25th Benelux Meeting

on

Systems and Control

March 13 – 15, 2006

Heeze, The Netherlands

Book of Abstracts

On Rantzer's density function

Gjerrit Meinsma
Department of Applied Mathematics, University of Twente
P.O. Box 217, 7500 AE Enschede, The Netherlands
Email: g.meinsma@math.utwente.nl

1 Introduction

In 2001 and following years Rantzer introduced a new stability characterization [1, 2, 3]. It is dual to Lyapunov stability. It is very elegant with many interesting particle-flow interpretations and it has the striking feature that it can be used for systems with multiple equilibrium points (or sets).

In this talk we will interpret Rantzer's main theorem and re-address the inverse problem (already solved [2, 3]). The inverse problem — well studied for Lyapunov functions — is about the fundamental property that a system is stable (in some sense) *only* if a Lyapunov function exists, or in the present context, *only* if a Rantzer density function exists.

2 The theorem

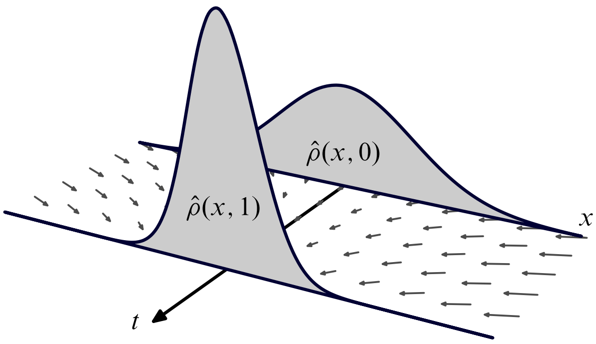
Consider a system $\dot{x}(t) = f(x(t))$ with $f : I\!R^n \to I\!R^n$, $t \in I\!R$ and f continuously differentiable with $f(0) = 0$. Suppose that there is a (density) function $\rho : I\!R^n \backslash \{0\} \to I\!R$ that is continuously differentiable and

1. $\rho(x) \geq 0$ for every $x \neq 0$,
2. $\nabla \cdot (f\rho)(x) > 0$ for almost every $x \neq 0$,
3. $f(x)\rho(x)/|x|$ is absolutely integrable on $|x| > R$ for some large enough R,

then the system is *almost globally stable* meaning that

$$\lim_{t\to\infty} x(t) = 0 \quad \text{for almost all } x(0).$$

A function $\rho(x)$ with the above properties will be referred to as a *Rantzer density function* for the system $\dot{x}(t) = f(x(t))$. Such functions $\rho(x)$ are unbounded around $x = 0$.



3 An interpretation

A fluid interpretation of this result is as follows. Consider a time-dependent density $\hat{\rho}(x,t)$ of state "particles" x. Under the system equation $\dot{x}(t) = f(x(t))$ the particles move and as a result the density of particles changes (the figure on the left explains the idea for a one-dimensional x). The well known continuity equation states that the change is governed by the equation $\partial\hat{\rho}/\partial t = -\nabla \cdot (f\hat{\rho})$. If particles are also generated at some (time-independent) rate $g(x)$ then the continuity equation becomes

$$\frac{\partial\hat{\rho}}{\partial t} = g - \nabla \cdot (f\hat{\rho}).$$

In Rantzer's theorem the density function $\hat{\rho}$ was assumed stationary so that $g = \nabla \cdot (f\hat{\rho})$. An interpretation of the theorem now is that if almost everywhere particles are generated ($g := \nabla \cdot (f\rho) > 0$) while the density of particles remains constant $\rho(x) \geq 0$ then almost all particles leave any set where ρ is defined. They must end up in the "sink" $x = 0$. The third condition is a technical condition to preclude finite escape time and to cope with non-compact state sets.

How to find such ρ? Return to the case that no particles are generated ($g = 0$). Under some conditions on $\hat{\rho}(x,0)$ the $\hat{\rho}(x,t)$ converges — as the figure suggests — to a delta function as $t \to \infty$ if the system is almost globally stable. Then ρ *defined* as

$$\rho(x) = \int_0^\infty \hat{\rho}(x,t)\,\mathrm{d}t \qquad (x \neq 0)$$

exists and satisfies $\nabla \cdot (f\rho)(x) = \hat{\rho}(x,0)$. Hence any such choice of initial $\hat{\rho}(x,0)$ that further is > 0 almost everywhere results in a Rantzer density function $\rho(x)$. This is the inverse theorem, be it that a proper proof seems to require some nontrivial technicalities.

References

[1] A. Rantzer, A dual to Lyapunov's theorem. *Systems and Control*, 42(3), pp. 161–168, 2001.

[2] A. Rantzer, An converse theorem for density functions. *Proceedings of the 41th Conference on Decision and Control*, 2002.

[3] A. Rantzer and S. Hedlund, Duality between cost and density in optimal control. *Proceedings of the 42th Conference on Decision and Control*, 2003.