by $\mathscr{X}_\infty$ the configuration space. Then, it is possible to define an operator $\Phi(t) : \mathscr{X}_\infty \to \mathscr{X}_\infty$ such that

$$(\alpha_E, \alpha_M)(t) = \Phi(t)(\alpha_E, \alpha_M)(0)$$

for each $t \geq 0$. It is possible to prove that $\Phi(t)$ is a family of bounded and continuous operators which is called C_0-semi-group on $\mathscr{X}$, [157]. The operator Φ gives the solutions of the set of PDE (4.105) once initial and boundary conditions are specified. For every $\chi \in \mathscr{X}_\infty$, denote by

$$\gamma(\chi) := \bigcup_{t \geq 0} \Phi(t)\chi \tag{6.4}$$

the set of all the orbits of (4.105) through χ, and by

$$\omega(\chi) := \left\{ \bar{\chi} \in \mathscr{X}_\infty \mid \bar{\chi} = \lim_{n \to \infty} \Phi(t_n)\chi, \text{ with } t_n \to \infty \text{ as } n \to \infty \right\}$$

the (possibly empty) ω-limit set of χ. It is possible to prove that $\omega(\chi)$ is always positively invariant, i.e. $\Phi(t)\omega(\chi) \subset \omega(\chi)$, and closed. Moreover, from classical topological dynamics we take the following result, [117].

Theorem 6.1. *If $\chi \in \mathscr{X}_\infty$ and $\gamma(\chi)$ is precompact*[1]*, then $\omega(\chi)$ is nonempty, compact, connected. Moreover,*

$$\lim_{t \to \infty} d(\Phi(t)\chi, \omega(\chi)) = 0$$

where, given $\bar{\chi} \in \mathscr{X}_\infty$ and $\Omega \subset \mathscr{X}_\infty$, $d(\bar{\chi}, \Omega)$ denotes the "distance" from $\bar{\chi}$ to Ω, that is

$$d(\bar{\chi}, \Omega) = \inf_{\omega \in \Omega} \|\bar{\chi} - \omega\|$$

This theorem characterizes the asymptotic behavior of the distributed parameter systems once the ω-limit set is calculated. Based on this result, it is possible to state the La Salle's theorem.

Theorem 6.2 (La Salle's theorem). *Denote by H_∞ a continuous Lyapunov function for the system (4.105), that is for $\Phi(t)$, and by $\mathscr{B}$ the largest invariant subset of*

$$\left\{ \chi \in \mathscr{X}_\infty \mid \dot{H}_\infty(\chi) = 0 \right\}$$

that is $\Phi(t)\mathscr{B} = \mathscr{B}$ for all $t \geq 0$. If $\chi \in \mathscr{X}_\infty$ and $\gamma(\chi)$ is precompact, then

$$\lim_{t \to \infty} d(\Phi(t)\chi, \mathscr{B}) = 0$$

An immediate consequence is expressed by the following corollary.

Proposition 6.1. *Consider a distributed parameter system and denote by χ^* an equilibrium point and by H_∞ a candidate Lyapunov function (the Hamiltonian in the case of dpH systems). If the largest invariant subset of*

$$\left\{ \chi \in \mathscr{X}_\infty \mid \dot{H}_\infty(\chi) = 0 \right\}$$

[1] See [56, 117].

equals $\{\chi^*\}$*, then* χ^* *is asymptotically stable.*

6.3 Control by damping injection

6.3.1 Basic results

Consider the distributed port Hamiltonian system (4.105) extended to include the distributed port variables $(f^d, e^d) \in \Omega^{n-d}(Z) \times \Omega^d(Z)$, with $d \geq 0$, along the spatial domain Z as in Sect. 4.2.6.1, and for which (f_b, e_b) are the boundary power variables, defined on ∂Z. Denote by $\chi = (\alpha_p, \alpha_q)$ the state of the distributed port-Hamiltonian system and by χ^* a desired equilibrium configuration. As in finite dimensions, if the energy function (Hamiltonian) H of the system is characterized by a minimum in χ^*, then it is possible to drive the system in the desired configuration by interconnecting a controller, that behaves as a dissipative element, to the plant.

If the controller is interconnected on the boundary of the spatial domain, we can speak about *boundary control* of the distributed parameter system (more precisely, about damping injection through the boundary). If the controller is interconnected along the distributed port, we can speak about *distributed control* of the infinite dimensional system (distributed damping injection). Consider the map

$$R_d : \Omega^{n-d}(Z) \times Z \to \Omega^d(Z)$$

and suppose that it is possible to find $\bar{Z} \subset Z$ such that $R_d(\cdot, z) = 0$ if $z \in Z \setminus \bar{Z}$. Furthermore, suppose that

$$\int_{\bar{Z}} R_d(e_d) \wedge e_d \geq 0$$

for every $e_d \in \Omega^d(Z)$. In the same way, consider the map

$$R_b : \Omega^{n-q}(\partial Z) \times \partial Z \to \Omega^{n-p}(\partial Z)$$

and suppose that it is possible to find $\bar{\partial Z} \subset \partial Z$ such that $R_b(\cdot, z) = 0$ if $z \in \partial Z \setminus \bar{\partial Z}$ and that

$$\int_{\bar{\partial Z}} R_b(e_b) \wedge e_b \geq 0$$

Distributed dissipation can be added if the controller can impose the following relations between ditributed effort and flow on Z:

$$f_d = -R_d(e_d) \tag{6.5}$$

while boundary damping injection is introduced if

$$f_b = -R_b(e_b) \tag{6.6}$$

on ∂Z. Suppose that the power flow through $Z \setminus \bar{Z}$ and $\partial Z \setminus \bar{\partial Z}$ is equal to zero or, equivalently, that

$$f_d = 0 \text{ on } Z \setminus \bar{Z} \qquad \langle e_b \mid f_b \rangle = 0 \text{ on } \bar{\partial Z}$$

From the energy balance equation (4.107) and (4.132), we have that

$$\frac{\mathrm{d}H}{\mathrm{d}t} \leq -\int_{\bar{Z}} R_d(e_d) \wedge e_d - \int_{\bar{\partial Z}} R_b(e_b) \wedge e_b \leq 0$$

Consequently, the energy function is non-increasing along system trajectories and it reaches a steady state configuration when

$$\begin{aligned} R_d G^* \begin{bmatrix} \delta_p H \\ \delta_q H \end{bmatrix} &= 0 \qquad (\text{on } \bar{Z}) \\ R_b(e_b) &= 0 \qquad (\text{on } \bar{\partial Z}) \end{aligned} \tag{6.7}$$

with G defined in (4.128). Denote by $\mathscr{B}$ the set of configuration χ compatible with relations (6.7). Consequently, from Proposition 6.1, it is possible to state the following proposition.

Proposition 6.2. *Consider the distributed port Hamiltonian system (4.105), extended with the distributed port as in Sect. 4.2.6.1, thus characterized by the Stokes-Dirac structure (4.127), and the control laws (6.5) and (6.6). If the largest invariant subset of*

$$\left\{ \chi \mid \dot{H}(\chi) = 0 \right\} \cap \mathscr{B}$$

equals $\{\chi^*\}$*, then the configuration* χ^* *is asymptotically stable.*

Proof. The proof follows immediately from the La Salle invariance principle. Note the similarities with the statement of Proposition 5.4.

6.3.2 Control of the Timoshenko beam by damping injection

In this section, some considerations about control by damping injection applied to the Timoshenko beam are presented. In order to be as general as possible, consider the dpH formulation of the Timoshenko beam with distributed port (4.161). The energy functional (4.150) assumes its minimum in the *zero* configuration, i.e. when

$$p_t = 0 \qquad p_r = 0 \qquad \varepsilon_t = 0 \qquad \varepsilon_r = 0 \tag{6.8}$$

or, equivalently, when

$$w(t,z) = \alpha^* z + d^* \qquad \phi(t,z) = \alpha^* \tag{6.9}$$

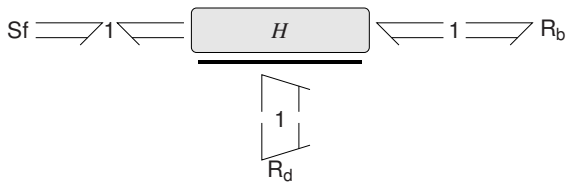


Fig. 6.1 Control by damping injection of a flexible beam.

where the constants α^* and d^* are determined by the boundary conditions on w and ϕ. In (6.9), α^* represents the rotation angle of the beam around the point $x = z$, while d^* is the vertical displacement in $z = 0$.

If dissipation is introduced by the control action, it is possible to drive the state of the beam to the configuration where the (open loop) energy functional (4.150) assumes its minimum. As discussed in Sect. 6.3.1, the controller can interact with the system through the border and/or the distributed port and the energy dissipation can be introduced by *terminating* these ports with a *dissipative element*, i.e. by a generalized *impedance*. Clearly, it is the control algorithm that, in some sense, simulates the desired impedance. In Fig. 6.1, the interconnection of the Timoshenko beam with a distributed and a boundary controller in $z = L$ is presented.

In order to simplify some stability proofs that will be presented in the remaining part of this section, it is important to characterize the behavior of the Timoshenko beam equation when the energy function becomes constant and when the boundary conditions are equal to zero. We give this important remark, [56, 117].

Remark 6.1. Consider the dpH model of the Timoshenko beam (4.157). The only invariant solution compatible with $\dot{H} = 0$ and with the boundary conditions

$$\begin{cases} f_b^t(0) = f_b^r(0) = 0 \\ e_b^t(L) = e_b^r(L) = 0 \end{cases} \quad \text{or} \quad \begin{cases} f_b^t(L) = f_b^r(L) = 0 \\ e_b^t(0) = e_b^r(0) = 0 \end{cases}$$

is the zero solution (6.8).

Note 6.1. More precisely, Remark 6.1 should be extended in order to contain also informations about the *observability* of the Timoshenko beam model, as discussed in [56]. These conditions can be interpreted as a generalization of the definition of detectibility to the infinite dimensional case.

6.3.2.1 Boundary control

Suppose that a finite dimensional controller can be interconnected to the beam in $z = L$ and that the beam can interact with the environment in $z = 0$. Moreover, suppose that no interaction can take place through the distributed port. The last hypothesis means that, in (4.161), it can be assumed that

$$f_d^t(t,z) = 0 \qquad f_d^r(t,z) = 0$$

The controller is designed in order to act as if a dissipative element is connected to the power port of the beam in $z = L$. This is possible if the following relation between flow an effort in $z = L$ holds:

$$\begin{cases} f_b^t(t,L) = -b^t(t) * e_b^t(t,L) \\ f_b^r(t,L) = -b^r(t) * e_b^r(t,L) \end{cases} \Leftrightarrow \begin{cases} \left.\frac{1}{\rho} * p_t\right|_{z=L} = -b^t(\cdot) * K * \varepsilon_t|_{z=L} \\ \left.\frac{1}{I_\rho} * p_r\right|_{z=L} = -b^r(\cdot) * EI * \varepsilon_r|_{z=L} \end{cases} \tag{6.10}$$

with $b^t > 0$ and $b^r > 0$ smooth functions of t. In this way, the energy balance equation (4.162) becomes

$$\frac{\mathrm{d}H}{\mathrm{d}t}(t) = -b^t(t)\left[K * \varepsilon_t\,|_{z=L}\right]^2 - b^r(t)\left[EI * \varepsilon_r\,|_{z=L}\right]^2 + \\ + \left[e_b^t(t,0) f_b^t(t,0) + e_b^r(t,0) f_b^r(t,0)\right] \tag{6.11}$$

If, for example, the boundary conditions in $x = 0$ are

$$w(t,0) = 0 \qquad\qquad \phi(t,0) = 0 \tag{6.12}$$

and, consequently

$$f_b^t(t,0) = f_b^r(t,0) = 0$$

then (6.11) becomes

$$\frac{\mathrm{d}H}{\mathrm{d}t}(t) = -b^t(t)\left[K * \varepsilon_t\,|_{z=L}\right]^2 - b^r(t)\left[EI * \varepsilon_r\,|_{z=L}\right]^2 \leq 0$$

So, it is possible to state the following proposition.

Proposition 6.3. *Consider port Hamiltonian formulation of the Timoshenko beam (4.157) and suppose that the boundary conditions in $z = 0$ are given by (6.12) and that the controller (6.10) is interconnected to the beam in $z = L$. Then, the final configuration is (6.9), with $\alpha^* = 0$ and $d^* = 0$, i.e. $w(t,z) = 0$ and $\phi(t,z) = 0$.*

Proof. The proof is immediate from Remark 6.1 and Proposition 6.2. Furthermore, it is necessary that $\alpha^* = 0$ and $d^* = 0$ in (6.9), in order to be compatible with the boundary conditions (6.12).

Note 6.2. These results were already presented in [102] using a different approach. The proposed control law was written in the following form:

$$\frac{\partial w}{\partial t}(t,L) = -b^t(t) \cdot K\left[\frac{\partial w}{\partial z}(t,L) - \phi(t,L)\right]$$
$$\frac{\partial \phi}{\partial t}(t,L) = -b^r(t) \cdot EI\frac{\partial \phi}{\partial z}(t,L)$$

which is clearly equivalent to (6.10). The main advantage in approaching the problem within the port Hamiltonian framework is that both the way the control law is

deduced and the proof of its stability can be presented in a more intuitive (in some sense *physical*) and elegant way. The same considerations hold for the distributed control of the beam by damping injection presented in the next section: also in this case, the same results were already presented in [66], but with a different approach.

6.3.2.2 Distributed control

Following the same ideas presented in the previous section, it is possible to extend the control by damping injection to the case in which the interaction between system and controller takes place through a distributed port. In this case, the (distributed) power port has to be terminated by a desired impedance implemented by a *distributed* controller. In other words, in this section it is shown how stabilize the Timoshenko beam with a locally distributed control based on an extension to the infinite dimensional case of the damping injection control technique.

Assume that $b_d^t(t,z)$ and $b_d^r(t,z)$ are smooth functions on Z and suppose that it is possible to find $\bar{Z} \subset Z$ and $b_0 > 0$ such that $b_d^t(\cdot,z) \geq b_0^t$ and $b_d^r(\cdot,z) \geq b_0^r > 0$ if $z \in \bar{Z} \subset Z$. Dissipation can be introduced through the distributed port if the *controller* imposes the following relation between flows and efforts on Z:

$$\begin{cases} f_d^t = -b_d^t * e_d^t \\ f_d^r = -b_d^r * e_d^r \end{cases} \quad \Leftrightarrow \quad \begin{cases} f_d^t = -\dfrac{b_d^t}{\rho}\, p_t \\ f_d^r = -\dfrac{b_d^r}{I_\rho}\, p_r \end{cases} \tag{6.13}$$

and, clearly, the closed-loop system is described by the following set of PDEs:

$$\begin{aligned} \rho \frac{\partial^2 w}{\partial t^2} - K\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \phi}{\partial x}\right) + b_d^t \frac{\partial w}{\partial t} = 0 \\ I_\rho \frac{\partial^2 \phi}{\partial t^2} - EI\frac{\partial^2 \phi}{\partial x^2} + K\left(\frac{\partial w}{\partial x} - \phi\right) + b_d^r \frac{\partial \phi}{\partial t} = 0 \end{aligned}$$

in which the boundary conditions have still to be specified. Moreover, the energy balance (4.162) becomes

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}t} &= \int_{\partial Z} \left(e_b^t \wedge f_b^t + e_b^r \wedge f_b^r\right) + \int_{\bar{Z}} \left(e_d^t \wedge f_d^t + e_d^r \wedge f_d^r\right) \\ &= \int_{\partial Z} \left(e_b^t \wedge f_b^t + e_b^r \wedge f_b^r\right) - \int_{\bar{Z}} \left(b_d^t\, e_d^t \wedge *e_d^t + b_d^r\, e_d^r \wedge *e_d^r\right) \end{aligned} \tag{6.14}$$

Assume, for simplicity, that the beam is clamped in $z = 0$, that is

$$w(t,0) = 0 \qquad\qquad \phi(t,0) = 0 \tag{6.15}$$

and that there is no force/torque acting on $z = L$. Moreover, the boundary conditions, i.e. the values assumed by the power variables on ∂Z, are given by

$$\begin{cases} f_b^t(t,0) = 0 \\ f_b^r(t,0) = 0 \end{cases} \qquad \begin{cases} e_b^t(t,L) = 0 \\ e_b^r(t,L) = 0 \end{cases} \tag{6.16}$$

From (6.14), the energy balance relation (4.162) becomes

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\int_{\bar{Z}} \left(b_d^t \, e_d^t \wedge *e_d^t + b_d^r \, e_d^r \wedge *e_d^r \right) = \\ = -\int_{\bar{Z}} \left[\frac{1}{b_d^t} \left(\frac{\partial w}{\partial t} \right)^2 + \frac{1}{b_d^r} \left(\frac{\partial \phi}{\partial t} \right)^2 \right] \mathrm{d}z \leq 0 \tag{6.17}$$

So, it is possible to state the following proposition.

Proposition 6.4. *Consider the Timoshenko beam with distributed port (4.161) and suppose that the boundary conditions are given by (6.15) or (6.16). Then, the distributed control action (6.13) asymptotically stabilizes the system in* $w(t,z) = 0$ *and* $\phi(t,x) = 0$.

Proof. From (6.17), we have that $\dot{H} = 0$ if $\varepsilon_t = \varepsilon_r = 0$ and $p_t = p_r = 0$ on $\bar{Z}$. Consequently, from Proposition 6.3 and from the boundary conditions (6.15) or (6.16), we deduce that also on $Z \setminus \bar{Z}$ we have $\varepsilon_t = \varepsilon_r = 0$ and $p_t = p_r = 0$. The only configuration compatible with this *energy* configuration and the boundary conditions (6.15) is clearly $w(t,z) = 0$ and $\phi(t,z) = 0$.

Note 6.3. It is important to point out that, from a mathematical point of view, the most difficult point in the analysis of the stability of the proposed control schemes is the proof of Remark 6.1 which characterizes the invariants solutions of the Timoshenko beam equations for zero boundary conditions. As regard the La Salle theorem, which is the second mathematical tool widely used in the proposed stability proofs, the key point is the study of the $\gamma(\chi)$ set (6.4). More details on these problems and the rigorous way to solve them, as usual, in [56, 117].

6.4 Control by interconnection and energy shaping

6.4.1 General considerations

The control by damping injection can be fruitfully applied when the open-loop energy function is characterized by a minimum at the desired final configuration. Then, by interconnecting a controller that behaves as a generalized impedance, it is possible to increase the amount of dissipated energy and then reaching the minimum of energy. Then, the desired equilibrium configuration is asymptotically stabilized. Problems arise when the equilibrium is chosen in a non-minimum energy configuration. As discussed in Sect. 5.3 and in Sect. 5.6 for finite dimensional port Hamiltonian systems, it is necessary to develop a controller that properly shapes the energy

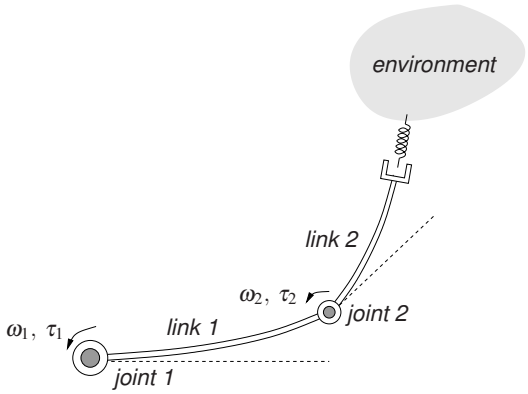


Fig. 6.2 A 2-dof robot with flexible links.

of the systems, thus providing a closed-loop system characterized by a Hamiltonian function with a minimum in the desired equilibrium point. The idea is the generalization of this well-established control methodology to the distributed parameter case.

In this section, the stabilization problem for *mixed finite and infinite dimensional port Hamiltonian systems* is discussed. These are dynamical systems resulting from the power conserving interconnection of finite and infinite dimensional Hamiltonian systems. This is a quite common situation. Consider, for example, the *simple* stabilization of a generic infinite dimensional system: the control law can be applied to the plant only by means of finite dimensional *actuators*. We deduce that, whatever situation is considered, the resulting closed-loop system is given by the interconnection of two main subsystems, the finite dimensional and the infinite dimensional ones. In particular, we suppose that the finite dimensional controller can act on the finite dimensional plant through a distributed parameter system, which leads to a generalization of the results discussed in [171].

An example is given in Fig. 6.2 which represents a 2-dof flexible robot. In this case, the stabilization problem can be stated as follows: the controller acting on joint-1 has to impose the control law $\tau_1 = \tau_1(\omega_1)$ in order to properly stabilize the position of joint-2 in the plane. Note that the regulator (finite dimensional) can act on the system to be controller only through link-1, which is an infinite dimensional systems. In the same way, the position of the end-effector can be stabilized by properly defining the control action $\tau_2 = \tau_2(\omega_2)$ at joint-2 which modifies the position of the *gripper* thanks to the flexible link 2.

Generally speaking, the open-loop energy function can be shaped by acting on the energy function of the controller. As in the finite dimensional case, the key point is to robustly relate the state variable of the controller by means of Casimir functions (in infinite dimensions, it is more correct to speak about *Casimir functional*) to the state variable of the system to be stabilized. In this way, the regulator energy function, which is freely assignable, becomes a function of the configuration of the plant and, then, it can be easily shaped in order to solve the regulation problem.

This Section is organized as follows. In Sect. 6.4.2, the interconnection of finite and infinite dimensional systems is studied and it is shown that the resulting interconnection can be again described by a Dirac structure and the model by a port Hamiltonian system. Then, the problem of the achievable Dirac structure that can be obtained by interconnecting a to-be-designed finite dimensional controller to a given plant belonging to a particular class of m-pH systems is finally studied, together with its implications on the control of port Hamiltonian systems. The regulation problem becomes a central issue in Sect. 6.4.5.2, where necessary and sufficient conditions for the existence of Casimir functions for this particular class of m-pH systems are deduced. So, in Sect. 6.4.5.2, under some further hypothesis on the distributed parameter subsystem, the control by interconnection methodology is generalized to deal with mixed port-Hamiltonian systems. Finally, a simple example is discussed in Sect. 6.4.5.6.

6.4.2 Interconnections of Dirac structures for mixed port-Hamiltonian systems

Denote by $\mathscr{D}_1$ and $\mathscr{D}_2$ two Dirac structures and suppose that they are interconnected to each other via a Stokes'-Dirac structure, denoted by $\mathscr{D}_\infty$. As far as concerns the Stokes'-Dirac structure, the simple case $p = q = n = 1$ in Proposition 4.3 is considered throughout, even if the results can be extended, if not easily, to the higher dimensional case. An immediate example of the case $p = q = n = 1$ is that of a transmission line (cf. Sect. 4.1.2.2).

Consider $\mathscr{D}_1$ on the product space $\mathscr{F}_1 \times \mathscr{F}_0$ of two linear spaces $\mathscr{F}_1$ and $\mathscr{F}_0$, and the Stokes'-Dirac structure $\mathscr{D}_\infty$ on the product space $\mathscr{F}_0 \times \mathscr{F}_{p,q} \times \mathscr{F}_l$, where $\mathscr{F}_0$ and $\mathscr{F}_l$ are linear spaces (representing the space of boundary variables of the Stokes'-Dirac structure) and $\mathscr{F}_{p,q}$ in an infinite dimensional function space, with p and q representing the two different physical energy domains interacting with each other. The linear space $\mathscr{F}_0$ is the space of shared flow variables and its dual $\mathscr{F}_0^*$, the space of shared effort variables between $\mathscr{D}_1$ and $\mathscr{D}_\infty$. Finally, suppose that $\mathscr{D}_2$ is defined on the product space $\mathscr{F}_l \times \mathscr{F}_2$ of two linear spaces, where $\mathscr{F}_l$ is the space of shared flow variables while its dual $\mathscr{F}_l^*$ is the space of shared effort variables between $\mathscr{D}_2$ and $\mathscr{D}_\infty$.

The interconnections between $\mathscr{D}_1$ and $\mathscr{D}_\infty$ is defined as:

$$
\begin{aligned}
&\mathscr{D}_1 \parallel \mathscr{D}_\infty := \\
&\quad := \Big\{ (f_1, e_1, f_p, f_q, e_p, e_q, f_l, e_l) \in \mathscr{F}_1 \times \mathscr{F}_1^* \times \mathscr{F}_{p,q} \times \mathscr{F}_{p,q}^* \times \mathscr{F}_l \times \mathscr{F}_l^* \ \mid \\
&\qquad\qquad \exists (f_0, e_0) \in \mathscr{F}_0 \times \mathscr{F}_0^* \text{ s.t. } (f_1, e_1, f_0, e_0) \in \mathscr{D}_1 \text{ and} \\
&\qquad\qquad\qquad\qquad (-f_0, e_0, f_p, f_q, e_p, e_q, f_l, e_l) \in \mathscr{D}_\infty \Big\}
\end{aligned}
$$

while the interconnection between $\mathscr{D}_\infty$ and $\mathscr{D}_2$ as

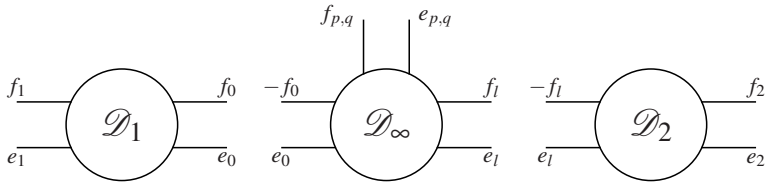


Fig. 6.3 $\mathscr{D}_1 \parallel \mathscr{D}_\infty \parallel \mathscr{D}_2$.

$$\begin{aligned}
&\mathscr{D}_\infty \parallel \mathscr{D}_2 := \\
&:= \Big\{ (-f_0, e_0, f_p, f_q, e_p, e_q, f_2, e_2) \in \mathscr{F}_0 \times \mathscr{F}_0^* \times \mathscr{F}_{p,q} \times \mathscr{F}_{p,q}^* \times \mathscr{F}_2 \times \mathscr{F}_2^* \mid \\
&\qquad \exists (f_l, e_l) \in \mathscr{F}_l \times \mathscr{F}_l^* \text{ s.t. } (-f_0, e_0, f_p, f_q, e_p, e_q, f_l, e_l) \in \mathscr{D}_\infty \\
&\qquad\qquad \text{and } (-f_l, e_l, f_2, e_2) \in \mathscr{D}_\infty \Big\}
\end{aligned}$$

Hence, the total interconnection of $\mathscr{D}_1$, $\mathscr{D}_\infty$ and $\mathscr{D}_2$ can be defined as (see also Fig. 6.3):

$$\begin{aligned}
&\mathscr{D}_1 \parallel \mathscr{D}_\infty \parallel \mathscr{D}_2 := \\
&:= \Big\{ (f_1, e_1, f_p, f_q, e_p, e_q, f_2, e_2) \in \mathscr{F}_1 \times \mathscr{F}_1^* \times \mathscr{F}_{p,q} \times \mathscr{F}_{p,q}^* \times \mathscr{F}_2 \times \mathscr{F}_2^* \mid \\
&\qquad \exists (f_0, e_0) \in \mathscr{F}_0 \times \mathscr{F}_0^* \text{ s.t. } (f_1, e_1, f_0, e_0) \in \mathscr{D}_1 \text{ and} \\
&\qquad\qquad (-f_0, e_0, f_p, f_q, e_p, e_q, f_l, e_l) \in \mathscr{D}_\infty \text{ and} \\
&\qquad \exists (f_l, e_l) \in \mathscr{F}_l \times \mathscr{F}_l^* \text{ s.t. } (-f_0, e_0, f_p, f_q, e_p, e_q, f_l, e_l) \in \mathscr{D}_\infty \\
&\qquad\qquad \text{and } (-f_l, e_l, f_2, e_2) \in \mathscr{D}_\infty \Big\}
\end{aligned}$$

This yields the following bilinear form (+pairing operator) on $\mathscr{F}_1 \times \mathscr{F}_1^* \times \mathscr{F}_{p,q} \times \mathscr{F}_{p,q}^* \times \mathscr{F}_2 \times \mathscr{F}_2^*$:

$$\begin{aligned}
&\ll (f_1^a, f_p^a, f_q^a, f_2^a, e_1^a, e_p^a, e_q^a, e_2^a), (f_1^b, f_p^b, f_q^b, f_2^b, e_1^b, e_p^b, e_q^b, e_2^b) \gg := \\
&\qquad := \left\langle e_1^b \mid f_1^a \right\rangle + \left\langle e_1^a \mid f_1^b \right\rangle + \left\langle e_2^a \mid f_2^b \right\rangle + \left\langle e_2^b \mid f_2^a \right\rangle + \\
&\qquad\qquad + \int_Z \left[e_p^a \wedge f_p^b + e_p^b \wedge f_p^a + e_q^b \wedge f_q^a + e_q^a \wedge f_q^b \right] \qquad (6.18)
\end{aligned}$$

Theorem 6.3. *Let $\mathscr{D}_1$, $\mathscr{D}_2$ and $\mathscr{D}_\infty$ be Dirac structures as said above, which are defined with respect to $\mathscr{F}_1 \times \mathscr{F}_1^* \times \mathscr{F}_0 \times \mathscr{F}_0^*$, $\mathscr{F}_l \times \mathscr{F}_l^* \times \mathscr{F}_2 \times \mathscr{F}_2^*$ and $\mathscr{F}_0 \times \mathscr{F}_0^* \times \mathscr{F}_{p,q} \times \mathscr{F}_{p,q}^* \times \mathscr{F}_l \times \mathscr{F}_l^*$. Then, $\mathscr{D} = \mathscr{D}_1 \parallel \mathscr{D}_\infty \parallel \mathscr{D}_2$ is a Dirac structure defined with respect to the bilinear form on $\mathscr{F}_1 \times \mathscr{F}_1^* \times \mathscr{F}_{p,q} \times \mathscr{F}_{p,q}^* \times \mathscr{F}_2 \times \mathscr{F}_2^*$ given by (6.18).*

Proof. Since $\mathscr{D}_1$, $\mathscr{D}_2$ and $\mathscr{D}_\infty$ individually are Dirac structures, on $\mathscr{F}_1 \times \mathscr{F}_1^* \times \mathscr{F}_0 \times \mathscr{F}_0^*$ the +pairing is defined as

$$\ll (f_1^a, f_0^a, e_1^a, e_0^a), (f_1^b, f_0^b, e_1^b, e_0^b) \gg := \left\langle e_1^b \mid f_1^a \right\rangle + \left\langle e_1^a \mid f_1^b \right\rangle + \\ + \left\langle e_0^b \mid f_0^a \right\rangle + \left\langle e_0^a \mid f_0^b \right\rangle \quad (6.19)$$

and $\mathscr{D}_1 = \mathscr{D}_1{}^{\perp}$ with respect to the bilinear form (6.19). Similarly, on $\mathscr{F}_2 \times \mathscr{F}_2^* \times \mathscr{F}_l \times \mathscr{F}_l^*$, the +pairing is defined as

$$\ll (-f_l^a, e_l^a, f_2^a, e_2^a), (-f_l^b, e_l^b, f_2^b, e_2^b) \gg := \left\langle e_2^b \mid f_2^a \right\rangle + \left\langle e_2^a \mid f_2^b \right\rangle - \\ - \left\langle e_l^b \mid f_l^a \right\rangle - \left\langle e_l^a \mid f_l^b \right\rangle \quad (6.20)$$

and $\mathscr{D}_2 = \mathscr{D}_2{}^{\perp}$ with respect to the bilinear form (6.20). Finally, on $\mathscr{F}_0 \times \mathscr{F}_0^* \times \mathscr{F}_{p,q} \times \mathscr{F}_{p,q}^* \times \mathscr{F}_l \times \mathscr{F}_l^*$ the +pairing operator takes the following form:

$$\ll (f_p^a, f_q^a, f_b^a, e_p^a, e_q^a, e_b^a), (f_p^b, f_q^b, f_b^b, e_p^b, e_q^b, e_b^b) \gg := \\ := \int_Z \left[e_p^a \wedge f_p^b + e_p^b \wedge f_p^a + e_q^b \wedge f_q^a + e_q^a \wedge f_q^b \right] + \\ + \left\langle e_l^a \mid f_l^b \right\rangle + \left\langle e_l^b \mid f_l^a \right\rangle - \left\langle e_0^a \mid f_0^b \right\rangle + \left\langle e_0^b \mid f_0^a \right\rangle \quad (6.21)$$

and $\mathscr{D}_\infty = \mathscr{D}^{\perp}{}_\infty$ with respect to the bilinear form (6.21). The proof is completed into two step. In the first one, it is proved that $\mathscr{D} \subset \mathscr{D}^{\perp}$, while in the second one that $\mathscr{D}^{\perp} \subset \mathscr{D}$.

- $\mathscr{D} \subset \mathscr{D}^{\perp}$: let
$$(f_1^a, f_p^a, f_q^a, f_2^a, e_1^a, e_p^a, e_q^a, e_2^a) \in \mathscr{D}$$
and consider any other
$$(f_1^b, f_p^b, f_q^b, f_2^b, e_1^b, e_p^b, e_q^b, e_2^b) \in \mathscr{D}$$
Then, it is possible to find (f_0^a, e_0^a) and (f_l^a, e_l^a) such that
$$(f_1^a, e_1^a, f_0^a, e_0^a) \in \mathscr{D}_1$$
$$(-f_l^a, e_l^a, f_2^a, e_2^a) \in \mathscr{D}_2$$
$$(-f_0^a, e_0^a, f_p^a, f_q^a, e_p^a, e_q^a, f_l^a, e_l^a) \in \mathscr{D}_\infty$$
Furthermore, it is also possible to find (f_0^b, e_0^b) and (f_l^b, e_l^b) such that
$$(f_1^b, e_1^b, f_0^b, e_0^b) \in \mathscr{D}_1$$
$$(-f_0^b, e_0^b, f_p^b, f_q^b, e_p^b, e_q^b, f_l^b, e_l^b) \in \mathscr{D}_\infty$$
$$(-f_l^b, e_l^b, f_2^b, e_2^b) \in \mathscr{D}_2$$
Since $\mathscr{D}_\infty$ is a Dirac structure with respect to (6.21), we have that

$$\int_Z \left[e_p^a \wedge f_p^b + e_p^b \wedge f_p^a + e_q^b \wedge f_q^a + e_q^a \wedge f_q^b \right] = -\left\langle e_l^a \mid f_l^b \right\rangle - \left\langle e_l^b \mid f_l^a \right\rangle + \left\langle e_0^a \mid f_0^b \right\rangle - \left\langle e_0^b \mid f_0^a \right\rangle \quad (6.22)$$

Substituting (6.22) in (6.18) and using the fact that the bilinear from (6.19) is zero on $\mathscr{D}_1$ and (6.20) is zero on $\mathscr{D}_2$, we obtain

$$\left\langle e_1^b \mid f_1^a \right\rangle + \left\langle e_1^a \mid f_1^b \right\rangle \left\langle e_2^b \mid f_2^a \right\rangle + \left\langle e_2^a \mid f_2^b \right\rangle + \int_Z \left[e_p^a \wedge f_p^b + e_p^b \wedge f_p^a + e_q^b \wedge f_q^a + e_q^a \wedge f_q^b \right] = 0$$

and hence $\mathscr{D} \subset \mathscr{D}^\perp$.

- $\mathscr{D}^\perp \subset \mathscr{D}$: we know that the flow and effort variables of $\mathscr{D}_\infty$ are related as

$$\mathscr{D}_\infty := \left\{ (f,e) \in \mathscr{F} \times \mathscr{F}^* \mid \begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e_p|_{\partial Z} \\ e_q|_{\partial Z} \end{bmatrix} \right\} \quad (6.23)$$

Let $(f_1^a, f_p^a, f_q^a, f_2^a, e_1^a, e_p^a, e_q^a, e_2^a) \in \mathscr{D}^\perp$. Then, for all

$$(f_1^b, f_p^b, f_q^b, f_2^b, e_1^b, e_p^b, e_q^b, e_2^b) \in \mathscr{D}$$

the right side of equation (6.18) is zero. Now, consider the vectors

$$(f_1^b, f_p^b, f_q^b, f_2^b, e_1^b, e_p^b, e_q^b, e_2^b) \in \mathscr{D}$$

with $f_1^b = f_2^b = e_1^b = e_2^b = 0$ and also $f_0^b = e_0^b = f_l^b = e_l^b = 0$. Then, from (6.23) and (6.18) we have that

$$\int_Z \left[e_p^a \wedge \mathrm{d}e_q^b + e_p^b \wedge f_p^a + e_q^b \wedge f_q^a + e_q^a \wedge \mathrm{d}e_p^b \right] = 0$$

This implies (see [184, Theorem 2.1]) that

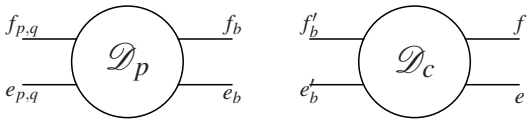
$$f_p^a = \mathrm{d}e_q^a \qquad f_q^a = \mathrm{d}e_p^a \quad (6.24)$$

By substituting (6.24) in (6.18), we have that

$$\left\langle e_1^b \mid f_1^a \right\rangle + \left\langle e_1^a \mid f_1^b \right\rangle + \left\langle e_2^b \mid f_2^a \right\rangle + \left\langle e_2^a \mid f_2^b \right\rangle + \int_Z \left[e_p^a \wedge \mathrm{d}e_q^b + e_p^b \wedge \mathrm{d}e_q^a + e_q^b \wedge \mathrm{d}e_p^a + e_q^a \wedge \mathrm{d}e_p^b \right] = 0$$

This yields by Stokes' theorem:

Fig. 6.4 $\mathscr{D}_p \parallel \mathscr{D}_c$.



$$\left\langle e_1^b \mid f_1^a \right\rangle + \left\langle e_1^a \mid f_1^b \right\rangle + \left\langle e_2^b \mid f_2^a \right\rangle + \left\langle e_2^a \mid f_2^b \right\rangle + \left[\left\langle e_p^a \mid e_q^b \right\rangle + \left\langle e_p^b \mid e_q^a \right\rangle \right]_0^l = 0$$

for all e_p and e_q. Expanding the above and substituting for the boundary conditions

$$\begin{aligned} \left\langle e_1^b \mid f_1^a \right\rangle + \left\langle e_1^a \mid f_1^b \right\rangle + \left\langle e_0^a \mid f_0^b \right\rangle + \left\langle e_0^b \mid f_0^a \right\rangle + \\ + \left\langle e_2^b \mid f_2^a \right\rangle + \left\langle e_2^a \mid f_2^b \right\rangle - \left\langle e_l^a \mid f_l^b \right\rangle - \left\langle e_l^b \mid f_l^a \right\rangle = 0 \end{aligned} \quad (6.25)$$

since $(f_0^b, e_0^b, f_l^b, e_l^b)$ are arbitrary. With $f_l^b = e_l^b = f_2^b = e_2^b = 0$ the above equation reduces to

$$\left\langle e_1^b \mid f_1^a \right\rangle + \left\langle e_1^a \mid f_1^b \right\rangle + \left\langle e_0^a \mid f_0^b \right\rangle + \left\langle e_0^b \mid f_0^a \right\rangle = 0$$

which implies that $(f_1^a, e_1^a, f_0^a, e_0^a) \in \mathscr{D}_1$. By using similar arguments (with $f_l^b = e_l^b = f_1^b = e_1^b = 0$), (6.25) reduces to

$$\left\langle e_2^b \mid f_2^a \right\rangle + \left\langle e_2^a \mid f_2^b \right\rangle - \left\langle e_l^a \mid f_l^b \right\rangle - \left\langle e_l^b \mid f_l^a \right\rangle = 0$$

implying $(-f_l^a, e_l^a, f_2^a, e_2^a) \in \mathscr{D}_2$ and hence $\mathscr{D}^\perp \subset \mathscr{D}$, completing the proof.

6.4.3 Achievable Dirac structures for mixed port-Hamiltonian systems

As discussed in Sect. 5.4, it is important to investigate which closed-loop port-Hamiltonian systems can be achieved by interconnecting a plant port-Hamiltonian system P (which in our case is an infinite dimensional port-Hamiltonian system) with a controller port-Hamiltonian system(s) (which here are port-Hamiltonian systems connected at the boundaries of P) or, equivalently, what closed-loop Dirac structures can be achieved. In particular, given a Stokes'-Dirac structure $\mathscr{D}_\infty$ (which is considered as the plant Dirac structure $\mathscr{D}_p$) and to be designed Dirac structures $\mathscr{D}_1$ and $\mathscr{D}_2$ (which comprise the controller Dirac structure $\mathscr{D}_c$), the following Theorem provides necessary and sufficient conditions which are able to describe all the achievable Dirac structures in the form $\mathscr{D}_1 \parallel \mathscr{D}_\infty \parallel \mathscr{D}_2$ (or $\mathscr{D}_p \parallel \mathscr{D}_c$). Please, refer to Fig. 6.4 for the symbol used. In particular, we have

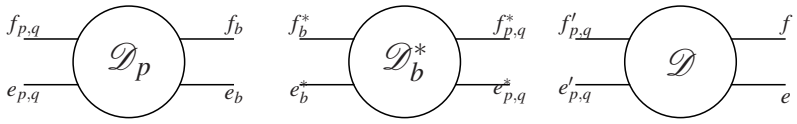


Fig. 6.5 $\mathscr{D} = \mathscr{D}_p \parallel \mathscr{D}_p^* \parallel \mathscr{D}$.

$$f_b = \begin{bmatrix} -f_0 \\ f_l \end{bmatrix} \quad e_b = \begin{bmatrix} e_0 \\ e_l \end{bmatrix} \quad f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad f_b' = \begin{bmatrix} f_0 \\ -f_l \end{bmatrix} \quad e_b = \begin{bmatrix} e_0 \\ e_l \end{bmatrix}$$

such that $(f_1, e_1, f_0, e_0) \in \mathscr{D}_1$ and $(-f_l, e_l, f_2, e_2) \in \mathscr{D}_2$.

Theorem 6.4. *Given any plant Dirac structure $\mathscr{D}_p$, a certain interconnected $\mathscr{D} = \mathscr{D}_p \parallel \mathscr{D}_c$ can be achieved by a proper choice of the controller Dirac structure $\mathscr{D}_c$ if and only if the following two conditions are satisfied*

$$\begin{aligned} \mathscr{D}_p^0 &\subset \mathscr{D}^0 \\ \mathscr{D}^\pi &\subset \mathscr{D}_p^\pi \end{aligned} \tag{6.26}$$

where

$$\begin{aligned} \mathscr{D}_p^0 &:= \Big\{ (f_p, f_q, e_p, e_q) \mid (f_p, f_q, e_p, e_q, 0, 0) \in \mathscr{D}_p \Big\} \\ \mathscr{D}_p^\pi &:= \Big\{ (f_p, f_q, e_p, e_q) \mid \exists (f_b, e_b) \text{ s.t. } (f_p, f_q, e_p, e_q, f_b, e_b) \in \mathscr{D}_p \Big\} \\ \mathscr{D}^0 &:= \Big\{ (f_p, f_q, e_p, e_q) \mid (f_p, f_q, e_p, e_q, 0, 0) \in \mathscr{D} \Big\} \\ \mathscr{D}^\pi &:= \Big\{ (f_p, f_q, e_p, e_q) \mid \exists (f, e) \text{ s.t. } (f_p, f_q, e_p, e_q, f, e) \in \mathscr{D} \Big\} \end{aligned} \tag{6.27}$$

Proof. As in [181] the proof is based on the "copy" of $\mathscr{D}_p$ (see Fig. 6.5) defined as follows:

$$\mathscr{D}_p^* := \Big\{ (f_p, f_q, e_p, e_q, f_b, e_b) \mid (-f_p, -f_q, e_p, e_q - f_b, e_b) \in \mathscr{D}_p \Big\}$$

The necessity of (6.26) and (6.27) is obvious and proof of sufficiency follows the same procedure as in [181] by taking the controller Dirac structure $\mathscr{D}_c = \mathscr{D}_p^* \parallel \mathscr{D}_c$ and first proving $\mathscr{D} \subset \mathscr{D}_p \parallel \mathscr{D}_c$, and then $\mathscr{D}_p \parallel \mathscr{D}_c \subset \mathscr{D}$. Hence, we omit the proof here.

Remark 6.2. The resulting controller in the above theorem could be either a finite-dimensional or an infinite-dimensional controller, depending on the choice of the (f, e) variables. Since the point of discussion here is on mixed finite and infinite-dimensional systems we consider finite dimensional controllers for the rest of the section.

Remark 6.3. We can also consider other mixed cases where we can take $\mathscr{D}_p$ as the interconnection of the Stokes' Dirac structure and a Dirac structure (finite-

dimensional) connected to one of its boundary, and $\mathscr{D}_c$ would then be a Dirac structure (again finite dimensional) interconnected to the other end of the Stokes'-Dirac structure. This is the case if we want to control a plant which is interconnected to a controller through a infinite dimensional system, which is also one of the cases we consider in the next section.

6.4.4 Control by Casimir generation

Like in the finite-dimensional case an important application of the above theory concerns the characterization of the Casimir functions of the closed-loop system by interconnecting a given plant port-Hamiltonian system with associated Dirac structure $\mathscr{D}_p$ with a controller port-Hamiltonian system with associated Dirac structure $\mathscr{D}_c$. This constitutes in the finite-dimensional case a cornerstone for passivity based control of port-Hamiltonian systems, [160, 171, 180].

A *Casimir function* $C : \mathscr{X} \to \mathbb{R}$ of a port-Hamiltonian system has been introduced in Sect. 2.6.2 as a function which is constant along all trajectories of the port-Hamiltonian system, irrespective of the Hamiltonian H. Similarly to Sect. 5.4, we consider now the question of characterizing the set of achievable Casimirs in the mixed finite and infinite dimensional case for the closed-loop system $\mathscr{D}_p \parallel \mathscr{D}_c$, where $\mathscr{D}_p$ is the given Dirac structure of the plant port-Hamiltonian system with Hamiltonian H, and $\mathscr{D}_c$ is the controller Dirac structure. We here consider the case where $\mathscr{D}_p$ is a Stokes' Dirac structure and investigate as to what are the achievable Casimirs. Again for simplicity we consider the case $p = q = n = 1$.

Consider the notation in Fig. 6.4, and assume that the ports (f_p, f_q, e_p, e_q) are connected to the (given) energy storing elements of the plant port-Hamiltonian system, that is

$$\begin{cases} f_p = -\dfrac{\partial \alpha_p}{\partial t} \\ e_p = \delta_p H \end{cases} \qquad \text{and} \qquad \begin{cases} f_q = -\dfrac{\partial \alpha_q}{\partial t} \\ e_q = \delta_q H \end{cases}$$

while (f, e) are connected to the (to be designed) energy storing elements of the controller port-Hamiltonian system(s). From the power variable description of an infinite-dimensional port-Hamiltonian system, the Casimir functions are determined by the subspace

$$\left\{ (e_p, e_q) \in \mathscr{F}^*_{p,q} \mid (0, 0, e_p, e_q) \in \mathscr{D}_\infty \right\}$$

In this situation, the achievable Casimir functions are functions $C(x, \xi)$ such that $\frac{\partial^{\mathrm{T}} C}{\partial x}(x)$ belongs to the space

$$P_C = \left\{ (e_p, e_q) \mid \exists\, \mathscr{D}_c \text{ s.t } \exists e \text{ s.t } (0, 0, e_p, e_q, 0, e) \in \mathscr{D}_p \parallel \mathscr{D}_c \right\} \tag{6.28}$$

where again, as in the previous section, the controller Dirac structure $\mathscr{D}_c$ comprises the Dirac structures $\mathscr{D}_1$ and $\mathscr{D}_2$ and

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

as reported in Fig. 6.4. Similar to the finite dimensional case (c.f. Theorem 5.4 and [181]), the following Theorem addresses the question of characterizing the achievable Casimirs of the closed-loop system, regarded as functions of the plant state x, by finding a characterization of the space P_C.

Theorem 6.5. *The space P_C defined in (6.28) is equal to the linear space*

$$\tilde{P} = \left\{ (e_p, e_q) \mid \exists (f_b, e_b) \text{ s.t. } (0,0,e_p,e_q,f_b,e_b) \in \mathscr{D}_p \right\}$$

Proof. The inclusion $P_C \subset \tilde{P}$ is obvious, and taking the controller Dirac structure $\mathscr{D}_c = \mathscr{D}_p^*$, the second inclusion $\tilde{P} \subset P_C$ is obtained. Since for all $e_p, e_q \in \tilde{P}$, we have $f_p = f_q = 0$ and $(0,0,e_p,e_q,-f_b,e_b) \in \mathscr{D}_p^*$. Moreover, in view of (6.23), this would mean that the space $\tilde{P}$ is such that e_p and e_q are constants as functions of the spatial variable, which in addition would mean $f_0 = f_l$ and $e_0 = e_l$, thus resulting in finite dimensional controllers.

Example 6.1 (continued). In case of the transmission line the Casimir gradients belong to the following space

$$P_C = \left\{ (e_E, e_M) \mid \exists (f_b, e_b) \text{ s.t } \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} e_E \\ e_M \end{bmatrix} \right\}$$

with $e_p \mid_{\partial Z} = e_p$ and $e_q \mid_{\partial Z} = e_q$, and where E and M denote the electric and magnetic energy domains respectively. This means that the Casimir functions are such that

$$\frac{\partial}{\partial z} \delta_E C = 0 \qquad\qquad \frac{\partial}{\partial z} \delta_M C = 0$$

or, in other words

$$\delta_E C \text{ and } \delta_M C \text{ are constant as functions of } z.$$

Thus, Casimirs are of the form

$$k_1 \int_0^l q(z,t) \mathrm{d}z \qquad\qquad k_2 \int_0^l \phi(z,t) \mathrm{d}z$$

with k_1 and k_2 being constants.

Next, we consider the case where $\mathscr{D}_p$ is the interconnection of a Stokes'-Dirac structure with a Dirac structure interconnected to one of its boundary. In terms of Fig. 6.4, this would mean $\mathscr{D}_p := \mathscr{D}_1 \parallel \mathscr{D}_\infty$ and then characterize the set of achievable Casimirs for the closed-loop system $\mathscr{D}_p \parallel \mathscr{D}_c$ (see also Fig. 6.6). We assume that the ports in (f_1, e_1) are connected to the energy storing elements of $\mathscr{D}_1$, that is

$$f_1 = -\dot{x} \qquad\qquad e_1 = \frac{\partial H}{\partial x}$$

the ports in $(f_{p,q}, e_{p,q})$ are connected to the (given) energy storing elements of $\mathscr{D}_\infty$, that is

$$\begin{cases} f_p = -\dfrac{\partial \alpha_p}{\partial t} \\ e_p = \delta_p H \end{cases} \quad \text{and} \quad \begin{cases} f_q = -\dfrac{\partial \alpha_q}{\partial t} \\ e_q = \delta_q H \end{cases}$$

and (f_2, e_2) are connected to the energy storing elements of the (to be designed) controller port-Hamiltonian system $\mathscr{D}_2$, that is

$$f_2 = -\dot{\xi} \qquad\qquad e_2 = \frac{\partial H}{\partial \xi}$$

Then, the achievable Casimirs are functions $C(x,\xi)$ such that $\frac{\partial C}{\partial x}(x,\xi)$ belongs to the space

$$P_C = \left\{ (e_1, e_p, e_q) \;\mid\; \exists\, \mathscr{D}_c \text{ s.t. } \exists e_2 \text{ s.t. } (0, e_1, 0, e_p, e_q, 0, e_2) \in \mathscr{D}_p \parallel \mathscr{D}_c \right\} \tag{6.29}$$

Again as above, the following Theorem characterizes the set of achievable Casimirs of the closed-loop system, regarded as functions of the plant state x, by finding a characterization of the space P_C.

Theorem 6.6. *The space P_C defined in (6.29) is equal to the linear space*

$$\tilde{P} = \left\{ (e_1, e_p, e_q) \mid \exists (f_l, e_l) : (0, e_1, 0, 0, e_p, e_q, f_l, e_l) \in \mathscr{D}_p \right\}$$

Proof. The inclusion $P_C \subset \tilde{P}$ is obvious, and by taking the controller Dirac structure $\mathscr{D}_c = \mathscr{D}_1^*$, the inclusion $\tilde{P} \subset P_C$ is obtained immediately. $\mathscr{D}_1^*$ is defined as

$$\mathscr{D}_1^* = \left\{ (f_1, e_1, f_0, e_0) \;\mid\; (-f_1, e_1, -f_0, e_0) \in \mathscr{D}_1 \right\}$$

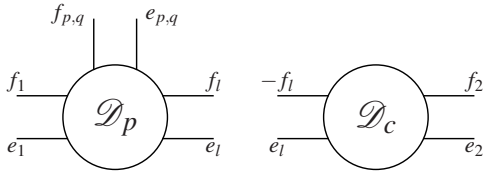
This is also equivalent to considering $\mathscr{D}_c = \mathscr{D}_p^*$ $(= \mathscr{D}_1^* \parallel \mathscr{D}_\infty^*)$ as in the previous theorem where $f_p = f_q = 0$ would again mean that the space $\tilde{P}$ is such that e_p and e_q are constants as functions of the spatial variable. This fact, in addition, would mean $f_0 = f_l$ and $e_0 = e_l$, hence we can take $\mathscr{D}_c = \mathscr{D}_1^*$.

Example 6.2. Consider the case as in Fig. 6.6, where the Dirac structure of the plant is given by

$$\begin{bmatrix} f_1 \\ f_p \\ f_q \end{bmatrix} = -\begin{bmatrix} J(x) & 0 & 0 \\ 0 & 0 & \mathrm{d} \\ 0 & \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_p \\ e_q \end{bmatrix} - \begin{bmatrix} g(x) \\ 0 \\ 0 \end{bmatrix} e_{p0} \qquad \begin{bmatrix} f_l \\ e_l \end{bmatrix} = \begin{bmatrix} -e_{ql} \\ e_{pl} \end{bmatrix}$$
$$e_0 = g^{\mathrm{T}}(x) e_1$$

In this case

Fig. 6.6 $\mathscr{D}_p \parallel \mathscr{D}_c$.



$$P_C = \left\{ (e_1, e_{p,q}) \mid \exists (f_l, e_l) \text{ s.t. } 0 = J(x)e_1 + g(x)e_{p0} \text{ and } 0 = \frac{\partial}{\partial z} e_p;\ 0 = \frac{\partial}{\partial z} e_q \right\}$$

Then the Casimir functions are such that they satisfy the following set of equations

$$\begin{aligned} J(x)\frac{\partial C}{\partial x}(x) + g(x)\delta_p C\,|_l &= 0 \\ \mathrm{d}\delta_p C = \mathrm{d}\delta_q C &= 0 \\ f_l &= -\delta_q C\,|_l \\ e_l &= \delta_p C\,|_l \end{aligned}$$

The first line of the above equation corresponds to the finite dimensional subsystem of the plant Dirac structure, and the Casimirs (as a function of the plant state) are similar to those in the finite dimensional plant-controller interconnection. The remaining linese correspond to the conditions for the Casimir functionals due to the infinite dimensional part of the plant Dirac structure and if the infinite dimensional subsystem is a transmission line then the conditions on the functional are the same as in Example 6.1.

6.4.5 Control by interconnection of a class of mixed port-Hamiltonian systems

6.4.5.1 Class of mixed port-Hamiltonian systems under study

By generalizing the framework proposed in the previous sections, here the stabilization problem for a port Hamiltonian system made of the interconnection of two finite dimensional and a distributed parameter system is approached. In particular, we suppose that the finite dimensional controller can act on the finite dimensional plant through a set of transmission lines, as discussed in Sect. 6.4.3 and in Sect. 6.4.4. We can say that this section is a generalization of the result presented there and in [171], where only a single lossless transmission line is considered.

The problem is approached in a different way, that is by determining necessary and sufficient conditions under which a functional defined on the mixed configuration space could be a structural invariant (i.e. Casimir functional) for the closed-loop system even in presence of dissipative effects in the distributed parameter subsystem. Then, the controller state variable is related to the state variable of the plant

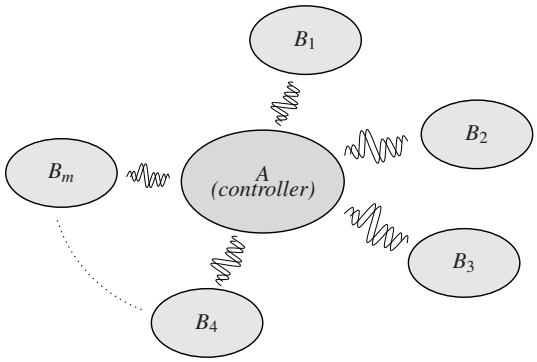


Fig. 6.7 An example of mixed port-Hamiltonian systems. System B is given by the m finite dimensional SISO port Hamiltonian systems $B_1, \ldots, B_m$.

by means of a Casimir functional, thus implementing a *structural* state feedback law. In this way, whatever energy function is chosen for the controller, it results in a function of the plant variables and the energy shaping procedure can be easily completed.

More in details, it is supposed to interconnect system

$$\begin{cases} \dot{x}_a = [J_a(x_a) - R_a(x_a)] \dfrac{\partial H_a}{\partial x_a} + G_a(x_a) u_a \\ y_a = G_a^{\mathrm{T}}(x_a) \dfrac{\partial H_a}{\partial x_a} \end{cases} \tag{6.30}$$

and

$$\begin{cases} \dot{x}_b = [J_b(x_b) - R_b(x_b)] \dfrac{\partial H_b}{\partial x_b} + G_b(x_b) u_b \\ y_b = G_b^{\mathrm{T}}(x_b) \dfrac{\partial H_b}{\partial x_b} \end{cases} \tag{6.31}$$

by means of m distributed port Hamiltonian systems, each of them modeling a one dimensional wave (e.g. telegrapher equation, 1D fluid, vibrating string). In (6.30) and (6.31), denote by $\mathscr{X}_a$ and $\mathscr{X}_b$ the state space of system A and B respectively, with $\dim \mathscr{X}_a = n_a$ and $\dim \mathscr{X}_b = n_b$, while $H_a : \mathscr{X}_a \to \mathbb{R}$ and $H_b : \mathscr{X}_b \to \mathbb{R}$ are the Hamiltonian functions, bounded from below. Moreover, suppose that $J_a(x_a) = -J_a^{\mathrm{T}}(x_a)$ and $R_a(x_a) = R_a^{\mathrm{T}}(x_a)$ for every $x_a \in \mathscr{X}_a$, that $J_b(x_b) = -J_b^{\mathrm{T}}(x_b)$ and $R_b(x_b) = R_b^{\mathrm{T}}(x_b)$ for every $x_b \in \mathscr{X}_b$ and that $\dim \mathscr{U}_a = \dim \mathscr{U}_b = m$.

As reported in Fig. 6.7, the controller (system A) acts on a set of m single-input, single-output finite dimensional port Hamiltonian systems $B_1, \ldots, B_m$ that can be modeled by means of an unique port Hamiltonian systems B with appropriate order. Based on the model discussed in Sect. 4.1.3.2, the most general port Hamiltonian description of a transmission line with dissipation is given by:

$$\begin{aligned}\begin{bmatrix}-\partial_t \alpha_{E,i}\\ -\partial_t \alpha_{M,i}\end{bmatrix} &= \left\{\begin{bmatrix}0 & \mathrm{d}\\ \mathrm{d} & 0\end{bmatrix} + \begin{bmatrix}G_i * & 0\\ 0 & R_i *\end{bmatrix}\right\}\begin{bmatrix}\delta_{E,i} H_{\infty,i}\\ \delta_{M,i} H_{\infty,i}\end{bmatrix}\\ \begin{bmatrix}f_{b,i}\\ e_{b,i}\end{bmatrix} &= \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\delta_{E,i} H_{\infty,i}\,|_{\partial \mathscr{D}_i}\\ \delta_{M,i} H_{\infty,i}\,|_{\partial \mathscr{D}_i}\end{bmatrix}\end{aligned} \tag{6.32}$$

where i denotes the transmission line under consideration, $(\alpha_{E,i}, \alpha_{M,i}) \in \mathscr{X}_{\infty,i}$ are the state variables, with $\mathscr{X}_{\infty,i} := \Omega^1(Z_i) \times \Omega^1(Z_i)$ and $Z_i := [0, \ell_i]$ the spatial domain, being ℓ_i the length of the i-th line, $(f_{b,i}, e_{b,i}) \in \Omega^0(\partial Z_i) \times \Omega^0(\partial Z_i)$ are the power conjugated boundary variables and H_i the total energy, which can be expressed, in the simplest case, by means of the following quadratic functional:

$$H_{\infty,i}(\alpha_{E,i}, \alpha_{M,i}) = \frac{1}{2}\int_{Z_i}\left[\frac{1}{C_i}\alpha_{E,i} \wedge *\alpha_{E,i} + \frac{1}{L_i}\alpha_{M,i} \wedge *\alpha_{M,i}\right]$$

Furthermore, C_i, L_i, R_i and G_i are, respectively, the distributed capacitance, inductance, resistance and admittance of the i-th transmission line. Clearly, the set of m transmission lines can be treated as a single port-Hamiltonian system with state space $\mathscr{X}_\infty := \mathscr{X}_{\infty,1} \times \cdots \times \mathscr{X}_{\infty,m}$ and total Hamiltonian

$$H_\infty(\alpha_{E,1}, \ldots, \alpha_{E,m}, \alpha_{M,1}, \ldots, \alpha_{M,m}) = \sum_{i=1}^{m} H_{\infty,i}(\alpha_{E,i}, \alpha_{M.i})$$

In fact, if

$$\begin{aligned}\alpha_E &:= \begin{bmatrix}\alpha_{E,1} & \cdots & \alpha_{E,m}\end{bmatrix}^{\mathrm{T}} & \alpha_M &:= \begin{bmatrix}\alpha_{M,1} & \cdots & \alpha_{M,m}\end{bmatrix}^{\mathrm{T}}\\ \delta_E H_\infty &:= \begin{bmatrix}\delta_{E,1} H_{\infty,1} & \cdots & \delta_{E,m} H_{\infty,m}\end{bmatrix}^{\mathrm{T}} & \delta_M H_\infty &:= \begin{bmatrix}\delta_{M,1} H_{\infty,1} & \cdots & \delta_{M,m} H_{\infty,m}\end{bmatrix}^{\mathrm{T}}\\ f_b &:= \begin{bmatrix}f_{b,1} & \cdots & f_{b,m}\end{bmatrix}^{\mathrm{T}} & e_b &:= \begin{bmatrix}e_{b,1} & \cdots & e_{b,m}\end{bmatrix}^{\mathrm{T}}\end{aligned}$$

then the set of m transmission lines (6.32) can be written in a compact form as

$$\begin{aligned}\begin{bmatrix}-\partial_t \alpha_E\\ -\partial_t \alpha_M\end{bmatrix} &= \left\{\begin{bmatrix}0 & \mathrm{d}\\ \mathrm{d} & 0\end{bmatrix} + \begin{bmatrix}G * & 0\\ 0 & R *\end{bmatrix}\right\}\begin{bmatrix}\delta_E H_\infty\\ \delta_M H_\infty\end{bmatrix}\\ \begin{bmatrix}f_b\\ e_b\end{bmatrix} &= \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\delta_E H_\infty\,|_{\partial Z}\\ \delta_M H_\infty\,|_{\partial Z}\end{bmatrix}\end{aligned}$$

where $R := \mathrm{diag}(R_1, \ldots, R_m)$ and $G := \mathrm{diag}(G_1, \ldots, G_m)$.

Suppose to interconnect systems (6.30) and (6.31) by means of the m transmission lines (6.32) in a power conserving way. An admissible interconnection law is

$$u_a = \begin{bmatrix}e_{b,1}(0)\\ \vdots\\ e_{b,m}(0)\end{bmatrix} \quad y_a = -\begin{bmatrix}f_{b,1}(0)\\ \vdots\\ f_{b,m}(0)\end{bmatrix} \quad u_b = \begin{bmatrix}f_{b,1}(\ell_1)\\ \vdots\\ f_{b,m}(\ell_m)\end{bmatrix} \quad y_b = \begin{bmatrix}e_{b,1}(\ell_1)\\ \vdots\\ e_{b,m}(\ell_m)\end{bmatrix} \tag{6.33}$$

Note that, if $\ell_i = 0$ for every $i = 1, \ldots, m$, then (6.33) reduces to

$$\begin{cases} u_b = -y_a \\ y_b = u_a \end{cases} \tag{6.34}$$

that is the standard feedback/power conserving interconnection. The resulting system is a mixed finite and infinite dimensional port Hamiltonian system, with configuration space

$$\mathscr{X}_{cl} := \mathscr{X}_a \times \mathscr{X}_b \times \mathscr{X}_\infty \tag{6.35}$$

total Hamiltonian

$$H_{cl}(x_a, x_b, \alpha_{E,1}, \ldots \alpha_{M,m}) := H_a(x_a) + H_b(x_b) + H_\infty(\alpha_{E,1}, \ldots, \alpha_{M,m}) \tag{6.36}$$

and whose dynamics is described by means of the following set of ODEs and PDEs:

$$\begin{aligned} \begin{bmatrix} \dot{x}_a \\ \dot{x}_b \\ \dot{\alpha}_E \\ \dot{\alpha}_M \end{bmatrix} &= \begin{bmatrix} J_a(x_a) - R_a(x_a) & 0 & G_a(x_a) \cdot |_0 & 0 \\ 0 & J_b(x_b) - R_b(x_b) & 0 & -G_b(x_b) \cdot |_{\ell_{1,\ldots,m}} \\ 0 & 0 & -G* & -\mathrm{d} \\ 0 & 0 & -\mathrm{d} & -R* \end{bmatrix} \begin{bmatrix} \partial_{x_a} H_a \\ \partial_{x_b} H_b \\ \delta_E H_\infty \\ \delta_M H_\infty \end{bmatrix} \\ 0 &= \begin{bmatrix} G_a^{\mathrm{T}}(x_a) \partial_{x_a} H_a \\ G_b^{\mathrm{T}}(x_b) \partial_{x_b} H_b \end{bmatrix} - \begin{bmatrix} 0 & \cdot |_0 \\ \cdot |_{\ell_{1,\ldots,m}} & 0 \end{bmatrix} \begin{bmatrix} \delta_E H_\infty \\ \delta_M H_\infty \end{bmatrix} \end{aligned} \tag{6.37}$$

It is easy to verify that (6.37) satisfies the following power balance relation:

$$\frac{\mathrm{d}H_{cl}}{\mathrm{d}t} \leq -\left(\frac{\partial^{\mathrm{T}} H_a}{\partial x_a} R_a \frac{\partial H_a}{\partial x_a} + \frac{\partial^{\mathrm{T}} H_b}{\partial x_b} R_b \frac{\partial H_b}{\partial x_b} \right) \leq 0$$

6.4.5.2 Casimir functions for mixed port-Hamiltonian systems

As discussed in Sect. 5.3 for the finite dimensional case and in Sect. 6.4.4 in the distributed parameter framework, the applicability of the control by interconnection and energy shaping relies on the possibility of relating the controller state variables to the state variables of the plant by means of Casimir functions. Equivalently, we can say that the controller structure is chosen in order to constrain the closed-loop trajectory to evolve on a particular sub-manifold of the whole state space. Differently from Sect. 6.4.4 and, in particular, from Example 6.1, the set of *admissible* Casimir functions is not determined by finding the space P_C but directly from the definition itself of structural invariant. Since a Casimir function $C : \mathscr{X}_{cl} \to \mathbb{R}$ of the port-Hamiltonian system (6.37) is a function which is constant along all trajectories of the port-Hamiltonian system, irrespective of the Hamiltonian H, as discussed in Sect. 2.6.2 for the finite dimensional case or in Sect. 6.4.4 in the case of mixed or distributed port-Hamiltonian systems, following an analogous procedure as in [119, 121], we have that

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{\partial^{\mathrm{T}} C}{\partial x_a}\dot{x}_a + \frac{\partial^{\mathrm{T}} C}{\partial x_b}\dot{x}_b + \sum_{i=1}^{m}\int_{Z_i}\left[\frac{\partial \alpha_{E,i}}{\partial t}\wedge\delta_{E,i}C + \frac{\partial \alpha_{M,i}}{\partial t}\wedge\delta_{M,i}C\right]$$

Then, from (6.30), (6.31), (6.32) and (6.33) we obtain that

$$\begin{aligned}\frac{\mathrm{d}C}{\mathrm{d}t} = {} & \frac{\partial^{\mathrm{T}} C}{\partial x_a}[J_a - R_a]\frac{\partial H_a}{\partial x_a} + \frac{\partial^{\mathrm{T}} C}{\partial x_a}G_a\begin{bmatrix}\delta_{E,1}H_{\infty,1}\mid_0\\ \vdots \\ \delta_{E,m}H_{\infty,m}\mid_0\end{bmatrix} + \\ & + \frac{\partial^{\mathrm{T}} C}{\partial x_b}[J_b - R_b]\frac{\partial H_b}{\partial x_b} + \frac{\partial^{\mathrm{T}} C}{\partial x_b}G_b\begin{bmatrix}-\delta_{M,1}H_{\infty,1}\mid_{\ell_1}\\ \vdots \\ -\delta_{M,m}H_{\infty,m}\mid_{\ell_m}\end{bmatrix} + \\ & + \sum_{i=1}^{m}\int_{Z_i}\Big[-(\mathrm{d}\delta_{M,i}H_{\infty,i} + G_i{*}\delta_{E,i}H_{\infty,i})\wedge\delta_{E,i}C - \\ & \qquad -(\mathrm{d}\delta_{E,i}H_{\infty,i} + R_i{*}\delta_{M,i}H_{\infty,i})\wedge\delta_{M,i}C\Big] \end{aligned} \tag{6.38}$$

Since $\mathrm{d}(\alpha\wedge\beta) = \mathrm{d}\alpha\wedge\beta + \alpha\wedge\mathrm{d}\beta$ and $(\kappa{*}\alpha)\wedge\beta = \alpha\wedge(\kappa{*}\beta)$ when $\alpha,\beta\in\Omega^0(Z)$ and $\kappa\in\mathbb{R}$, the integral term in (6.38) becomes:

$$\begin{aligned}\sum_{i=1}^{m}\int_{Z_i}[\delta_{E,i}H_{\infty,i}\wedge(\mathrm{d}\delta_{M,i}C - G_i{*}\delta_{E,i}C) + \delta_{M,i}H_{\infty,i}\wedge(\mathrm{d}\delta_{E,i}C - R_i{*}\delta_{M,i}C)] - \\ -\sum_{i=1}^{m}\int_{Z_i}[\mathrm{d}(\delta_{M,i}H_{\infty,i}\wedge\delta_{E,i}C) + \mathrm{d}(\delta_{E,i}H_{\infty,i}\wedge\delta_{M,i}C)]\end{aligned} \tag{6.39}$$

From the Stokes' theorem, we have that

$$\int_{Z_i}\mathrm{d}(\delta H\wedge\delta C) = \int_{\partial Z_i}\delta H\mid_{\partial Z_i}\wedge\delta C\mid_{\partial Z_i}$$

Then, from (6.38) and (6.39), we can write that

$$\begin{aligned}
\frac{\mathrm{d}C}{\mathrm{d}t} = &\left\{ \frac{\partial^{\mathrm{T}} C}{\partial x_a} [J_a - R_a] + [\delta_{E,1} C\,|_0 \ \cdots\ \delta_{E,m} C\,|_0]\, G_a^{\mathrm{T}} \right\} \frac{\partial H_a}{\partial x_a} + \\
&+ \left\{ \frac{\partial^{\mathrm{T}} C}{\partial x_b} [J_b - R_b] - \left[\delta_{M,1} C\,|_{\ell_1} \ \cdots\ \delta_{M,m} C\,|_{\ell_m}\right] G_b^{\mathrm{T}} \right\} \frac{\partial H_b}{\partial x_b} + \\
&+ \sum_{i=1}^{m} \int_{Z_i} [\delta_{E,i} H_{\infty,i} \wedge (\mathrm{d}\delta_{M,i} C - G_i * \delta_{E,i} C) + \delta_{M,i} H_{\infty,i} \wedge (\mathrm{d}\delta_{E,i} C - R_i * \delta_{M,i} C)] + \\
&+ \left\{ \frac{\partial^{\mathrm{T}} C}{\partial x_a} G_a + [\delta_{M,1} C\,|_0 \ \cdots\ \delta_{M,m} C\,|_0] \right\} \begin{bmatrix} \delta_{E,1} H_{\infty,1}\,|_0 \\ \vdots \\ \delta_{E,m} H_{\infty,m}\,|_0 \end{bmatrix} + \\
&+ \left\{ \frac{\partial^{\mathrm{T}} C}{\partial x_b} G_b + \left[\delta_{E,1} C\,|_{\ell_1} \ \cdots\ \delta_{E,m} C\,|_{\ell_m}\right] \right\} \begin{bmatrix} -\delta_{M,1} H_{\infty,1}\,|_{\ell_1} \\ \vdots \\ -\delta_{M,m} H_{\infty,m}\,|_{\ell_m} \end{bmatrix}
\end{aligned}$$

which has to be equal to zero for every Hamiltonian function H_a, H_b and H_∞. Therefore, the following set of conditions has to be satisfied:

$$\frac{\partial^{\mathrm{T}} C}{\partial x_a} [J_a - R_a] + \left[\delta_{E,1} C\,|_0 \cdots \delta_{E,m} C\,|_0\right] G_a^{\mathrm{T}} = 0 \tag{6.40}$$

$$\frac{\partial^{\mathrm{T}} C}{\partial x_b} [J_b - R_b] - \left[\delta_{M,1} C\,|_{\ell_1} \cdots \delta_{M,m} C\,|_{\ell_m}\right] G_b^{\mathrm{T}} = 0 \tag{6.41}$$

$$\frac{\partial^{\mathrm{T}} C}{\partial x_a} G_a + \left[\delta_{M,1} C\,|_0 \cdots \delta_{M,m} C\,|_0\right] = 0 \tag{6.42}$$

$$\frac{\partial^{\mathrm{T}} C}{\partial x_b} G_b + \left[\delta_{E,1} C\,|_{\ell_1} \cdots \delta_{E,m} C\,|_{\ell_m}\right] = 0 \tag{6.43}$$

$$\mathrm{d}\delta_{M,i} C - G_i * \delta_{E,i} C = 0 \tag{6.44}$$

$$\mathrm{d}\delta_{E,i} C - R_i * \delta_{M,i} C = 0 \tag{6.45}$$

where (6.44) and (6.45) have to hold for every $i = 1, \ldots, m$. These conditions are a generalization of the classical definition of Casimir function reported in Sect. 2.6.2 and in [180]. In conclusion, the following proposition has been proved.

Proposition 6.5. *Consider the mixed finite and infinite dimensional port Hamiltonian system (6.37), for which $\mathscr{X}_{cl}$ is the configuration space, defined in (6.35), and $\mathscr{H}_{cl}$ is the Hamiltonian, defined in (6.36). Then, a functional $C : \mathscr{X}_{cl} \to \mathbb{R}$ is a Casimir functional if and only if conditions (6.40)–(6.45) are satisfied.*

Remark 6.4. Suppose that the m transmission lines are lossless, that is $R_i = G_i = 0$ for every $i = 1, \ldots, m$. Then, from (6.44) and (6.45), we deduce that C is a Casimir functional if and only if relations (6.40)–(6.43) hold and

$$\begin{cases} \mathrm{d}\delta_{E,i} C = 0 \\ \mathrm{d}\delta_{M,i} C = 0 \end{cases} \qquad i = 1, \ldots, m \tag{6.46}$$

Consequently, it is necessary that $\delta_{E,i}C$ and $\delta_{M,i}C$ are constant on Z_i as function of $z \in Z_i$, and anticipated in Example 6.1. Then, from (6.46), we have that

$$\begin{cases} \delta_{E,i}C = \delta_{E,i}C\,|_0 = \delta_{E,i}C\,|_{\ell_i} \\ \delta_{M,i}C = \delta_{M,i}C\,|_0 = \delta_{M,i}C\,|_{\ell_i} \end{cases} \qquad i = 1,\ldots,m \tag{6.47}$$

Then, from (6.47) and by combining (6.40) with (6.43) and (6.41) with (6.42), we deduce that C is Casimir functional if satisfies relation (6.46) and

$$\begin{aligned} &\frac{\partial^{\mathrm{T}} C}{\partial x_a}\left[J_a(x_a) - R_a(x_a)\right] - \frac{\partial^{\mathrm{T}} C}{\partial x_b} G_b(x_b) G_a^{\mathrm{T}}(x_a) = 0 \\ &\frac{\partial^{\mathrm{T}} C}{\partial x_b}\left[J_b(x_b) - R_b(x_b)\right] + \frac{\partial^{\mathrm{T}} C}{\partial x_a} G_a(x_a) G_b^{\mathrm{T}}(x_b) = 0 \end{aligned} \tag{6.48}$$

Note that (6.48) are the necessary and sufficient conditions for the existence of Casimir functions in the finite dimensional case, when the interconnection law is *purely algebraic* and given by (6.34). They are also necessary and sufficient conditions under the hypothesis that the interconnecting infinite dimensional system is lossless.

6.4.5.3 Initial remarks on the control of mixed port-Hamiltonian systems by energy shaping

Consider the system (6.31) and denote by x_b^* a desired equilibrium point. As discussed in Sect. 5.5, in finite dimensions the stabilization of (6.31) in x_b^* by means of the controller (6.30) can be solved by interconnecting both the systems according to (6.34) and looking for Casimir functions of the resulting closed-loop system in the form $C_i(x_a, x_b) = x_{a,i} - F_i(x_b)$, $i = 1,\ldots,n_a$, which have to satisfy conditions (6.48). In this way, since $\dot{C}_i = 0$, we have that $x_a = F(x_b) + \kappa$ for every energy function H_a and H_b. This relation defines, then, a *structural* state feedback law. Furthermore, H_a, which is freely assignable, can be expressed as a function of x_b: the problem of shaping the closed-loop energy in order to introduce a minimum in x_b^* can be solved by properly choosing H_a. Finally, if dissipation is added, then this new minimum is reached (cf. Sect. 5.3 or Sect. 5.5).

The stabilization of the system (6.37) can be stated as follows. Denote by $(\chi^*, x_b^*) \in \mathscr{X}_\infty \times \mathscr{X}_b$ a desired equilibrium configuration, where χ^* is a configuration of the infinite dimensional system that is *compatible* with the desired equilibrium point x_b^* of the finite dimensional sub-system. In order to stabilize the configuration (χ^*, x_b^*), it is necessary to chose the finite dimensional controller (6.30) so that the open-loop energy function

$$\sum_{i=1}^{m} H_{\infty,i}(\alpha_{E,i}, \alpha_{M,i}) + H_b(x_b) = H_\infty(\alpha_{E,1},\ldots,\alpha_{M,m}) + H_b(x_b)$$

can be shaped by acting on H_a. As in the finite dimensional case (cf. Sect. 5.3), a possible solution can be to robustly relate the controller state variable with the plant state variable by means of a set of Casimir functions. If

$$C_i(x_{a,i},x_b,\alpha_{E,1},\ldots,\alpha_{M,m}) = x_{a,i} - F_i(x_b) - \mathscr{F}_i(\alpha_{E,1},\ldots,\alpha_{M,m}) \tag{6.49}$$

with $i = 1,\ldots,n_a$, are a set of Casimir functionals for (6.37), then, independently from the energy functions H_a, H_b and H_∞, we have that:

$$x_{a,i} = F_i(x_b) + \mathscr{F}_i(\alpha_{E,1},\ldots,\alpha_{E,m},\alpha_{M,1},\ldots,\alpha_{M,m}) + \kappa_i \tag{6.50}$$

with $i = 1,\ldots,n_a$. The constants κ_i depend only on the initial conditions and can be set to zero if the initial state is known (see in Sect. 5.5.2 how to eliminate this constraint). In this way, the controller state variable is expressed as function of the state variable of the system (6.31) and of the configuration of the m transmission lines (6.32). Consequently, the closed-loop energy function (6.36) becomes:

$$\begin{aligned} H_{cl}(x_a,x_b,\alpha_{E,1},\ldots,\alpha_{E,m},\alpha_{M,1},\ldots,\alpha_{M,m}) &= H_b(x_b) + \sum_{i=1}^{m} H_{\infty,i}(\alpha_{E,i},\alpha_{M,i}) + \\ &+ H_a\left(F_1\left(x_b\right) + \mathscr{F}_1\left(\alpha_{E,1},\ldots,\alpha_{M,m}\right),\ldots,F_{n_a}\left(x_b\right) + \mathscr{F}_{n_a}\left(\alpha_{E,1},\ldots,\alpha_{M,m}\right)\right) \end{aligned}$$

where H_a can be freely chosen in order to introduce a minimum in (χ^*,x_b^*). The n_a functionals (6.49) are Casimir functionals for (6.37) if and only if conditions (6.40)–(6.45) are satisfied. In particular, (6.44) and (6.45) can be written as

$$\begin{cases} \mathrm{d}\delta_{M,j}\mathscr{F}_i - G_j * \delta_{E,j}\mathscr{F}_i = 0 \\ \mathrm{d}\delta_{E,j}\mathscr{F}_i - R_j * \delta_{M,j}\mathscr{F}_i = 0 \end{cases} \qquad i = 1,\ldots,n_a;\ j = 1,\ldots,m$$

which is a system of partial differential equations that has to be solved for every $\mathscr{F}_i$.

6.4.5.4 The lossless transmission lines case

As in the finite dimensional case (cf. Sect. 5.4.6), dissipation introduces strong constraints on the applicability of passivity-based control techniques or, equivalently, on the admissible Casimir functions for the closed-loop system. Clearly, these limitations are present also when dealing with mixed port-Hamiltonian systems. Consider, for the moment, that the infinite dimensional subsystem is lossless, as already discussed in Example 6.1 and in Remark 6.4. In this case, condition (6.46) becomes

$$\begin{cases} \mathrm{d}\delta_{E,i}\mathscr{F}_j = 0 \\ \mathrm{d}\delta_{M,i}\mathscr{F}_j = 0 \end{cases} \qquad i = 1,\ldots,m;\ j = 1,\ldots,n_a$$

which expresses the fact that $\delta_{E,j}\mathscr{F}_i$ and $\delta_{M,j}\mathscr{F}_i$ are constant along Z_j. Then, we have that, for every $i = 1,\ldots,n_a$ and $j = 1,\ldots,m$

$$\begin{cases} \delta_{E,j}\mathscr{F}_i = \delta_{E,j}\mathscr{F}_i\,|_0 = \delta_{E,j}\mathscr{F}_i\,|_{\ell_j} \\ \delta_{M,j}\mathscr{F}_i = \delta_{M,j}\mathscr{F}_i\,|_0 = \delta_{M,j}\mathscr{F}_i\,|_{\ell_j} \end{cases} \tag{6.51}$$

From (6.49) and (6.51), conditions (6.40)–(6.43) can be written as:

$$[J_a - R_a] = \begin{bmatrix} \delta_{E,1}\mathscr{F}_1 & \cdots & \delta_{E,m}\mathscr{F}_1 \\ \vdots & \ddots & \vdots \\ \delta_{E,1}\mathscr{F}_{n_a} & \cdots & \delta_{E,m}\mathscr{F}_{n_a} \end{bmatrix} G_a^{\mathrm{T}} \tag{6.52}$$

$$\frac{\partial^{\mathrm{T}} F}{\partial x_b}[J_b - R_b] = \begin{bmatrix} \delta_{M,1}\mathscr{F}_1 & \cdots & \delta_{M,m}\mathscr{F}_1 \\ \vdots & \ddots & \vdots \\ \delta_{M,1}\mathscr{F}_{n_a} & \cdots & \delta_{M,m}\mathscr{F}_{n_a} \end{bmatrix} G_b^{\mathrm{T}} \tag{6.53}$$

$$G_a = \begin{bmatrix} \delta_{M,1}\mathscr{F}_1 & \cdots & \delta_{M,m}\mathscr{F}_1 \\ \vdots & \ddots & \vdots \\ \delta_{M,1}\mathscr{F}_{n_a} & \cdots & \delta_{M,m}\mathscr{F}_{n_a} \end{bmatrix} \tag{6.54}$$

$$\frac{\partial^{\mathrm{T}} F}{\partial x_b} G_b = \begin{bmatrix} \delta_{E,1}\mathscr{F}_1 & \cdots & \delta_{E,m}\mathscr{F}_1 \\ \vdots & \ddots & \vdots \\ \delta_{E,1}\mathscr{F}_{n_a} & \cdots & \delta_{E,m}\mathscr{F}_{n_a} \end{bmatrix} \tag{6.55}$$

By substitution of (6.54) in (6.52), and of (6.55) in (6.53), after a post-multiplication by $\frac{\partial F}{\partial x_b}$, we deduce that

$$J_a + R_a = \frac{\partial^{\mathrm{T}} F}{\partial x_b}[J_b - R_b]\frac{\partial F}{\partial x_b} \tag{6.56}$$

Since J_a and J_b are skew-symmetric and R_a and R_b are symmetric and positive definite, we deduce that, necessarily,

$$J_a = \frac{\partial^{\mathrm{T}} F}{\partial x_b} J_b \frac{\partial F}{\partial x_b} \tag{6.57}$$

$$R_a = 0 \tag{6.58}$$

Furthermore, from (6.56) and (6.58), we deduce that

$$R_b \frac{\partial F}{\partial x_b} = 0 \tag{6.59}$$

and from (6.53), (6.54) and (6.59) that

$$\frac{\partial^{\mathrm{T}} F}{\partial x_b} J_b = G_a G_b^{\mathrm{T}} \tag{6.60}$$

In conclusion, the following proposition has been proved.

Proposition 6.6. *Consider the mixed port-Hamiltonian system (6.37) and suppose that the m transmission lines are lossless, that is $R_i = G_i = 0$, $i = 1,\dots,m$. Then, the n_a functionals (6.49) are Casimirs for this system if the conditions (6.51), (6.54), (6.55) and (6.57)–(6.60) are satisfied.*

Note that conditions (6.57)–(6.60), involving the finite dimensional subsystem of (6.37), are the same required in the finite dimensional energy Casimir method, [180]. Furthermore, Proposition 6.6 generalizes the results presented in [171].

6.4.5.5 Introducing dissipation in the transmission lines

Problems arise when the infinite dimensional system is *not* lossless. For simplicity, in (6.32) suppose that $G_i = 0$ and $R_i \neq 0$, $i = 1,\dots,m$. Then, from (6.49), conditions (6.44) and (6.45) can be written as the following set of ODEs in the spatial variable z

$$\begin{cases} \mathrm{d}\delta_{M,j}\mathscr{F}_i = 0 \\ \mathrm{d}\delta_{E,j}\mathscr{F}_i = R_j * \delta_{M,j}\mathscr{F}_i \end{cases} \qquad i = 1,\dots,n_a;\;\; j = 1,\dots,m$$

whose solution is given by

$$\begin{cases} \delta_{M,j}\mathscr{F}_i = M_{ij} \\ \delta_{E,j}\mathscr{S}_i = R_j M_{ij} z + E_{ij} \end{cases} \qquad i = 1,\dots,n_a;\;\; j = 1,\dots,m$$

where $E_{ij}, M_{ij} \in \mathbb{R}$ are constants to be specified and $z \in Z_j$. Consequently, in (6.49), we have that for every $i = 1,\dots,n_a$

$$\mathscr{F}_i(\alpha_{E,1},\dots,\alpha_{M,m}) = \sum_{j=1}^{m} \int_{Z_j} \left[M_{ij}\alpha_{M,j} + (R_j M_{ij} z + E_{ij})\,\alpha_{E,j} \right] \tag{6.61}$$

If $E := [E_{ij}]$, $M := [M_{ij}]$ and $L := \mathrm{diag}(\ell_1,\cdots,\ell_m)$, then conditions (6.40)–(6.43) can be written as

$$[J_a - R_a] = E G_a^{\mathrm{T}} \tag{6.62}$$

$$\frac{\partial^{\mathrm{T}} F}{\partial x_b}\,[J_b - R_b] = M G_b^{\mathrm{T}} \tag{6.63}$$

$$G_a = M \tag{6.64}$$

$$\frac{\partial^{\mathrm{T}} F}{\partial x_b} G_b = -\,[MRL + E] \tag{6.65}$$

From (6.63) and (6.64), we have that

$$\frac{\partial^{\mathrm{T}} F}{\partial x_b}\,[J_b - R_b]\,\frac{\partial F}{\partial x_b} = G_a G_b^{\mathrm{T}}\,\frac{\partial F}{\partial x_b} \tag{6.66}$$

Then, combining (6.62), (6.63) and (6.66), it can be obtained that

$$\frac{\partial^{\mathrm{T}} F}{\partial x_b}\left[J_b - R_b\right]\frac{\partial F}{\partial x_b} = J_a + R_a - G_a R L G_a^{\mathrm{T}} \tag{6.67}$$

Since J_a and J_b are skew-symmetric and R_a and R_b are symmetric and positive definite, the controller interconnection and damping matrices can be chosen as

$$J_a = \frac{\partial^{\mathrm{T}} F}{\partial x_b} J_b \frac{\partial F}{\partial x_b} \tag{6.68}$$

$$R_a = G_a R L G_a^{\mathrm{T}} \tag{6.69}$$

Consequently, in (6.49), the functions $F_i(\cdot)$, $i = 1,\ldots,n_a$ taking into account the finite dimensional part of the port-Hamiltonian system (6.37) are the solution of the following PDE:

$$R_b \frac{\partial F}{\partial x_b} = 0 \tag{6.70}$$

This is the same equation that has to be solved either for the finite dimensional and mixed port-Hamiltonian system (with lossless transmission lines). Finally the input matrix G_a of the controller is the solution of the following equation, resulting from (6.63), (6.64) and (6.70):

$$\frac{\partial^{\mathrm{T}} F}{\partial x_b} J_b = G_a G_b^{\mathrm{T}} \tag{6.71}$$

The same result holding in finite dimensions and when the distributed parameter sub-system is lossless has been obtained again. Then, the following proposition has been proved.

Proposition 6.7. *Consider the mixed port-Hamiltonian system (6.37) and suppose that $R_i \neq 0$ and $G_i = 0$, $i = 1,\ldots,m$. Then, the n_a functionals (6.49) are Casimir functionals for this system if each $\mathscr{F}_i$ is given as in (6.61), $i = 1,\ldots,n_a$, and conditions (6.68)-(6.71) are satisfied.*

Remark 6.5. The presence of dissipative phenomena in the port-Hamiltonian system (6.32) obviously modifies the expression of the Casimir functions (6.49) of the closed-loop system. It is important to note, however, that, if $x_{a,i} - F_i(\cdot)$, $i = 1,\ldots,n_a$, are Casimir functions in finite dimensions, then the same functions S_i appear in the expression of the Casimir functions (6.49) for the m-pH system if the damping matrix R_a of the controller is chosen according to (6.69). What happens is that the contribution to the structural invariants (6.49) of the finite dimensional part is still the solution of the PDE (6.70), which appears also in finite dimensions; furthermore, in order to *compensate* the effect of dissipation in the m-pH system (6.32) which interconnects plant (system B) and controller (system A), it is necessary to introduce dissipation, expressed by (6.69) also in the controller. Note that the damping matrix R_a of the controller depends on the overall dissipation inside the set of transmission lines.

An analogous result holds if, in (6.32), it is supposed that $G_i \neq 0$ and $R_i = 0$, $i = 1,\ldots,n_a$. In this case, conditions (6.44) and (6.45) transform into the following

set of ODEs in the spatial variable z:

$$\begin{cases} \mathrm{d}\delta_{M,j}\mathscr{F}_i = G_j * \delta_{E,j}\mathscr{F}_i \\ \mathrm{d}\delta_{E,j}\mathscr{F}_i = 0 \end{cases} \qquad i = 1,\dots,n_a;\ j = 1,\dots,m$$

whose solution is given by

$$\begin{cases} \delta_{M,j}\mathscr{F}_i = G_j E_{ij} z + M_{ij} \\ \delta_{E,j}\mathscr{F}_i = E_{ij} \end{cases} \qquad i = 1,\dots,n_a;\ j = 1,\dots,m \tag{6.72}$$

with $E_{ij}, M_{ij} \in \mathbb{R}$. In (6.49), we have that for every $i = 1,\dots,n_a$

$$\mathscr{F}_i(\alpha_{E,1},\dots,\alpha_{M,m}) = \sum_{j=1}^{m} \int_{Z_j} \left[E_{ij}\alpha_{E,j} + (G_j E_{ij} z + M_{ij})\,\alpha_{M,j} \right] \tag{6.73}$$

and conditions (6.40)–(6.43) can be written as

$$[J_a - R_a] = E G_a^{\mathrm{T}} \tag{6.74}$$

$$\frac{\partial^{\mathrm{T}} F}{\partial x_b}\,[J_b - R_b] = [EGL + M]\,G_b^{\mathrm{T}} \tag{6.75}$$

$$G_a = M \tag{6.76}$$

$$\frac{\partial^{\mathrm{T}} F}{\partial x_b} G_b = -E \tag{6.77}$$

It is possible to verify that

$$\frac{\partial^{\mathrm{T}} F}{\partial x_b}\,[J_b - R_b]\,\frac{\partial F}{\partial x_b} = J_a + R_a - \frac{\partial^{\mathrm{T}} F}{\partial x_b} G_b G L G_b^{\mathrm{T}} \frac{\partial F}{\partial x_b} \tag{6.78}$$

Consequently, if the interconnection and damping matrices of the controller are

$$J_a = \frac{\partial^{\mathrm{T}} F}{\partial x_b} J_b \frac{\partial F}{\partial x_b} \tag{6.79}$$

$$R_a = \frac{\partial^{\mathrm{T}} F}{\partial x_b} G_b G L G_b^{\mathrm{T}} \frac{\partial F}{\partial x_b} \tag{6.80}$$

then the functions F_i are again the solutions of the set of PDEs (6.70), which is the same as in finite dimensions. In this case, the input matrix G_a is solution of the following equation:

$$\frac{\partial^{\mathrm{T}} F}{\partial x_b}\left[J_b + G_b G L G_b^{\mathrm{T}}\right] = G_a G_b^{\mathrm{T}} \tag{6.81}$$

Also in this case, it is necessary to introduce dissipation in the controller by means of R_a in order to compensate the loss of energy in the transmission lines.

Proposition 6.8. *Consider the port-Hamiltonian system (6.37) and suppose that* $R_i = 0$ *and* $G_i \neq 0$, $i = 1,\ldots,m$. *Then, the* n_a *functionals (6.49) are Casimirs for this system if each* $\mathscr{F}_i$ *is given as in (6.73),* $i = 1,\ldots,n_a$, *and conditions (6.79)-(6.81) are satisfied.*

Once it is possible to choose the controller interconnection, damping and input matrices in order to render the set of functions (6.49) structural invariants for the closed-loop system (6.37), the energy function of the controller (6.30), that is its Hamiltonian function H_a can be chosen in order to introduce a minimum in the desired equilibrium configuration $(\chi^*, x_b^*) \in \mathscr{X}_\infty \times \mathscr{X}_b$ in the closed-loop energy function (6.36). At this point, (asymptotic) stability can be verified by following the method proposed in [200] and based on Arnold's first and second stability theorem for linear and nonlinear distributed parameter systems or by applying the generalization of La Salle's Invariance Principle to infinite dimensions proposed in [117].

In the next section, the stabilization of a simple mixed port-Hamiltonian system for which the stability proof relies on the application of the method proposed in [200] is presented. This example is a generalization of the case study presented in [171]. The same technique is also applied to the stabilization of the Timoshenko beam by interconnection and energy shaping discussed in Sect. 6.5.

6.4.5.6 Stabilization of a simple mixed port-Hamiltonian system by energy shaping

Consider the series RLC circuit whose port Hamiltonian model is given by

$$\begin{cases} \begin{bmatrix} \dot{x}_{b,1} \\ \dot{x}_{b,2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -R \end{bmatrix} \begin{bmatrix} \partial_{x_{b,1}} H_b \\ \partial_{x_{b,2}} H_b \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_b \\ y_b = \dfrac{\partial H_b}{\partial x_{b,2}} \end{cases} \tag{6.82}$$

in which $x_b = [x_{b,1}\ x_{b,2}]^{\mathrm{T}}$ is the state variable, with $x_{b,1}$ the charge stored in the capacitor and $x_{b,2}$ the flux in the inductance, and

$$H_b(x_{b,1}, x_{b,2}) = \frac{1}{2}\frac{x_{b,1}^2}{C_b} + \frac{1}{2}\frac{x_{b,2}^2}{L_b}$$

the total energy. As reported in Fig. 6.8, suppose to interconnect system (6.82) to a transmission line in $z = \ell$, where ℓ denotes the length of the line. Furthermore, suppose that $G \neq 0$ and that $R = 0$ in (6.32). Finally, suppose to interconnect the controller (6.30) to the transmission line in $z = 0$. If (f_0, e_0) and (f_ℓ, e_ℓ) are the power conjugated port variables at $z = 0$ and $z = \ell$ respectively, then, after a port dualization[2] at both sides of the line in order to give physical consistency to efforts

[2] The port dualization changes the role of efforts and flows variables in the transmission line. Differently from Sect. 6.4.5.2 and Sect. 6.4.5.3, in which the finite dimensional systems are supposed

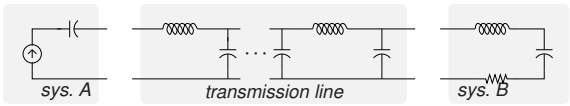


Fig. 6.8 An example of mixed port-Hamiltonian system.

and flows, the interconnection law (6.33) can be written as

$$\begin{cases} y_a = -e_0 \\ u_a = f_0 \end{cases} \qquad \begin{cases} y_b = f_\ell \\ u_b = e_\ell \end{cases}$$

Furthermore, denote by χ the state of the resulting mixed port-Hamiltonian system, by C_∞ the distributed capacitance and by L_∞ the distributed inductance of the line. Consequently, the total closed-loop energy function becomes

$$H_{cl}(\chi) = \frac{1}{2}\frac{x_{b,1}^2}{C_b} + \frac{1}{2}\frac{x_{b,2}^2}{L_b} + H_a(x_a) + \frac{1}{2}\int_0^\ell \left(\frac{\alpha_E^2}{C_\infty} + \frac{\alpha_M^2}{L_\infty}\right) \mathrm{d}z \tag{6.83}$$

In order to apply the control by interconnection methodology, it is necessary to find the Casimir functions of the form (6.49) for the closed-loop dynamics. Since it is desired to *conserve* the finite dimensional part of the Casimir function even in presence of an interconnection involving a distributed port-Hamiltonian systems, condition (6.70) has to be satisfied. As in the finite dimensional case, we obtain that

$$\frac{\partial F}{\partial x_{b,1}} = 1 \qquad \text{and} \qquad \frac{\partial F}{\partial x_{b,2}} = 0$$

which implies that $F(x_b) = x_{b,1}$. Moreover, as far as the functional $\mathscr{F}(\alpha_E, \alpha_M)$ is concerned, we obtain that

$$\delta_M \mathscr{F}(\alpha_E, \alpha_M) = E \qquad \text{and} \qquad \delta_E \mathscr{F}(\alpha_E, \alpha_M) = GEz + M \tag{6.84}$$

where $E, M \in \mathbb{R}$ are constants that will be specified later on. It is easy to verify that conditions (6.62)-(6.65) can be satisfied by choosing $J_a = 0$, $R_a = G\ell$ and $G_a = -1$ and, consequently, $E = 1$ and $M = -G\ell$ in (6.84), as presented in Proposition 6.7. Then, a Casimir function for the closed-loop system is

$$C(x_a, x_b, \alpha_E, \alpha_M) = x_a - x_{b,1} - \int_0^\ell [\alpha_E + G(z-\ell)\,\alpha_M]\,\mathrm{d}z \tag{6.85}$$

and the closed-loop energy function (6.83) can be written as

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to have an effort as input, in this case the input signal is a flow (i.e. a current). The immediate consequence is that, even if it is supposed that $G \neq 0$ and $R = 0$, the result concerning the existence of Casimir functions in the form (6.49) that has to be used is the one presented in Proposition 6.7.

$$H_{cl}(\chi) = \frac{1}{2}\frac{x_{b,1}^2}{C_b} + \frac{1}{2}\frac{x_{b,2}^2}{L_b} + \\ + H_a\left(x_{b,1} + \int_0^\ell [\alpha_E + G(z-\ell)\,\alpha_M]\,\mathrm{d}z + \kappa\right) + \frac{1}{2}\int_0^\ell \left(\frac{\alpha_E^2}{C_\infty} + \frac{\alpha_M^2}{L_\infty}\right)\mathrm{d}z$$

since, from (6.85), we have that

$$x_a = x_{b,1} + \int_0^\ell [\alpha_E + G(z-\ell)\,\alpha_M]\,\mathrm{d}z + \kappa \tag{6.86}$$

where $\kappa \in \mathbb{R}$. If the initial configuration is known, it is possible to assume $\kappa = 0$. Denote by χ^* a desired equilibrium configuration, that is

$$\chi^* = \left[x_{b,1}^*\ x_{b,2}^*\ \alpha_E^*\ \alpha_M^*\right]^{\mathrm{T}} = \left[x_{b,1}^*\ 0\ C_\infty\frac{x_{b,1}^*}{C_b}\ L_\infty\frac{x_{b,1}^*}{C_b}G(\ell-z)\right]^{\mathrm{T}} \tag{6.87}$$

and by x_a^* the value of x_a given by (6.85) and evaluated in χ^*. Then, in the remaining part of this section, it will be shown that, by selecting

$$H_a(x_a) = \frac{1}{2}\frac{1}{C_a}\tilde{x}_a^2 + k\tilde{x}_a \tag{6.88}$$

with $\tilde{x}_a := x_a - x_a^*$, $C_a > 0$ and $k \in \mathbb{R}$ to be specified, it is possible to shape the closed-loop energy in such a way that it has a minimum at the equilibrium point χ^*. Furthermore, it will be verified that this configuration is stable in the sense Definition 6.1. This property can be verified by following the procedure illustrated in Sect. 6.2.1. First of all, it is necessary to verify that $\nabla H_{cl}(\chi^*) = 0$, relation that holds if in (6.88) we chose $k = -\frac{x_1^*}{C}$. In fact

$$\nabla H_{cl}(\chi^*) = \begin{bmatrix} \frac{x_1^*}{C} + k \\ 0 \\ \frac{\alpha_E^*}{C_\infty} + k \\ \frac{\alpha_M^*}{L_\infty} + kG(z-\ell) \end{bmatrix} = 0$$

Then, it is necessary to compute the nonlinear functional $\mathscr{N}(\Delta\chi) := H_{cl}(\chi^* + \Delta\chi) - H_{cl}(\chi^*)$, which is proportional to the second variation of H_{cl} evaluated in χ^*. It can be obtained that

$$\mathscr{N}(\Delta\chi) = \frac{1}{2}\frac{\Delta x_{b,1}^2}{C_b} + \frac{1}{2}\frac{\Delta x_{b,2}^2}{L_b} + \int_0^\ell \left(\frac{\Delta\alpha_E^2}{C_\infty} + \frac{\Delta\alpha_M^2}{L_\infty}\right)\mathrm{d}z + \\ + \frac{1}{2}\frac{1}{C_a}\left(\Delta x_{b,1} + \int_0^\ell [\Delta\alpha_E + G(z-\ell)\Delta\alpha_M]\,\mathrm{d}z\right)^2$$

The stability proof is completed if it is possible to find $\alpha, \gamma_1, \gamma_2 > 0$ such that

$$\gamma_1 \|\Delta\chi\|^2 \leq \mathscr{N}(\Delta\chi) \leq \gamma_2 \|\Delta\chi\|^{\alpha} \tag{6.89}$$

where $\|\cdot\|$ is a suitable norm on $\mathscr{X}_{cl}$. If the following norm is assumed:

$$\|\chi\| := \left(\Delta x_1^2 + \Delta x_2^2 + \int_0^\ell \Delta\alpha_E^2 \mathrm{d}z + \int_0^\ell \Delta\alpha_M^2 \mathrm{d}z\right)^{\frac{1}{2}}$$

the constant γ_1 can be easily estimated in

$$\gamma_1 = \frac{1}{2}\min\left\{\frac{1}{C_b}, \frac{1}{L_b}, \frac{1}{C_\infty}, \frac{1}{L_\infty}\right\}$$

Moreover, note that

$$\left|\Delta x_{b,1}\right| \left|\int_0^\ell [\Delta\alpha_E + G(z-\ell)\Delta\alpha_M]\,\mathrm{d}z\right| \leq \frac{1}{2}\Delta x_{b,1}^2 + \\ + \frac{1}{2}\left(\int_0^\ell [\Delta\alpha_E + G(z-\ell)\Delta\alpha_M]\,\mathrm{d}z\right)^2$$

and

$$\frac{1}{2}\left(\int_0^\ell [\Delta\alpha_E + G(z-\ell)\Delta\alpha_M]\,\mathrm{d}z\right)^2 \leq \ell\left(\int_0^\ell \Delta\alpha_E^2 \mathrm{d}z + G^2\ell^2\int_0^\ell \Delta\alpha_M^2 \mathrm{d}z\right)$$

Consequently, it is possible to chose

$$\gamma_2 = \frac{1}{2}\max\left\{\frac{1}{C_b} + 2\frac{1}{C_a}, \frac{1}{L_b}, \frac{1}{C_\infty} + 2\frac{\ell}{C_a}, \frac{1}{L_\infty} + 2\frac{G^2\ell^3}{C_a}\right\}$$

and $\alpha = 2$ in order to complete stability proof of the desired configuration χ^*. In other words, the following proposition has been proved.

Proposition 6.9. *Consider the mixed port-Hamiltonian system of Fig. 6.8, where the transmission line is not lossless (i.e. $R = 0$ and $G \neq 0$). If the controller interconnection, damping and input matrices are equal to 0, $G\ell$ and -1 respectively, then, if the controller energy function is given by (6.88), the configuration (6.87) is stable in the sense of Definition 6.1.*

6.5 Control by energy shaping of the Timoshenko beam

6.5.1 Model of the plant

Consider the mechanical system of Fig. 6.9, in which a flexible beam, modeled according to the Timoshenko theory and whose port-Hamiltonian model is given

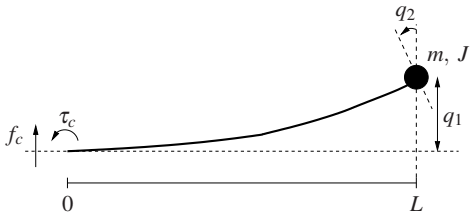


Fig. 6.9 Flexible link with mass in $z = L$.

by (4.157), is connected to a rigid body with mass m and inertia momentum J in $z = L$ and to a controller in $z = 0$. The controller acts on the system with a force f_c and a torque τ_c. Since the Timoshenko model of the beam is valid only for small deformations, it is possible to assume that the motion of the mass is the combination of a rotational and of a translational motion along $z = L$. The port Hamiltonian model of the mass is given by:

$$\begin{aligned}\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} &= \left\{\begin{bmatrix}0 & I\\ -I & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\ 0 & D\end{bmatrix}\right\}\begin{bmatrix}\frac{\partial H}{\partial q}\\ \frac{\partial H}{\partial p}\end{bmatrix} + \begin{bmatrix}0\\ I\end{bmatrix}\mathbf{f}\\ \mathbf{e} &= \frac{\partial H}{\partial p}\end{aligned} \tag{6.90}$$

where $q = [q_1,\ q_2]^{\mathrm{T}} \in \mathbb{R}^2$ are the generalized coordinates, with q_1 the distance from the equilibrium configuration and q_2 the rotation angle, p are the generalized momenta, $\mathbf{f}, \mathbf{e} \in \mathbb{R}^2$ are the port variables and

$$H(p,q) := \frac{1}{2}p^{\mathrm{T}}M^{-1}p + \frac{1}{2}q^{\mathrm{T}}Kq = \frac{1}{2}\left(\frac{p_1^2}{m} + \frac{p_2^2}{J}\right) + \frac{1}{2}\left(k_1 q_1^2 + k_2 q_2^2\right) \tag{6.91}$$

is the total energy (Hamiltonian) function, with $k_1, k_2 > 0$. It is assumed that a translational and rotational spring is acting on the mass, with center of stiffness in $(0,0)$.

As far as the controller is concerned, we assume that it can be modeled by means of the following finite dimensional port Hamiltonian systems

$$\begin{aligned}\begin{bmatrix}\dot{q}_c\\ \dot{p}_c\end{bmatrix} &= \left\{\begin{bmatrix}0 & I\\ -I & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\ 0 & D_c\end{bmatrix}\right\}\begin{bmatrix}\frac{\partial H_c}{\partial q_c}\\ \frac{\partial H_c}{\partial p_c}\end{bmatrix} + \begin{bmatrix}0\\ G_c\end{bmatrix}\mathbf{f}_c\\ \mathbf{e}_c &= G_c^{\mathrm{T}}\frac{\partial H_c}{\partial p_c}\end{aligned} \tag{6.92}$$

where $q_c \in \mathbb{R}^2$ are the generalized coordinates, p_c the generalized momenta and $\mathbf{f}_c, \mathbf{e}_c \in \mathbb{R}^2$ the power conjugated port variables. Moreover, $H_c(q_c, p_c)$ is the Hamiltonian and it will be specified in the remaining part of this section in order to drive the whole system in a desired equilibrium configuration. The *pseudo*-bond graph representation of the closed loop system made of the Timoshenko beam, the mass in $z = L$ and the finite dimensional port Hamiltonian controller acting in $z = 0$ is given in Fig. 6.10. Then, the interconnections constraints between the port variables of the

Fig. 6.10 *Pseudo*-bond graph representation of the closed-loop system.

subsystems are given by the following power-preserving relations:

$$\begin{cases} \left[f_b^t(L)\ f_f^r(L)\right]^{\mathrm{T}} = -\mathbf{e} \\ \left[e_b^t(L)\ e_b^r(L)\right]^{\mathrm{T}} = \mathbf{f} \end{cases} \qquad \begin{cases} \left[f_b^t(0)\ f_f^r(0)\right]^{\mathrm{T}} = \mathbf{e}_c \\ \left[e_b^t(0)\ e_b^r(0)\right]^{\mathrm{T}} = \mathbf{f}_c \end{cases} \tag{6.93}$$

From (4.157), (6.90), (6.92) and (6.93), it is possible to obtain the mixed finite and infinite dimensional port Hamiltonian representation of the closed-loop system. The total energy H_{cl} is defined in the extended space

$$\mathscr{X}_{cl} := \underbrace{\mathbb{R}^4 \times \mathbb{R}^4}_{\mathscr{X}} \times \underbrace{\Omega^1(Z) \times \Omega^1(Z) \times \Omega^1(Z) \times \Omega^1(Z)}_{\mathscr{X}_\infty} \tag{6.94}$$

and it is given by the sum of the energy functions of the subsystems, that is

$$H_{cl} := H + H_c + H \tag{6.95}$$

Moreover, it is easy to verify that the energy rate is equal to

$$\frac{\mathrm{d}H_{cl}}{\mathrm{d}t} = -\left(\frac{\partial^{\mathrm{T}} H_{cl}}{\partial p} D \frac{\partial H_{cl}}{\partial p} + \frac{\partial^{\mathrm{T}} H_{cl}}{\partial p_c} D_c \frac{\partial H_{cl}}{\partial p_c}\right) \leq 0$$

where D_c and H_c have to be designed in order to drive the system in the desired equilibrium position, which is still to be specified. Following the same procedure of Sect. 6.4 (see Sect. 6.4.5.6, in particular), the idea is to *shape* the total energy H_{cl} by properly choosing the controller Hamiltonian H_c in order to have a new minimum of energy in the desired configuration that can be reached if some dissipative effect is introduced. The first step is find the Casimir functionals of the closed-loop system.

6.5.2 Casimir functionals for the closed-loop system

Denote by $C : \mathscr{X} \times \mathscr{X}_\infty \to \mathbb{R}$ a scalar function defined on the extended state space (6.94). This is a Casimir function for closed-loop system reported in Fig. 6.10, if along system trajectories $\dot{C} = 0$ for every Hamiltonian of the system H_{cl}. Clearly,

$$\begin{aligned} \frac{\mathrm{d}C}{\mathrm{d}t} = \frac{\partial^{\mathrm{T}} C}{\partial q}\dot{q} + \frac{\partial^{\mathrm{T}} C}{\partial p}\dot{p} + \frac{\partial^{\mathrm{T}} C}{\partial q_c}\dot{q}_c + \frac{\partial^{\mathrm{T}} C}{\partial p_c}\dot{p}_c + \\ + \int_Z \left(\frac{\partial p_t}{\partial t} \wedge \delta_{p_t} C + \frac{\partial p_r}{\partial t} \wedge \delta_{p_r} C + \frac{\partial \varepsilon_t}{\partial t} \wedge \delta_{\varepsilon_t} C + \frac{\partial \varepsilon_r}{\partial t} \wedge \delta_{\varepsilon_r} C\right) \end{aligned}$$

and, from (4.157), (6.90), (6.92) and the interconnection constraints (6.93), we obtain

$$\begin{aligned}\frac{\mathrm{d}C}{\mathrm{d}t} &= \frac{\partial^{\mathrm{T}}C}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial^{\mathrm{T}}C}{\partial p}\left\{-\frac{\partial H}{\partial q} - D\frac{\partial H}{\partial p} + \left[\delta_{\varepsilon_t}H\,|_{z=L}\ \delta_{\varepsilon_r}H\,|_{z=L}\right]^{\mathrm{T}}\right\} + \\ &+ \frac{\partial^{\mathrm{T}}C}{\partial q_c}\frac{\partial H_c}{\partial p_c} + \frac{\partial^{\mathrm{T}}C}{\partial p_c}\left\{-\frac{\partial H_c}{\partial q_c} - D_c\frac{\partial H_c}{\partial p_c} + G_c\left[\delta_{\varepsilon_t}H\,|_{z=0}\ \delta_{\varepsilon_r}H\,|_{z=0}\right]^{\mathrm{T}}\right\} + \\ &+ \int_Z [\mathrm{d}\delta_{\varepsilon_t}H \wedge \delta_{p_t}C + (\mathrm{d}\delta_{\varepsilon_r}H + *\delta_{\varepsilon_t}H) \wedge \delta_{p_r}C] + \\ &+ \int_Z [(\mathrm{d}\delta_{p_t}H - *\delta_{p_r}H) \wedge \delta_{\varepsilon_r}C + \mathrm{d}\delta_{p_r}H \wedge \delta_{\varepsilon_t}C] \end{aligned} \tag{6.96}$$

Since $\mathrm{d}\delta H \wedge \delta C = \mathrm{d}(\delta H \wedge \delta C) - \delta H \wedge \mathrm{d}\delta C$ and $*\delta H \wedge \delta C = *\delta H \wedge \delta C$, the integral term in (6.96) is equal to

$$\begin{aligned}&\int_Z [\mathrm{d}(\delta_{\varepsilon_t}H \wedge \delta_{p_t}C) + \mathrm{d}(\delta_{\varepsilon_r}H \wedge \delta_{p_r}C) + \mathrm{d}(\delta_{p_t}H \wedge \delta_{\varepsilon_t}C) + \mathrm{d}(\delta_{p_r}H \wedge \delta_{\varepsilon_r}C)] - \\ &- \int_Z [\delta_{p_t}H \wedge \mathrm{d}\delta_{\varepsilon_t}C + \delta_{p_r}H \wedge (\mathrm{d}\delta_{\varepsilon_r}C + *\delta_{\varepsilon_t}C)] - \\ &- \int_Z [\delta_{\varepsilon_t}H \wedge (\mathrm{d}\delta_{p_t}C - *\delta_{p_r}C) + \delta_{\varepsilon_r}H \wedge \mathrm{d}\delta_{p_r}C] \end{aligned} \tag{6.97}$$

where, from Stokes theorem, the first term can be written as

$$\begin{aligned}&\int_{\partial Z} [\delta_{\varepsilon_t}H\,|_{\partial Z} \wedge \delta_{p_t}C\,|_{\partial Z} + \cdots + \delta_{p_r}H\,|_{\partial Z} \wedge \delta_{\varepsilon_r}C\,|_{\partial Z}] = \\ &= \int_{\partial Z} \left[\delta_{p_t}C\,|_{\partial Z}\ \delta_{p_r}C\,|_{\partial Z}\right]\left[\delta_{\varepsilon_t}H\,|_{\partial Z}\ \delta_{\varepsilon_r}H\,|_{\partial Z}\right]^{\mathrm{T}} + \\ &+ \int_{\partial Z} \left[\delta_{\varepsilon_t}C\,|_{\partial Z}\ \delta_{\varepsilon_r}C\,|_{\partial Z}\right]\left[\delta_{p_t}H\,|_{\partial Z}\ \delta_{p_r}H\,|_{\partial Z}\right]^{\mathrm{T}} \end{aligned} \tag{6.98}$$

From (6.90), (6.92) and the interconnection constraints (6.93), we have that

$$\begin{bmatrix}\delta_{p_t}H\,|_{z=L} \\ \delta_{p_r}H\,|_{z=L}\end{bmatrix} = -\mathbf{e} = -\frac{\partial H}{\partial p} \qquad \begin{bmatrix}\delta_{p_t}H\,|_{z=0} \\ \delta_{p_r}H\,|_{z=0}\end{bmatrix} = \mathbf{e}_c = G_c^{\mathrm{T}}\frac{\partial H_c}{\partial p_c}$$

Then, combining (6.96) with (6.97) and (6.98), we obtain that

$$
\begin{aligned}
\frac{\mathrm{d}C}{\mathrm{d}t} = &- \frac{\partial^{\mathrm{T}} C}{\partial p}\frac{\partial H}{\partial q} - \frac{\partial^{\mathrm{T}} C}{\partial p_c}\frac{\partial H_c}{\partial q_c} - \\
&- \left\{ -\frac{\partial^{\mathrm{T}} C}{\partial q} + \frac{\partial^{\mathrm{T}} C}{\partial p} D + \left[\delta_{\varepsilon_t} C\,|_{z=L}\ \delta_{\varepsilon_r} C\,|_{z=L}\right] \right\} \frac{\partial H}{\partial p} - \\
&- \left\{ -\frac{\partial^{\mathrm{T}} C}{\partial q_c} + \frac{\partial^{\mathrm{T}} C}{\partial p_c} D_c + \left[\delta_{\varepsilon_t} C\,|_{z=0}\ \delta_{\varepsilon_r} C\,|_{z=0}\right] G_c^{\mathrm{T}} \right\} \frac{\partial H_c}{\partial p_c} + \\
&+ \left\{ \frac{\partial^{\mathrm{T}} C}{\partial p} + \left[\delta_{p_t} C\,|_{z=L}\ \delta_{p_r} C\,|_{z=L}\right] \right\} \left[\delta_{\varepsilon_t} H\,|_{z=L}\ \delta_{\varepsilon_r} H\,|_{z=L}\right]^{\mathrm{T}} + \\
&+ \left\{ \frac{\partial^{\mathrm{T}} C}{\partial p_c} - \left[\delta_{p_t} C\,|_{z=0}\ \delta_{p_r} C\,|_{z=0}\right] \right\} \left[\delta_{\varepsilon_t} H\,|_{z=0}\ \delta_{\varepsilon_r} H\,|_{z=0}\right]^{\mathrm{T}} - \\
&- \int_Z \left[\delta_{p_t} H \wedge \mathrm{d}\delta_{\varepsilon_t} C + \delta_{p_r} H \wedge (\mathrm{d}\delta_{\varepsilon_r} C + *\delta_{\varepsilon_t} C)\right] - \\
&- \int_Z \left[\delta_{\varepsilon_t} H \wedge (\mathrm{d}\delta_{p_t} C - *\delta_{p_r} C) + \delta_{\varepsilon_r} H \wedge \mathrm{d}\delta_{p_r} C\right]
\end{aligned}
$$

that has to be equal to zero for every Hamiltonian H, H_c and H. This is true if and only if

$$
\begin{aligned}
\mathrm{d}\delta_{\varepsilon_t} C = 0 \qquad & \qquad \frac{\partial C}{\partial p} = 0 \\
\mathrm{d}\delta_{p_t} C - *\delta_{p_r} C = 0 \qquad & \qquad \frac{\partial C}{\partial p_c} = 0 \\
\mathrm{d}\delta_{\varepsilon_r} C + *\delta_{\varepsilon_t} C = 0 \qquad & \qquad \begin{bmatrix} \delta_{p_t} C\,|_{z=L} \\ \delta_{p_r} C\,|_{z=L} \end{bmatrix} = 0 \\
\mathrm{d}\delta_{p_r} C = 0 \qquad & \qquad \begin{bmatrix} \delta_{p_t} C\,|_{z=0} \\ \delta_{p_r} C\,|_{z=0} \end{bmatrix} = 0 \\
\frac{\partial C}{\partial q} = \begin{bmatrix} \delta_{\varepsilon_t} C\,|_{z=L} \\ \delta_{\varepsilon_r} C\,|_{z=L} \end{bmatrix} \qquad & \qquad \frac{\partial C}{\partial q_c} = G_c \begin{bmatrix} \delta_{\varepsilon_t} C\,|_{z=0} \\ \delta_{\varepsilon_r} C\,|_{z=0} \end{bmatrix}
\end{aligned}
\tag{6.99}
$$

In other words, the following proposition has been proved, [118, 119].

Proposition 6.10. *Consider the mixed port-Hamiltonian system of Fig. 6.10, which results from the power conserving interconnection (6.93) of the (4.157), (6.90) and (6.92). If $\mathscr{X} \times \mathscr{X}_\infty$ is the* extended *state space of the system, introduced in (6.94), then $C : \mathscr{X} \times \mathscr{X}_\infty \to \mathbb{R}$ is a Casimir for the closed-loop system if and only if relations (6.99) hold.*

6.5.3 Control by energy shaping of the Timoshenko beam

In order to control the flexible beam with the finite dimensional controller (6.92), the first step is to find Casimir functionals for the closed-loop system that can relate the state variables of the controller q to the state *variables* that describe the configuration

of the flexible beam and of the mass connected to its extremity. In particular, we are looking for some functionals $\tilde{C}_i$, $i = 1,2$, such that

$$C_i(q,p,q_c,p_c,p_t,p_r,\varepsilon_t,\varepsilon_r) := q_{c,i} - \tilde{C}_i(q,p,p_c,p_t,p_r,\varepsilon_t,\varepsilon_r), \quad \text{with } i = 1,2$$

are Casimir functionals for the closed loop system, i.e. satisfying the conditions of Proposition 6.10. At first, from (6.99), it is immediate to note that every Casimir functional cannot depend on p and p_c. Moreover, since it is necessary that $\mathrm{d}\delta_{\varepsilon_t} C_i = 0$ and $\mathrm{d}\delta_{p_r} C_i = 0$, we deduce that $\delta_{\varepsilon_t} C_i$ and $\delta_{p_r} C_i$ have to be constant as function on z on Z and their value will be determined by the boundary conditions on C_i. Since, from (6.92), $\delta_{p_r} C_i \mid_{\partial Z} = 0$, we deduce that $\delta_{p_r} C_i = 0$ on Z. But $\mathrm{d}\delta_{p_t} C_i = *\delta_{p_r} C_i = 0$, then, from the boundary conditions, we deduce that also $\delta_{p_t} C_i = 0$ on Z. As a consequence, all the admissible Casimir functionals are also independent from p_t and p_r. In other words, we are interested in finding Casimirs in the following form:

$$C_i(q,q_c,\varepsilon_t,\varepsilon_r) := q_{c,i} - \tilde{C}_i(q,\varepsilon_t,\varepsilon_r), \qquad \text{with } i = 1,2$$

Assuming $G_c = I$, we have that

$$\frac{\partial C_1}{\partial q_c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \delta_{\varepsilon_t} C_1 \mid_{z=0} \\ \delta_{\varepsilon_r} C_1 \mid_{z=0} \end{bmatrix} \tag{6.100}$$

and, consequently, $\delta_{\varepsilon_t} C_1 = 1$ on Z. From (6.99), we have that $\mathrm{d}\delta_{\varepsilon_r} C_1 = -*\delta_{\varepsilon_t} C_1 = -*1 = -\mathrm{d}z$; then, $\delta_{\varepsilon_r} C_1 = -z + c_1$, where c_1 is determined by the boundary conditions. Since, from (6.100), $\delta_{\varepsilon_r} C_1 \mid_{z=0} = 0$, then $c_1 = 0$; moreover, we deduce that $\delta_{\varepsilon_r} C \mid_{z=L} = -L$, relation that introduces a new boundary condition in $z = L$. A consequence is that

$$\frac{\partial C_1}{\partial q} = \begin{bmatrix} \delta_{\varepsilon_t} C_1 \mid_{z=L} \\ \delta_{\varepsilon_r} C_1 \mid_{z=L} \end{bmatrix} = \begin{bmatrix} 1 \\ -L \end{bmatrix}$$

The first conclusion is that

$$C_1(q,q_c,\varepsilon_t,\varepsilon_r) = q_{c,1} - (Lq_2 - q_1) - \int_Z (z\varepsilon_r - \varepsilon_t) \tag{6.101}$$

is a Casimir for the closed loop system. Following the same procedure, it is possible to calculate C_2. From (6.99), we have that

$$\frac{\partial C_2}{\partial q_c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \delta_{\varepsilon_t} C_2 \mid_{z=0} \\ \delta_{\varepsilon_r} C_2 \mid_{z=0} \end{bmatrix}$$

and then $\delta_{\varepsilon_t} C_2 = 0$ on Z; moreover, $\mathrm{d}\delta_{\varepsilon_r} C_2 = 0$ and, consequently, $\delta_{\varepsilon_r} C_2 = 1$ on Z since (6.95) holds. Again from (6.99), we deduce that

$$\frac{\partial C_2}{\partial q} = \begin{bmatrix} \delta_{\varepsilon_t} C_2 \mid_{z=L} \\ \delta_{\varepsilon_r} C_2 \mid_{z=L} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So we can state that

$$C_2(q,q_c,\varepsilon_t,\varepsilon_r) = q_{c,2} + q_2 + \int_Z \varepsilon_r \tag{6.102}$$

is another Casimir functionals for the closed loop system. In conclusion, the following proposition has been proved, [118, 119].

Proposition 6.11. *Consider the port-Hamiltonian system resulting from the power conserving interconnection (6.93) of (4.157), (6.90) and (6.92) which is represented in Fig. 6.10. Then, this systems is characterized by the Casimir functions (6.101) and (6.102).*

Note 6.4. Since C_i, $i = 1,2$, are Casimir functionals, they are invariant for the system of Fig. 6.10. Then, for every energy function H_c of the controller, we have that

$$q_{c,1} = (Lq_2 - q_1) + \int_Z (z\varepsilon_r - \varepsilon_t) + C_1 \qquad q_{c,2} = -q_2 - \int_Z \varepsilon_r + C_2 \tag{6.103}$$

where C_1 and C_2 depend on the initial conditions. If the initial configuration of the system is known, then it is possible to assume these constants equal to zero. Since H_c is an arbitrary function of q_c, it is possible to *shape* the total energy function of the closed-loop system in order to have a minimum of energy in a desired configuration: if some dissipation effect is present, the new equilibrium configuration will be reached.

Suppose that $(q^*,0)$, with $q^* = [q_1^*\ q_2^*]^{\mathrm{T}}$, is the desired equilibrium configuration of the mass (6.90). Then, the corresponding equilibrium configuration of the beam can be calculated as the solution of (4.157) with

$$\frac{\partial p_t}{\partial t} = \frac{\partial p_r}{\partial t} = \frac{\partial \varepsilon_t}{\partial t} = \frac{\partial \varepsilon_r}{\partial t} = 0 \qquad \text{on } Z$$

and with boundary conditions in $z = L$ given by

$$\begin{bmatrix} f_b^t(L) \\ f_b^r(L) \end{bmatrix} = \frac{\partial H}{\partial p}(q^*,0) = 0 \qquad \begin{bmatrix} e_b^t(L) \\ e_b^r(L) \end{bmatrix} = \frac{\partial H}{\partial q}(q^*,0) = \begin{bmatrix} k_1 q_1^* \\ k_2 q_2^* \end{bmatrix} \tag{6.104}$$

From (4.157), we have that the equilibrium configuration has to satisfy the following system of PDEs

$$\begin{cases} \mathrm{d}\delta_{\varepsilon_t} H = 0 \\ *\delta_{\varepsilon_t} H + \mathrm{d}\delta_{\varepsilon_r} H = 0 \end{cases}$$

whose solution, compatible with the boundary conditions (6.104), is equal to

$$\varepsilon_t{}^*(x,t) = \frac{k_1}{K} q_1^* \qquad \varepsilon_r{}^*(x,t) = \frac{k_1 q_1^*}{EI}(L-z) + \frac{k_2 q_2^*}{EI} \tag{6.105}$$

Furthermore, at the equilibrium, it is easy to compute that $p_t = p_t{}^* = 0$ and that $p_r = p_r{}^* = 0$. From (6.103) and (6.105), define

$$
\begin{aligned}
q^*_{c,1} &= q_{c,1}(q^*_1, q^*_2, \varepsilon_t{}^*, \varepsilon_r{}^*) = Lq^*_2 - q^*_1 + \int_0^L (z\varepsilon_r{}^* - \varepsilon_t{}^*)\,\mathrm{d}z \\
&= Lq^*_2 - q^*_1 + \int_0^L \left[\frac{k_1 q^*_1}{EI}(L-z)z + \frac{k_2 q^*_2}{EI}z - \frac{k_1}{K}q^*_1\right]\mathrm{d}z \\
&= \left(\frac{k_1}{EI}\frac{L^3}{6} - \frac{k_1}{K}L - 1\right)q^*_1 + \left(\frac{k_2}{EI}\frac{L^2}{2} + L\right)q^*_2 \\
q^*_{c,2} &= q_{c,2}(q^*_2, \varepsilon_r{}^*) = -q^*_2 - \int_0^L \varepsilon_r \mathrm{d}z \\
&= -q^*_2 - \int_0^L \left[\frac{k_1 q^*_1}{EI}(L-z) + \frac{k_2 q^*_2}{EI}\right]\mathrm{d}z \\
&= -\frac{k_1}{EI}\frac{L^2}{2}q^*_1 - \left(\frac{k_2}{EI}L + 1\right)q^*_2
\end{aligned}
$$

Note that, at the equilibrium, $p_c = p^*_c = 0$. The energy function H_c of the controller (6.92) will be chosen in such a way that a minimum in the closed-loop energy function is introduced at $\chi^* = (q^*, p^*, p^*_c, p_t{}^*, p_r{}^*, \varepsilon_t{}^*, \varepsilon_r{}^*)$. In the remaining part of this section it will proved that, by choosing the controller energy as

$$
\begin{aligned}
H_c(p_c, q_c) = \frac{1}{2}p_c^{\mathrm{T}} M_c^{-1} p_c + \frac{1}{2}K_{c,1}(q_{c,1} - q^*_{c,1})^2 + \\
+ \frac{1}{2}K_{c,2}(q_{c,2} - q^*_{c,2})^2 + \Psi_1(q_{c,1}) + \Psi(q_{c,2})
\end{aligned} \tag{6.106}
$$

with $M_c = M_c^{\mathrm{T}} > 0$, $K_{c,1}$, $K_{c,1} > 0$ and Ψ_1, Ψ_2 functions still to be specified, the configuration χ^* is stable. The stability proof follows the Arnold first stability theorem discussed in Sect. 6.2.1.

Denote by χ the state variable of the closed-loop system. From (4.158), (6.91) and (6.106), the total energy function is given by

$$
\begin{aligned}
H_{cl}(\chi) = &\frac{1}{2}\left(\frac{p_1^2}{m} + \frac{p_2^2}{J}\right) + \frac{1}{2}\left(k_1 q_1^2 + k_2 q_2^2\right) + \\
&+ \frac{1}{2}\int_Z \left(\frac{1}{\rho}p_t \wedge *p_t + \frac{1}{I_\rho}p_r \wedge *p_r + K\varepsilon_t \wedge *\varepsilon_t + EI\varepsilon_r \wedge *\varepsilon_r\right) + \\
&+ \frac{1}{2}p_c^{\mathrm{T}} M_c^{-1} p_c + \frac{1}{2}K_{c,1}(q_{c,1} - q^*_{c,1})^2 + \frac{1}{2}K_{c,2}(q_{c,2} - q^*_{c,2})^2 + \\
&+ \Psi_1(q_{c,1}) + \Psi(q_{c,2})
\end{aligned}
$$

The first step in the stability proof is to find under which conditions, that is for what particular choice of the functions Ψ_1 and Ψ_2, relation (6.1) is satisfied. We have that

$$\nabla H_{cl}(\chi) = \begin{bmatrix} \frac{\partial H_{cl}}{\partial p} \\ \frac{\partial H_{cl}}{\partial q} \\ \frac{\partial H_{cl}}{\partial p_t} \\ \frac{\partial H_{cl}}{\partial p_r} \\ \frac{\partial H_{cl}}{\partial \varepsilon_t} \\ \frac{\partial H_{cl}}{\partial \varepsilon_r} \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial p} \\ \frac{\partial}{\partial q}(H+H_c) \\ \delta_{p_t} H \\ \delta_{p_r} H \\ \delta_{\varepsilon_t}(H+H_c) \\ \delta_{\varepsilon_r}(H+H_c) \end{bmatrix}$$

Clearly,

$$\frac{\partial H}{\partial p}(\chi^*) = 0 \qquad \frac{\partial H_c}{\partial p_c}(\chi^*) = 0 \qquad \delta_{p_t} H(\chi*) = 0 \qquad \delta_{p_r} H(\chi^*) = 0$$

Furthermore,

$$\begin{aligned} \frac{\partial H_{cl}}{\partial q_1} &= k_1 q_1 - K_{c,1}\left(q_{c,1} - q_{c,1}^*\right) - \frac{\partial \Psi_1}{\partial q_{c,1}} \\ \frac{\partial H_{cl}}{\partial q_2} &= k_2 q_2 + K_{c,1}\left(q_{c,1} - q_{c,1}^*\right) L - K_{c,2}\left(q_{c,2} - q_{c,2}^*\right) + \frac{\partial \Psi_1}{\partial q_{c,1}} L - \frac{\partial \Psi_2}{\partial q_{c,2}} \end{aligned}$$

and

$$\begin{aligned} \delta_{\varepsilon_t} H_{cl} &= K * \varepsilon_t - K_{c,1}\left(q_{c,1} - q_{c,1}^*\right) - \frac{\partial \Psi_1}{\partial q_{c,1}} \\ \delta_{\varepsilon_r} H_{cl} &= EI * \varepsilon_r + K_{c,1}\left(q_{c,1} - q_{c,1}^*\right) z - K_{c,2}\left(q_{c,2} - q_{c,2}^*\right) + \frac{\partial \Psi_1}{\partial q_{c,1}} z - \frac{\partial \Psi_2}{\partial q_{c,2}} \end{aligned}$$

then $\nabla H_{cl}(\chi^*) = 0$ if

$$\begin{cases} \Psi_1(q_{c,1}) = k_1 q_1^* q_{c,1} + \psi_{c,1} \\ \Psi_2(q_{c,2}) = (k_2 q_2^* + k_1 q_1^* L)\, q_{c,2} + \psi_{c,2} \end{cases}$$

with $\psi_{c,1}$ and $\psi_{c,2}$ arbitrary constants. Once the equilibrium is assigned in χ^*, it is necessary to verify the convexity condition (6.3) in χ^* on the nonlinear functional $\mathscr{N}$ defined as in (6.2). With some simple but boring calculation, it can be obtained that

$$\begin{aligned} \|\Delta\chi\| =& \frac{1}{2}\Delta p^{\mathrm{T}} M^{-1} \Delta p + \frac{1}{2}\Delta p_c^{\mathrm{T}} M_c^{-1} \Delta p_c + \frac{1}{2} k_1 \Delta q_1^2 + \frac{1}{2} k_2 \Delta q_2^2 \\ &+ \frac{1}{2}\int_0^L \left(\frac{1}{\rho}\Delta p_t \wedge * p_t + \frac{1}{I_\rho}\Delta p_r \wedge * p_r + K \Delta\varepsilon_t \wedge * \Delta\varepsilon_t + EI \Delta\varepsilon_r \wedge * \Delta\varepsilon_r \right) \\ &+ \frac{1}{2} K_{c,1} \left[L\Delta q_2 - \Delta q_1 + \int_0^L (z\Delta\varepsilon_r - \Delta\varepsilon_t) \right]^2 + \frac{1}{2} K_{c,2} \left[\Delta q_2 + \int_0^L \Delta\varepsilon_r \right]^2 \end{aligned}$$

The convexity condition (6.3) requires a norm in order to be verified. As already discussed in Sect. 6.4.5.6, a possible choice can be

$$\begin{aligned}\|\chi\|^2 =& \frac{1}{2}\Delta p^{\mathrm{T}}\Delta p + \frac{1}{2}\Delta p_c^{\mathrm{T}}\Delta p_c + \frac{1}{2}\Delta q_1^2 + \frac{1}{2}\Delta q_2^2 \\ &+ \frac{1}{2}\int_0^L (\Delta p_t \wedge *p_t + \Delta p_r \wedge *p_r + \Delta\varepsilon_t \wedge *\Delta\varepsilon_t + \Delta\varepsilon_r \wedge *\Delta\varepsilon_r)\end{aligned}$$

In (6.3), a possible choice for γ_1 can be

$$\frac{1}{2}\min\left\{|M^{-1}|,\ |M_c^{-1}|,\ k_1,\ k_2,\ \frac{1}{\rho},\ \frac{1}{I_\rho},\ K,\ EI\right\}$$

Moreover, if

$$\tilde{\gamma}_2 = \frac{1}{2}\max\left\{|M^{-1}|,\ |M_c^{-1}|,\ k_1,\ k_2,\ \frac{1}{\rho},\ \frac{1}{I_\rho},\ K,\ EI,\ K_{c,1},\ K_{c,2}\right\}$$

we have that

$$\mathscr{N}(\Delta\chi) \le \tilde{\gamma}_2\|\Delta\chi\| + \tilde{\gamma}_2\left[L\Delta q_2 - \Delta q_1 + \int_0^L (x\Delta\varepsilon_r - \Delta\varepsilon_t)\right]^2 + \tilde{\gamma}_2\left[\Delta q_2 + \int_0^L \Delta\varepsilon_r\right]^2$$

Since

$$\begin{aligned}&\left[L\Delta q_2 - \Delta q_1 + \int_0^L (z\Delta\varepsilon_r - \Delta\varepsilon_t)\right]^2 \le \\ &\qquad \le 2\left(L\Delta q_2 - \Delta q_1\right)^2 + 2\left[\int_0^L (z\Delta\varepsilon_r - \Delta\varepsilon_t)\right]^2 \le \\ &\le 4\left(|\Delta q_1|^2 + L^2|\Delta q_2|^2\right) + 4\left(\int_0^L x\Delta\varepsilon_r\right)^2 + 4\left(\int_0^L \Delta\varepsilon_t\right)^2 \le \\ &\le 4\left(|\Delta q_1|^2 + L^2|\Delta q_2|^2\right) + 4L\left(\int_0^L \Delta\varepsilon_t \wedge *\Delta\varepsilon_t + L\int_0^L \Delta\varepsilon_r \wedge *\Delta\varepsilon_r\right)\end{aligned}$$

and

$$\left[\Delta q_2 + \int_0^L \Delta\varepsilon_r\right]^2 \le 2|\Delta q_2|^2 + 2\left(\int_0^L \Delta\varepsilon_r\right)^2 \le 2|\Delta q_2|^2 + 2L\int_0^L \Delta\varepsilon_r \wedge *\Delta\varepsilon_r$$

it is possible to satisfy (6.3) by choosing $\alpha = 2$ and

$$\gamma_2 = \tilde{\gamma}_2 \cdot \max\left\{2,\ 2L^2 + 2L + 1\right\}$$

which completes the stability proof. In other words, the following proposition has been proved.

Proposition 6.12. *Consider the port-Hamiltonian system which results from the power conserving interconnection (6.93) of (4.157), (6.90) and (6.91). If in (6.92) it is assumed that $G_c = I$ and H_c is chosen according to (6.106), then the configuration χ^* is stable in the sense of Lyapunov, i.e. in the sense of Definition 6.1.*

An analogous result can be proved in the case that the mass at the extremity is under the effect of a gravitational field [120], that is the Hamiltonian function (6.91) is replaced by

$$H(p,q) = \frac{1}{2}\left(\frac{p_1^2}{m} + \frac{p_2^2}{J}\right) + mgq_1$$

Then, the desired equilibrium configuration of the beam expressed in terms of energy variables are

$$p_t{}^*(t,z) = p_r{}^*(t,z) = 0 \qquad \varepsilon_t{}^*(t,z) = -\frac{mg}{K} \qquad \varepsilon_r{}^*(t,z) = \frac{mg}{EI}(z-L) \tag{6.107}$$

It is important to note that the effect of the gravity field on the flexible beam have been neglected in order to make the model of the infinite dimensional subsystem easier. On the other hand, if these effects have to be taken into account, it is necessary to introduce a port along the spatial domain of the beam, as reported in (4.161), that is interconnected to a *distributed* source of effort equal to $-\rho g$, the gravity force per unit length. If q_1^* and q_2^* are the desired position and rotational angle of the mass, that is $w^*(t,L) = q_1^*$ and $\phi^*(t,L) = q_2^*$, the desired configuration (6.107) of the beam can be easily expressed in terms of its vertical displacement and cross section rotation:

$$\begin{aligned} w^*(t,z) &= \frac{mg}{6EI}(z-L)^3 + \left(q_2^* - \frac{mg}{K}\right)(z-L) + q_1^* \\ \phi^*(t,z) &= \frac{mg}{2EI}(z-L)^2 + q_2^* \end{aligned}$$

The energy function H_c of the controller (6.92) will be developed in order to regulate the closed-loop system in the configuration $\chi^* = (q^*, p^*, p_c^*, p_t{}^*, p_r{}^*, \varepsilon_t{}^*, \varepsilon_r{}^*)$. By choosing the controller energy as

$$\begin{aligned} H_c(q_c,p_c) := \frac{1}{2}p_c^{\mathrm{T}}M_c^{-1}p_c + \frac{1}{2}K_{c,1}(q_{c,1}-q_{c,1}^*)^2 + \\ + \frac{1}{2}K_{c,2}(q_{c,2}-q_{c,2}^*)^2 + \Psi_1(q_{c,1}-q_{c,1}^*) + \Psi_2(q_{c,2}-q_{c,2}^*) \end{aligned}$$

with $M_c = M_c^{\mathrm{T}} > 0$, $K_{c,1}, K_{c,2} > 0$, $q_{c,1}^*$ and $q_{c,2}^*$ given by (6.103) evaluated in the equilibrium configuration, Ψ_1 and Ψ_2 to be properly chosen, the configuration χ^* previously introduced is stable. If $H_{cl} = H + H_c + H$ is the closed loop energy function, by simple calculations, it is possible to prove that $\nabla H_{cl}(\chi^*) = 0$ if

$$\Psi_1(q_{c,1}-q_{c,1}^*) = -mg(q_{c,1}-q_{c,1}^*) \qquad \Psi_2(q_{c,2}-q_{c,2}^*) = -mgL(q_{c,2}-q_{c,2}^*)$$

Moreover, also $H_{cl}(\chi^*) = 0$. Then, the linear functional (6.2) becomes

$$\begin{aligned}\mathscr{N}(\Delta\chi) = \frac{1}{2}\Delta p^{\mathrm{T}} M^{-1}\Delta p + \frac{1}{2}\Delta p_c^{\mathrm{T}} M_c^{-1}\Delta p_c + \\ + \frac{1}{2}\int_Z \left(\frac{\Delta p_t{}^2}{\rho} + \frac{\Delta p_r{}^2}{I_\rho} + K\Delta\varepsilon_t{}^2 + EI\Delta\varepsilon_r{}^2\right)\mathrm{d}z \\ + \frac{1}{2}K_{c,1}\left(L\Delta q_2 - \Delta q_1 + \int_Z (z\Delta\varepsilon_r - \Delta\varepsilon_t)\mathrm{d}z\right)^2 + \\ + \frac{1}{2}K_{c,2}\left(\Delta q_2 + \int_Z \Delta\varepsilon_r \mathrm{d}z\right)^2 \end{aligned} \tag{6.108}$$

The upper bound for $\mathscr{N}$, required in (6.3), is obtained by choosing $\alpha = 2$ and

$$\gamma_2 = \frac{1}{2}\max\left\{\left|M^{-1}\right|, \left|M_c{}^{-1}\right|, \frac{1}{\rho}, \frac{1}{I_\rho}, \xi_2^a, \xi_2^b, \xi_2^c\right\}$$

where

$$\begin{aligned}\xi_2^a &:= K + 4LK_{c,1}K + 4LK_{c,1} \\ \xi_2^b &:= EI + 2LK_{c,2} + 4L^3K_{c,1} \\ \xi_2^c &:= 2(K_{c,1} + L^2K_{c,2})\end{aligned}$$

As regard the lower bound on $\mathscr{N}$, note that

$$\left(\Delta q_2 + \int_Z \Delta\varepsilon_r \mathrm{d}z\right)^2 \geq \frac{1}{2}\Delta q_2^2 - \left(\int_Z \Delta\varepsilon_r \mathrm{d}z\right)^2 \geq \frac{1}{2}\Delta q_2^2 - L\int_Z \Delta\varepsilon_r{}^2 \mathrm{d}z$$

and, in the same way, that

$$\begin{aligned}\left(L\Delta q_2 - \Delta q_1 + \int_Z (z\Delta\varepsilon_r - \Delta\varepsilon_t)\mathrm{d}z\right)^2 \geq \\ \geq \frac{1}{2}\Delta q_1^2 - 2L^2\Delta q_2^2 - 4L\int_Z \Delta\varepsilon_t{}^2\mathrm{d}z - 4L^3\int_Z \Delta\varepsilon_r{}^2\mathrm{d}z\end{aligned}$$

If $K_{c,1}$ and $K_{c,2}$ are chosen such that

$$\begin{aligned}\xi_1^a &:= K - 4LK_{c,1} > 0 \\ \xi_1^b &:= EI - 4L^3K_{c,1} - LK_{c,2} > 0 \\ \xi_1^c &:= \frac{1}{2}K_{c,2} - 2L^2K_{c,1} > 0\end{aligned} \tag{6.109}$$

then (6.3) can be satisfied if

$$\gamma_1 = \frac{1}{2}\min\left\{\left|M^{-1}\right|, \left|M_c{}^{-1}\right|, \frac{1}{\rho}, \frac{1}{I_\rho}, \frac{1}{2}K_{c,1}, \xi_1^a, \xi_1^b, \xi_1^c\right\}$$

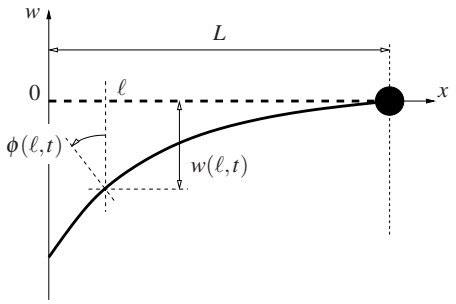


Fig. 6.11 Initial (dashed) and final (solid) configurations of the system. The system has been stabilized around the configuration for which $q_1^* = q_2^* = 0$, starting from the *zero* initial condition.

where ξ_1^a, ξ_1^b and ξ_1^c are given in (6.109).

Simulation results showing the validity of the latest developed controller are presented. In Fig. 6.11, the initial (dashed) and final (solid) configurations assumed by the beam are presented. The aim of the simulations is to show how it is possible to stabilize the whole system around the configuration for which $q_1^* = q_2^* = 0$, starting from the *zero* initial condition. As far as the beam parameters are concerned, let assume that $L = 1.0\,\mathrm{m}$, $\rho = 1.0\,\mathrm{kg/m}$, $I_\rho = 1.0\,\mathrm{kg \cdot m}$, $K = 10^3\,\mathrm{N}$ and $EI = 10\,\mathrm{N \cdot m^2}$. The simulation of the Timoshenko model of the beam (which is an infinite dimensional dynamical system) is possible once an approximated finite dimensional model is provided. In this case, the discrete model of the Timoshenko beam proposed in [89] is used. In particular, the beam has been divided in $n = 20$ identical parts, whose bond-graph model is described in details in [89]. By discretization of the PDE equation describing the dynamics of the Timoshenko beam, a class of finite dimensional approximation models can be obtained depending on the value of a free parameter $\alpha \in [0,1]$. This parameter determines the accuracy of the approximation model, which is maximum for $\alpha = 1/2$. This is the value assumed in the proposed simulations.

Two sets of simulations have been carried out. As already presented in Sect. 6.5.1, the controller is interconnected to the beam in $z = 0$, where it can apply a torque and a force. In this way, the extremity of the beam in $z = 0$ can translate along w and rotate (see again Fig. 6.11), thus allowing the regulation of both vertical displacement and rotation of the mass interconnected in $z = L$. In the first simulation, the values $m = 10^{-2}\,\mathrm{kg}$ and $J = 10^{-2}\,\mathrm{kg \cdot m^2}$ have been assumed for the mass at the free-end, and the corresponding results are reported in Figures 6.12a and 6.13a. The parameters of the controller have been chosen as $K_{c,1} = 1\,\mathrm{N/m}$, $K_{c,2} = 1\,\mathrm{N}$, $D_c = \mathrm{diag}\left(3\,\mathrm{N \cdot s^2/m},\ 3\,\mathrm{N \cdot s^2}\right)$ and $M_c = \mathrm{diag}\left(10^{-3}\,\mathrm{kg},\ 10^{-3}\,\mathrm{kg \cdot m^2}\right)$. In the second case, the values $m = 0.3\,\mathrm{kg}$ and $J = 0.2\,\mathrm{kg \cdot m^2}$ have been assigned to the mass. The results are shown in Figures 6.12b and 6.13b. In this case, the control parameters have been chosen in order to decrease the settling time of the closed-loop system. In particular, it has been assumed that $K_{c,1} = 4 \cdot 10^{-2}\,\mathrm{N/m}$, $K_{c,2} = 10^{-1}\,\mathrm{N}$, $D_c = \mathrm{diag}\left(10\,\mathrm{N \cdot s^2/m},\ 5\,\mathrm{N \cdot s^2}\right)$ and $M_c = \mathrm{diag}\left(10^{-3}\,\mathrm{kg},\ 10^{-3}\,\mathrm{kg \cdot m^2}\right)$.

In Figures 6.12a and 6.13a, the time evolution of the deflection of the beam from the initial configuration is given. In particular, the position deviation w of the beam in $z = L$, $z = 0.75L$, $z = 0.50L$, $z = 0.25L$ and $z = 0$ is represented. As presented

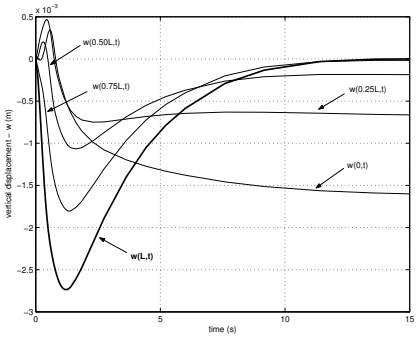


Fig. 6.12a Deflection w of the beam in $z = L$, $z = 0.75\,L$, $z = 0.50\,L$, $z = 0.25L$ and $z = 0$, with $m = 0.01\,\text{kg}$.

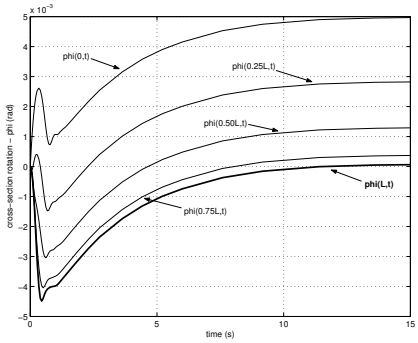


Fig. 6.12b Rotation ϕ of the beam cross-section in $z = L$, $z = 0.75\,L$, $z = 0.50\,L$, $z = 0.25\,L$ and $z = 0$, with $m = 0.01\,\text{kg}$.

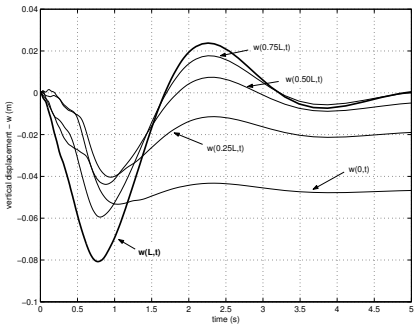


Fig. 6.13a Deflection w of the beam in $z = L$, $z = 0.75\,L$, $z = 0.50\,L$, $z = 0.25L$ and $z = 0$, with $m = 0.3\,\text{kg}$.

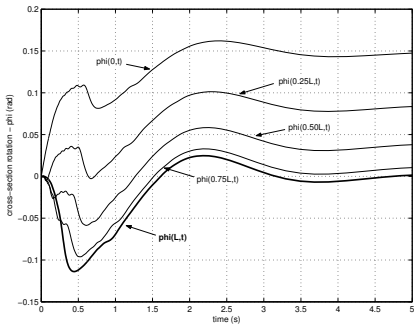


Fig. 6.13b Rotation ϕ of the beam cross-section in $z = L$, $z = 0.75\,L$, $z = 0.50\,L$, $z = 0.25\,L$ and $z = 0$, with $m = 0.3\,\text{kg}$.

in Fig. 6.11, the vertical displacement of the mass is, clearly, equal to the deflection of the beam in $z = L$. Moreover, in Figures 6.12b and 6.13b, the time evolution of the rotation of the beam cross section ϕ is given. Again, only the values in $z = L$, $z = 0.75\,L$, $z = 0.50\,L$, $z = 0.25\,L$ and $z = 0$ are reported. Clearly, the rotation of the mass at the extremity of the beam is equal to the rotation of the beam cross section in $z = L$.

Appendix A
Model Transformations and Analysis using Bond Graphs

P. C. Breedveld

Abstract This appendix illustrates the main procedures for model transformations and analysis using the bond graph formalism.

A.1 Model transformations

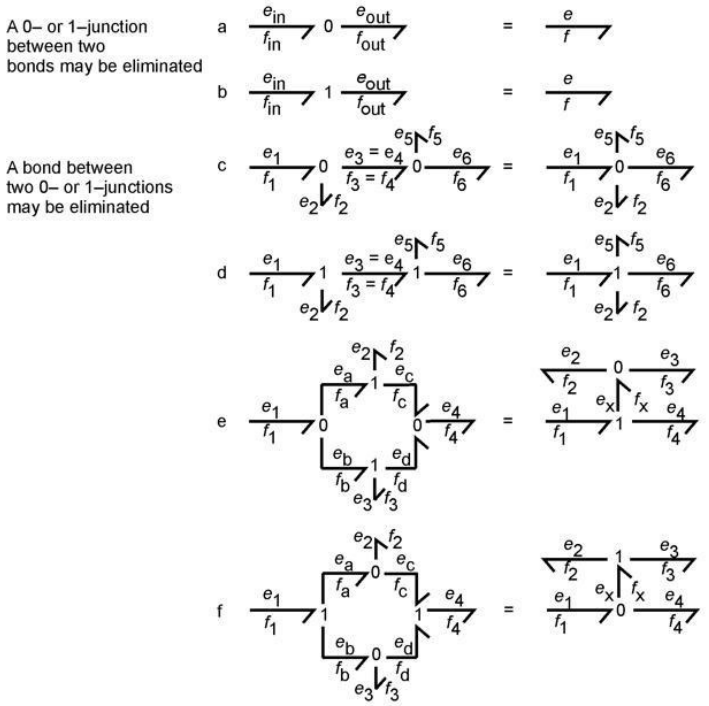
A.1.1 Conversion of an ideal physical model into a bond graph

A *standard procedure* for the *translation of an ideal physical model (IPM) into a bond graph* contains the following steps (specifics between brackets apply to the mechanical domain that is treated in a dual way due to the common choice of variables, i.e. the force-voltage analogy):

1. Identify the domains that are present.
2. Choose a *reference* effort (velocity *reference* and direction) for each of the domains (degrees of freedom).
3. Identify and label the other points with common effort (velocity) in the model.
4. Identify, classify and accordingly label the ports of the basic one- and two-port elements: C, I, GY, etc. in the model. A label consists of a node type and a unique identifier, e.g. in the linear case the constitutive parameter connected by a colon.
5. Identify the efforts (velocities) and effort differences (relative velocities) of all ports identified in the previous step.
6. Represent each effort by a 0-junction, (each velocity by a 1-junction). Use a 1-junction (0-junction) and bonds to construct a relation between each effort difference (relative velocity) and the composing efforts (velocities) as follows, taking care that each effort difference (relative velocity) is explicitly represented by a 0-junction (1-junction), see Fig. A.1.
7. Connect all ports identified in steps 4 and 5 to the corresponding junctions. Note that after this step all of the ports identified in step 4 are directly connected to a 0-junction (1-junction) only.

Fig. A.1 Construction of effort and flow differences.

Table A.1 Equivalence rules for simple junction structures.



8. (optional) Simplify the bond graph where necessary according to the equivalence rules in Table A.1.

These steps not only support this translation process, but also give a better insight when dynamic models are directly written in terms of a bond graph as well. In that case steps 1, 3, 4 and 5 should be changed from a translation of already made modeling choices into the modeling choices themselves.

This process is merely intended to establish a link between familiar representations and a bond graph representation. It does not suggest that the use of bond graphs to support modeling always takes place on the basis of an existing IPM. On the contrary, the process of making modeling choices is supported best by direct application of the bond graph representation, especially when it is causally augmented

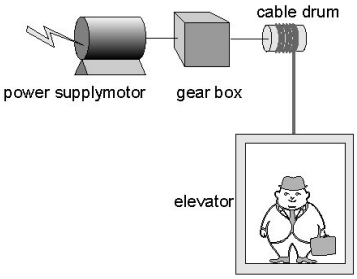


Fig. A.2a Sketch of an elevator system.

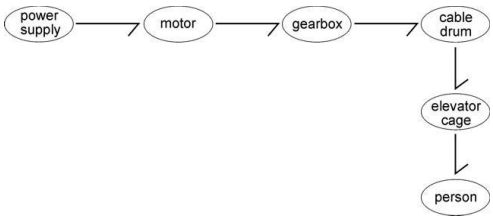
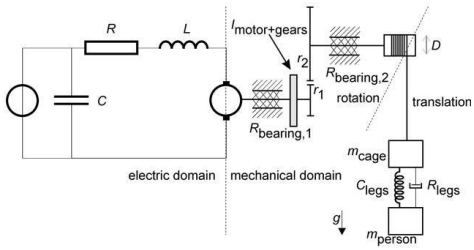


Fig. A.2b Word bond graph.

Fig. A.3 Iconic diagram of the IPM of the elevator system (step 1).



as will be shown in section , where the advantage of a bond graph as an alternative model 'view' will be illustrated.

A.1.2 Example: systematic conversion of a simple electromechanical system model into a bond graph representation.

In order to demonstrate the relationship between the domain-independent bond graph representation of a model and a domain dependent representation, e.g. the iconic diagram of the model of the elevator system in Fig. A.2a, a systematic conversion is demonstrated in Figures A.3 through A.10 following the eight steps described in the previous section. This conversion ignores the automatic feedback on modeling decisions from a bond graph, as the modeling decisions have already been made.

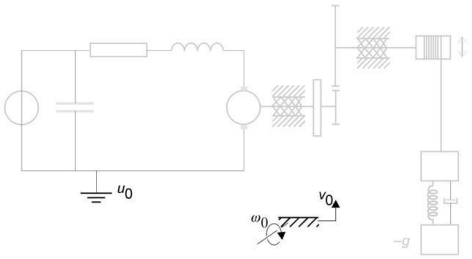


Fig. A.4 The IPM with references (step 2).

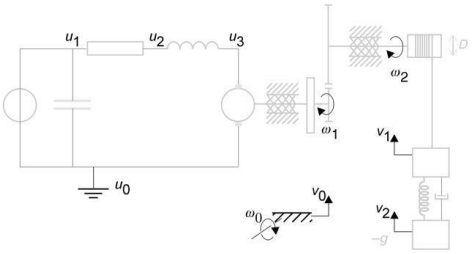


Fig. A.5 The IPM with relevant voltagees and (angular velocities indicated (step 3).

Fig. A.6 Representation of voltages by 0-junctions and of velocities by 1-junctions (step 4).

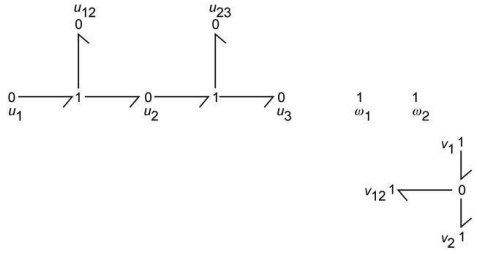


Fig. A.7 Construction of the difference variables (steps 5 and 6).

A.1.3 Conversion of causal bond graphs into block diagrams

As a causal bond represents a bi-lateral signal flow with fixed directions, a causal bond graph (e.g. Fig. A.11) can be expanded into a block diagram in three to four steps:

1. All node symbols are encircled and all bonds are expanded into bilateral signal flows according to the assigned causality (Fig. A.12).
2. All constitutive relations of each node are written into block diagram form, according to the assigned causality of each port. 0-junctions are represented by a signal-node for the efforts and a summation for the flows, while 1-junctions

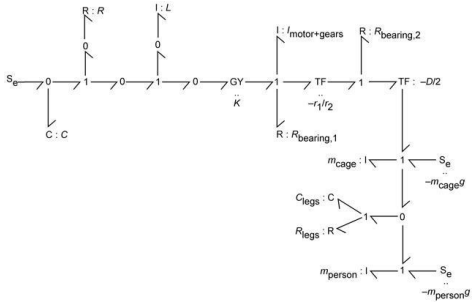


Fig. A.8 Complete bond graph (step 7).

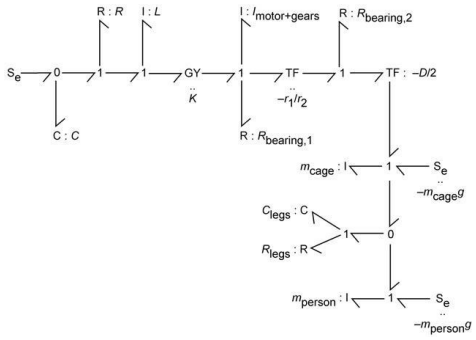


Fig. A.9 Simplification of the bond graph (step 8a).

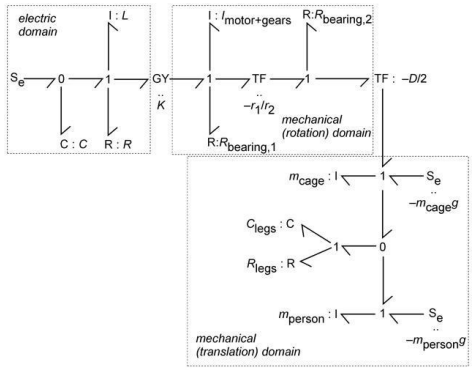


Fig. A.10 Simplification of the bond graph (step 8b).

are represented by a signal-node for the flows and a summation for the efforts (Fig. A.13).

3. All signals entering a summation resulting from a junction are given a sign corresponding to the half-arrow direction: if, while traveling from causal input to causal output, the bond orientation does not change (this does not exclude an orientation opposite to the signal direction!), then a plus sign is added representing a positive contribution to the summation; by contrast if the bond orientation does change, then a minus sign is added representing a negative contribution to the summation (Fig. A.14). In principle, a complete block diagram is obtained at this

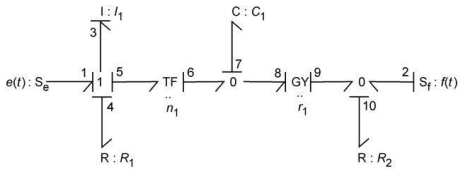


Fig. A.11 Causal bond graph.

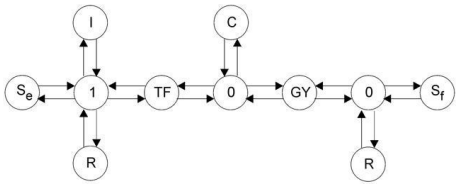


Fig. A.12 Expansion of causal bonds into bilateral signals.

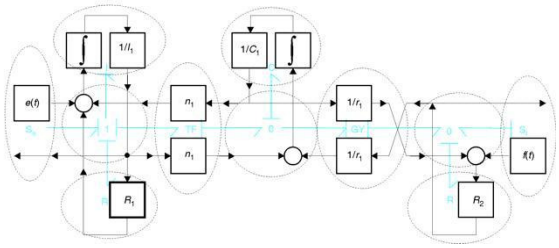


Fig. A.13 Expansion of the nodes into operational blocks.

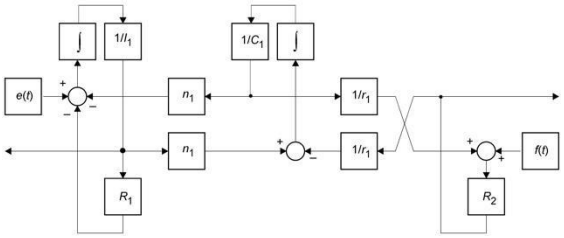


Fig. A.14 Addition of signs to the summations.

point. However, its topology is not common due to the location of the conjugate signals. This may be omitted in the next step.

4. (optional) Redraw the block diagram in such a way that the inputs are at the left-hand side and the outputs (observed variables) are at the right-hand side (Fig. A.15) with the integrators in the forward path. The block diagram may be manipulated according to the standard rules for block diagrams as to obtain a canonical form.

The procedure to obtain a signal flow graph is completely analogous to the above procedure as all operations represented by blocks, including the signs of the summations, are combined as much as possible and then written next to an edge, while all summations become nodes, as signal nodes can be distinguished from signal summation points by observing the signal directions (signal node has only one input, summation has only one output).

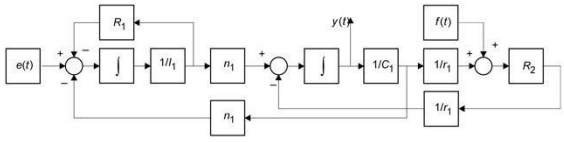


Fig. A.15 Conversion into conventional form.

A.1.4 Generation of a set of mixed algebraic and differential equations

An arbitrary bond graph with n bonds contains $2n$ conjugate power variables, $2n$ ports and $2n$ corresponding port relations (constitutive relations). If a bond graph is made causal, the order in which the causal strokes are assigned to the bonds can uniquely label the bonds and their corresponding efforts and flows by using the sequence numbers of this process as indices. Next the constitutive relation of each port is written in the form that corresponds to the assigned causality. This results in a mixed set of $2n$ algebraic and first-order differential equations in an assignment statement form. Note that the differential equations that belong to storage ports in preferred integral causality have a time derivative at the left-hand side of the assignment statements, indicating a 'postponed' integration, if it were. During numerical simulation, this integration is performed by the *numerical integration* method to allow for the next model evaluation step.

The switched junctions have the same causal port properties as the regular junctions, but no acausal form of the constitutive relations exists, while it necessarily contains 'if-then-else' statements that can only be written after causality has been assigned.

The algebraic relations can be used to eliminate all the variables that do not represent the state of a storage port or an input variable, thus resulting in a set of ordinary differential equations (ODE) if all storage ports have preferred causality or in a set of differential and algebraic equations (DAE) if there are dependent storage ports.

If the elimination of the algebraic relations is done by hand, the following three intermediate steps are advised:

1. Eliminate all variables that are dependent on the identities of junctions (0, 1) and sources ((M)Se, (M)Sf);
2. Eliminate all variables that are related by the algebraic port relations of all the ports that are not junction ports and not source ports ((M)R(S), (M)TF, (M)GY);
3. Eliminate all variables that are related by the port relations of all the ports that are junction summations (0, 1);
4. If an algebraic loop is present (active arbitrary causality) choose a variable in this loop to write an implicit algebraic relation and solve symbolically if possible. Otherwise the use of an implicit numerical method is required;
5. If present and possible, eliminate a differentiated *state variable* at the right-hand side of the relations symbolically if possible. Otherwise the use of an implicit numerical method is required.

For example, the bond graph in Fig. A.11 contains 10 bonds, 20 equations and 20 variables of which two are state variables, such that 18 variables have to be eliminated. There are 9 identities (2 source and $3+2+2=7$ junction ports), 6 multiplications (2×2 transducer $+2$ R) and 3 summations (3 junctions) resulting in the 18 necessary algebraic relations. The final result, assuming linearity of the elements, is

$$\frac{\mathrm{d}f_3}{\mathrm{d}t} = \frac{1}{I_1}\left(e(t) - R_1 f_3 - n_1 e_7\right) = -\frac{R_1}{I_1} f_3 - \frac{n_1}{I_1} e_7 + \frac{1}{I_1} e(t)$$
$$\frac{\mathrm{d}e_7}{\mathrm{d}t} = \frac{1}{C_1}\left(n_1 f_3 - \frac{R_2}{r_1}\left(\frac{1}{r_1} e_7 - f(t)\right)\right) = \frac{n_1}{C_1} f_3 - \frac{R_2}{C_1 r_1^2} e_7 + \frac{R_2}{C_1 r_1} f(t)$$

or in matrix form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} f_3 \\ e_7 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{I_1} & -\frac{n_1}{I_1} \\ \frac{n_1}{C_1} & -\frac{R_2}{C_1 r_1^2} \end{bmatrix}\begin{bmatrix} f_3 \\ e_7 \end{bmatrix} + \begin{bmatrix} \frac{1}{I_1} \\ \frac{R_2}{C_1 r_1} \end{bmatrix}\begin{bmatrix} e(t) \\ f(t) \end{bmatrix} \tag{A.1}$$

A.2 Linear analysis

A.2.1 Introduction

Even though it may support a frequently encountered, persistent misapprehension that the port-based approach and its bond graph representation require the restriction that all constitutive relations of the nodes of a bond graph should be linear, this assumption will be made in the next section, but only to show the link between bond graphs and commonly used linear analysis techniques in system dynamics. However, it should be strongly emphasized that much of the linear analysis directly applied to the bond graph representation can be qualitatively generalized to the nonlinear case, as it still provides an insight. At the least, it gives an impression about small-signal behavior near an operating point of a nonlinear system model.

Direct application of the wide range of linear analysis techniques on a bond graph should serve a purpose in the sense that it provides some additional information. If this is not the case, there is no need to change from a conventional model representation already obtained, like a set of linear state equations, into a bond graph representation.

If all constitutive relations of the nodes of a bond graph are assumed to be linear, the bond graph represents a linear system model and each elementary node other than the junctions (and the unit gyrator called symplectic gyrator) can be characterized by one parameter (C, I, R, TF, GY) or input(signal) ((M)Se, (M)Sf). In case of (external) modulation, the linear system model becomes time-variant (MR, MTF, MGY). Note that internal modulation causes nonlinearity and cannot occur in the linear case.

Table A.2 Impedance and admittance formulations of 1-port elements and corresponding gains.

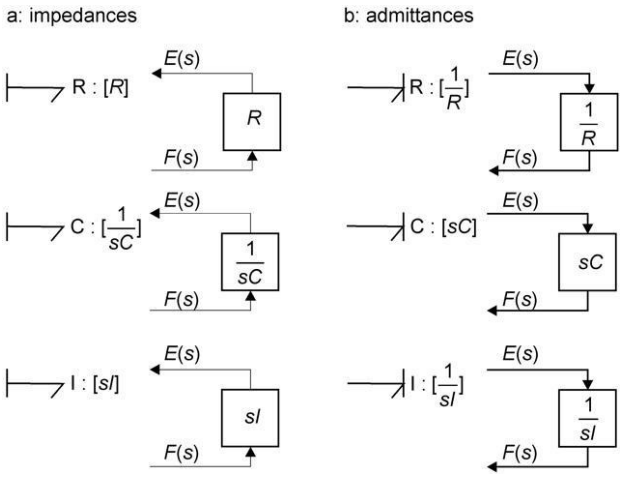
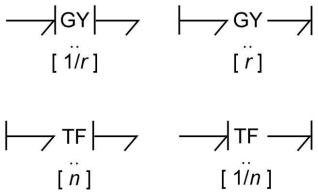


Table A.3 Gains 2-port elements in various causal forms.



Given that a causal, linear bond graph is equivalent with any other linear system representation, it can be used to support all kinds of linear analysis. The conversion of a bond graph into a block diagram, a signal flow graph or a set of differential equations was already discussed. This makes clear that any linear analysis technique that exists for these kinds of models formulations can be directly applied to a causal, linear bond graph as well. In particular transmission matrices and Mason's loop rule can be used to derive transfer functions between a chosen input and a chosen output in case of tree and chain structures [36]. As the identification of signal loops takes place in a bond graph via the causal paths, there is an immediate connection during modeling between the properties of a transfer function and the physical properties. The advantage of applying these techniques directly to the bond graph is that the relation of certain aspects of the linear analysis or the transfer function in particular with the physical structure can be directly observed and used to create or to adapt to desired behavior. This not only supports modeling decisions, but also allows insight in how physical changes can be made to obtain a required transfer.

In particular, an impedance analysis will be discussed, as it provides a means to directly generate port equivalent compositions and decompositions. For linear analysis purposes, it is often useful to write the gain related to a node directly in the bond

Table A.4 Composition rules for junctions and 1-ports.

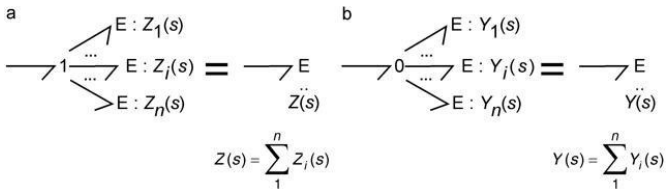
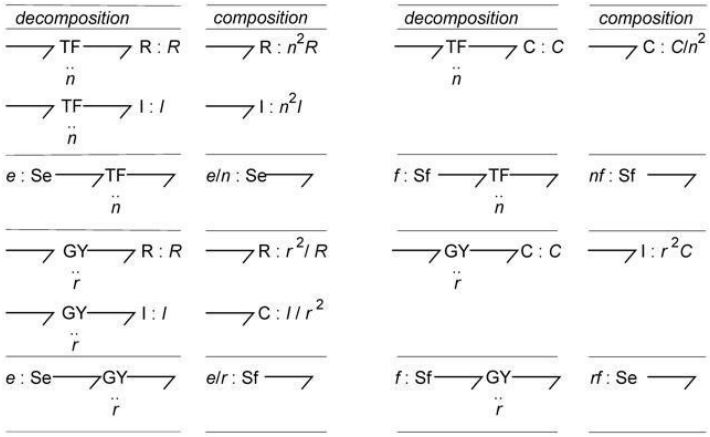


Table A.5 Composition rules for 2-ports and 1-ports.

Table A.6 (De-)composition rules involving transducers.



graph. In order to distinguish this notation from the regular notation of characteristic parameters (:) or generating functions (::) the gains, in which differentiations and integrations are replaced by the Laplace operator s and $1/s$ respectively, are placed between square brackets ([]).

A.2.2 Impedance analysis using bond graphs

A port of an element in effort-out causality can be characterized by an impedance, while a port with flow-out causality is best described by an admittance.

Tables A.2 and A.3 provide listings of the possible gains that characterize the basic elements, both in impedance and in admittance form. Table A.4 illustrates that (de-)composition operations involving a 1-junction are best performed in impedance

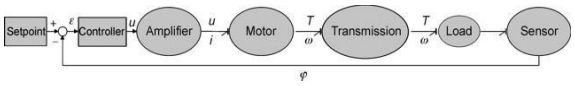


Fig. A.16 Word bond graph of a simple servo system.

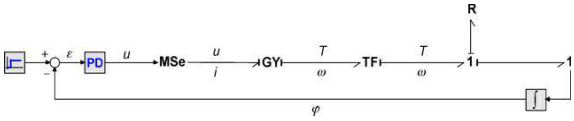


Fig. A.17 Bond graph of the dominant behaviors.

form, while (de-)composition operations involving a 0-junction are best performed in admittance form as this leads to simple summation operations. Tables A.5 and A.6 list some elementary (de-)composition rules and the results for basic elements respectively.

A.3 Port-based modeling and simulation of dynamic behavior of physical systems in terms of bond graphs: a simple example

The model structure of a simple servo system is generated in order to give an impression of the port-based approach and the feedback on modeling decisions provided by the causal analysis. First a word bond graph is drawn at the component level, combined with a block diagram representation of setpoint, controller and closed loop (Fig. A.16). This gives an impression of the important domains and the corresponding variables of interest. Next the components are replaced by the nodes of a bond graph that represent the dominant behavior of each of the components (Fig. A.17). The causality shows that, apart from the dynamics of the controller and the integration in the position sensor the drive system model has no dynamics: the imposed voltage directly determines the servo speed (1^{st} order system). In order to add some dynamic behavior, the resistance and inductance of the motor circuit, the friction and inertia of the rotor are added as well as the inertia of the load (Fig. A.18). The causality not only shows that the rigid connection between the rotor inertia and the load inertia makes them dependent, but also that the inductance of the motor circuit forms a second order loop (causal path) with the mechanical inertia (rotor & load) via the gyrator (3^{rd} order system). Fig. A.19 demonstrates that modeling the torsion of the drive system resolves the dependency between rotor and load inertia, but creates a new second order loop (5^{th} order system). Fig. A.20 shows that changing from a voltage control of the motor to current control not only suppresses its electrical time constant, but also the second order loop between inertia and inductance via the gyrator.

Figures A.16 through A.20 are all screen dumps of models that can directly be simulated when relevant parameters are selected. The information of all these steps supports the modeling process, depending on the problem context.

The introduction to bond graph concepts and notation in previous sections has already discussed many links between the representation and the modeling process. It

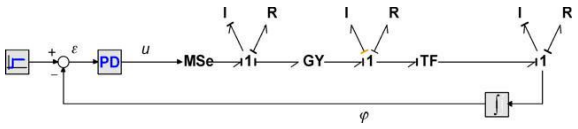


Fig. A.18 Bond graph of the dominant behaviors and some important dynamics.

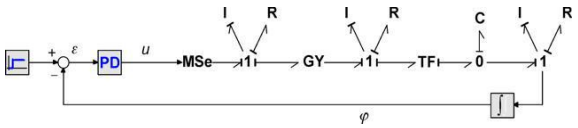


Fig. A.19 Addition of the torsion of the drive system resolves the dependency between rotor and load inertia.

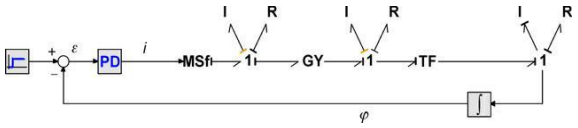


Fig. A.20 Current-control of the motor suppresses the electrical time constant as well as the second order loop via the GY.

cannot be sufficiently emphasized that modeling is a decision process that is different each time, but which can be supported by looking for conceptual structure based on universal principles as well as a direct link with computational issues, which provides direct feedback on modeling decisions. The bond graph notation supports this due to its domain independence and its ability to represent conceptual and computational information simultaneously. A true understanding of these features is only obtained by sufficient practicing.

Appendix B
Mathematical Background

B.1 Linear algebra and differential geometry

This appendix presents a short intuitive overview of some of the mathematical concepts used in this book. More detailed, precise, and extended treatments can be found in many textbooks, such as [111, 208] for linear algebra and vector spaces, [41, 68, 186] for differential geometry and manifolds, and [175, 187] for Lie groups.

B.1.1 Duality in vector spaces

We start with two basic definitions of mappings and some possible properties. These properties are illustrated in the figures B.1a-B.1d.

Definition B.1 (mappings). A mapping f between two sets A and B associates exactly one element of B to each element of A. We denote it abstractly as $f : A \to B$, and its action on an element $a \in A$ as $f(a) \mapsto b$ with $b \in B$. The set A is called the domain of f, and the set B its co-domain. The set of all $b \in B$ such that there exists an $a \in A$ with $f(a) \mapsto b$ is called the range of f.

Definition B.2 (surjective, injective, bijective). A mapping $f : A \to B$ is surjective (or *onto*), if its range is equal to its co-domain. It is injective (or *one-to-one*) if for every b in its range, there is exactly one $a \in A$ such that $f(a) \mapsto b$. A mapping is bijective (or *one-to-one and onto*) if it is injective and surjective.

In addition, a *diffeomorphism* is a mapping between $\mathbb{R}^n$ and $\mathbb{R}^n$ that is injective, continuously differentiable, and has a continuously differentiable inverse. Some examples of different types of mappings $f : \mathbb{R} \to \mathbb{R}$ are the functions $f(x) \mapsto x^2$ (not surjective, not injective), $f(x) \mapsto \tan(x)$ (surjective, not injective), $f(x) \mapsto e^x$ (not surjective, injective), and $f(x) \mapsto x^3$ (bijective).

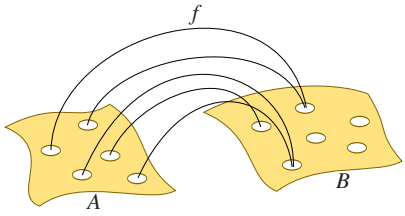


Fig. B.1a Examples of mappings $f : A \to B$: not surjective and not injective.

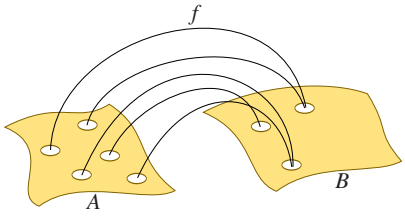


Fig. B.1b Examples of mappings $f : A \to B$: surjective and not injective.

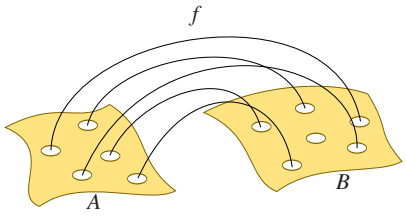


Fig. B.1c Examples of mappings $f : A \to B$: not surjective and injective.

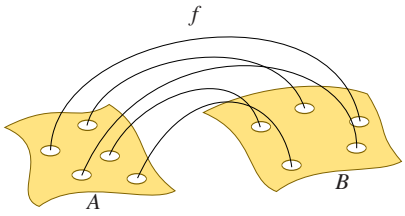


Fig. B.1d Examples of mappings $f : A \to B$: surjective and injective.

Definition B.3 (vector space). A real vector space V is a set of elements (called vectors), one element called the identity (or zero-vector $\overrightarrow{0}$), and two operations $\oplus$ (addition of two vectors) and $\cdot$ (multiplication of a vector by a scalar), such that

- for all two elements $v_1, v_2 \in V$, also $v_1 \oplus v_2 \in V$;
- for all elements $v \in V$ and $x \in \mathbb{R}$, also $x \cdot v \in V$;
- for all elements $v \in V$, there exists a $v^{-1} \in V$ such that $v \oplus v^{-1} = \overrightarrow{0}$;

and such that the following properties hold for all $v_1, v_2, v_3 \in V$ and $x_1, x_2 \in \mathbb{R}$.

$$
\begin{aligned}
v_1 \oplus \overrightarrow{0} &= v_1 & (v_1 \oplus v_2) \oplus v_3 &= v_1 \oplus (v_2 \oplus v_3) \\
1 \cdot v_1 &= v_1 & (x_1 + x_2) \cdot v_1 &= (x_1 \cdot v_1) \oplus (x_2 \cdot v_1) \\
x_1 \cdot (x_2 \cdot v_1) &= (x_1 x_2) \cdot v_1 & x_1 \cdot (v_1 \oplus v_2) &= (x_1 \cdot v_1) \oplus (x_1 \cdot v_2)
\end{aligned}
$$

where $x_1 + x_2$ and $x_1 x_2$ are standard addition and multiplication of real numbers.

The abstract definition of a vector space includes many different spaces with a linear structure. Not only an obvious example like the space of all velocity vectors of a point mass is a vector space, but, for example, also the space of all 2×3 matrices, if we take element-wise addition as the $\oplus$ operator and the zero-matrix as the identity element.

Since a vector space is closed under addition and scalar multiplication, we can search for the smallest number of elements $e_i \in V$ such that any element of V can be constructed by addition and scalar multiplication of the elements e_i. If we can find $n < \infty$ elements e_i that accomplish this, then the vector space is said to be n-dimensional, and the e_i elements are called a basis of the vector space. The dimension n of a vector space is unique, but the choice of basis e_i is not. By definition, we

can express any element $v \in V$ as a linear combination of the basis elements, i.e. as

$$v = \sum_{i=1}^{n} v^i e_i = v^1 e_1 + v^2 e_2 + \ldots + v^n e_n \tag{B.1}$$

the n numbers $v^i \in \mathbb{R}$ can serve as coordinates for V, as they define a bijective mapping between $\mathbb{R}^n$ and V.

Definition B.4 (dual vector space). The dual space V^* of a vector space V is the space of all linear mappings (called co-vectors) from V to $\mathbb{R}$, i.e. all mappings $f : V \to \mathbb{R}$ such that for all $v_i \in V$ and $x_i \in \mathbb{R}$.

$$f\left((x_1 \cdot v_1) \oplus \ldots \oplus (x_k \cdot v_k)\right) = x_1 f(v_1) + \ldots + x_k f(v_k) \tag{B.2}$$

Definition B.5 (dual product). The dual product is the natural pairing of an element $v \in V$ and an element $f \in V^*$ as $\langle f \mid v \rangle := f(v) \in \mathbb{R}$

If we choose a basis e_i for V, we can express any element $v \in V$ as (B.1), and hence from (B.2) we see that the mapping of v by an element $f \in V^*$ can be written as

$$\langle f \mid v \rangle = f(v) = f\left(\sum_{i=1}^{n} v^i e_i\right) = \sum_{i=1}^{n} v^i f(e_i) \tag{B.3}$$

i.e. as a linear combination of the mappings of the basis elements. This shows that a dual element f is fully defined by how it maps the basis elements, and, since each element $f(e_i)$ is a single real number, it shows that the dimension of V^* is equal to the dimension of V, i.e. the number of basis elements e_i. It also suggests a basis for the dual space V^*, which we denote by e^j and is defined by the condition

$$\left\langle e^j \mid e_i \right\rangle = e^j(e_i) = \delta_i^j := \begin{cases} 1 & \text{when } i = j \\ 0 & \text{when } i \neq j \end{cases} \tag{B.4}$$

where δ_j^i is the Kronecker delta. Any $f \in V^*$ can then be written as a linear combination of the basis elements e^i

$$f = \sum_{j=1}^{n} f_j e^j = f_1 e^1 + f_2 e^2 + \ldots + f_n e^n \tag{B.5}$$

with the numbers $f_j \in \mathbb{R}$ again defining coordinates for V^* in the basis e^j. With these choices of bases, computing the dual product (B.3) becomes

$$\langle f \mid v \rangle = \left(\sum_{j=1}^{n} f_j e^j\right)\left(\sum_{i=1}^{n} v^i e_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} f_j v^i e^j(e_i) = \sum_{i=1}^{n} f_i v^i \tag{B.6}$$

so, for these choices of bases, computing the dual product of a vector and a co-vector simply means summing the pair-wise products of their coordinates.

Vector spaces and their duals define an interesting mathematical structure, but they can also be used to represent a physical structure, for example as follows. Consider a robotic mechanism with n joints. The space of velocities $\dot{q}$ (at a point q) forms a vector space V, and we can choose a basis for example as $e_i = \dot{q}^i$, i.e. each basis element describes the unit-velocity of one joint, and zero velocity of the other joints. This vector space V automatically induces a dual space V^* of abstract linear operators mapping a velocity to a real number. We can just ignore this dual space, but we can also think of it as the space of all collocated joint torques, i.e. the n-dimensional space with elements τ and basis elements $e^i = \tau_i$. From the structure of the vector space and its dual, we can pair elements as

$$\langle \tau \mid \dot{q} \rangle = \sum_{i=1}^{n} \tau_i \dot{q}^i \tag{B.7}$$

such that applying τ to $\dot{q}$ produces a real number. The reason for choosing this interpretation of a vector space and its dual becomes clear when we interpret also this real number: the dual product represents the mechanical power flowing into the system when it is moving with velocity $\dot{q}$ and with applied torques τ.

Associating the abstract mathematical concept of (dual) vector spaces to the practical physical concept of collocated power variables (force and velocity) can help reasoning about the physical concepts. The mathematical structure constrains computations to make sense. For example, computing the power as (B.7) only makes sense when τ and $\dot{q}$ are collocated, which is equivalent to V and V^* having dual bases as defined in (B.4). In this way, keeping the mathematical structure between physical variables in mind can help to avoid mistakes.

Definition B.6 (tensor). Given a vector space V and its dual V^*, a tensor T is a mapping of the form

$$T : \underbrace{V^* \times \cdots \times V^*}_{p \text{ times}} \times \underbrace{V \times \cdots \times V}_{q \text{ times}} \to \mathbb{R} \tag{B.8}$$

that is linear in all its arguments. The tensor T is said to have order $p+q$, order p contra-variant and order q co-variant, and is called a type (p,q) tensor.

Tensors are linear operators that map vectors and co-vectors to $\mathbb{R}$, and as such, are generalizations of the concepts of vectors and co-vectors. In fact, a co-vector is a type $(0,1)$ tensor, since it maps a vector (an element of one copy of V) to $\mathbb{R}$. Similarly, a vector is a type $(1,0)$ tensor, since it maps a co-vector (an element of one copy of V^*) to $\mathbb{R}$. Both mappings are defined by the dual product.

Basis elements and the corresponding coordinates for tensors can be constructed from coordinates for vectors and co-vectors, simply by taking the appropriate coordinates for each of the arguments. The $(i_1, \ldots, i_p, j_1, \ldots, j_q)$th coordinate of a type (p,q) tensor T, i.e. the result of applying T to the basis vectors $e^1, \ldots, e^p$ and

co-vectors $e_1, \ldots, e_q$, is denoted by $T^{i_1,\ldots,i_p}_{j_1,\ldots,j_q}$. This convention of writing the contra-variant indices as superscripts and the co-variant indices as subscripts can be useful to quickly assess the type of tensor from its representation in coordinates.

A metric tensor, often denoted by g, is a symmetric positive-definite type $(0,2)$ tensor. It is symmetric in the sense that $g(v,w) = g(w,v)$ for any two vectors v,w, and positive-definite in the sense that $g(v,v) > 0$ for all vectors v except the zero-vector (in which case it returns zero by linearity of tensors). With a metric-tensor, we define the inner product between two vectors v and w as

$$\langle v,w\rangle := g(v,w) = g(w,v) = \sum_{i,j} g_{ij} v^i w^j \tag{B.9}$$

the length of a vector v as

$$\|v\| := \sqrt{\langle v,v\rangle} = \sqrt{g(v,v)} = \sqrt{\sum_{i,j} g_{ij} v^i v^j} \tag{B.10}$$

and the cosine of the angle between two nonzero vectors v and w as

$$\cos(\angle(v,w)) = \frac{\langle v,w\rangle}{\|v\|\,\|w\|} \tag{B.11}$$

When $\langle v,w\rangle = 0$, the vectors v and w are said to be orthogonal in the metric g.

We can again relate the mathematical concept of a metric to physical variables. In this case, we take e.g. again the space of velocities $\dot{q}$ as the vector space V, and now the mass matrix M as a metric tensor, since it is indeed symmetric and positive definite. Then, when we apply the tensor M to two copies of $\dot{q}$, that is, we multiply the matrix with the vectors as $\dot{q}^{\mathrm{T}} M \dot{q}$, we obtain a number that represents the physical quantity of twice the kinetic co-energy associated with the velocity $\dot{q}$.

We can also define a new tensor by applying the metric tensor only to one copy of V. The resulting tensor maps one tangent vector to $\mathbb{R}$, and is hence a tensor of type $(0,1)$ – a co-vector. If we again take the physical example of a robot with velocity $\dot{q}$ and mass matrix M, the new tensor is $M\dot{q}$ – the generalized momentum (co) vector. Hence, we have seen two interpretations of the dual vector space: one as the space of forces, and one as the space of generalized momenta.

The distinction between the different types of tensors allow to assess what operations between them are possible. For example, a metric tensor is an operator mapping two vectors to $\mathbb{R}$, and hence it does not make mathematical sense to apply them to co-vectors, even though the coordinates of a metric tensor (represented by an $n \times n$ matrix) can be multiplied by the coordinates of a co-vector (represented by an n-dimensional column vector).

B.1.2 Manifolds

The configuration space of a system is generally represented by an abstract space that is not directly equal to $\mathbb{R}^n$ for some suitable n. For example, Sect. 3.2 shows how the configuration of a rigid body is described by an element of $SE(3)$, and how using $\mathbb{R}^6$ (six numbers, e.g. including Euler angles) leads to singularities and other numerical problems that are not present in physics.

Differential geometry is a field of mathematics that makes exact the global properties of a configuration space such as $SE(3)$, while still allowing to do computations locally using real numbers. In this section, we give a very brief intuitive overview of the idea of differential geometry and how it can help to use some concepts from this field. The central concept in differential geometry is the concept of a manifold. We can think intuitively of a manifold as some kind of abstract space (such as $SE(3)$) that locally looks like $\mathbb{R}^n$. More precise, if we take a point in the abstract space, then the space around this point can locally be described by coordinates in an open subset of $\mathbb{R}^n$. An example of such a manifold is the surface of the earth, which globally is (more or less) a sphere, and locally (at each point) can be described by coordinates in $\mathbb{R}^2$, i.e. a flat chart. These charts, unfortunately, are not global, due to the topology of the sphere. Once it becomes clear that a space is a manifold, i.e. once we have found enough local coordinate charts to $\mathbb{R}^n$ to cover the whole space, we can define global objects on the manifold (such as functions) by defining them first locally for each chart, and then checking certain compatibility conditions between the charts. These compatibility conditions ensure, for example, that a certain point in the abstract space, with two different coordinates in two different local charts, still has the same function value.

A manifold is called differentiable, if the mappings that change coordinates between different charts are diffeomorphisms. At each point p of a differentiable manifold $\mathscr{M}$, the space of all tangent vectors is called the tangent space, denoted $T_p\mathscr{M}$. This space is a linear vector space of dimension n, and describes all possible directions around p. The union of the tangent spaces over all points of $\mathscr{M}$ is called the tangent bundle, denoted $T\mathscr{M}$. An element of the tangent bundle consists of a point $p \in \mathscr{M}$ plus a vector in $T_p\mathscr{M}$: the tangent bundle is hence $2n$ dimensional. The fact that the tangent space is a vector space allows to generalize the linear algebra concepts from Sect. B.1.1 to the setting of differential geometry. Since the tangent vector space at every point p has a dual, denoted by $T_p^*\mathscr{M}$, we can define the co-tangent or dual tangent bundle $T^*\mathscr{M}$ as the union of all dual tangent spaces. The concept of a tensor can be generalized to a *tensor field*, which is an object, defined on the manifold, that at each point p maps copies of the tangent space $T_p\mathscr{M}$ and dual tangent space $T_p^*\mathscr{M}$ to a real number, and that varies smoothly over $\mathscr{M}$. Note that tensor fields only operate on vectors and co-vectors that are elements of tangent and co-tangent space at the same point p.

An example of a tensor field is a vector field, which is a tensor field of type $(1,0)$ that assigns to each point of the manifold a tangent vector. Fig. B.2a shows an example. It also shows how, from a vector field, we can define its integral curves as the curves with velocity vector at all points equal to the value of the vector field

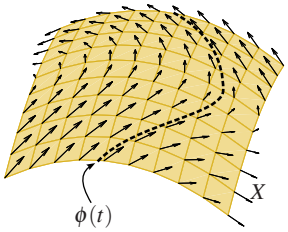


Fig. B.2a Vector field X and one of its integral curves $\phi(t)$.

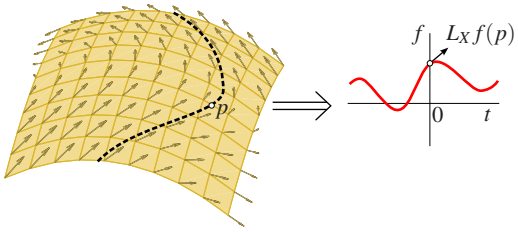


Fig. B.2b A function f and its Lie derivative along X. The curve $\phi(t)$ is parameterized by t with $\phi(0) = p$.

at those points. Integral curves can be interpreted as the trajectories of a particle flowing along the vector field.

Given a function $f : \mathscr{M} \to \mathbb{R}$ of the points of the manifold, this function is obviously also defined for the points of the integral curves. If the function is differentiable, we define its Lie derivative along the vector field X at a point p as

$$(L_X f)(p) := \frac{\mathrm{d}}{\mathrm{d}t}\left(f(\phi(t))\right)\Big|_{t=0} \tag{B.12}$$

where $\phi(t)$ is an integral curve of X with $\phi(0) = p$. It can be shown that this expression is independent of the choice of integral curve. An example of a Lie derivative is shown in Fig. B.2b. From two vector fields X and Y, we can also define a third vector field Z as the unique vector field such that for all functions f

$$L_Z f = L_{[X,Y]} f = L_X(L_Y f) - L_Y(L_X f) \tag{B.13}$$

This new vector field is called the Lie bracket of the two vector fields. It roughly represents the velocity when moving a little along X, then a little along Y, then a little along $-X$, and finally a little along $-Y$.

Another example of a tensor field is a metric tensor field, which assigns to each point of the manifold a metric tensor, i.e. a symmetric positive definite type $(0,2)$ tensor. Such a tensor defines the metric concepts (dot-product, length, and angle) for tangent vectors at all points of the manifold.

B.1.3 Geometric structures

A *geometric structure* on a manifold is, loosely speaking, an object (structure) that is defined on all points of the manifold and that does not depend on the chosen coordinate systems for the manifold. The word 'geometric' refers to the fact that you can usually picture these structures as geometric entities (subspaces, planes, lines, etcetera) attached in some way to the manifold. Examples of geometric structures were already given in the previous section. For example, the tangent bundle is a geo-

metric structure, as it is defined independently from the coordinates on the manifold as the space of all vectors tangent to the manifold. Similarly, a certain vector field on a manifold is a geometric structure, as the elements of the field attach certain 'arrows' to all points of the manifold. Of course, the *parameterization* of geometric structures does depend on the coordinates, in the sense that the elements that belong to the structure are expressed in different coordinates for different coordinate systems on the manifold, but these expressions are such that the elements of the structure themselves are invariant under coordinate changes. For example, the coordinates of elements of the tangent bundle may change under coordinate changes, but the geometrical 'arrows' that they represent remain the same.

Another example of a geometric structure on a manifold is a *vector bundle*, which assigns to each point of the manifold a vector space (not necessarily the tangent space), in such a way that the vector spaces at different points have the same structure, such that the vector spaces at different points can be mapped to each other by isomorphisms. A direct example of a vector bundle is the tangent bundle on a manifold, but another example is the Dirac structure, as defined in Definition 2.2.

Finally, the *Whitney sum* of two vector bundles B_1 and B_2 is defined as the vector bundle $B_1 \oplus B_2$ with vector space at each point equal to the direct sum of the two vector spaces of B_1 and B_2 at that point. In other words, if two vector bundles B_i assign to each point of the manifold a vector space V_i, then the Whitney sum $B_1 \oplus B_2$ assigns to each point of the manifold a vector space $V_1 \oplus V_2$, the direct sum of V_1 and V_2.

B.1.4 Lie groups and algebras

Definition B.7 (group). A group G is a set S together with a binary operator

$$\bullet : S \times S \to S$$

and an element $I \in S$, such that for all $s_1, s_2, s_3 \in S$ we have

$$\text{identity element:} \qquad s_1 \bullet I = I \bullet s_1 = s_1 \tag{B.14}$$

$$\text{associativity:} \qquad (s_1 \bullet s_2) \bullet s_3 = s_1 \bullet (s_2 \bullet s_3) \tag{B.15}$$

$$\text{inverse element:} \qquad \exists\, s_1^{-1} \in S \text{ such that } s_1 \bullet s_1^{-1} = s_1^{-1} \bullet s_1 = I \tag{B.16}$$

If also $s_1 \bullet s_2 = s_2 \bullet s_1$ (commutativity property), the group is called abelian.

A simple example of a group is the set $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ together with the standard addition operator and identity element 0. It can be checked that this is indeed a group: adding zero to any element of the set indeed gives that same element, addition is associative, and for each element, the inverse is simply the negation of that element. Since summation is even commutative, this group is also abelian.

Definition B.8 (Lie group). A Lie group $\mathscr{G}$ is a manifold that is also a group, i.e. it has a binary operator $\bullet : \mathscr{G} \times \mathscr{G} \to \mathscr{G}$ and an identity element $I \in \mathscr{G}$ that satisfy the group properties.

A Lie group is basically a group with a differentiable structure, which allows to talk about curves, velocities, tangent spaces, etcetera. We discuss a few examples of Lie groups that are useful for robotics, i.e. examples that describe positions and orientations in space. Since these groups are generally abstract, we also discuss matrix representations, i.e. sets of matrices with certain properties, which, together with the usual matrix multiplication as binary operator and the identity matrix as identity element, can be related one-to-one to abstract elements of the group. The matrix representations can be used in numerical computations as a type of singularity-free (though redundant) set of coordinates.

Example B.1 (Translation). The group of all translations in n dimensions is denoted by $T(n)$, e.g. the group of translations in three dimensions is denoted by $T(3)$. Clearly these groups can be directly identified with $\mathbb{R}^n$, and so a matrix representation of $T(n)$ would be the space of n-dimensional column-vectors p_n, together with vector addition as the the binary operator, and the zero vector as the identity element. Another possible representation is as the set of all $(n+1 \times n+1)$ dimensional matrices structured as

$$\begin{bmatrix} I_n & p_n \\ 0 & 1 \end{bmatrix} \tag{B.17}$$

with p_n the translation vector, together with matrix multiplication as binary operator and the identity matrix as identity element. This method looks cumbersome and the matrix representation is highly redundant, but it proves useful when translations are combined with rotations.

Example B.2 (Fixed-axis rotation). The space of rotations around a fixed axis forms a group under the binary operator of combining rotations by performing one rotation after the other. The group is denoted by $SO(2)$ and can be identified with the circle $\mathscr{S}^1$. The group is one-dimensional, and in practice, it is often described by a single real number (the angle of rotation). This, however, neglects the fact that a full 360° rotation does not change the group element, although it does change the angle; this is the difference between a circle and a straight line. Instead, the group of rotations can be described by the set of special orthogonal 2×2 matrices (whence the name $SO(2)$), meaning 2×2 orthogonal matrices with determinant $+1$. These matrices have the form

$$R = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \tag{B.18}$$

where ϕ is the angle of rotation. Together with matrix multiplication as the binary operator, and the identity matrix ($\phi = 0$) as the identity element, these matrices form a complete representation of the group of fixed-axis rotations.

Example B.3 (Spatial rotation). The space of free rotations around any axis in three dimensions forms a group, and is denoted by $SO(3)$. The group is three-dimensional, and is often represented locally by three angles, called the Euler angles, that describe three consecutive rotations around three (local) axes. Such a parameterization, however, has singularities, which results in non-smooth behavior of the coordinates around singularities. Instead, rotations can be fully and uniquely identified with the set of all special orthogonal 3×3 matrices (whence the name $SO(3)$), meaning 3×3 orthogonal matrices with determinant $+1$.

Another representation of the group of spatial rotations is by unit quaternions. In this representation, a vector of the form

$$q = \left[\cos(\tfrac{\theta}{2})\ n_1 \sin(\tfrac{\theta}{2})\ n_2 \sin(\tfrac{\theta}{2})\ n_3 \sin(\tfrac{\theta}{2})\right] \tag{B.19}$$

is used to describe rotation around an axis $n = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix}$ with angle θ. The axis n is constrained to have unit norm (in the Euclidean sense), which means that also the vector q has unit norm. This representation is singularity free, but it doubly covers $SO(3)$, since the rotation angles α and $\alpha + 360^\circ$ (for some α) define the same rotation, but are represented by different vectors q.

Example B.4 (Planar motion). We can combine the group of two-dimensional translations, i.e. translations in a plane, with the group of fixed-axis rotations and take the fixed axis to be orthogonal (in the Euclidean sense) to the translational plane. The resulting object is again a Lie group, and it describes all planar motions, that is, the set of all possible ways that an object can be positioned in a plane. This group is called the special Euclidean group of dimension two, denoted $SE(2)$.

As representation of this group, we can simple use a combination of a two-dimensional vector p to describe translation and a matrix R of the form (B.18) for the translation. This choice is often made in literature, but computations in this representation are cumbersome, since two consecutive motions need to be combined as

$$R_{13} = R_{23} R_{12} \qquad\qquad p_{13} = p_{23} + R_{23} p_{12} \tag{B.20}$$

which leads to long and tedious equations for multiple consecutive motions. Instead, we combine translation and rotation in one so-called homogeneous matrix of the form

$$H = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \tag{B.21}$$

where R is the rotation matrix (B.18). Consecutive planar motions can now be represented by simple matrix multiplications of the corresponding homogeneous matrices. The matrix representation of a translation as (B.17) is a special case of (B.21) for zero rotation, $R = I$.

Example B.5 (Three-dimensional motion). Similar to the planar situation, we can combine the group of translations in three dimensions $T(3)$ with the group of free-

axis rotations $SO(3)$. The result is the special Euclidean group in three dimensions, or, $SE(3)$, that describes the space of all possible relative positions and orientations in three-dimensional space. It can also be represented by a matrix of the form (B.21) but now with R a three-dimensional rotation matrix, and p a three-dimensional translation vector. Again, consecutive motions are simply represented by matrix multiplication of the appropriate homogeneous matrices.

The examples show how many useful transformations are actually Lie groups, and that these can be represented globally and without singularities by matrices with the appropriate properties. The realization that these transformations are Lie groups allows to perform certain useful operations on them. First, because of the group structure, we can take an element of the group and combine it with another element of the group. This is called (left or right) translation, since effectively it transports one element of the group to another place, by means of group multiplication. In particular, since every element of a group has an inverse, we can transport a group element to the identity of the group. This transport can be done in two ways, either by pre- or post-multiplication with the inverse. Secondly, since a Lie group has a differentiable manifold structure, we can talk about continuous and differentiable curves in the group, which represent smooth consecutive transformations, i.e. smooth motions of an object. The derivatives of these curves represent the (angular, linear, or combined) velocities of the moving objects.

Combining these two aspects (the group aspect and the manifold aspect), we can transport a curve $\gamma(t)$ near an element $A \in \mathcal{G}$ to a curve near the identity by applying A^{-1} to every element of the curve. We can then take the derivative of the transformed curve to obtain an element of the tangent space $T_I\mathcal{G}$ at the identity. Depending on whether left or right translation is chosen, different velocity vectors are obtained. Since in this way, velocity vectors at any point $A \in \mathcal{G}$ can be transported to the tangent space at the identity, this tangent space provides a common vector space which allows to compare and add different velocities. Furthermore, the tangent space at the identity can be given the structure of a Lie algebra, which is defined as follows.

Definition B.9 (Lie algebra). A Lie algebra is a vector space together with a binary operator $[,] : V \times V \to V$ (called Lie bracket or commutator), that satisfies the following properties for all $v_1, v_2, v_3 \in V$ and $a_1, a_2 \in \mathbb{R}$

$$\text{bilinearity:} \quad \begin{cases} [a_1 v_1 + a_2 v_2, v_3] = a_1[v_1, v_3] + a_2[v_2, v_3] \\ [v_1, a_1 v_2 + a_2 v_3] = a_1[v_1, v_2] + a_2[v_1, v_3] \end{cases} \tag{B.22}$$

$$\text{skew-symmetry:} \quad [v_1, v_2] = -[v_2, v_1] \tag{B.23}$$

$$\text{Jacobi's identity:} \quad [v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0 \tag{B.24}$$

An example of a Lie algebra is the vector space V of all $n \times n$ matrices with the Lie bracket defined as $[A, B] := AB - BA$ for $A, B \in V$.

In the case of a Lie group, the tangent space at the identity can be given the structue of a Lie group by defining the appropriate Lie bracket on it, and this tangent space is thus usually called the Lie algebra of the group, and is denoted by $\mathfrak{g} := T_I\mathcal{G}$.

Finally, as shown in Sect. 3.2, the tangent vectors at the identity of a group can have a clear physical interpretation, much more than tangent vectors at general points $A \in \mathscr{G}$.

B.2 Legendre transforms and co-energy

A concise overview of homogeneous functions and Legendre transformations is given and applied to energy functions, thus leading to the concept of co-energy. Various properties of energy and co-energy related to storage are shortly discussed and related to a physical interpretation. Finally some domain specific forms of co-energy are discussed.

B.2.1 Homogeneous functions and Euler's theorem

A function $F(x)$, with $x = x_1, \ldots, x_k$, is homogeneous of order (or degree) n if $F(\alpha x) = \alpha^n F(x)$. Define $y_i(x) = \frac{\partial F}{\partial x_i}$, then $y_i(x)$ or

$$y_i(\alpha x) = \frac{\partial F(\alpha x)}{\partial \alpha x_i} = \frac{\alpha^n}{\alpha} \frac{\partial F(x)}{\partial x_i} = \alpha^{n-1} y_i(\underline{x}) \tag{B.25}$$

is homogeneous of order $(n-1)$. For a homogeneous function, Euler's theorem holds:

$$\sum i = 1^k \frac{\partial F}{\partial x_i} x_i = nF(x) \quad \text{or} \quad F(x) = \frac{1}{n} \sum_{i=1}^{k} y_i x_i = \frac{1}{n} y^{\mathrm{T}} \cdot x \tag{B.26}$$

with $y = y_1, \ldots, y_K$. By definition:

$$\mathrm{d}F = \sum_{i=1}^{k} \frac{\partial F}{\partial x_i} \cdot \mathrm{d}x_i = \sum_{i=1}^{k} y_i \mathrm{d}x_i = y^{\mathrm{T}} \cdot \mathrm{d}x \tag{B.27}$$

but also

$$\mathrm{d}F = \mathrm{d}\left(\frac{1}{n} y^{\mathrm{T}} \cdot x \right) = \frac{1}{n} y^{\mathrm{T}} \cdot \mathrm{d}x \tag{B.28}$$

Hence:

$$\mathrm{d}y^{\mathrm{T}} \cdot x = (n-1) y^{\mathrm{T}} \cdot \mathrm{d}x \;\rightarrow\; \begin{cases} \text{for } n = 1: & \mathrm{d}y^{\mathrm{T}} \cdot x = 0 \\ \text{for } n \neq 1: & \mathrm{d}F = \dfrac{1}{n-1} \mathrm{d}y^{\mathrm{T}} \cdot x \end{cases} \tag{B.29}$$

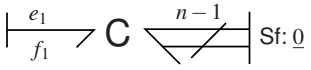


Fig. B.3 Bond graph representation of a storage element with $n-1$ ports constrained.

B.2.2 Homogeneous energy functions

The *energy* of a system with k *state variables* q_i is $E(q) = E(q_1, \ldots, q_k)$. If q_i is an *extensive* state variable, this means that $E(\alpha q) = \alpha E(q) = \alpha^1 E(q)$. Hence $E(q)$ is *first order* (i.e., $n = 1$) homogeneous, so $e_i(q) = \frac{\partial E}{\partial q_i}$ is *zero-th* order (i.e., $n - 1 = 0$) homogeneous, which means that $e_i(q)$ is an *intensive* variable, i.e. $e_i(\alpha q) = \alpha^0 e_i(q) = e_i(q)$. This also means that in case $n = 1$ and $k = 1$, $e(q)$ is constant, i.e. $\frac{\partial e}{\partial q} = \frac{\mathrm{d}e}{\mathrm{d}q} = 0$, which changes the behavior of this element into that of a *source*. In order to enable storage, storage elements = multiports $(k > 1)$. Hence, the common '1-port storage element' = n-port storage element with flows of $n-1$ ports kept zero, i.e. the corresponding $n-1$ states remain constant and *are not recognized as states*. Such a state is often considered a parameter: if $E(q_1, q_2, \ldots, q_n)|\mathrm{d}q_i = 0\ \forall i \neq 1 = E'(q_1)$, then $E'(q_i)$ is not necessarily first-order homogeneous in q_1. This can be represented in bond graph form as in Fig. B.3.

For $n = 1$ and k *independent extensities* there are only $k-1$ *independent intensities*, because for $n = 1$ we find a generalized *Gibbs' fundamental relation*:

$$E(q) = e^{\mathrm{T}} \cdot q$$

By definition, already $\mathrm{d}E = e^{\mathrm{T}} \cdot \mathrm{d}q$, and combining these equations results in the generalized *Gibbs-Duhem relation*:

$$\mathrm{d}e^{\mathrm{T}} \cdot q = 0 \tag{B.30}$$

B.2.3 Legendre transform

A *Legendre transform* of a homogeneous function $F(x)$ with respect to x_i is defined as

$$L\{F(x)\}_{x_i} = L_{x_i} = F(x) - y_i x_i \tag{B.31}$$

where $y_i = \frac{\partial F}{\partial x_i}$. Moreover, the *total Legendre transform* of $F(x)$ is

$$L\{F(x\} = L = F(x) - \sum_{i=1}^{k} y_i x_i \tag{B.32}$$

Note that for $n = 1$, $L = 0$. Now

$$\mathrm{d}L_{x_i} = \mathrm{d}F - \mathrm{d}(y_i x_i) = \mathrm{d}F - y_i \mathrm{d}x_i - \mathrm{d}_i \mathrm{d}y_i = \sum_{j \neq i} y_j \mathrm{d}x_j - \mathrm{d}_i \mathrm{d}y_i \tag{B.33}$$

or $L_{x_i} = L_{x_i}(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_k)$, which means that x_i is replaced by y_i as *independent* variable or 'coordinate'! Hence $L = L(y)$ and $\mathrm{d}L = -\sum_{i=1}^{k} x_i \mathrm{d}y_i = -\mathrm{d}y^{\mathrm{T}} \cdot x$.

B.2.4 Co-energy function

The *co-energy* $E^*_{q_i}$ of $E(q)$ with respect to q_i is by definition:

$$E^*_{q_i} = -L_{q_i} = E^*_{q_i}(q_1, \ldots, q_{i-1}, e_i, q_{i+1}, \ldots, q_k) \tag{B.34}$$

Hence $E(q) + E^*_{q_i}(\ldots, e_i, \ldots) = e_i q_i$. The *total co-energy* $E^*(e)$ of $E(e)$ is $E^* = -L$, hence $E(q) + E^*(e) = e^{\mathrm{T}} \cdot q$. For $n = 1$, we have that

$$E^*(e) = 0$$

thus confirming the earlier conclusion that there are only $k - 1$ independent e_i. For $n = 2$,

$$E(q) = E^*(e) = \frac{1}{2} e^{\mathrm{T}} \cdot q$$

and for $n = 3$,

$$E(q) = \frac{1}{3} e^{\mathrm{T}} \cdot q \qquad\qquad E^*(e) = \frac{2}{3} e^{\mathrm{T}} \cdot q$$

B.2.5 Relations for co-energy functions

$$\mathrm{d}E^*_{q_i} = \mathrm{d}e_i q_i - \sum_{j \neq 1} e_j \mathrm{d}q_j \tag{B.35}$$

$$\mathrm{d}E^* = \sum_{i=1}^{k} \mathrm{d}e_i q_i = \mathrm{d}e^{\mathrm{T}} \cdot q = (n-1) e^{\mathrm{T}} \cdot \mathrm{d}q = (n-1)\mathrm{d}E \tag{B.36}$$

$$E^* = (n-1)E = \frac{n-1}{n} e^{\mathrm{T}} \cdot q = \left(1 - \frac{1}{n}\right) e^{\mathrm{T}} \cdot q \tag{B.37}$$

Table B.1 Extensities q_i and intensities $e_i(q)$, with internal energy U.

extensities q_i	**intensities** $e_i(q) = \frac{\partial E}{\partial q_i}$
entropy S	temperature $T = \frac{\partial U}{\partial S}$
volume V	pressure $p = -\frac{\partial U}{\partial V}$
total mole number N	total material potential $\mu_{\text{tot}} = \frac{\partial U}{\partial N}$
mole number per i-th species N_i	chemical potential $\mu_i = \frac{\partial U}{\partial N_i}$

B.2.6 Legendre transforms in simple thermodynamics

In thermodynamics, the Legendre transforms appear in the following relations. The meaning of the involved quantities is explained in Table B.1.

$$\begin{aligned}
\text{Free energy } F\text{:}\quad & F = L_S = U - TS = -pV + \sum_{i=1}^{m-1} \mu_i N_i + \mu^{\text{tot}} \cdot N \\
& \mathrm{d}F = -S\mathrm{d}T - p\mathrm{d}V + \sum_{i=1}^{m-1} \mu_i \mathrm{d}N_i + \mu^{\text{tot}} \cdot \mathrm{d}N \\
& F(T,V,N,N_i) \quad (= N_i f(T,v,c)) \\
\text{Enthalpy } H\text{:}\quad & H = L_V = U - (-pV) = U + pV \\
& \mathrm{d}H = T\mathrm{d}S + V\mathrm{d}p + \sum_{i=1}^{m-1} \mu_i \cdot \mathrm{d}N_i + \mu^{\text{tot}} \cdot \mathrm{d}N \\
& H(S,p,N,N_i) \quad (= N_i h(s,p,c)) \\
\text{Gibbs free enthalpy } G\text{:}\quad & G = L_{S,V} = U - TS - (-pV) = \mu^{\text{tot}} \cdot N + \sum_{i=1}^{m-1} \mu_i N_i \\
& \mathrm{d}G = -S\mathrm{d}T + V\mathrm{d}p + \sum_{i=1}^{m-1} \mu_i \mathrm{d}N_i + \mu^{\text{tot}} \cdot \mathrm{d}N \\
& G(T,p,N,N_i) \quad (= N_i g(T,p,c)) \\
\text{For } m = 1\text{:}\quad & g = \mu^{\text{tot}}(T,p)
\end{aligned}$$

B.2.7 Legendre transforms and causality

If an effort is *forced* on a port of a C-element (i.e. with *derivative causality* or *flow causality*), this means that the roles of e and q are interchanged in the set of independent variables, which means that the energy has to be *Legendre transformed* in order to continue to serve as a generating function for the constitutive relations.

Fig. B.4a Bond graph associated to $\mathrm{d}F = u\mathrm{d}q$.

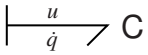


Fig. B.4b Bond graph associated to $P = \frac{\mathrm{d}F}{\mathrm{d}t}$

Such a transformation is particularly useful when the *effort e* is *constant* (e.g. an electrical capacitor in an isothermal environment with $T = T_{\mathrm{const}}$):

- $\mathrm{d}F = u\mathrm{d}q - S\mathrm{d}T = u\mathrm{d}q$, or as the bond graph of Fig. B.4a;
- $P = u \cdot \dot{q} = \frac{\mathrm{d}F}{\mathrm{d}t}$, or as the bond graph of Fig. B.4b.

B.2.8 Constitutive relations

The function $e_i(q)$ is called *constitutive relation*, also called *constitutive equation*. *constitutive law*, *state equation*, *characteristic equation*, etcetera. If $e_i(q)$ is *linear*, i.e. first order homogeneous, then $E(q)$ is second order homogeneous, i.e. $E(q)$ is *quadratic*. In this case, and only in this case:

$$E(\alpha q) = \alpha^2 E(q)$$
$$E(q) = \frac{1}{2} e^{\mathrm{T}} \cdot q$$
$$\mathrm{d}e^{\mathrm{T}} \cdot q = e^{\mathrm{T}} \cdot \mathrm{d}q = \mathrm{d}E$$

B.2.9 Maxwell reciprocity

From the principles of energy conservation (*first law*) can be derived that

$$\frac{\partial^2 E}{\partial q_i \partial q_j} = \frac{\partial^2 E}{\partial q_j \partial q_i}$$
$$\frac{\partial e_j}{\partial q_i} = \frac{\partial e_i}{\partial q_j}$$

i.e. the *Jacobian* matrix of the constitutive relation is *symmetric*. This is called *Maxwell reciprocity* or *Maxwell symmetry*.

B.2.10 Intrinsic stability

Intrinsic stability requires that this *Jacobian* is *positive-definite*

$$\det\left(\frac{\partial e}{\partial q}\right) > 0$$

and that the *diagonal elements* of the Jacobian are *positive*

$$\frac{\partial e_i}{\partial q_i} > 0 \qquad \forall\, i$$

B.2.11 Legendre transforms in mechanics

In mechanical systems with kinetic energy T, potential energy V, displacements x, momenta p, velocities v, and forces F, the following holds:

$$\begin{aligned}
\text{Hamiltonian } H\text{:} \quad & E(q) = H(x,p) = T + V \\
\text{Lagrangian } L\text{:} \quad & H_p^* = -L_p = v^{\mathrm{T}} \cdot p - H \\
& = (T + T^*) - (T + V) = T^* - V = L(x,v) \\
& \text{with } v = \frac{\partial H}{\partial p} \\
\text{co-Hamiltonian:} \quad & H_{x,p}^* = v^{\mathrm{T}} \cdot p + F^{\mathrm{T}} \cdot q - H \\
& = (T + T^*) + (V + V^*) - (T + V) \\
& = T^* + V^* = H^*(F,v) \\
\text{co-Lagrangian or Hertzian:} \quad & H_x^* = F^{\mathrm{T}} \cdot q - H = (V + V^*) - (T + V) \\
& = V^* - T = -L^*(F,p) \\
& \text{with } F = \frac{\partial H}{\partial x}
\end{aligned}$$

B.2.12 Legendre transforms in electrical circuits

In electrical circuits with capacitor charger q, voltages u, coil flux linkages Φ, and currents i, we have

$$\begin{aligned}
E(q,\Phi) &= E_C(q) + E_L(\Phi) \\
E^*(u,i) &= u^{\mathrm{T}} \cdot q + i^{\mathrm{T}} \cdot \Phi - E
\end{aligned}$$

Only in the linear case, it follows that $E^* = E$.

Appendix C
Nomenclature and Symbols appearing in Sect. 3.4

Abstract This appendix summarizes the notation (i.e nomenclature, symbols and vector/tensor operators) adopted in Sect. 3.4.

Nomenclature

A	surface area or affinity
a	activity defined with respect to the state of pure component
$(AB)^{\mp}$	activated complex
A, B	chemical symbol
C	molar concentration
c	specific heat capacity
$\mathbf{d}$	driving force for diffusion
D	diffusion coefficient
$\mathbf{e}$	effort variable for diffusion
E	activation energy of a chemical reaction
$\mathbf{f}$	flux of scalar quantity per unit of surface area
F	total flux of a scalar quantity
G, g	total and specific Gibbs free energy
$\mathbf{g}$	external body force per unit of mass
H, h	total and specific enthalpy
$\hbar$, $\hbar'$	total energy per mass unit for a thermodynamic system
$\bar{h}$	Planck constant
$\mathbf{I}$	unit tensor of order 2
$\bar{k}$	Boltzmann constant
k	chemical rate constant
K	equilibrium constant

l	distance
M, $\bar{M}$	total and molar mass
N	number of moles or number of components in a mixture
$\mathbf{p}$	momentum per mass unit
P	pressure
Pr	chemical symbol of a product
r, $\tilde{r}$	rate of a chemical reaction by unit of volume and its linearized counterpart
R	number of independent chemical reactions and ideal gas constant
Re	chemical symbol of a reactant
r	radial position
S, s	total and specific entropy
T	absolute temperature
Tr	trace of a tensor
t	time
U, u	total and specific internal energy
V, v	total and specific volume
$\mathbf{v}$, v	velocity and its modulus
W, w	total or work per unit of volume
Y, y	total and specific scalar quantity
$\mathbf{y}$	specific vectorial quantity

For a total quantity Y, the specific quantity per mass unit is y, and the specific quantity per mole unit is $\bar{y}$. $\Delta_r Y$ is the variation of Y associated to a chemical reaction.

Greek symbols

α	heat transfer coefficient or conductance by unit of surface area
θ_1, θ_2	parameters in an equation of state
φ	equation of state
κ	constant for a linearized rate equation or volume viscosity
Φ	tensor of the momentum flux
τ	shear stress tensor
σ	source term per unit of volume
η	viscosity
μ	chemical potential
ρ	density or mass concentration
χ	extent of a chemical reaction or molar fraction
ν	stoechiometric coefficient
γ	activity coefficient defined with respect to the state of pure component
λ, λ_0	heat conductivity and a parameter in the heat conductivity expression

ω mass fraction
Ψ potential energy per mass unit
Δ difference
δ small quantity
Σ total source term

Subscripts

ext surrounding
f forward
$\hbar$, $\hbar'$ total energy
i, j indexes for components
k index for chemical reaction
l index for pipes
m mixtures
P constant pressure
q heat or thermal energy
p momentum
r reverse
ref origin or reference state
s entropy
u internal energy
v constant volume
w non-thermal energy
y scalar quantity
$\mp$ activated state

Superscripts

a anti-symmetric
b boundary
e excess
eq equilibrium
f forward
k index for chemical reactions
R flux expressed with respect to a moving frame
$Reac$ due to a chemical reaction
Rev reversibly
r reverse

s	symmetric
Sc	scalar
T	thermal diffusion coefficient (Soret effect)
$Tens$	tensorial
$Vect$	vectorial
$*$	pure component in the same state then a mixture
id	ideal solution
ig	ideal gas
0	standard state

Vectors and tensors notation

We give here only the notations of the main quantities that are manipulated in Sect. 3.4 defined in a cartesian system of coordinates (x_1, x_2, x_3). For a more complete presentation of this topic, see for example [5, 21] and Sect. 4.2.1 (and the references therein) for their geometric, coordinate-free interpretation in terms of differential forms.

The gradient of a scalar quantity y, i.e. ∇y, is a vector of which the coordinates are

$$\nabla y = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \frac{\partial y}{\partial x_3} \end{bmatrix}$$

By extension, one can also define the gradient of a vector $\mathbf{y} = [y_1\ y_2\ y_3]^{\mathrm{T}}$, i.e. $\nabla \mathbf{y}$, by the second order tensor:

$$\nabla \mathbf{y} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \frac{\partial y_3}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_3}{\partial x_2} \\ \frac{\partial y_1}{\partial x_3} & \frac{\partial y_2}{\partial x_3} & \frac{\partial y_3}{\partial x_3} \end{bmatrix}$$

The divergence of a vector $\mathbf{y}$ is a scalar quantity:

$$\nabla \cdot \mathbf{y} = \frac{\partial y_1}{\partial x_1} + \frac{\partial y_2}{\partial x_2} + \frac{\partial y_3}{\partial x_3}$$

An element of τ, the viscous part of the momentum flux (or shear stress tensor) is defined as follows: τ_{ij} is the force for unit of surface area exerted in the direction i on a plane perpendicular to direction j. The divergence of τ is given by:

$$\nabla \cdot \tau = \begin{bmatrix} \frac{\partial \tau_{11}}{\partial x_1} & \frac{\partial \tau_{21}}{\partial x_2} & \frac{\partial \tau_{31}}{\partial x_3} \\ \frac{\partial \tau_{12}}{\partial x_1} & \frac{\partial \tau_{22}}{\partial x_2} & \frac{\partial \tau_{32}}{\partial x_3} \\ \frac{\partial \tau_{13}}{\partial x_1} & \frac{\partial \tau_{23}}{\partial x_2} & \frac{\partial \tau_{33}}{\partial x_3} \end{bmatrix}$$

The convected momentum **vp** is given by the following tensor:

$$\mathbf{vp} = \begin{bmatrix} v_1 p_1 & v_2 p_1 & v_3 p_1 \\ v_1 p_2 & v_2 p_2 & v_3 p_2 \\ v_1 p_3 & v_2 p_3 & v_3 p_3 \end{bmatrix}$$

The flux of work per unit of surface are done by the tensor τ is given by:

$$\tau \cdot \mathbf{v} = \begin{bmatrix} \tau_{11} v_1 & \tau_{21} v_1 & \tau_{31} v_1 \\ \tau_{12} v_2 & \tau_{22} v_2 & \tau_{32} v_2 \\ \tau_{13} v_3 & \tau_{23} v_3 & \tau_{33} v_3 \end{bmatrix}$$

The power dissipation per unit of volume due to viscous forces is given by:

$$\tau : \mathbf{v} = \sum_{i=1}^{3} \sum_{j=1}^{3} \tau_{ji} \frac{\partial v_i}{\partial x_j}$$

Author's Biographies

Carles Batlle Arnau was born in Vilanova i la Geltrú (Catalonia) in 1961. He graduated in Physics in 1984 from Barcelona University, and received the Ph.D. degree in Theoretical Physics in 1988 from the same institution, under the supervision of Professor Joaquim Gomis. He was a post-doc fellow at the Physics Department of Princeton University from september 1988 to august 1989, and entered the Technical University of Catalonia (UPC) in 1989 as a lecturer. Since 1993 he is an Associate Professor with the Department of Applied Mathematics IV of the UPC. He has taught several first and second year math courses for engineering students, as well as courses for math and graduate students. His current research interests include energy-based modeling and control of power electronics and electromechanical systems, fuel cells and open parameter distributed systems.

Peter Breedveld is an associate professor with tenure at the University of Twente, Netherlands, where he received a B.Sc. in 1976, an M.Sc. in 1979 and a Ph.D. in 1984. He has been a visiting professor at the University of Texas at Austin in 1985 and at the Massachusetts Institute of Technology in 1992-1993. He is or has been an industrial consultant. He initiated the development of the modeling and simulation tool that is now commercially available under the name 20-sim. In 1990 he received a Ford Research grant for his work in the area of physical system modeling and the design of computer aids for this purpose. He is an associate editor of the *Journal of the Franklin Institute*, SCS *Simulation* and *Mathematical and Computer Modeling of Dynamical Systems*. His scientific interests are: Integrated modeling, control and design of physical systems; graphical model representations (bond graphs); generalized thermodynamics; computer-aided modeling, simulation, analysis and design; dynamics of spatial mechanisms; mechatronics; generalized networks; numerical methods; applied fluid mechanics; applied electromagnetism; qualitative physics;

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Herman Bruyninckx received the M.Math. degree and the M.Sc. degrees in computer science engineering and in mechatronics in 1984, 1987, and 1988, respectively, and the Ph.D. degree in mechanical engineering in 1995, all from the Katholieke Universiteit (K.U.) Leuven, Leuven, Belgium. His thesis was on "Kinematic Models for Robot Compliant Motion with Identification of Uncertainties." He is currently a Professor at the Department of Mechanical Engineering, K.U. Leuven, and held Visiting Research positions at the Grasp Laboratory, University of Pennsylvania, Philadelphia, the Robotics Laboratory of Stanford University, and theKungl TekniskaHogskolan in Stockholm, in 1996, 1999, and 2002, respectively. In 2001, he started the Free Software project Orocos, to support his research interest, and to facilitate industrial exploitation. Since 2007, he has been a Coordinator of the European Robotics Research Network EURON. His current research interests include online estimation in sensor-based robot tasks, kinematics of robots, geometric foundations of robotics, compliant robot motion, force control, active sensing, and motion planning.

Francoise Couenne was graduated as engineer in automatic control from the Institut Polytechnique de Grenoble (France) in 1983. She received in 1986 her Ph.D. degree in automatic control from the same institute. She has a CNRS (french National Center for Scientific Research) Research Associate position and is presently based at LAGEP-UMR CNRS 5007 (Laboratory of Automatic Control and Chemical Engineering) at University of Lyon. Her current research deals with modelling, network modelling using bond graphs, port-Hamiltonian systems, model reduction and control of distributed parameter systems for physical chemical processes.

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Enric Fossas Colet graduated in Mathematics in 1981 and received the Ph.D. degree in Mathematics in 1986, both from Universitat de Barcelona. In 1986 he joined the Department of Applied Mathematics, Universitat Politècnica de Catalunya. In 1999 he moved to the Institute of Industrial and Control Engineering (IOC), Universitat Politècnica de Catalunya and to the Department of Automatic Control & Computer Engineering, where he is presently a full professor. He served as head of department from 1993 to 1999 and as director of the IOC from 2003 to 2009. His research interests include nonlinear control (theory and applications), and in particular variable structure systems, with applications to electronic power converters.

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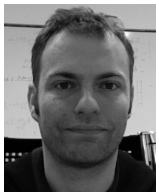
Laurent Lefèvre was born in Leuven, Belgium, the 3rd of November 1970. He got his B.Sc. in engineering in 1990 from Université Libre de Bruxelles (Ecole Polytechnique de Bruxelles) and his M.Sc. in applied mathematics in 1993 from Université Catholique de Louvain (Faculté de Sciences Appliquées), both in Belgium. Then he went to Lille (France) where he got his Ph.D. in automatic control in 1999 from Ecole Centrale de Lille and Université des Sciences et Techniques de Lille. Laurent Lefèvre worked then as associate researcher within the Centre for systems engineering and applied mechanics (CESAME, Université de Louvain) in 1999 and then joined the Laboratoire d'Automatique de Grenoble (now Gipsa Lab, from Grenoble Institute of Technology). He is now with the Laboratoire d'Automatique et de Génie des Procédés (LAGEP, Université Claude Bernard - Lyon I and CNRS) since 2003. His current research interests concern distributed parameter systems modelling, reduction and control, specially structured approaches based on physical modelling assumptions. His current research topics are port-Hamiltonian formulations, Lattice-Boltzmann methods, geometric reduction schemes, passivity based approach and energy shaping techniques for control, shallow water systems, chemical reactors, diffusion, adsorption and extrusion processes.



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