Bus holding of electric vehicles: an exact optimization approach

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ABSTRACT
In high-frequency electric bus services with more than five buses per hour, bus operators strive to increase the service regularity by minimizing the deviation between the planned and actual headways. In this pursue, bus operators apply corrective control strategies, such as bus holding(s) at control point stops. This study expands the traditional headway-based models of bus holding to cater for the planned arrival times of electric buses at the respective charging points. To this end, this study models the bus holding problem for electric buses considering the scheduled charging times in the objective function. The resulting mathematical program is not smooth and cannot be solved to global optimality. To rectify this, this study introduces slack variables and proposes a problem reformulation into a convex, nonlinear program that can be easily solved to global optimality with exact optimization methods. The proposed bus holding logic is shown to reduce significantly the delayed arrivals at the charging points (up to 34%), while the average passenger waiting times exhibit a marginal increase when compared to the existing control methods (up to 1.08%).

Keywords: bus holding; service regularity; electric buses; real-time control; nonlinear programming.

INTRODUCTION

Electric buses typically have an almost identical charging capacity and fixed charging sites. Notwithstanding this, planning the overall electric bus system is challenging given that a pool of electric buses might be in need of a limited number of charging sites Zheng et al. (*1*). This requires meticulous planning of the charging schedules of all operational buses which, oftentimes, takes into consideration the urban power network structure strength Pan and Zhang (*2*). The latter is advisable because electric buses require significantly more energy than private electric vehicles, thus impacting the power grid Srinivasaraghavan and Khaligh, Foster et al., Clement-Nyns et al., Richardson (*3, 4, 5, 6*).

Studies on planning the charging operations have demonstrated the importance of meeting the planned schedules for both the performance of the power grid and the efficiency of the bus operations. For instance, Su and Chow (*7*) proposed an algorithm for optimally managing a large number of plug-in hybrid electric vehicles charged at a municipal parking station. In addition, Leou and Hung (*8*) proposed a mathematical model of controlled charging, which includes the operational guidelines of the bus company, the capacity, and the energy charges of the charging station.

While there is an extensive body of works on planning the charging of the charging station(s) and scheduling the bus timetables accordingly, the optimal charging plans can be disrupted during the actual operations due to the inherent uncertainty of bus travel times Gkiotsalitis and Stathopoulos, Gkiotsalitis and Van Berkum (*9, 10*). For instance, exogenous factors such as traffic congestion and traffic light cycles might increase the waiting times of passengers and delay the arrival of the bus at the charging point Sun and Hickman, Gkiotsalitis and Alesiani (*11, 12*). The latter might result in the disruption of the charging schedules with several negative effects, such as: (i) charging with an increased energy price; (ii) delay or refuse of charging because the charging point is occupied by another bus; (iii) disruption of future bus dispatches/crew schedules; and (iv) excess strain on the power grid.

This motivates our work: we propose a bus holding control method with the dual objective of (a) minimizing the deviation between the actual and planned headways to improve the regularity of the bus operations, and (b) adhering to the pre-planned charging schedule by arriving at the charging point(s) on time. We note here that a charging schedule is the schedule of the arrival times of all electric buses operating in the course of one day at the respective charging point(s).

In this study, we will focus on bus holding which is the most prominent control method and does not lead to the refusal of passenger boardings Ibarra-Rojas et al., Sánchez et al. (*13, 14*). Bus holding has a similar effect as the decrease of traveling speed between bus stops and is implemented at particular control point stops where a bus is held if it has come too close to its preceding one Van Oort et al. (*15*). Since we target urban environments with several electric buses, we note that in such context buses operate under high frequencies (i.e., more than 5 buses per hour per route) where the main objective is to adhere to the planned headways Trompet et al., Gkiotsalitis et al. (*16, 17*).

The remainder of this paper is structured as follows: in section 2, we provide a literature review on the corrective control method of bus holding. Section 3 provides our problem definition and our mathematical program. It also introduces adjustments to the main mathematical program for deriving a reliable bus holding solution under the precense of travel time uncertainty. Section 4 demonstrates the solution of our mathematical program in a hypothetical scenario. In addition, it tests the performance of our proposed holding method against the closed-form bus holding method of Fu and Yang (*18*) in a simulation environment. Section 5 discusses the results of our work and its current limitations. Finally, section 6 concludes our work and draws the future research directions.

LITERATURE ON BUS HOLDING

Passengers expect to experience waiting times at stops which are at most equal to the scheduled headways of the daily timetable Randall et al. (*19*). Notwithstanding this, the travel time variability during the actual operations results in bus bunching and excessive passenger waiting Turnquist, Daganzo (*20, 21*).

Control methods for bus bunching have been studied since the early 1970s Osuna and Newell (*22*). Nevertheless, the bus bunching problem remains a prominent research topic because of its inherent complexity. Osuna and Newell (*22*) and Newell (*23*) introduced specific key performance indicators to monitor the average passenger waiting times at stops. To minimize the average waiting time of passengers, Newell (*23*) considered one control point at which buses can be intentionally held.

Typical objectives of bus holding methods are headway adherence Rossetti and Turitto, Gkiotsalitis, Gkiotsalitis and Cats (*24, 25, 26*), headway regularity Daganzo, Bartholdi III and Eisenstein (*21, 27*), and the minimization of passenger waiting and in-vehicle times Delgado et al., Delgado et al., Sáez et al., Gkiotsalitis et al. (*28, 29, 30, 31*). It should be noted here that, as a general practice, buses are not held at every stop because this will increase the passenger inconvenience. On the contrary, bus holding is only allowed at a pre-determined sub-set of important bus stops, known as control points Cats et al. (*32*).

In bus holding, two different directions of research have emerged. One prominent direction models the bus holding problem as a rolling horizon problem where decisions about the holding times of a number of trips are made simultaneously Eberlein et al., Gkiotsalitis and Maslekar (*33, 34*). In this line of research, the bus holding problem is typically modeled as a multivariate mathematical program where collective decisions are made. The second direction is based on closed-form functions of bus arrival times that determine the holding times by considering the differences between the actual and target headways Fu and Yang, Bartholdi III and Eisenstein, Daganzo and Pilachowski, Xuan et al., Berrebi et al., Berrebi et al. (*18, 27, 35, 36, 37, 38*). Works in the second direction prevent bus bunching from the onset and can return a globally optimal solution, unlike the more detailed multivariate mathematical programs of rolling horizon methods for which one should typically resort to heuristics.

Rolling Horizon Methods

Examples of works on bus holding in rolling horizons are the Ph.D. thesis of Eberlein (*39*) and the works of Eberlein et al., Shen and Wilson, Sánchez-Martínez et al. (*33, 40, 41*). Such works determine simultaneously the holding times of all buses that are expected to operate within a rolling horizon. The optimized holding times can be updated later in time when subsequent buses arrive at the control point.

Eberlein et al. (*33*) assumed that interstation travel times and passenger arrival rates are constant in rolling horizons with short time duration. The holding problem of all running buses in a rolling horizon was modeled as a quadratic program with the objective to minimize the total passenger waiting times. Sánchez-Martínez et al. (*41*) formulated a mathematical model to produce a plan of holding times that cater for expected changes in running times and demand. Its effectiveness was evaluated within a simulation environment.

Delgado et al. (*28*) developed a mathematical program that incorporates vehicle-capacity constraints. Their objective was to minimize the total travel times experienced by all passengers in the system resulting in a non-convex, quadratic objective function which cannot be always solved to global optimality. In a later work, Delgado et al. (*29*) investigated two control policies applied within a rolling horizon framework: (i) vehicle holding, which can be applied at any stop, and (ii) holding combined with boarding limits, in which the number of boarding passengers at any stop can be limited in order to increase operational speed. The respective mathematical programs were solved using MINOS on an Intel Core2 Duo @ 2.66 GHz with reported running times in the range of 3.8 s - 5.2 s.

Sáez et al. (*30*) utilized a dynamic objective function and a predictive model of the bus system to make decisions on bus holding and stop-skipping (known also as expressing). The uncertain passenger demand was included in the model as a disturbance. The resulting optimization problem was NP-Hard and was solved using an ad hoc implementation of a Genetic Algorithm. Gkiotsalitis and Maslekar (*34*) used also an algorithm from the area of evolutionary optimization to solve an NP-Hard program that suggests holding times which minimize the waiting times of passengers and account for regulatory constraints.

Zolfaghari et al. (*42*) developed a mathematical control model for holding using real-time information of the locations of buses along a specified route. The model was solved with the simulated annealing metaheuristic. Hickman (*43*) used the stochastic model developed by Marguier (*44*) to derive the trajectories of buses on a single route. Using Marguier's model, Hickman (*43*) developed a bus holding algorithm that is applied each time a bus arrives at the control point stop. To this end, Marguier's model was used to approximate the trajectories of all "upstream" buses. After that, the bus holding time was selected using a line search method because obtaining an analytic solution was not possible given the complexity of deriving the first-order conditions of the optimization problem.

Most of the above-mentioned models resort to (meta)heuristics to solve the respective multivariate mathematical programs because of their inherent complexity.

Closed-form (threshold-based) Models

Typically, closed-form expressions are used to determine the holding times by considering the differences between the actual and the target headways. Fu and Yang (*18*) tested two of the most common closed-form expressions for bus holding: (i) the one-headway-based control where a bus is held at a control point stop if its time headway with its preceding bus is lower than a pre-defined threshold; and (ii) the two-headway-based control that considers the time headway of a bus with both its preceding and following bus.

Similarly, Sun and Hickman (*45*) set the holding time of a bus to zero if its predicted headway with its following bus is less than or equal to the minimum headway. When the actual vehicle headway is less than the prescribed minimum headway, the following vehicle will be delayed until the minimum headway requirement is satisfied.

Daganzo and Pilachowski (*35*) proposed an adaptive control scheme that adjusts a bus cruising speed in real-time based on both its front and rear spacings. In line with other closed-form approaches, it had a simple and decentralized logic enabling to correct the effect of traffic disruptions in real-time.

Bartholdi III and Eisenstein (*27*) proposed an analytic bus holding solution that changes the headway of each newly arrived bus at a control point stop to the weighted average of its former headway and the former headway of the trailing bus. This approach tends to re-equalize the headways after a disturbance. The major difference of Bartholdi III and Eisenstein (*27*) from the previously described works is that it merely normalizes the headways and does not adhere to a target headway value. Thus, it does not use a headway threshold to trigger the bus holding.

Finally, Berrebi et al., Berrebi et al. (*37, 38*) proposed a method consisting of identifying probabilistically the bus that will be the most delayed upon its arrival at a control point stop. Then, they held each preceding bus to prevent the lagging bus from departing with a big gap. Van Oort et al. (*15*) also tested schedule-based and headway-based holding strategies where the solution was expressed as a closed-form expression of arrival times and scheduled headways. They tested the importance of setting a maximum holding time and a reliability buffer time in tram line 9 in The Hague.

Contribution

From the above studies, bus holding control methods focus overwhelmingly on improving the bus operations (whether this means maintaining the target headways, reducing the passenger waiting times or limiting the in-vehicle travel times). Therefore, we identify a main research gap: there is a lack of bus holding studies that consider the improvement of the service regularity and, at the same time, cater for the charging requirements of electric buses.

This study tries to fill this research gap by integrating the planned charging times of electric buses to the bus holding problem and incorporating both aspects in a mathematical program.

The incremental contributions of this work to the state-of-the-art are:

- the development of a bus holding control model applicable to electric buses with scheduled charging times and uncertain interstation travel times;
- the exploration of the properties of the resulting mathematical program and the introduction of a reformulated program which is proved to be convex;
- the introduction of reliable bus holding solutions that account for the interstation travel time uncertainty and unstable headway dynamics.

PROBLEM DEFINITION AND MATHEMATICAL PROGRAM

Problem Definition

The bus holding decision of a bus trip i at a control point stop s is made when it has completed all its boardings/alightings and is ready to depart. In our problem, this holding decision should take into consideration not only the regularity of the service, but also the planned charging time of the bus. As in the majority of works in real-time bus holding (see Daganzo, Bartholdi III and Eisenstein, Daganzo and Pilachowski, Hickman (*21, 27, 35, 43*)), we do not consider the effect of overcrowding when holding a bus. This is a reasonable assumption because the scheduling of the vehicle capacity is already performed at the tactical planning stage and considers a capacity buffer in case of increased passenger demand due to holding Gkiotsalitis and Cats (*46*). Besides, the vehicle capacity problem cannot be addressed in real-time control because it results in non-convex mathematical programs that do not have a globally optimal solution and are computationally intractable (see Delgado et al. (*29*)).

Proceeding to a formal description of our model, we introduce the following nomenclature:

NOMENCLATURE

Sets

$S = \langle 1, 2, ... \rangle$	ordered set of bus stops
$I = \langle 1, 2, ... \rangle$	ordered set of running bus trips

Indices

s	bus stop
i	bus trip

Parameters

H_0	target headway
c	holding control parameter, where $c \in \mathbb{R} : 0 \leq c \leq 1$
$T_{i,s}$	earliest possible time after which trip i can depart from stop s
$\mathbb{E}[t_{i,s}]$	expected travel time of trip i from stop s to the charging location
$\text{Var}[t_{i,s}]$	travel time variation of trip i from stop s to the charging location
ρ_i	scheduled charging time of bus trip i at the charging location. If a trip i does not require charging, $\rho_i = +\infty$
M	a very large number

Decision Variable

$d_{i,s}$	determined departure time of trip i at stop s

At time instance $T_{i,s}$, bus trip i is ready to depart from the control point s. When we make a holding decision, our objective is to hold the bus trip i at stop s to maintain the target headway, H_0, with its preceding bus trip, $i-1$. At the same time, our holding decision, $d_{i,s} - T_{i,s} \geq 0$, should not result in a delay that will postpone the scheduled charging time, ρ_i, of bus trip i at the charging station. Note that a bus holding decision is an individualistic decision in the sense that it considers the optimal option for the trip that is ready to depart from the control point stop (see Fu and Yang (*18*)).

The time instance $T_{i,s}$ when a bus trip i is ready to depart from control point stop s is presented in Fig.1. In the time-space diagram of Fig.1 we present the realized and expected trajectories of bus trip i and its preceding trip, $i-1$.

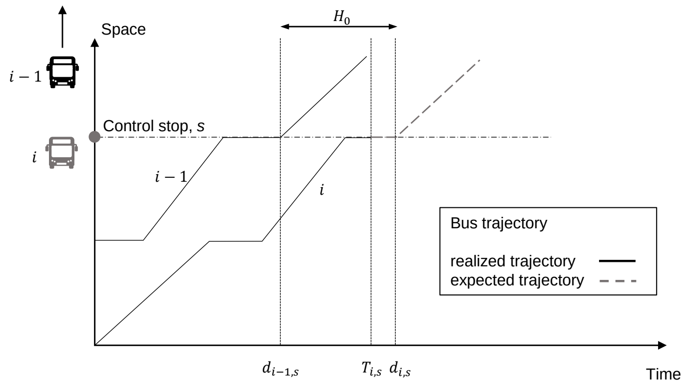

FIGURE 1 : Illustration of bus trajectories in a time-space diagram

Bus trip i is held at stop s if it is closer to its preceding bus, $i-1$, than the target headway, H_0. That is, if $T_{i,s} - d_{i-1,s} < H_0$. If this is the case, we hold trip i for time $(d_{i-1,s} + H_0) - T_{i,s}$ at the control point stop to eliminate the deviation between the actual and the target headway. In the opposite case where $T_{i,s} - d_{i-1,s} > H_0$, bus trip i will depart as soon as possible because its actual headway with the preceding bus trip $i-1$ is greater than H_0, indicating that it is left behind. This yields the control logic:

$$d_{i,s} = \begin{cases} d_{i-1,s} + H_0 \text{ if } T_{i,s} < d_{i-1,s} + H_0 \\ T_{i,s} \text{ otherwise} \end{cases} \tag{1}$$

where $d_{i,s}$ is the determined departure time of trip i from the control point s and $d_{i,s} - T_{i,s}$ the resulting holding time. The expected travel time of bus trip i from stop s to the charging location is $\mathbb{E}[t_{i,s}]$. If bus trip i needs to reach its charging location before time ρ_i for charging as planned, then:

$$d_{i,s} + \mathbb{E}[t_{i,s}] \leq \rho_i \tag{2}$$

In addition, trip i cannot depart prior to $T_{i,s}$ which is the time when trip i has completed the boardings/alighting at stop s. This yields:

$$T_{i,s} \leq d_{i,s} \tag{3}$$

Note that from the above constraints, the constraint of Eq.(3) is a physical, hard constraint and cannot be violated (i.e., trip i cannot depart from stop s if it has not completed its boardings/alightings). Note also that constraints (1), (2) cannot be satisfied in all cases because they conflict with the physical constraint of Eq.(3).

Mathematical Program

Since constraints Eq.(1), (2), (3) cannot be always satisfied, a hierarchy between soft and hard constraints should be established. Obviously, Eq.(3) is a hard constraint and has the highest priority because a bus cannot depart if it has not completed its boardings/alightings.

Lowest in the hierarchy is the equality constraint of Eq.(1) which determines the optimal bus departure from the control point and is more of an objective rather than a constraint. Therefore, we relax it by reformulating it as a problem objective. For this, we introduce the dummy variable:

$$\mu_{i,s} = \begin{cases} 1 \text{ if } T_{i,s} < d_{i-1,s} + H_0 \\ 0 \text{ otherwise} \end{cases} \tag{4}$$

In addition, we introduce the objective function:

$$f(d_{i,s}) = \mu_{i,s}\big(d_{i,s} - (d_{i-1,s} + H_0)\big)^2 + (1-\mu_{i,s})\big(d_{i,s} - T_{i,s}\big)^2 \tag{5}$$

which returns the optimal value of $d_{i,s}$ when it is minimized. This value is equivalent to the value of the equality constraint of Eq.(1) if constraints Eq.(2) and (3) are satisfied. The proof of this claim is provided below.

Theorem 3.1. *Provided that constraints Eq.(2) and (3) are satisfied, the minimizer of $f(d_{i,s})$ is equal to $d_{i-1,s} + H_0$ if $T_{i,s} < d_{i-1,s} + H_0$ and $T_{i,s}$ if $T_{i,s} \geq d_{i-1,s} + H_0$.*

Proof. Assuming that Eq.(2) and (3) are satisfied, the global optimum of function f over its domain $[0, +\infty)$ is $d_{i,s}$ such that $f(d_{i,s}) \leq f(x)$, $\forall x \in [0, +\infty)$. Given that the devised function f is a smooth quadratic function with continuous first and second-order derivatives for all feasible $x \in [0, +\infty)$, it has a stationary point when $\frac{\partial f}{\partial x} = 0 \Rightarrow 2\mu_{i,s}(x - (d_{i-1,s} + H_0)) + 2(1-\mu_{i,s})(x - T_{i,s}) = 0$. The stationary point when $T_{i,s} < d_{i-1,s} + H_0$ is $2 \cdot 1(x - (d_{i-1,s} + H_0)) + 2 \cdot 0(x - T_{i,s}) = 0 \Rightarrow x = (d_{i-1,s} + H_0)$. When $T_{i,s} \geq d_{i-1,s} + H_0$, we have $\mu_{i,s} = 0$ and the stationary point becomes $x = T_{i,s}$. The second-order derivative of f is $\frac{\partial^2 f}{\partial x^2} = 2\mu_{i,s} + 2(1-\mu_{i,s}) = 2 > 0$; thus the stationary point is the minimizer of f in $[0, +\infty)$. Hence, the minimizer of f is $d_{i-1,s} + H_0$ if $T_{i,s} < d_{i-1,s} + H_0$ and $T_{i,s}$ if $T_{i,s} \geq d_{i-1,s} + H_0$. □

Theorem 3.1 proves that the minimization of our objective function f returns the values of Eq.(1) provided that all other constraints are satisfied. Therefore, we can replace Eq.(1) with f and solve the following mathematical program to obtain the optimal holding solution of bus trip i at control point s:

$$\begin{aligned} (Q_{i,s}): \quad & \min_{d_{i,s}} \quad f(d_{i,s}) \\ & \text{s.t.:} \quad d_{i,s} \in \mathscr{F} = \{d_{i,s} \mid d_{i,s} \text{ satisfies Eq. (2), (3)}\} \end{aligned} \tag{6}$$

Solving the mathematical program $(Q_{i,s})$ is equivalent to enforcing the (in)equality constraints Eq.(1),(2),(3) when constraints Eq.(2),(3) are satisfied for the optimal value of $d_{i,s}$. The advantage of mathematical program $(Q_{i,s})$ is that it still computes an optimal value for $d_{i,s}$ even if such solution does not result in meeting the target headway. That is, a trade-off between meeting the planned charging time and adhering to the target headway is established. Toward this end, the objective function $f(d_{i,s})$ will seek solutions close to $d_{i-1,s} + H_0$ if $\mu_{i,s} = 1$ and $T_{i,s}$ if $\mu_{i,s} = 0$ because any deviations from those values are progressively penalized with a squared penalty.

Finally, as we previously discussed, there exist cases where the constraint of meeting the scheduled charging time, Eq.(2), and the hard constraint of Eq.(3) are conflicting. Thus, they cannot be satisfied simultaneously. Since Eq.(3) is a hard, physical constraint, we relax Eq.(2) which becomes a *soft* constraint and is allowed to be violated under certain circumstances.

Soft constraints are typically treated as penalty terms and are added to the objective function Voss (47). In this way, program $(Q_{i,s})$ that, under certain circumstances, has no feasible solution can be transformed to the feasible program $(\hat{Q}_{i,s})$. This is achieved by relaxing the inequality constraint of Eq.(2) and adding a penalty for its violation to the objective function. To this end, its relative importance is weighted by introducing a very large number $M \in \mathbb{R}_{\geq 0}$ which ensures that the satisfaction of the constraint is prioritized over the objective function f:

$$\begin{aligned}(\hat{Q}_{i,s}): \quad & \min_{d_{i,s}} \quad f(d_{i,s}) + M\max(d_{i,s} + \mathbb{E}[t_{i,s}] - \rho_i, 0) \\ & \text{s.t.:} \quad d_{i,s} \in \mathscr{F} = \{\, d_{i,s} \mid T_{i,s} \leq d_{i,s} \,\}\end{aligned} \tag{7}$$

The penalty term $M\max(d_{i,s} + \mathbb{E}[t_{i,s}] - \rho_i, 0)$ ensures that the soft constraint $d_{i,s} + \mathbb{E}[t_{i,s}] \leq \rho_i$ is prioritized over $f(d_{i,s})$. Indeed, if $d_{i,s} + \mathbb{E}[t_{i,s}] \leq \rho_i$ for some $d_{i,s}$, then this solution does not add any penalty to the objective function since $M\max(d_{i,s} + \mathbb{E}[t_{i,s}] - \rho_i, 0) = 0$. In contrast, when $d_{i,s} + \mathbb{E}[t_{i,s}] > \rho_i$ for some $d_{i,s}$, then the penalty term penalizes the objective function by a very large number $M(d_{i,s} + \mathbb{E}[t_{i,s}] - \rho_i)$ and directs the program towards another solution that reduces the value of $M\max(d_{i,s} + \mathbb{E}[t_{i,s}] - \rho_i, 0)$.

Note that the "max" term of $M\max(d_{i,s} + \mathbb{E}[t_{i,s}] - \rho_i, 0)$ makes the new objective function of program $\hat{Q}_{i,s}$ non-smooth and the program cannot be always solved to global optimality. To rectify this, we implement the "max" penalty by introducing a new variable ν that, due to its bounds and the direction of optimization, will take the value $\max(d_{i,s} + \mathbb{E}[t_{i,s}] - \rho_i, 0)$ at the solution. The reformulated program is:

$$\begin{aligned}(\tilde{Q}_{i,s}): \quad & \min_{\nu, d_{i,s}} \quad f(d_{i,s}) + M\nu \\ & \text{s.t.:} \quad d_{i,s} \geq T_{i,s} \\ & \qquad\quad \nu \geq 0 \\ & \qquad\quad \nu \geq d_{i,s} + \mathbb{E}[t_{i,s}] - \rho_i\end{aligned} \tag{8}$$

which can be solved to global optimality and has a unique solution, as shown in Theorem 3.2.

Theorem 3.2. *A local minimizer of $(\tilde{Q}_{i,s})$ is also its unique global minimizer.*

Proof. A local minimizer of $(\tilde{Q}_{i,s})$ is the unique global minimizer of $(\tilde{Q}_{i,s})$ if the objective function is strictly convex and the feasible region is a convex set. The feasible region is defined by linear inequalities and is a polyhedron (thus, it is also a *convex set*). Further, we prove that the objective function $f(d_{i,s}) + M\nu$ is strictly convex with respect to $d_{i,s}, \nu$. Let $\tilde{f}(d_{i,s}, \nu) \doteq f(d_{i,s}) + M\nu$. Let also $x \equiv d_{i,s}$ for simplifying the notation. Then, the *Hessian* matrix of $\tilde{f}$ reads:

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 \tilde{f}(x,\nu)}{\partial x^2} & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial x \partial \nu} \\ \\ \frac{\partial^2 \tilde{f}(x,\nu)}{\partial \nu \partial x} & \frac{\partial^2 \tilde{f}(x,\nu)}{\partial \nu^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ \\ 0 & 0 \end{bmatrix}$$

To prove the strict convexity of $\tilde{f}$, we should prove that the Hessian matrix is positive definite. That is, $\mathbf{z}^\intercal \mathbf{H} \mathbf{z}$ is positive for every column vector $\mathbf{z} \in \mathbb{R}^2 \setminus \{0, 0\}$. This yields

$$
\begin{aligned}
\mathbf{z}^\intercal \mathbf{H} \mathbf{z} &= \begin{bmatrix} z_1 & z_2 \end{bmatrix} \mathbf{H} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\
&= \begin{bmatrix} (2z_1) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 2z_1^2
\end{aligned}
$$

which is positive for any possible values of $z_1, z_2 \in \mathbb{R} \setminus \{0\}$. Thus, $\tilde{f}$ is strictly convex and this completes our proof. □

Finding a Reliable Solution

Until now, our bus holding solution considers the expected travel time, $\mathbb{E}[t_{i,s}]$, of bus trip i from the control point stop s to the charging station. Nevertheless, due to the interstation travel time uncertainty, our bus trip might arrive late at the charging station. A common approach to account for the inherent uncertainty of travel times is the computation of a *reliable* solution. In our search for a reliable solution, we seek a solution $d_{i,s}$ for which we can be confident that bus trip i will arrive at the charging point before time ρ_i. This can be achieved by considering the y-th percentile, t_s^y, of the total travel time from stop s to the charging point. The y-th percentile can be the 90th, 95th, 99th or another percentile according to the needs of the bus operator. It is defined as:

$$
t_s^y : Pr(t_{i,s} \leq t_s^y) = \int_{-\infty}^{t_s^y} \phi(t_{i,s})\, dt_{i,s} = y\% \tag{9}
$$

where $\phi(t_{i,s})$ is the probability density function of the random variable $t_{i,s}$. I.e., if y is the 95th percentile, then $t_s^{y=95} : Pr(t_{i,s} \leq t_s^y) = \int_{-\infty}^{t_s^y} \phi(t_{i,s})\, dt_{i,s} = 95\%$.

Consequently, the average travel time $\mathbb{E}[t_{i,s}]$ in program $\tilde{Q}_{i,s}$ can be substituted by the y-th percentile since it will only exceed that value at (100-y)% of the cases. With this substitution, the reliable solution is found by solving the following program:

$$
\begin{aligned}
(\tilde{P}_{i,s}): \quad & \min_{\nu, d_{i,s}} \quad f(d_{i,s}) + M\nu \\
& \text{s.t.:} \quad d_{i,s} \geq T_{i,s} \\
& \qquad\quad \nu \geq 0 \\
& \qquad\quad \nu \geq d_{i,s} + t_s^y - \rho_i
\end{aligned} \tag{10}
$$

Note that program $\tilde{P}_{i,s}$ inherits the properties of $\tilde{Q}_{i,s}$ because its only difference is the replacement of $\mathbb{E}[t_{i,s}]$ with t_s^y. Thus, the *reliable* bus holding problem can be also solved to global optimality with exact optimization methods.

NUMERICAL EXPERIMENTS

Demonstration

To describe the mechanism behind our constrol logic, we perform a small demonstration. In our demonstration, we use an idealized scenario to manifest the solution of our model and its underlying control logic. In our idealized scenario, trip i arrives at control stop s and completes its boardings/alightings at time $T_{i,s} = 1500$s. The parameter values of our scenario are: $d_{i-1,s} = 1000$ s, $T_{i,s} = 1500$ s, $H_0 = 600$ s, and $\mathbb{E}[t_{i,s}] = 3000s$. For those parameter values it is evident that $T_{i,s} < d_{i-1,s} + H_0$, and thus $\mu_{i,s} = 1$.

Let us now display the decisions of our proposed control logic for different values of planned charging times, ρ_i. To perform this task, we report the respective globally optimal solution of our mathematical program $\tilde{Q}_{i,s}$ for each value of ρ_i. Mathematical program $\tilde{Q}_{i,s}$ is solved in a general-purpose computer with Intel Core i7-455 7700HQ CPU @ 2.80GHz and 16 GB RAM using CPLEX 12.8. The results are summarized in Table 1.

TABLE 1 : Globally Optimal Holding decisions for different values of ρ_i

ρ_i	Solving $\tilde{Q}_{i,s}$ with CPLEX		
	ν	$d_{i,s}$	Comp. time
4800 s	0 s	1600 s	0.02 s
4600 s	0 s	1600 s	0.02 s
4550 s	0 s	1550 s	0.02 s
4500 s	0 s	1500 s	0.02 s
4200 s	300 s	1500 s	0.02 s

For $\rho_i = 4800$ s, we plot the objective function of program $\tilde{Q}_{i,s}$ in the region $\mathscr{F} = \{\, d_{i,s} \geq T_{i,s} \,\}$ (see Fig.2). From Fig.2 it is evident that the optimal solution is $d_{i,s} = 1600$ s. Note that for $d_{i,s} > 1800$ s, $\nu > 0$ because $d_{i,s} + t_s^y - \rho_i > 0$ and the objective function is penalized by the product $M\nu$.

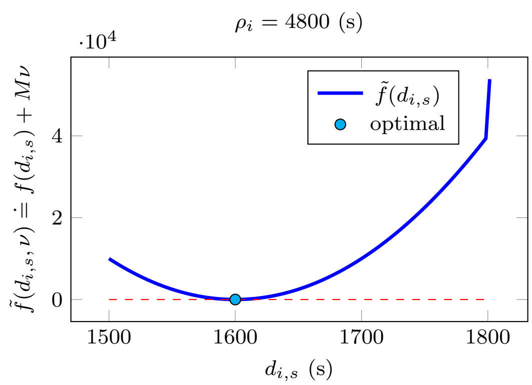

FIGURE 2 : Performance of the objective function in the region $\mathscr{F} = \{\, \nu = 0, d_{i,s} \geq T_{i,s} \,\}$ for $\rho_i = 4800$ s.

For $\rho_i = 4600$ s, we also plot the objective function of program $\tilde{Q}_{i,s}$ in the region $\mathscr{F} = \{\, d_{i,s} \geq T_{i,s} \,\}$ (see Fig.3). For $d_{i,s} > 1600$ s, the objective function cost exhibits a sharp increase because the constraint $\nu \geq d_{i,s} + \mathbb{E}[t_{i,s}] - \rho_i$ is activated and the objective function is penalized by the large value of $M\nu$.

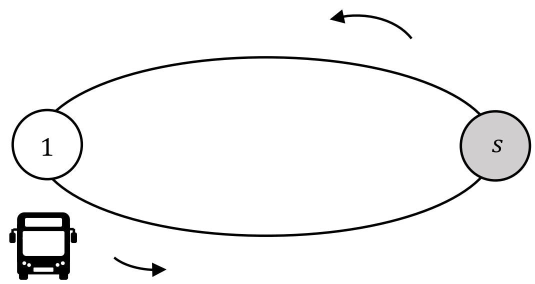

FIGURE 4 : Illustrative example of circular bus line with control point stop, s

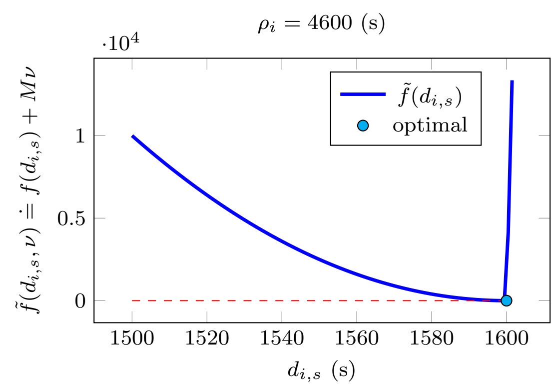

FIGURE 3 : Performance of the objective function in the region $\mathscr{F} = \{\ \nu = 0, d_{i,s} \geq T_{i,s}\ \}$ for $\rho_i = 4600$ s.

For $\rho_i = 4550$ s, the constraint $\nu \geq d_{i,s} + \mathbb{E}[t_{i,s}] - \rho_i$ is active (where active means that $\nu = d_{i,s} + \mathbb{E}[t_{i,s}] - \rho_i$) for $d_{i,s} = 1550$ s. Hence, $d_{i,s} = 1550$ s is the optimal solution of program $\tilde{Q}_{i,s}$ instead of the solution $d_{i,s} = 1600$ s that does not consider the adherence to the planned charging time.

Now, for $\rho_i = 4500$ s, we have an active constraint $\nu = d_{i,s} + \mathbb{E}[t_{i,s}] - \rho_i$ when $d_{i,s} = T_{i,s} = 1500$ s. Finally, for $\rho_i = 4200$ s, bus i is already delayed and cannot meet its scheduled charging time because, even if $d_{i,s}$ admits its minimum possible value, $T_{i,s} + \mathbb{E}[t_{i,s}] > \rho_i$. Therefore, the solution of $\tilde{Q}_{i,s}$ is indeed $d_{i,s} = 1500$ s with $\nu = 300$ s.

Simulation-based Evaluation of our Control Logic

Holding a bus every time it arrives a control point stop is a local-level decision that might have broader impacts on the entire chain of running trips. For this reason, we investigate the impact of such decisions.

To evaluate our control logic in a systematic manner, we build a simulation of an idealized, circular bus line operated by electric buses. We consider a short time period of the day with 10 bus trips. Trips start from stop 1 and complete their service at the same stop, which happens to be a charging point. Buses can be held at the control point stop, $s = 2$. The idealized line is presented in Fig.4.

Bus trips $i = \{1, 2, ..., 10\}$ are dispatched every 6 minutes (frequency of 10 trips per hour) and the target headway H_0 is 6 min, or, equivalently, $H_0 = 360$ s. In addition, our control logic computes a *reliable* holding solution every time a bus trip is ready to depart the control point stop considering the y-th percentile of the total travel time from stop s to the charging point, t_s^y.

In the current implementation of our simulation system, we do not explicitly model signal control and other traffic. Instead, we consider the interstation bus travel times (speed) as random variables that follow a probability distribution that can differ from case study to case study. This simulation approach has been widely used for validation purposes in bus holding control studies because the selection of an appropriate travel time distribution is able to capture the effects of signal control and traffic congestion Bartholdi III and Eisenstein, Andersson et al. (*27, 48*).

In past literature, normal and lognormal distributions are commonly used to model interstation travel times Fu and Yang, Berrebi et al., Gkiotsalitis and Maslekar (*18, 38, 49*). In this study, we assume a normal distribution as in Fu and Yang (*18*). Therefore, the realization of each interstation travel time from stop j to stop $j+1$ is $t_{i,j} \sim \mathcal{N}(\mathbb{E}(t_{i,j}), \text{Var}(t_{i,j}))$ where $\mathbb{E}(t_{i,j})$ is the expected interstation travel time and $\text{Var}(t_{i,j})$ its variance for the respective $j = \{1, s\}$. Given that a sampled travel time from a normal distribution can assume a negative value, we bound the lower value of the interstation travel time $t_{i,j}$ by $t_{i,j}^{min} \in \mathbb{R}_{>0}$, where $t_{i,j}^{min}$ is the minimum possible travel time under free flow conditions. That is to say, in our simulation scenario the interstation travel times, $\tilde{t}_{i,j}$, are sampled from the following probability distribution:

$$\tilde{t}_{i,j} = \max\{t_{i,s}^{min}, t_{i,j} \sim \mathcal{N}\left(\mathbb{E}(t_{i,j}), \text{Var}(t_{i,j})\right)\} \tag{11}$$

Furthermore, our simulation tests assume that passenger arrivals at each bus stop follow a Poisson process, as in Fu and Yang, Newell and Potts (*18, 50*). This is a reasonable assumption because several studies have shown that passengers do not coordinate their arrivals at bus stops with the expected arrival times of buses in high-frequency services Hickman, Ding et al. (*43, 51*). In addition, as in Gkiotsalitis and Cats, Ding et al. (*46, 51*), the hourly passenger boarding and alighting rates in the simulation tests are assumed to be constant.

In the simulation analysis, four performance measures are used to evaluate the general effects of our control strategy that seeks to minimize the deviation between the actual and the target headways while meeting the charging schedules. These are: (i) the average passenger waiting times, (ii) the average bus travel time, (iii) the missed scheduled charging(s), and (iv) the overall charging delay.

The average passenger waiting times are calculated according to the well-known formula of Newell and Potts (*50*):

$$\mathbb{E}[W] \doteq \frac{\mathbb{E}[H]}{2} + \frac{\text{Var}[H]}{2\mathbb{E}[H]} \tag{12}$$

where $\mathbb{E}[W]$ is the average passenger waiting time and $\text{Var}[H]$ the headway variance at the control point stop. For our simulation analysis, we assume the following values (Table 2).

TABLE 2 : Parameter values of the simulated line

Parameter	Value
$\mathbb{E}(t_{i,1}), \sqrt{\text{Var}(t_{i,1})}$	1700 s, 100 s
$\mathbb{E}(t_{i,s}), \sqrt{\text{Var}(t_{i,s})}$	1000 s, 100 s
$t_{i,1}^{min}$	1500 s
$t_{i,s}^{min}$	800 s
H_0	360 s
t_s^y	1200 s
c	1.0

In addition, the scheduled dispatching and charging times of each trip are presented in Table 3.

TABLE 3 : Dispatching and scheduled charging times of the 10 simulated trips

Trip, i	Dispatching time	Scheduled charging time, ρ_i
1	0 s	2900 s
2	360 s	3260 s
3	720 s	3980 s
4	1080 s	4340 s
5	1440 s	4700 s
6	1800 s	5060 s
7	2160 s	5420 s
8	2520 s	5780 s
9	2880 s	6140 s
10	3240 s	6500 s

Then, we perform a comparative analysis using as a benchmark the control logic of Fu and Yang (*18*). In the comparative analysis, the control logic of Fu and Yang (*18*) and the control logic proposed in this study are applied in the same simulation scenario. Similar to the vast majority of works in bus holding, the control logic of Fu and Yang (*18*) is deterministic and does not consider the scheduled charging times of electric buses. The logic of Fu and Yang (*18*) is summarized as:

$$d_{i,s} = \begin{cases} d_{i-1,s} + H_0 \text{ if } T_{i,s} < d_{i-1,s} + cH_0 \\ T_{i,s} \text{ otherwise} \end{cases} \tag{13}$$

where $c \in \mathbb{R} : 0 \leq c \leq 1$ and is a holding control parameter.

Evaluation

To reduce the comparison bias of the comparative analysis, we run 1,000 Monte Carlo simulations with repeated random sampling. For this reason, at each simulation we sample the interstation travel times using Eq.(11) and apply the respective control logic given the underlying status of the simulated operations.

The average values of the four performance measures after using the respective holding control measures at each simulation are summarized in Table 4. The improvement (deterioration) of the performance indicators when applying our proposed control logic instead of the control logic of Fu and Yang (*18*) is provided in Fig.5.

TABLE 4 : Average performance of the two control logics in 1,000 simulation scenarios

Key Performance Indicator	Control Logic of Fu and Yang (18)	Proposed Control Logic
Average Passenger Waiting Time (s)	182.4	184.4
Average Trip Travel Time (s)	4996	4887
Missed Chargings	4	1
Overall Charging Delay (s)	96.59	63.52

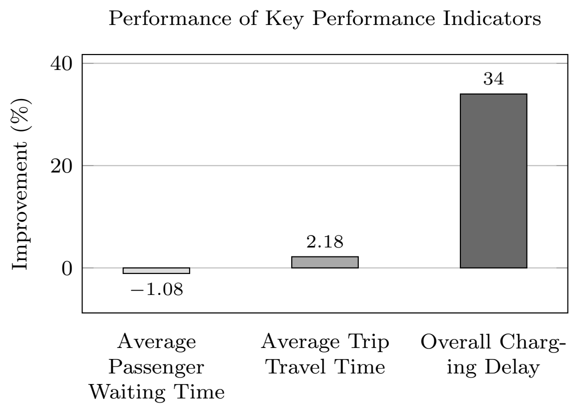

FIGURE 5 : Average improvement (deterioration) of the main Key Performance Indicators when applying the proposed control logic.

DISCUSSION

Evaluation results

As shown in Fig.5, the average passenger waiting time is increased by 1.08% when applying our control logic. Therefore, passengers will have to wait 2 seconds more (on average). This is expected because, unlike past studies that seek to minimize the average passenger waiting times Fu and Yang, Daganzo, Berrebi et al. (*18*, *21*, *37*), our control logic prioritizes the adherence to the charging schedule.

From our evaluation, one can observe that the deterioration of average passenger waiting times is minimal compared to the potential gains from arriving at the charging points on time (i.e., the improvement of charging delays stands at 34%). This is one of the main findings of our evaluation and underlines the potential benefit of our proposed control logic.

In addition, our control logic has the following indirect positive effects:

- the total trip travel time is reduced by 2.18% (on average). Hence, the in-vehicle travel times of passengers are also reduced;

- bus trips are completed earlier; thus, it is more probable that the planned dispatches of future trips operated by the same buses will not be postponed. This alleviates the negative effects of "schedule sliding" which is a key side-effect of bus holding control Daganzo (*21*).

The aforementioned indirect effects are crucial, and other simulation studies, such as Lin et al. (*52*), have also made it explicit that one should consider the trade-off between passenger wait times and other factors.

Limitations

To facilitate the reproducibility of our work, we state the main limitations of our control logic. Those limitations are:

- it can be only applied to high-frequency bus lines (more than 5 buses per hour) that operate under regularity-based schemes;
- it is suitable for correcting the effects of mild disruptions to the service regularity. In the case of severe disruptions, bus operators should consider more radical measures such as changes in the planned service provision and resource allocation;
- as in the vast majority of bus holding works Fu and Yang, Daganzo, Eberlein et al., Hickman (*18, 21, 33, 43*), our control logic is suitable in the context where service supply (capacity) is sufficient to ensure that there are no passengers who are unable to board due to overcrowding.

CONCLUSION

This work provided a control logic for bus holding of electric buses. The consideration of the scheduled charging times of electric buses added another dimension to the traditional bus holding problem and this resulted in a non-smooth mathematical program which is not always feasible. Introducing slack variables, we proposed a novel reformulation resulting in a mathematical program with a convex, quadratic objective function and linear inequality constraints.

Our reformulated program can be easily solved by exact optimization solution methods and it can be used to derive reliable solutions in the case of travel time uncertainty. After carrying out a systematic simulation-based analysis of the performance of our proposed bus holding(s), the following conclusions have been drawn:

- the charging time delay(s) can be reduced by up to 34% with a minimal trade-off of 1.08% increase in passenger waiting times;
- restraining the holding times due to the charging constraints can improve the total trip travel times and limit schedule sliding effects.

In future research, our approach can be expanded in a wide range of problems involving electric vehicles. For instance, with the proper modifications, future research can expand our method to railway operations that operate under regularity-based schemes.

Other advances could be the expansion towards using a two-headway-based logic to consider the headway with the preceding and the following bus when making a holding decision, and the incorporation of time-varying charging costs in the objective function.

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