

Closure coefficients in scale-free complex networks

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Abstract

The formation of triangles in complex networks is an important network property that has received tremendous attention. Recently, a new method to measure triadic closure was introduced: the closure coefficient. This statistic measures clustering from the head node of a triangle (instead of from the center node, as in the often studied clustering coefficient). We analyze the behavior of the local closure coefficient in two random graph models that create simple networks with power-law degrees: the hidden-variable model and the hyperbolic random graph. We show that the closure coefficient behaves significantly different in these simple random graph models than in the previously studied multigraph models. We also show that the closure coefficient can be related to the clustering coefficient and the average nearest neighbor degree.

1 Introduction

Networks describe the connectivity patterns of pairs of vertices. Examples of networks include social networks, the Internet, biological networks, the brain or communication networks. While these examples are very different in application, their connection patterns often share some characteristics. For example, in many real-world networks, vertices tend to cluster together in groups with relatively many edges between the group members [15] and networks often contain a large amount of triangles [32, 34].

Often the amount of triangles, or clustering, is measured in terms of the local clustering coefficient $c(k)$, the probability that two neighbors of a degree- k vertex are neighbors themselves. In most real-world networks, the function $c(k)$ decreases in k as some power law [33, 25, 31, 5, 10, 23, 29]. This indicates for example that two random friends of a popular person are less likely to know each other than two random friends of a less popular person. The local clustering curve of a network influences the spread of epidemic processes on a network, it contains information about its community structure, and is an important feature in classifying emails as spam or non spam, making it an important network property [27, 11, 3].

While the clustering coefficient turns out to be useful in some important network applications, other statistics that measure triadic closure may be more useful in other network applications. One promising such network statistic is the recently introduced the closure coefficient [37]. Where the clustering coefficient measures the fraction of times a vertex of degree k serves as the center of a triangle, the closure coefficient measures the fraction of times a vertex of degree k serves as the head of a triangle. Whereas the local clustering coefficient often decreases in k , the closure coefficient of many real-world networks is increasing instead [37], thus behaving substantially different from the local clustering coefficient. The closure coefficient was shown to be useful in link prediction [37] and other types of network prediction tasks [38]. Furthermore, the conductance of the neighborhood of a vertex can be linked to its closure coefficient [37], a statistic that is often used in community detection. Thus, the closure coefficient provides essential information about the tendency for network clustering which complements the information obtained from the clustering coefficient.

To be able to interpret the closure coefficient on real-world network data, it is crucial to understand its behavior in network null models, also known as random graph models. Whereas the behavior of the local clustering coefficient is well-understood in many types of random graphs [10, 23, 29, 22, 18], the behavior of the closure coefficient is yet unknown in most random graph models. Only for one such model, the configuration model, the local closure coefficient was analyzed and was shown to be proportional to k [37, 38]. However, the configuration model creates multigraphs: graphs with self-loops and multiple edges, whereas most real-world networks are simple networks. This makes the configuration model unfit for understanding real-world networks. Therefore, analyzing the behavior of the closure coefficient in random graph models that create simple networks instead is an important problem. However, imposing simplicity constraints on a random graph model often makes it more difficult to analyze, because of the non-trivial degree-degree correlations that arise from such simplicity constraints [19, 8, 35].

Degree-correlations often become more pronounced when the degree distribution is scale-free. Scale-free networks describe network connections with strong degree heterogeneity, often modeled as a power law where the proportion of vertices with k neighbors scales as $k^{-\tau}$. In many real-world networks, the power-law exponent τ was often found to lie between 2 and 3 [1, 12, 21, 33], so that the vertex degrees have a finite first and infinite second moment. These power laws make vertices of extremely high degrees (also called hubs) likely to be present, causing non-trivial degree-correlations [26, 19]. Power-law degrees and hubs significantly influence local network properties such as the presence of certain subgraphs like triangles and cliques [26, 31, 18].

In this paper, we focus on two substantially different simple network models that create power-law degrees and analyze their closure coefficient, taking these non-trivial degree correlations into account. We first investigate the closure coefficient in the hidden-variable model. In the absence of high-degree vertices, this model creates graphs that are similar to the configuration model [17]. With power-law degrees however, the degree-degree correlations make these two models significantly different [19]. We show that the degree-degree correlations that come with the simplicity constraint result in considerably different behavior for the local closure coefficient in the hidden-variable model than in the configuration model, where such correlations are not present.

The configuration model, as well as the hidden-variable model are locally tree-like and therefore do not contain many triangles in the large-network limit. Real-world networks on the other hand, often contain high levels of clustering [15]. We therefore investigate the closure coefficient in the hyperbolic model, which in recent years has emerged as an important scale-free network model [24, 2, 6, 14, 7]. The hyperbolic model creates a random graph by sampling vertices in hyperbolic space, and then connects pairs of vertices that are geometrically close. The hyperbolic model is mathematically tractable and produces networks that simultaneously possess two crucial characteristics of real-world networks: power-law degrees and clustering. Therefore, the behavior of the closure coefficient in the hyperbolic model could potentially serve as a benchmark for the behavior of the closure coefficient in real-world network data.

A second contribution of this paper is in relating the closure coefficient to two other frequently used network statistics: the clustering coefficient, which measures the presence of triangles from a different perspective and the average nearest neighbor degree of a vertex. Specifically, we show that the closure coefficient of a vertex can be related to a ratio of its clustering coefficient with its average nearest neighbor degree. Since the behavior of these statistics is known for many random graph models, this provides a simple method to investigate the closure coefficient of many network models.

We further show that the behavior of the local closure coefficient of a randomly chosen vertex of degree k may be significantly different from the average behavior over all vertices of degree k , particularly for small values of k . Since the closure coefficient of a vertex gives a bound on the conductance of its neighborhood, we propose to analyze the 'typical' closure coefficient of vertices of degree k instead of the local closure coefficient that averages over all vertices of degree k , since the latter is dominated by rare events where the closure coefficient is much higher than for almost all other vertices. Furthermore, the typical behavior of the closure coefficient is easier to analyze.

We also analyze the higher-order closure coefficient, measuring the closure of cliques of size l into larger cliques of size $l + 1$. These higher order closure coefficient bound the motif conductance

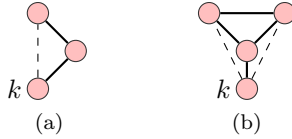


Figure 1: The closure coefficient investigates the probability that a two-path starting from a degree- k vertex closes into a triangle (with the dashed edge), illustrated in (a). (b) illustrates $H_3(k)$. The solid lines from a 3-wedge, and $H_3(k)$ computes the probability that the dashed lines are present.

of the neighborhood of a vertex, which is useful in community detection [37]. The behavior of this higher order local closure coefficient for any type of random graph model was yet unknown. In this paper, we provide the asymptotic behavior of the higher-order closure coefficient for both the hyperbolic model and the hidden-variable model. We obtain these asymptotics by optimizing the structure of network cliques, a method which has recently been proposed to analyze subgraphs and local clustering [29, 30].

We first present the definitions of the local closure coefficient as well as the higher-order closure coefficients in Section 2. We then analyze the behavior of the closure coefficient in the hidden-variable model in Section 3 and for the hyperbolic random graph in Section 4.

Notation We write $\xrightarrow{\mathbb{P}}$ for convergence in probability. We say that a sequence of events $(\mathcal{E}_n)_{n \geq 1}$ happens with high probability (w.h.p.) if $\lim_{n \rightarrow \infty} \mathbb{P}(\mathcal{E}_n) = 1$. Furthermore, we write $X_n = o_{\mathbb{P}}(g(n))$ if $X_n/g(n) \xrightarrow{\mathbb{P}} 0$. We say that $X_n \propto_{\mathbb{P}} g(n)$ if for any $h(n)$ such that $\lim_{n \rightarrow \infty} h(n) = \infty$,

$$\liminf_{n \rightarrow \infty} \mathbb{P}\left(\frac{X_n}{g(n)} < \frac{1}{h(n)}\right) = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \mathbb{P}\left(\frac{X_n}{g(n)} > h(n)\right) = 0. \quad (1)$$

Finally, we denote $[k] = 1, 2, \dots, k$.

2 Closure coefficients

The closure coefficient of vertex v , $H(v)$, is defined by [37]

$$H(v) = \frac{2\Delta_v}{W(v)}, \quad (2)$$

where Δ_v denotes the number of triangles attached to vertex v and $W(v)$ the number of two-paths from vertex v . The coefficient 2 accounts for the fact that each triangle contains two different length-2 paths from node v . Thus, the closure coefficient measures the fraction of two-paths that merge into triangles, as illustrated in Fig. 1a. The average local closure coefficient is then defined as the average of this quantity over all vertices of degree k ,

$$H^{(a)}(k) = \frac{1}{N_k} \sum_{i: d_i=k} \frac{\Delta_{v_i}}{W(i)}, \quad (3)$$

where N_k denotes the number of vertices of degree k , and d_i denotes the degree of vertex i .

2.1 Higher-order closure

Where the closure coefficient measures the fraction of two-paths merging into triangles, higher order closure coefficients measure the tendency of forming larger cliques.

We define an l -wedge from vertex v as an edge incident to vertex v connecting to some other vertex u , such that u is part of an l -clique [36]. See Fig. 1b for an example. Let $W^{(l)}(v)$ denote

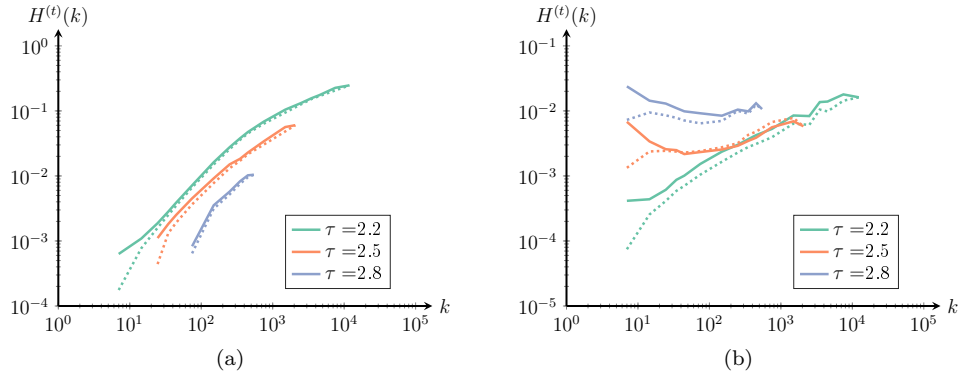


Figure 2: The solid line plots $kc(k)/a(k)$ whereas the dashed line plots $H^{(t)}(k)$, for $n = 10^5$ and various values of τ for a) the hidden variable model b) the hyperbolic random graph.

the number of l -wedges attached to vertex v . Furthermore, let $K_l(v)$ denote the number of cliques incident to vertex v . Then, the l -th order closure coefficient is defined as the fraction of l -wedges that close into $l + 1$ cliques containing vertex v . Formally,

$$H_l(v) = \frac{lK_{l+1}(v)}{W^{(l)}(v)}, \quad (4)$$

where the coefficient l accounts for the fact that each $l + 1$ clique contains l different l -wedges. Then $H_l^{(a)}(k)$ is again defined as the average of this quantity over all vertices of degree k .

2.2 Typical closure

For the local closure coefficient, the typical behavior of its numerator as well as the denominator can be analyzed easily. However, when taking the average of these random ratios over all vertices of degree k , some rare values with a large ratio may dominate the value of the average local closure coefficient. In this paper, we therefore focus on the ‘typical’ closure coefficient of vertices with degree k instead of its average. Typical closure can be interpreted as the behavior of the closure coefficient of a randomly chosen vertex of degree k and we denote the typical closure coefficient of a vertex of degree k by $H^{(t)}(k)$. Formally, let V_k denote a randomly chosen vertex of degree k . Then, we define the typical closure coefficient as

$$H^{(t)}(k) = \frac{\Delta_{V_k}}{W(V_k)}. \quad (5)$$

This typical closure is more informative than average closure, since it provides information on the closure of a randomly chosen vertex of degree k , instead of on the closure of some rare vertices of degree k with extremely high closure values. Furthermore, the typical closure coefficient is easier to analyze, as it allows to analyze the numerator and denominator separately. Similarly, we denote the typical higher-order closure coefficient by $H_l^{(t)}(k)$, which is defined as the higher-order closure coefficient of a randomly chosen vertex of degree k .

In the rest of this paper, we will investigate the asymptotic behavior of these clustering coefficient. That is, we investigate the behavior of $H^{(t)}(k_n)$ as the network size n tends to infinity, while k_n grows as a function of n .

2.3 Relation with $a(k)$ and $c(k)$

We now show that in many random graph models, the typical closure coefficient can be related to two other important network statistics: the average nearest neighbor degree $a(k)$ and the local

clustering coefficient $c(k)$. The average nearest neighbor degree $a(k)$ is defined as

$$a(k) = \frac{1}{kN_k} \sum_{i:d_i=k} \sum_{j \in \mathcal{N}_i} d_j, \quad (6)$$

where \mathcal{N}_i denotes the set of neighbors of vertex i and N_k again denotes the number of vertices of degree k .

The local clustering coefficient $c(k)$ measures the fraction of pairs of neighbors of a vertex that close into triangles. Thus, the local clustering coefficient measures a similar tendency as the local closure coefficient, with the difference that the clustering coefficient measures the fraction of triangles that are formed with v as the center of the triangle, whereas the closure coefficient measures the fraction of triangles with v as the head of the triangle. Formally, the local clustering coefficient of vertex v is defined as

$$c(v) = \frac{\Delta_v}{d_v(d_v - 1)/2}. \quad (7)$$

Usually, the local clustering coefficient is analyzed by averaging over all vertices of degree k , which is denoted by $c(k)$. A notable difference between $c(k)$ and the local closure coefficient is that whereas $c(k)$ has a random numerator (the number of triangles at a vertex of degree k) but a deterministic denominator ($k(k-1)/2$), the closure coefficient has both a random numerator and a random denominator, making the analysis of the closure coefficient more involved.

We now explain the relation between $H^{(t)}(k)$ and the two other network statistics $a(k)$ and $c(k)$. The average number of wedges from a vertex of degree k equals $k(a(k) - 1)$ (where the -1 accounts for the fact that one of the connections of a neighbor is used to connect to the degree k vertex itself). Also, the average number of triangles attached to a vertex of degree k equals $k^2 c(k)/2$.

Furthermore, for k sufficiently large, the number of wedges from a vertex of degree k concentrates around $k(a(k) - 1) \approx ka(k)$ in many random graph models [28, Proposition 2]. That is, if $W(V_k)$ denotes the number of wedges from a randomly chosen vertex of degree k , then in many random graph models

$$\frac{W(V_k)}{ka(k)} \xrightarrow{\mathbb{P}} 1. \quad (8)$$

Similarly, for k sufficiently large, the number of triangles attached to a vertex of degree k concentrates around $k^2 c(k)/2$ for several random graph models [29]. Thus, if Δ_{V_k} denotes the number of triangles attached to a randomly chosen vertex of degree k , then in many random graph models

$$\frac{\Delta_{V_k}}{k^2 c(k)/2} \xrightarrow{\mathbb{P}} 1. \quad (9)$$

This yields for the typical closure coefficient that

$$\frac{H^{(t)}(k)}{kc(k)/(2a(k))} \xrightarrow{\mathbb{P}} 1. \quad (10)$$

For many random graph models $c(k)$ and $a(k)$ are known. Furthermore, for k sufficiently large, the number of wedges and triangles concentrate around these values for many random graph models. Therefore, this gives an easy method to investigate the local closure coefficient of many types of random graph models when k is sufficiently large.

Figure 2 illustrates the relationship between $H^{(t)}(k)$, $a(k)$ and $c(k)$ for two different random graph models. For the hidden-variable model (which will be introduced in Section 3)), (10) also holds for small values of k , whereas for the hyperbolic model (which will be introduced in Section 4)), the fit only holds for larger values of k .

3 Hidden variable model

The hidden-variable model creates a random graph on n vertices. Every vertex i is equipped with a weight, h_i . These weights are drawn independently from the distribution

$$\rho(h) = Ch^{-\tau}, \quad (11)$$

for some $\tau \in (2, 3)$. Every pair of vertices (h, h') is then connected with probability $p(h, h')$. In this paper we work with (although many other choices are possible)

$$p(h, h') = \min\left(\frac{hh'}{\langle h \rangle n}, 1\right), \quad (12)$$

which is equivalent to the Chung-Lu model [9]. Here $\langle h \rangle$ denotes the expected vertex weight. In the large-network limit, the degree of vertex i will be close to its hidden variable h_i . Thus, the hidden-variable model creates simple, power-law random graphs in this manner.

3.1 Closure coefficient

We now analyze the closure coefficient in the hidden-variable model. The number of wedges attached to a vertex of degree k with $k \gg n^{(\tau-2)/(\tau-1)}$ scales as $ka(k)$ with high probability [28] in the hidden variable model. Furthermore, for $k \gg n^{(\tau-2)/(\tau-1)}$ [28],

$$\frac{a(k)}{n^{3-\tau}k^{\tau-3}} \xrightarrow{\mathbb{P}} \frac{C\langle h \rangle^{2-\tau}}{(3-\tau)(\tau-2)}. \quad (13)$$

Similarly, the number of triangles attached to a vertex of degree k with $k \gg n^{(\tau-2)/(\tau-1)}$ concentrates around $k^2c(k)$, where with high probability for $k \gg n^{(\tau-2)/(\tau-1)}$ [31]

$$c(k) = \begin{cases} n^{2-\tau} \log(n/k^2) \frac{C^2\langle h \rangle^{-\tau}}{(3-\tau)(\tau-2)} (1 + o_{\mathbb{P}}(1)) & n^{(\tau-2)/(\tau-1)} \ll k \ll \sqrt{n} \\ n^{5-2\tau} k^{2\tau-6} \frac{C^2\langle h \rangle^{3-2\tau}}{(3-\tau)^2(\tau-2)^2} (1 + o_{\mathbb{P}}(1)) & k \gg \sqrt{n}. \end{cases} \quad (14)$$

Thus, by (10), for $k \gg n^{(\tau-2)/(\tau-1)}$, the typical closure coefficient of a vertex of degree k in the hidden-variable model satisfies

$$H^{(t)}(k) = \begin{cases} n^{-1} k^{4-\tau} \log(n/k^2) C\langle h \rangle^{-2} (1 + o_{\mathbb{P}}(1)) & n^{(\tau-2)/(\tau-1)} \ll k \ll \sqrt{n} \\ n^{2-\tau} k^{\tau-2} \frac{\langle h \rangle^{1-\tau} C}{2(3-\tau)(\tau-2)} (1 + o_{\mathbb{P}}(1)) & k \gg \sqrt{n}. \end{cases} \quad (15)$$

Note that this behavior is significantly different from the $H^{(a)}(k) \propto k$ behavior found in the configuration model [37]. This difference is caused by the numerous multiple edges appearing in the configuration model for $\tau \in (2, 3)$, as well as by the difference between the behavior of the average closure coefficient versus its typical behavior.

Figure 3 shows that Eq. (15) indeed describes the asymptotic behavior of $H^{(t)}(k)$ well. Figure 4a shows that indeed the distribution of the closure coefficient is skew for small values of k , whereas it becomes less skew for larger values of k . This implies that the typical closure coefficient becomes closer to its average value when k gets larger.

3.2 Higher-order closure

We now investigate the typical higher-order closure coefficient of a vertex of degree k . To obtain this closure coefficient, it is convenient to study the typical higher-order closure coefficient of a vertex of *weight* h instead, which we denote by $\bar{H}_l^{(t)}(h)$. That is, when V_h denotes a randomly chosen vertex of weight h , then

$$\bar{H}_l^{(t)}(h) = \frac{lK_{l+1}(V_h)}{W^{(l)}(V_h)}. \quad (16)$$

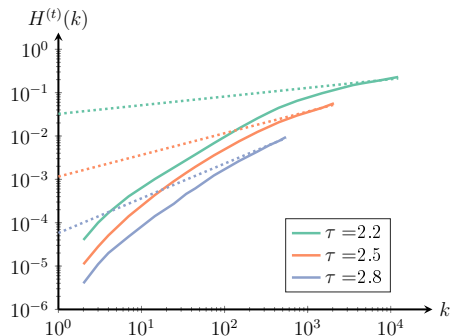


Figure 3: $H^{(t)}(k)$ averaged over 10^4 realizations of hidden-variable models with $n = 10^6$ and various values of τ . The dashed line gives the asymptotic slope for $k \gg \sqrt{n}$ predicted by Eq. (15).

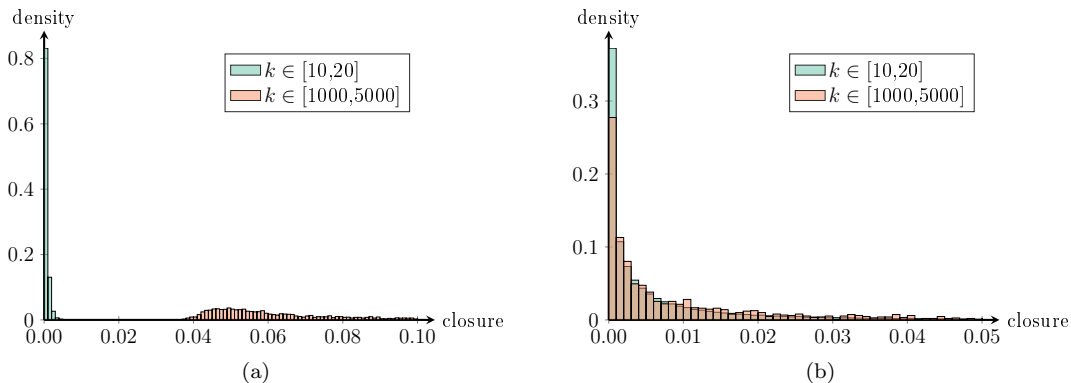


Figure 4: Density plot of closure coefficients for two ranges of k and $n = 10^6$, $\tau = 2.5$ in a) the hidden-variable model and b) the hyperbolic model.

This is convenient, as the connection probabilities in Eq. (12) are defined in terms of weights as well. In Appendix A we then show that because the degree of a vertex is tightly concentrated around its weight, our results for the higher-order closure coefficient of a vertex of given weight also hold for the typical higher-order closure coefficient of a vertex of *degree* k , $H_l^{(t)}(k)$.

Cliques attached to a vertex of weight h . First, we analyze the numerator of Eq. (16). Thus, we investigate the typical number of cliques of size $l + 1$ attached to a vertex of weight h . To obtain this, we find the optimal clique structure in terms of the weights of vertices involved in the clique as in [30, 29]. The probability that the vertex of weight h attaches to a clique of size l can be written as

$$\mathbb{P}(\text{clique}) = \int_{\mathbf{h}} \mathbb{P}(\text{clique with vertices of weights } \mathbf{h}) \rho(h_1) \rho(h_2) \dots \rho(h_l) d\mathbf{h} \quad (17)$$

where the integral is over all possible weight sequences $\mathbf{h} = (h_1, \dots, h_l)$ of the l vertices involved in the clique. We now let h_1, \dots, h_l scale as $n^{\alpha_1}, \dots, n^{\alpha_l}$ and find which weights give the largest contribution to the integrand in (17).

We compute the probability that a randomly chosen vertex of weight h forms a clique with l neighbors which have weights proportional to $(n^{\alpha_i})_{i \in [l]}$. By (12), the probability that these vertices form a clique with the weight- h vertex is proportional to

$$\mathbb{P}(\text{clique of vertices of weights } n^{\alpha_i} \text{ with } V_h) \propto \prod_{i < j} \min(n^{\alpha_1 + \alpha_j - 1}, 1) \prod_{j \in [l]} \min(h n^{\alpha_j - 1}, 1), \quad (18)$$

where the second product denotes the probability that all vertices connect to the weight- h vertex, and the first product denotes the probability that all other vertices connect. By (11), with high probability there are proportionally $n^{1+\alpha_i(1-\tau)}$ vertices of weight proportional to n^{α_i} . Thus, with high probability, the number of cliques containing the weight- h vertex and l vertices of weight $(n^{\alpha_i})_{i \in [l]}$ scales as

$$\begin{aligned} & \# \text{ cliques with } V_h \text{ and vertices of weights } (n^{\alpha_i})_{i \in [l]} \\ & \propto_{\mathbb{P}} n^{l+\sum_i \alpha_i(1-\tau)} \prod_{j \in [l]} \min(hn^{\alpha_j-1}, 1) \prod_{i < j} \min(n^{\alpha_1+\alpha_j-1}, 1). \end{aligned} \quad (19)$$

Similarly as in [20, Theorem 2.1], we can show that for $h \gg 1$,

$$K_{l+1}(V_h) \propto_{\mathbb{P}} \max_{\alpha_1, \dots, \alpha_l \in [0, 1/(\tau-1)]} n^{l+\sum_i \alpha_i(1-\tau)} \prod_{j \in [l]} \min(hn^{\alpha_j-1}, 1) \prod_{i < j} \min(n^{\alpha_1+\alpha_j-1}, 1), \quad (20)$$

if this equation has a unique maximizer over $\alpha_1, \dots, \alpha_l \in [0, 1/(\tau-1)]$. Here the constraint $\alpha \in [0, 1/(\tau-1)]$ comes from the fact that the maximal weight sampled from (11) is proportional to $n^{1/(\tau-1)}$ with high probability.

For $l \geq 3$ (19) is uniquely maximized at $\alpha_i = 1/2$ for all i . Plugging this optimizer into (20) shows that the number of complete graphs of size $l+1$ attached to a typical vertex of weight h scales as

$$K_{l+1}(V_h) \propto_{\mathbb{P}} n^{l(3-\tau)/2} \min(hn^{-1/2}, 1)^l. \quad (21)$$

Optimizing (19) not only gives the scaling of the number of cliques, it also gives the structure of the most likely clique attached to a weight- h vertex. That is, the fact that (19) is optimized by $\alpha_i = 1/2$ for all i shows that almost all cliques attached to vertices of weight h (and therefore degree close to h with high probability, as shown in Appendix A) are cliques where the weights (and therefore the degrees) of the other vertices involved are proportional to \sqrt{n} .

Wedges attached to a vertex of weight h . We now analyze the denominator of (4) and compute the typical number of l -wedges from a vertex of weight h . That is, we count the number of edges from the weight- h vertex attached to a complete graph of size l . To obtain this, we use a similar optimization problem as for the number of l -cliques. We now compute the probability that a vertex of weight h is attached to a clique with l vertices of weights proportional to $(n^{\alpha_i})_{i \in [l]}$. W.l.o.g. we assume that the weight- h vertex attaches to vertex 1 in the clique (the vertex with weight proportional to n^{α_1}). By (12), the probability that the weight- h vertex connects to vertex 1 and that vertex 1 forms a clique with the other $l-1$ vertices is given by

$$\begin{aligned} & \mathbb{P}(\text{weight-}h \text{ vertex attached to clique with weights } n^{\alpha_i}) \\ & \propto \min(hn^{\alpha_1-1}, 1) \prod_{1 \leq i < j \leq l} \min(n^{\alpha_1+\alpha_j-1}, 1). \end{aligned} \quad (22)$$

Here the first term denotes the probability that the weight- h vertex connects to the vertex of weight n^{α_1} , and the second term denotes the probability that vertex 1 forms a clique with the other $l-1$ vertices. Again, by (11), with high probability, there are $n^{1+\alpha_i(1-\tau)}$ vertices of weight proportional to n^{α_i} . Thus, with high probability, the number of l -wedges attached to the weight- h vertex where the other vertices have weights $(n^{\alpha_i})_{i \in [l]}$ is proportional to

$$\begin{aligned} & \# \text{ wedges } V_h \text{ with vertices of weights } (n^{\alpha_i})_{i \in [l]} \\ & \propto_{\mathbb{P}} n^{l+\sum_i \alpha_i(1-\tau)} \min(hn^{\alpha_1-1}, 1) \prod_{i < j} \min(n^{\alpha_1+\alpha_j-1}, 1). \end{aligned} \quad (23)$$

Again, similarly to (20), we can show that for $h \gg 1$,

$$W^{(l)}(V_h) \propto_{\mathbb{P}} \max_{\alpha_1, \dots, \alpha_l \in [0, 1/(\tau-1)]} n^{l+\sum_i \alpha_i(1-\tau)} \min(hn^{\alpha_1-1}, 1) \prod_{i < j} \min(n^{\alpha_1+\alpha_j-1}, 1), \quad (24)$$

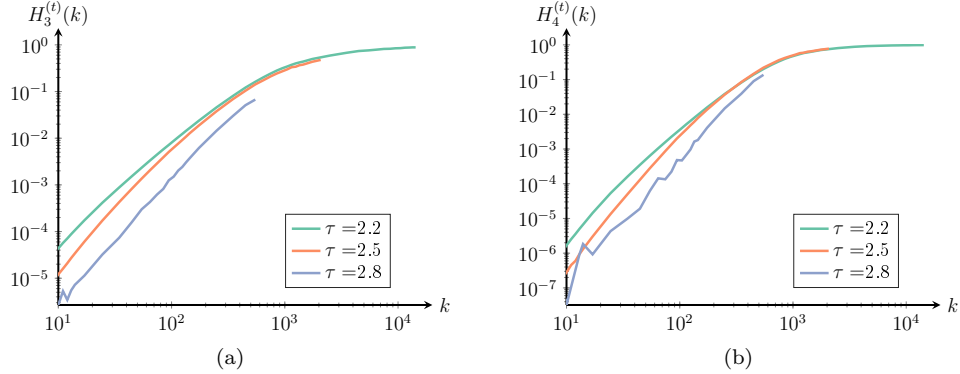


Figure 5: Higher-order closure coefficients in the hidden-variable model with $n = 10^6$ and various values of τ . a) $H_3(k)$, b) $H_4(k)$.

if this equation has a unique maximizer over $\alpha_1, \dots, \alpha_l \in [0, 1/(\tau - 1)]$. For $l \geq 3$, this equation is indeed uniquely maximized for $\alpha_i = 1/2$ for all i . Plugging this optimizer into (24) yields that the typical number of l -wedges attached to a vertex of degree k scales as

$$W^{(l)}(V_h) \propto_{\mathbb{P}} n^{l(3-\tau)/2} \min(hn^{-1/2}, 1). \quad (25)$$

Combining this with (21) shows that the typical higher-order closure coefficient satisfies for $l \geq 3$,

$$\bar{H}_l^{(t)}(h) \propto_{\mathbb{P}} \begin{cases} h^{l-1} n^{-(l-1)/2} & 1 \ll h \ll \sqrt{n} \\ 1 & h \gg \sqrt{n}. \end{cases} \quad (26)$$

In Appendix A we show that also

$$H_l^{(t)}(k) \propto_{\mathbb{P}} \begin{cases} k^{l-1} n^{-(l-1)/2} & 1 \ll k \ll \sqrt{n} \\ 1 & k \gg \sqrt{n}. \end{cases} \quad (27)$$

Interestingly, the closure coefficient becomes independent of k for $k \gg \sqrt{n}$. This is caused by the core of the hidden-variable model, where most vertices of degrees \sqrt{n} and higher form a giant clique. Once you find a clique in this core, it is very likely to form a larger clique with additional vertices of degree \sqrt{n} and higher, explaining the constant higher-order closure coefficient of these vertices.

Figure 5 shows the behavior of $H_3^{(t)}(k)$ and $H_4^{(t)}(k)$ in the hidden-variable model. Indeed, when k gets sufficiently large, the closure coefficient approaches a constant value, as predicted by Eq. 27. Note that for τ close to 3, this is less pronounced, as the largest degree in a power-law network scales as $n^{1/(\tau-1)}$, which is close to \sqrt{n} .

4 Hyperbolic model

The hyperbolic random graph [24] is a key model that creates simple power-law random graphs with non-trivial clustering. The presence of an underlying geometric space creates subgraph structures that are significantly different from those in the hidden-variable model.

The hyperbolic random graph samples n vertices in a disk of radius $R = 2 \log(n/\nu)$. The density of the radial coordinate r of a vertex $p = (r, \phi)$ is

$$\rho(r) = \beta \frac{\sinh(\beta r)}{\cosh(\beta R) - 1} \quad (28)$$

with $\beta = (\tau - 1)/2$. Here ν is a parameter that influences the average degree of the generated networks. The angle ϕ of p is sampled uniformly from $[0, 2\pi]$. Two vertices are connected if their hyperbolic distance is at most R . The hyperbolic distance of points $u = (r_u, \phi_u)$ and $v = (r_v, \phi_v)$ satisfies

$$\cosh(d(u, v)) = \cosh(r_u) \cosh(r_v) - \sinh(r_u) \sinh(r_v) \cos(\Delta\theta), \quad (29)$$

where $\Delta\theta$ denotes the angle between u and v . Two neighbors of a vertex are likely to be close to one another due to the geometric nature of the hyperbolic random graph. Therefore, the hyperbolic random graph contains many triangles [16]. Furthermore, the model generates scale-free networks with degree exponent τ [24] and small diameter [13].

4.1 Closure coefficient

We now investigate the typical closure coefficient of a vertex in the hyperbolic random graph using the relation between the closure coefficient and the clustering coefficient and average nearest neighbor degree (10). Similarly as in the hidden-variable model [28]

$$a(k) \propto_{\mathbb{P}} \begin{cases} n^{\frac{3-\tau}{\tau-1}} & k \ll n^{(\tau-2)/(\tau-1)} \\ n^{3-\tau} k^{\tau-3} & k \gg n^{(\tau-2)/(\tau-1)}. \end{cases} \quad (30)$$

Moreover, for $k \gg n^{(\tau-2)/(\tau-1)}$ the number of wedges attached to a vertex of degree k concentrates around $ka(k)$ [28]. Furthermore, in the hyperbolic model

$$c(k) \propto_{\mathbb{P}} \begin{cases} k^{-1} & \tau > 5/2 \\ k^{4-2\tau} & \tau < 5/2, k \ll \sqrt{n} \\ k^{2\tau-6} n^{5-2\tau} & \tau < 5/2, k \gg \sqrt{n}, \end{cases} \quad (31)$$

and the number of triangles concentrates around $k^2 c(k)$ for $k \gg 1$ [29].

Thus, for $k \gg n^{(\tau-2)/(\tau-1)}$, $H^{(t)}(k)$ satisfies

$$H^{(t)}(k) \propto_{\mathbb{P}} \begin{cases} n^{\tau-3} k^{3-\tau} & k \gg n^{\frac{\tau-2}{\tau-1}}, \tau > 5/2 \\ k^{8-3\tau} n^{\tau-3} & n^{\frac{\tau-2}{\tau-1}} \ll k \ll \sqrt{n}, \tau < 5/2, \\ k^{\tau-2} n^{2-\tau} & k \gg \sqrt{n}, \tau < 5/2. \end{cases} \quad (32)$$

This shows that the behavior of the closure coefficient undergoes a transition when $\tau = 5/2$. This transition is caused by the typical structure of a triangle in the hyperbolic model. Whereas for $\tau > 5/2$ a typical triangle from a vertex of degree k contains two other vertices of constant degree, for $\tau < 5/2$ it contains two other vertices of degree n/k when k is sufficiently large [29].

Furthermore, for τ close to 2 or 3, the slope of the closure coefficient in k becomes small, making it almost k -independent for large k and τ close to 2 or 3.

Figure 2b illustrates the behavior of $H^{(t)}(k)$ in the hyperbolic random graph. Figure 4b shows the density of the closure coefficient for the hyperbolic model for large as well as small values of k . For both large and small values, the distribution of the closure coefficient is skew, indicating a difference between the average and the typical closure coefficient.

4.2 Higher-order closure

We now analyze the higher-order closure coefficients of the hyperbolic random graph using similar methods as in Section 3. Here it is convenient to define *type* of a vertex as

$$w_i = e^{(R-r_i)/2}. \quad (33)$$

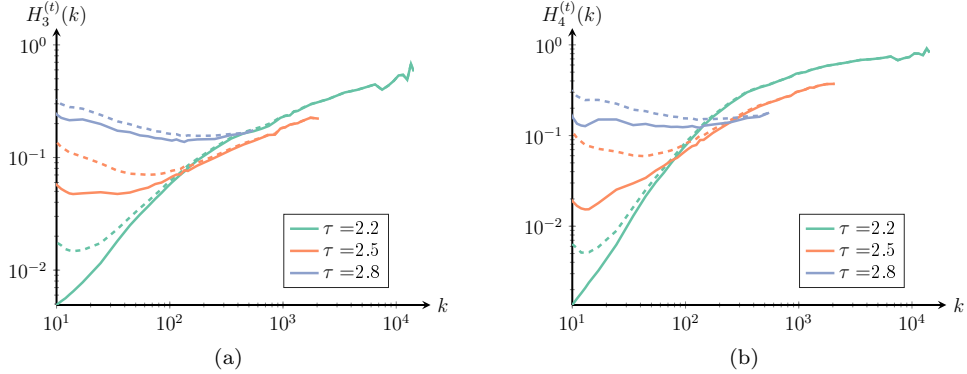


Figure 6: The solid line plots the median value of the closure coefficient whereas the dashed line its average value, for the hyperbolic random graph with $n = 10^5$ and various values of τ for a) $H_3(k)$, b) $H_4(k)$.

By [28], the type of a vertex is close to its degree. Therefore, we will first analyze the higher-order closure coefficient of a randomly chosen vertex with type w , which we denote by

$$\hat{H}_l^{(t)}(w) = \frac{lK_{l+1}(V_w)}{W^{(l)}(V_w)}, \quad (34)$$

where V_w is a randomly chosen vertex of type w . In Appendix B, we show that $\hat{H}_l^{(t)}(w)$ scales the same as $H_l^{(t)}(k)$ for $w = k$.

Cliques in the hyperbolic model with type- w vertex. We first compute the probability that a vertex of type w forms a clique with $l \geq 3$ vertices of types n^α . In the hyperbolic random graph, the probability that vertex i of type n^α connects to vertex j of type w satisfies [29]

$$\mathbb{P}(i \leftrightarrow j) \propto \min(wn^{\alpha-1}, 1). \quad (35)$$

Thus, the probability that the type- w vertex connects to the l vertices of type n^α scales as $\min(wn^{\alpha-1}, 1)^l$. W.l.o.g., assume that the type- w vertex is located at angle $\phi = 0$. Then, the angle of a randomly chosen neighbor of the type- w vertex of type proportional to n^α is uniformly distributed in an interval $[a, b]$, where $a \propto \min(wn^{\alpha-1}, 1)$ and $b \propto -\min(wn^{\alpha-1}, 1)$ [29]. These neighbors of the type- w vertex form a clique if they are sufficiently close. Two vertices of types proportional to n^α are connected if their relative angle is at most $2\nu n^{2\alpha-1}$ [4]. Thus, for the l vertices of type proportional to n^α to form a clique, their angles need to fall in the same interval of length proportional to $\min(n^{2\alpha-1}, 1)$.

The probability that l vertices have angular coordinates in a given interval of width proportional to $\min(n^{2\alpha-1}, 1)$ conditionally on the fact that they are uniformly distributed in some interval $[a, b]$ of width proportional to $\min(wn^{\alpha-1}, 1)$ is

$$\mathbb{P}(v_1 \leftrightarrow v_2) \propto \left(\min \left(\frac{\min(n^{2\alpha-1}, 1)}{\min(wn^{\alpha-1}, 1)}, 1 \right) \right)^{l-1} = \min(n^\alpha \max(n^{\alpha-1}, w^{-1}), 1)^{l-1}. \quad (36)$$

Here the power $l - 1$ comes from the fact that for l vertices to fall in the same interval, we can first choose the position of the first vertex. This vertex defines the center of this interval, and the other $l - 1$ vertices then have to be positioned in this interval.

Combining this with (35) yields

$$\mathbb{P}(l + 1\text{-clique with vertices of types } k, n^\alpha)$$

$$\propto \min(w n^{\alpha-1}, 1)^l \min(n^\alpha \max(n^{\alpha-1}, w^{-1}), 1)^{l-1}, \quad (37)$$

Since the type distribution of the hyperbolic random graph is also given by (11) [4, Lemma 1.3], the number of vertices of weights proportional to n^α is with high probability proportional to $n^{1+\alpha(1-\tau)}$, as in the hidden-variable model. Combining this with (37) yields that for $w \gg 1$

$$\begin{aligned} & \# \text{ cliques with } V_h \text{ and vertices of weights } (n^{\alpha_i})_{i \in [l]} \\ & \propto_{\mathbb{P}} n^{l+\alpha l(1-\tau)} \min(w n^{\alpha-1}, 1)^l \min(n^\alpha \max(n^{\alpha-1}, w^{-1}), 1)^{l-1}. \end{aligned} \quad (38)$$

Then, similarly to (20), for $w \gg 1$,

$$K_{l+1}(V_w) \propto_{\mathbb{P}} \max_{\alpha \in [0, 1/(\tau-1)]} n^{l+\alpha l(1-\tau)} \min(w n^{\alpha-1}, 1)^l \min(n^\alpha \max(n^{\alpha-1}, w^{-1}), 1)^{l-1}, \quad (39)$$

if this equation has a unique optimizer. The unique optimizer α^* is given by

$$n^{\alpha^*} = \begin{cases} n^0 & \tau > 3 - \frac{1}{l} \\ w & \tau < 3 - \frac{1}{l}, w \ll \sqrt{n} \\ n/w & 3 - \frac{2}{l} < \tau < 3 - \frac{1}{l}, w \gg \sqrt{n} \\ \sqrt{n} & \tau < 3 - \frac{2}{l}, w \gg \sqrt{n}. \end{cases} \quad (40)$$

Then, we obtain from (39)

$$K_{l+1}(V_w) \propto_{\mathbb{P}} \begin{cases} w & \tau > 3 - \frac{1}{l}, w \gg 1 \\ w^{l(3-\tau)} & \tau < 3 - \frac{1}{l}, 1 \ll w \ll \sqrt{n} \\ (n/w)^{l(3-\tau)-1} & 3 - \frac{2}{l} < \tau < 3 - \frac{1}{l}, w \gg \sqrt{n} \\ n^{(3-\tau)l/2} & \tau < 3 - \frac{2}{l}, w \gg \sqrt{n}. \end{cases} \quad (41)$$

Wedges in the hyperbolic model with type- w vertex. We now proceed to calculate the number of l -wedges attached to a randomly chosen vertex of type w . By (35) and the fact that the vertex types are distributed as (11), the number of neighbors of a vertex of type w that have type proportional to n^α scales as $n^{\alpha(1-\tau)+1} \min(w n^{\alpha-1}, 1)$ with high probability. Furthermore, with high probability, a vertex of type n^α is part of $K_l(n^\alpha)$ cliques, where $K_l(n^\alpha)$ is as in (41).

Thus, the number of l -wedges where the neighbor of the type- w vertex has type proportional to n^α satisfies for $w \gg 1$

$$\begin{aligned} & \# \text{ } l\text{-wedges attached to } V_w \text{ where neighbor of } V_w \text{ has type } n^\alpha \\ & \propto_{\mathbb{P}} n^{\alpha(1-\tau)+1} \min(w n^{\alpha-1}, 1) K_l(n^\alpha), \end{aligned} \quad (42)$$

where $K_l(n^\alpha)$ can be computed from (41). Similarly to (24), for $w \gg 1$,

$$W^{(l)}(V_w) \propto_{\mathbb{P}} \max_{\alpha \in [0, 1/(\tau-1)]} n^{\alpha(1-\tau)+1} \min(w n^{\alpha-1}, 1) K_l(n^\alpha), \quad (43)$$

if this has a unique optimizer over α . The unique optimizer α^* is given by

$$n^{\alpha^*} = \begin{cases} n^{1/(\tau-1)} & \tau > 3 - \frac{2}{l}, w \ll n^{\frac{\tau-2}{\tau-1}} \\ n/w & \tau > 3 - \frac{2}{l}, w \gg n^{\frac{\tau-2}{\tau-1}} \\ \sqrt{n} & \tau < 3 - \frac{2}{l} \end{cases} \quad (44)$$

Combining this with (43) results in

$$W^{(l)}(V_w) \propto_{\mathbb{P}} \begin{cases} w^{\tau-2} n^{3-\tau} & \tau > 3 - \frac{1}{l-1}, w \gg n^{\frac{\tau-2}{\tau-1}} \\ w n^{\frac{3-\tau}{\tau-1}} & \tau > 3 - \frac{1}{l-1}, 1 \ll w \ll n^{\frac{\tau-2}{\tau-1}} \\ w n^{\frac{1}{2}(l(3-\tau)-1)} & \tau < 3 - \frac{1}{l-1}, 1 \ll w \ll \sqrt{n} \\ n \left(\frac{n}{w}\right)^{l(3-\tau)-2} & 3 - \frac{2}{l} < \tau < 3 - \frac{1}{l-1}, w \gg \sqrt{n} \\ n^{\frac{1}{2}l(3-\tau)} & \tau < 3 - \frac{2}{l}, w \gg \sqrt{n}. \end{cases} \quad (45)$$

Using (34), we can now compute the higher-order closure coefficients for the hyperbolic random graph, resulting in

$$\hat{H}_l^{(t)}(w) \propto_{\mathbb{P}} \begin{cases} (w/n)^{3-\tau} & \tau < 3 - \frac{1}{l-1}, 1 \ll w \ll \sqrt{n} \\ 1 & \tau < 3 - \frac{1}{l-1}, w \gg \sqrt{n} \\ w^{l(3-\tau)-1} n^{\frac{\tau-3}{\tau-1}} & 3 - \frac{1}{l-1} < \tau < 3 - \frac{1}{l}, 1 \ll w \ll n^{\frac{\tau-2}{\tau-1}} \\ w^{2-\tau+l(3-\tau)} n^{3-\tau} & 3 - \frac{1}{l-1} < \tau < 3 - \frac{1}{l}, n^{\frac{\tau-2}{\tau-1}} \ll w \ll \sqrt{n} \\ (w/n)^{4-\tau-l(3-\tau)} & 3 - \frac{1}{l-1} < \tau < 3 - \frac{1}{l}, w \gg \sqrt{n} \\ n^{\frac{\tau-3}{\tau-1}} & \tau > 3 - \frac{1}{l}, 1 \ll w \ll n^{\frac{\tau-2}{\tau-1}} \\ (w/\sqrt{n})^{l(3-\tau)-1} & \tau > 3 - \frac{1}{l}, w \gg n^{\frac{\tau-2}{\tau-1}} \end{cases} \quad (46)$$

In Appendix B, we show that $H^{(t)}(k)$ follows the same scaling. In particular, this shows that for any value of $\tau \in (2, 3)$ there exists an $l^* \geq 2$ such that $H_l^{(t)}(k)$ is of constant order of magnitude for all $l \geq l^*$ and $k \gg \sqrt{n}$.

This implies that sufficiently large cliques linked to a vertex of degree $k \gg \sqrt{n}$ are likely to be part of larger cliques. This also occurs in the hidden-variable model (see Eq (27)). However, for the hidden-variable model, this already happens for cliques of size 3. In the hyperbolic random graph on the other hand, cliques of size $1 + 1/(3 - \tau) > 3$ for $\tau \in (2, 3)$ only start to be likely to be part of a larger clique. Eq. (46) also shows that $H_l(k)$ is always non-decreasing in k .

Figures 6a and 6b show the behavior of $H_3^{(t)}(k)$ and $H_4^{(t)}(k)$ in the hyperbolic random graph. These figures show that indeed the slope of $H_4^{(t)}(k)$ becomes very small when k gets large, as predicted by Eq. (46).

Furthermore, Figure 6 shows that the typical values of the higher-order closure coefficients indeed behave significantly different from their average values, as predicted in Section 2.2. Especially for small values of k , the average closure coefficients are much higher than their typical values. This indicates the presence of a few low-degree vertices with extremely large values of their higher-order closure coefficients that dominate the mean value of $H_l(k)$. The typical behavior of $H_l(k)$ is illustrated in Figure 6 by the median value of the local closure coefficient of vertices of degree k , which is less affected by such outliers. Therefore, analyzing the typical behavior of the local closure coefficient of a vertex of degree k results in a lower value than its average value.

5 Conclusion

In this paper, we have investigated the behavior of the closure coefficient in two random graph models that create simple, scale-free networks: the hidden-variable model and the hyperbolic random graph. For both models we have obtained asymptotic expressions of the closure coefficient as well as all higher-order closure coefficients.

The behavior of the closure coefficients in these random graph models is significantly different from its behavior that was found in previous research in the configuration model [37], a random graph model that creates multigraphs. This shows that the degree-correlations arising from the simplicity constraint in the models we have analyzed has a significant impact on this new measure for clustering.

The difference between the hyperbolic random graph and the hidden-variable model in terms of the behavior of their closure coefficients indicates that the presence of geometry mostly influences the presence of small cliques. In the hyperbolic model, these are more likely to be formed outside of the core of degree \sqrt{n} vertices and higher than in the hidden-variable model. Once the clique sizes are sufficiently large, they are usually formed in the core of high-degree vertices in both models.

In this paper, we have introduced a method to investigate the behavior of the higher order closure coefficient. We applied this method to two random graph models: one with an underlying geometry and one without. It would be interesting to apply this same method to investigate the behavior of these closure coefficients for other random graph models.

We investigated the scaling of the closure coefficient in terms of the network size n . It would be interesting to obtain more precise results on its behavior. We conjecture that the rescaled closure coefficient in the hidden-variable model converges in probability to a constant. Finding this constant can probably be achieved by using integral equations as in [20]. For the hyperbolic model, we believe that for k sufficiently large, the closure coefficients will also converge in probability to a constant. However, deriving this constant will be more difficult than for the hidden-variable model. Its precise behavior for small k is another interesting open question.

Furthermore, while we only investigated the behavior of the closure coefficient of a randomly chosen vertex of degree k , it would be interesting to see rigorous results for its average value over all vertices of degree k . Our simulations suggest that for small values of k , these two quantities may behave significantly different, but they seem to be close for larger values of k . Finding the value of k where both statistics behave similarly is therefore an interesting question for further research.

Another interesting direction for future work is investigating the typical motif conductance of vertices of degree k , as the conductance of a vertex has recently been shown to be related to its higher-order closure coefficients [37]. In particular, our results in (27) then and (46) give lower bounds on the l -clique conductance of neighborhoods of vertices of degree k . A low motif conductance of a vertex indicates it may be a good starting point for a community detection algorithm. Therefore, pursuing this line of research to investigate the motif conductance of these models more carefully could lead to better understanding of community detection.

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A From weights to degrees

Let $f(h, n)$ denote the scaling of the typical closure coefficient of a randomly chosen of weight h as predicted by (26). That is,

$$f(h, n) = \begin{cases} h^{l-1} n^{-(l-1)/2} & 1 \ll h \ll \sqrt{n} \\ 1 & h \gg \sqrt{n}. \end{cases} \quad (47)$$

We now show that $H_l^{(t)}(k)$ also scales as $f(h, n)$. Conditionally on the weights, the degree of a vertex v , D_v , is the sum of $n - 1$ independent indicators indicating the presence of an edge between vertex v and the other vertices. Furthermore, the connection probability (12) ensures that $\mathbb{E}[D_v | h_v] = h_v$. Thus, by the Chernoff bound, for any $\delta > 1$,

$$\mathbb{P}(D_v \geq w_v(1 + \delta) | w_v) \leq \exp(-\delta w_v/3). \quad (48)$$

Similarly,

$$\mathbb{P}(D_v \leq w_v(1 - \delta) | w_v) \leq \exp(-\delta w_v/2). \quad (49)$$

Furthermore,

$$\begin{aligned} \mathbb{P}(w_v < (1 - \varepsilon)k | D_v = k) &= \frac{\mathbb{P}(D_v = k, w_v < (1 - \varepsilon)k)}{\mathbb{P}(D_v = k)} \\ &= \frac{\mathbb{P}(D_v = k | w_v < (1 - \varepsilon)k) \mathbb{P}(w_v < (1 - \varepsilon)k)}{\mathbb{P}(D_v = k)} \\ &\leq \frac{\mathbb{P}(D_v \leq k | w_v < (1 - \varepsilon)k) \mathbb{P}(w_v < (1 - \varepsilon)k)}{\mathbb{P}(D_v = k)} \\ &\leq \frac{\mathbb{P}(D_v \leq k | w_v = (1 - \varepsilon)k) \mathbb{P}(w_v < (1 - \varepsilon)k)}{\mathbb{P}(D_v = k)}, \end{aligned} \quad (50)$$

where the last line follows because the probability that a vertex with weight h_1 has degree at most k is larger than the probability that a vertex of weight h_2 has degree at most k when $h_1 < h_2$. Since $\mathbb{P}(D_v = k) \propto k^{-\tau}$ and $\mathbb{P}(w_v < (1 - \varepsilon)k) \propto 1 - k^{1-\tau}$ by (11),

$$\mathbb{P}(w_v < (1 - \varepsilon)k \mid D_v = k) = O\left(k^\tau \exp\left(-\frac{k}{3(1 - \varepsilon)}\right)\right). \quad (51)$$

Similarly,

$$\mathbb{P}(w_v > (1 + \varepsilon)k \mid D_v = k) = O\left(k^\tau \exp\left(-\frac{k}{2(1 + \varepsilon)}\right)\right). \quad (52)$$

Let V_k denote a randomly chosen vertex of degree k . Then, for any $h(n)$ such that $\lim_{n \rightarrow \infty} h(n) = \infty$,

$$\begin{aligned} \mathbb{P}\left(\frac{H_l^{(t)}(V_k)}{f(k, n)} > h(n)\right) &\leq \mathbb{P}\left(\frac{H_l^{(t)}(V_k)}{f(k, n)} > h(n) \mid h_{V_k} \in [(1 - \varepsilon)k, (1 + \varepsilon)k]\right) \\ &\quad + \mathbb{P}(h_{V_k} \notin [(1 - \varepsilon)k, (1 + \varepsilon)k]), \end{aligned} \quad (53)$$

which tends to zero as $n \rightarrow \infty$ when $k \gg 1$. Similarly, we can prove that

$$\mathbb{P}\left(\frac{H_l^{(t)}(V_k)}{f(k, n)} < \frac{1}{h(n)}\right) \rightarrow 0$$

as $n \rightarrow \infty$ for $k \gg 1$. Thus,

$$H_l^{(t)}(k) \propto_{\mathbb{P}} \begin{cases} h^{l-1} n^{-(l-1)/2} & 1 \ll h \ll \sqrt{n} \\ 1 & h \gg \sqrt{n}. \end{cases} \quad (54)$$

B From types to degrees

Showing that $\hat{H}^{(t)}(w)$ has the same scaling as $H^{(t)}(k)$ in the hyperbolic model follows the same lines as the proof in Appendix B for the hidden-variable model. Again, let $\tilde{f}(w, n)$ denote the scaling of $\hat{H}^{(t)}(w)$, so that

$$\tilde{f}(w, n) \begin{cases} (w/n)^{3-\tau} & \tau < 3 - \frac{1}{l-1}, w \ll \sqrt{n} \\ 1 & \tau < 3 - \frac{1}{l-1}, w \gg \sqrt{n} \\ w^{l(3-\tau)-1} n^{\frac{\tau-3}{\tau-1}} & 3 - \frac{1}{l-1} < \tau < 3 - \frac{1}{l}, w \ll n^{\frac{\tau-2}{\tau-1}} \\ w^{2-\tau+l(3-\tau)} n^{3-\tau} & 3 - \frac{1}{l-1} < \tau < 3 - \frac{1}{l}, n^{\frac{\tau-2}{\tau-1}} \ll w \ll \sqrt{n} \\ (w/n)^{4-\tau-l(3-\tau)} & 3 - \frac{1}{l-1} < \tau < 3 - \frac{1}{l}, w \gg \sqrt{n} \\ n^{\frac{\tau-3}{\tau-1}} & \tau > 3 - \frac{1}{l}, w \ll n^{\frac{\tau-2}{\tau-1}} \\ (w/\sqrt{n})^{l(3-\tau)-1} & \tau > 3 - \frac{1}{l}, w \gg n^{\frac{\tau-2}{\tau-1}} \end{cases} \quad (55)$$

By, [28, Eq (112)], for any $\varepsilon > 0$ and $D_v \gg 1$,

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(w_v > (1 + \varepsilon) \frac{\pi(\tau - 2)}{2\nu(\tau - 1)} D_v\right) = 0 \quad (56)$$

and

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(w_v < (1 - \varepsilon) \frac{\pi(\tau - 2)}{2\nu(\tau - 1)} D_v\right) = 0. \quad (57)$$

Then, for any $h(n)$ such that $\lim_{n \rightarrow \infty} h(n) = \infty$,

$$\mathbb{P}\left(\frac{H_l^{(t)}(V_k)}{\tilde{f}(k, n)} > h(n)\right) \leq \mathbb{P}\left(\frac{H_l^{(t)}(V_k)}{f(k, n)} > h(n) \mid w_{V_k} \in \frac{\pi(\tau - 2)}{2\nu(\tau - 1)} k[(1 - \varepsilon), (1 + \varepsilon)]\right)$$

$$+ \mathbb{P} \left(w_{V_k} \notin \frac{\pi(\tau-2)}{2\nu(\tau-1)} k[(1-\varepsilon), (1+\varepsilon)] \right), \quad (58)$$

which tends to zero as $n \rightarrow \infty$. Similarly, we can show that $\mathbb{P} \left(\frac{H_l^{(t)}(V_k)}{f(k,n)} < 1/h(n) \right) \rightarrow 0$ as $n \rightarrow \infty$, which proves that $H^{(t)}(k) \propto_{\mathbb{P}} f(k,n)$.