Modelling Rigid and Flexible Bodies with Truss Elements

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Abstract. The truss element, due to its simplicity, can fulfill the need to model multibody systems in a way that reduces the size of the problems or improves the efficiency of calculations. The truss element can be used to model rigid and flexible bodies as well as several joints with a single truss element or with aggregates built up from a number of truss elements. With an extended mass description, planar binary rigid links or links that can undergo a uniform dilatation with pin joints can be modelled by a single truss element. Planar ternary elements can likewise be modelled by three truss elements. In three dimensions, a rigid body can be modelled by six truss elements along the edges of a tetrahedron, but also three truss elements can be combined to form a triangular membrane element or six truss elements to form a constant-strain finite solid element. Applications to two benchmark problems and a Delta robot are given.

1 Introduction

There is still a need to model multibody systems in a simple way, so that the size of the problems can be made as small as possible and the efficiency of calculations is high. The truss element, due to its ultimate simplicity, is a good candidate to contribute to alleviate this need. The truss element is indeed very versatile and can be used in much wider classes of systems than just trusses; also general rigid and flexible bodies can be modelled, as well as several joints, with truss elements only, although some additional elements are useful in special cases.

In this chapter, an extended mass description for a planar truss element will be described, which makes it possible to model binary rigid links with pin joints by a single truss element. This element can also be used if the body can undergo uniform dilatation as in auxetic bodies with a Poisson ratio close to -1. Furthermore, it is shown how rigid bodies and their joints can be modelled in two and three dimensions. The modelling techniques are illustrated by the application to the two benchmark problems from the handbook [1] and to a Delta robot.

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2 Planar Truss Element with Extended Mass Description

The well-known planar truss element, whose configuration is described by the Cartesian coordinates of its two end-points and has its extension or compression as its only deformation, has been introduced in multibody system dynamics by van der Werff [2]. Its nodal coordinates are the four Cartesian coordinates of its two end-points p and q, see Fig. 1(a), with position vectors $\boldsymbol{r}_p = [x_p, y_p]^{\mathrm{T}}$ and $\boldsymbol{r}_q = [x_q, y_q]^{\mathrm{T}}$, respectively, and hence its current length is given by $l = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2}$. The generalized strain is $\varepsilon = l - l_0$, where l_0 is the reference length. As the configuration of a rigid body attached to a planar truss element is fully determined by the four nodal coordinates, a mass matrix for the truss element which represents the complete inertia properties of the body modelled by this element can be derived. The same mass description applies to the case in which the body can undergo a uniform dilatation as in an auxetic metamaterial with a Poisson ratio close to −1 [3].

A point of the body can be identified by the dimensionless coordinates in a local coordinate frame, which has its origin in the point p, whose x-axis points towards point q and whose y-axis is perpendicular to this axis if it is rotated in the positive direction by a right angle. The local dimensionless coordinates are $\xi = x/l$ and $\eta = y/l$. A point on the body is then located at

$$\boldsymbol{r} = \begin{bmatrix} (1-\xi)x_p + \eta y_p + \xi x_q - \eta y_q \\ -\eta x_p + (1-\xi) y_p + \eta x_q + \xi y_q \end{bmatrix} \tag{1}$$

and its time derivative is

$$\dot{\boldsymbol{x}} = \begin{bmatrix} (1-\xi)\dot{x}_p + \eta \dot{y}_p + \xi \dot{x}_q - \eta \dot{y}_q \\ -\eta \dot{x}_p + (1-\xi) \dot{y}_p + \eta \dot{x}_q + \xi \dot{y}_q \end{bmatrix}. \tag{2}$$

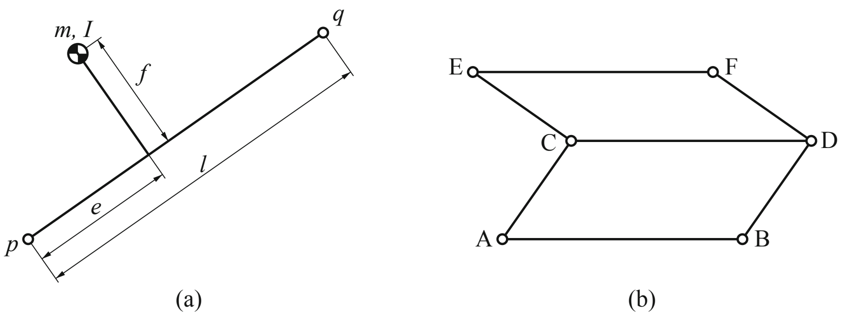

Fig. 1. Truss element with arbitrary mass distribution (a) and model of parallel joint (b)

With the mass of the body, m, the location of the centre of mass at $\xi = e/l$, $\eta = f/l$ and the moment of inertia with respect to the centre of mass I, the mass matrix associated with the nodal coordinates $[x_p, y_p, x_q, y_q]^{\mathrm{T}}$ is obtained as

$$\begin{aligned} \boldsymbol{M} &= \int \begin{bmatrix} 1-\xi & -\eta \\ \eta & 1-\xi \\ \xi & \eta \\ -\eta & \xi \end{bmatrix} \begin{bmatrix} 1-\xi & \eta & \xi & -\eta \\ -\eta & 1-\xi & \eta & \xi \end{bmatrix} \rho l^2 \mathrm{d}\xi \mathrm{d}\eta \\ &= \begin{bmatrix} \dfrac{I_q}{l^2} & 0 & -\dfrac{I_{\mathrm{red}}}{l^2} & -m\dfrac{f}{l} \\ 0 & \dfrac{I_q}{l^2} & m\dfrac{f}{l} & -\dfrac{I_{\mathrm{red}}}{l^2} \\ -\dfrac{I_{\mathrm{red}}}{l^2} & m\dfrac{f}{l} & \dfrac{I_p}{l^2} & 0 \\ -m\dfrac{f}{l} & -\dfrac{I_{\mathrm{red}}}{l^2} & 0 & \dfrac{I_p}{l^2} \end{bmatrix}, \end{aligned} \tag{3}$$

where ρ is the mass per unit of area, $I_p = I + me^2 + mf^2$ is the moment of inertia with respect to the point p, $I_q = I + m(l-e)^2 + mf^2$ is the moment of inertia with respect to the point q and $I_{\mathrm{red}} = I - me(l-e) + mf^2$ is a reduced moment of inertia. This mass matrix is constant, as m is a constant, e and f are proportional to l, and the moments of inertia are proportional to l^2. The mass matrix is also invariant under rotations of the coordinate system. This is the same mass matrix as is given for a planar body described with natural coordinates [4]. A general planar force system acting on the element in the case the body is rigid can be replaced by equivalent forces on the two nodal points. For a constant moment, these vary with the orientation.

A ternary joint connected by pin joints can simply be modelled by three truss elements in a triangle. The several planar joints can be obtained as follows. The planar revolute joint, or pin joint, can be trivially realized by connecting two elements. A fixed relative orientation can be realized by parallelograms, see Fig. 1(b), and a prismatic joint can be realized by a straight-line mechanism. It can be more convenient to model the last-mentioned joint by other elements, however.

Planar elastic bodies can be modelled by triangular finite elements modelled by three truss elements along their sides, which define constant strains within the element. Indeed, the Lagrangian strains along the directions of the sides are $[(\varepsilon_i + l_{i0})^2/l_{i0}^2 - 1]/2$, $(i = 1, 2, 3)$, from which the three independent component of the Green–Lagrange strain tensor can be calculated. The dual stress components from a constitutive relation can be transformed to generalized stresses for the element in a similar way, where for the sides that are common to two elements, the two contributions of the adjacent elements have to be added. This way of modelling has the advantage that elements share generalized strains and stresses, so these only have to be calculated and stored once. Higher-order elements can be obtained by combining several triangular elements, although the transformations become more complicated. In other respects, the procedure is similar to standard finite element formulations.

3 Spatial Systems

A spatial truss element can be defined in a way analogous to the planar case, with two nodal points with three Cartesian coordinates each, so $\boldsymbol{r}_p = [x_p, y_p, z_p]^{\mathrm{T}}$ and $\boldsymbol{r}_q = [x_q, y_q, z_q]^{\mathrm{T}}$; the current length is given by $l = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2 + (z_q - z_p)^2}$. The generalized strain is again the elongation of the element, $\varepsilon = l - l_0$. As a single truss element does not determine the orientation of a rigid body in space, more elements are needed. A minimal set would consist of three truss elements arranged in a triangle, but a more convenient model consists of six truss elements arranged along the edges of a tetrahedron. The mass distribution of a general rigid body can be replaced by four lumped masses at the vertices of a tetrahedron, which still results in a constant mass matrix. A further freedom arises if distributed masses for the truss elements are allowed, which still leads to a constant mass matrix, although it is no longer a diagonal matrix. With arbitrarily chosen vertices of the tetrahedron, a general mass distribution of a rigid body can be described by assigning appropriate values to the four lumped masses at the vertices and the six uniformly distributed masses, although some masses can be negative. The negative masses have no influence on the dynamics, because the resulting system mass matrix is still positive definite. The distribution of the masses can be obtained as follows. With homogeneous coordinates $[L_1, L_2, L_3, L_4]^{\mathrm{T}}$, $L_1 + L_2 + L_3 + L_4 = 1$, the position of a point in the body can be written as

$$\boldsymbol{r} = [L_1\boldsymbol{I}_3 \; L_2\boldsymbol{I}_3 \; L_3\boldsymbol{I}_3 \; L_4\boldsymbol{I}_3] \begin{bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{r}_3 \\ \boldsymbol{r}_4 \end{bmatrix}, \tag{4}$$

where $\boldsymbol{I}_3$ is the 3×3 identity matrix and $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$ and $\boldsymbol{r}_4$ are the position vectors of the vertices of the tetrahedron. The resulting mass matrix is a 4×4 block matrix with diagonal blocks given by the integrals

$$\boldsymbol{I}_3 \int L_i L_j \rho \mathrm{d}V = m_{ij}\boldsymbol{I}_3, \tag{5}$$

where ρ is the mass per unit of volume and $\mathrm{d}V$ is the volume element over which the integration takes place. Now the distributed masses of the truss elements connected between the nodes i and j are $6m_{ij}$, and the lumped masses are the diagonal elements minus one-third of the sum of the three distributed masses of the elements connected to the corresponding vertices; that is, the lumped mass at node i is $m_{ii} - 2\sum_{j \neq i} m_{ij}$. For a tetrahedron with a uniformly distributed mass m, we have $m/10$ on the diagonal and $m/20$ for the other elements of m_{ij}, which means that the distributed masses of the truss elements are all $3m/10$ and the lumped masses at the nodes are $-m/5$.

There are now six generalized strains for the body, so a constant strain can be represented. The aggregate of six truss elements for a rigid body has the same number of independent nodal coordinates as a beam, but the mass matrix is a constant.

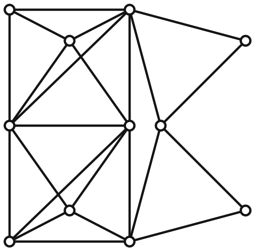

Fig. 2. Sarrus mechanism as a spatial prismatic joint, where the top can move in the vertical direction with respect to the bottom; the mechanism consists of two identical parts at an angle, in this a right angle

It is possible to combine three truss elements in a triangular membrane element, which can describe the three uniform deformations of the membrane. The mass matrix is still constant if all mass is concentrated in the plane of the membrane. The tetrahedra in space, as the triangles in the plane, can be used to construct constant-strain finite elements for general flexible bodies.

Several lower-pair joints can be modelled by truss elements only, although not all and some are complicated to realize. A spherical joint is realized by connecting two elements in a node. A revolute joint can be obtained by letting two tetrahedra have a common edge. A prismatic joint can be realized with a Sarrus mechanism, as shown in Fig. 2. A cylindrical joint is a combination of a revolute and a prismatic joint. Also a planar joint can be built up from three elementary joints. The screw pair cannot be modelled.

4 Example Problems

The usefulness of the truss element is shown in some test problems and examples. The first example is the motion of a planar seven-body mechanism used as a benchmark in [1]. The system can be modelled with ten truss elements and 18 nodal coordinates, of which eight are fixed, see Fig. 3. In order to overcome a singularity, it is better to model the system with nine truss elements and a rigid beam element, which gives an additional nodal coordinate. The original model used in [5] had eleven beam elements and one truss element, which led to a model with 31 nodal coordinates, so the new way of modelling the system leads to a reduction of the complexity. The resulting motion is exactly the same.

As a spatial example, the benchmark robot from [1] is modelled with truss elements only to show their possibilities. A schematic representation, not to scale, of the a model of the robot is shown in Fig. 4. The robot is fixed at the base point A and the points B and D can only move vertically. The truss element between the points A and B models the vertical degree of freedom, z_1, and the rotation about the vertical axis with the angle γ_1 is possible, which is the second degree of freedom of the robot. The first body is modelled by a tetrahedron of six truss elements represented by BCD. A massless intermediate tetrahedron is

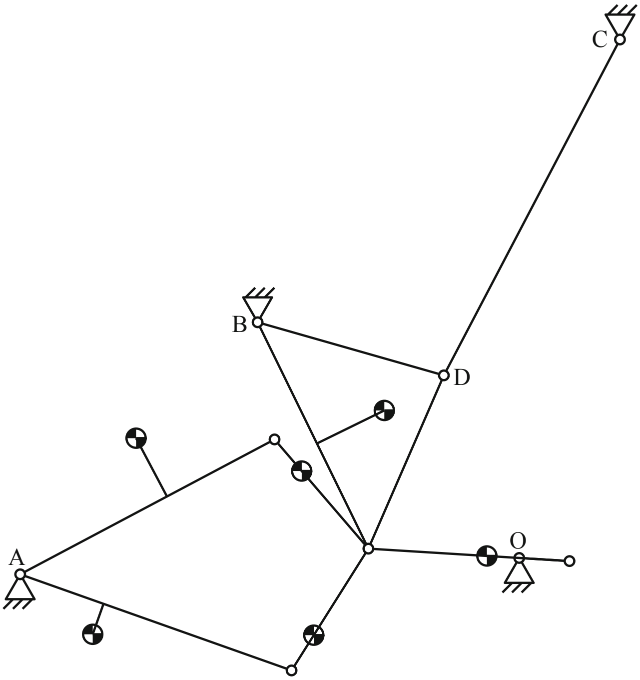

Fig. 3. Seven-body mechanism; the element between points C and D is a spring, the other elements are rigid; the centre of mass of the crank is not shown

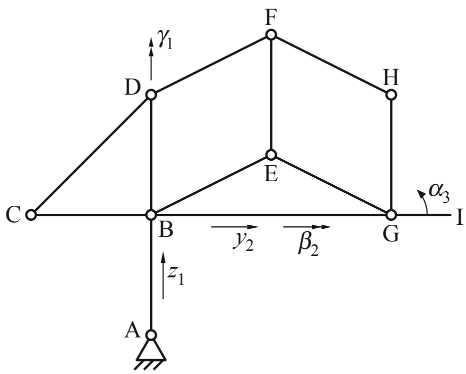

Fig. 4. Schematic representation of a model for the robotic manipulator

connected to this to make the rotation about the axis BC with the rotation angle β_2 possible. The linear motion along the axis BG with displacement y_2 is made possible by a Sarrus mechanism represented schematically by the parallelograms BDEFGH, which adds 32 truss elements to the model. The truss element BG

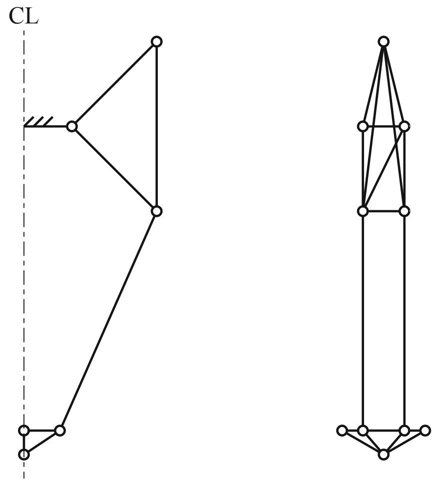

Fig. 5. Delta robot modelled with truss elements

is added as an actuator to model this degree of freedom. The second body is attached to GH by three additional truss elements and the end effector GI, which can rotate with an angle α_3 about the revolute joint G is modelled by five additional truss elements. In total, there are 53 truss elements and 63 nodal coordinates, of which seven are fixed. The degrees of freedom for the translational joints are the elongations of the two trusses, but as there are no angles in the model, the rotational degrees of freedom are replaced by convenient coordinates of three nodal points; this may lead to singularities during the simulation, but in the present example with limited angles, these do not occur.

The original model used in [5] only needed 43 coordinates, of which eight were fixed. The original as well as the present model are not the most compact models possible, so this gives a reasonably fair comparison. Although many elements and coordinates are needed, the presented model may still give rise to an efficient calculation procedure. Both models yield the same results, although in a different form.

Finally, a parallel manipulator, the Delta robot, is considered, see Fig. 5. It consists of three arms that connect a fixed base with a platform, of which one arm is shown in the figure. The other two arms are identical and rotated over $120\,^\circ$ about the centre line. Each arm consists of an upper arm connected to the base by a revolute actuated joint and a lower arm consisting of two rods arranged as the sides of a parallelogram connecting the upper arm to the moving platform. As each arm constrains one rotation of the platform and these rotations are independent, the platform can only translate. Each upper arm is modelled by nine truss elements arranged along the edges of a pyramid with a rectangular base, one of which edges is fixed to the base, and each lower arm is modelled by two truss elements. The platform is a regular hexagon modelled with 15 truss elements. One coordinate of each apex of the pyramids can be used as the degrees of freedom of the system. The model has 66 coordinates, of which 18 are fixed and three are degrees of freedom. The number of dependent coordinates, 45, is equal to the number of truss elements. A traditional way of modelling the system would use 85 coordinates, so the reduction in the number of coordinates is not so large, but the elements are simpler.

5 Conclusions

A general planar binary element with pin joints can be modelled by a single planar truss element with an appropriate generalization of its mass matrix. General applied forces can be reduced to equivalent nodal forces.

Quite complicated spatial mechanisms, such as robots, can be modelled by truss elements only, although this may lead to large collections of elements, which are, however, individually simple. It is nonetheless advisable to have other kinds of elements available. It has been shown that the inertial properties of a spatial rigid body can be modelled by six truss elements with uniform distributed mass arranged along the edges of a tetrahedron with nodes at general positions, with additional lumped masses at the nodes.

The modelling can be facilitated with some pre-processing tools, in particular the representation of moments and forces by equivalent nodal forces. The interpretation of results needs some post-processing, as angles are not directly available.

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