ON PROJECTIVE BUNDLE REPRESENTATIONS AND GAUGE EQUIVALENCE

HENK HOOGLAND

Department of Applied Mathematics, Twente University of Technology,
P.O. Box 217, 7500 AE Enschede, The Netherlands.

ABSTRACT

The projective equivalence relation for representations of symmetry groups in quantum mechanics is refined to a so-called gauge equivalence relation. An example is given where this leads to a physically more relevant classification of elementary quantum mechanical systems. It is indicated how this idea might be applied to projective bundle representations.

1. GAUGE EQUIVALENCE

Elementary quantum mechanical systems in a space(time) X with (covering) symmetry group G correspond to irreducible semi-unitary multiplier representations of G. The usual classification of such representations is based upon the projective equivalence relation

$$U'(g) = \nu(g)\ S\ U(g)\ S^{-1} \qquad (1)$$

where $\nu(g)$ is a phase factor and S is a semi-unitary transformation between the carrier Hilbert spaces. Obviously, this classification only depends on the abstract group structure of G. It does not depend on the action of G as a group of transformations in X. Hence, the elementary q.m. systems in two space(time)s X_1 and X_2 with isomorphic symmetry groups will be "isomorphically" classified. This leads to a paradox in the case that X_1 and X_2 have a different nature.

An example[1] of this situation is provided by the 2-dimensional Euclidean space X_1 and the $(1+1)$-dimensional Newton-Hooke spacetime X_2. The corresponding symmetry groups are isomorphic.Their isomorphism, however, is not compatible with their action as transformation group. Hence, it is a mathematical coincidence, which is physically irrelevant. So one can hardly expect any physical effect from that accidental group isomorphism. Nevertheless, its effect is that the elementary q.m. systems in X_1 and X_2 have isomorphic classifications, based upon the projective equivalence relation.

This paradox is solved by the recognition that the projective equivalence relation is too coarse for quantum mechanical purposes. There is too much freedom in the choice of the equivalence transformation S between the wave function spaces. One should allow only "trivial" transformations, from the q.m.

point of view. This consideration leads to the gauge transformations. These are, indeed, trivial in the sense that they leave the local quantities (probability densities) invariant, whereas arbitrary unitary transformations do not have that property. Therefore we introduce a finer equivalence relation, called gauge equivalence[1]. This also is given by equation (1) where, however, the semi-unitary S now shall be a gauge transformation, possibly combined with complex conjugation. A classification based upon this gauge equivalence relation does not only depend on the group structure of G but also on the space(time) X and its transformations under G. Hence, in different X_1 and X_2 with isomorphic symmetry groups one no longer automatically obtains "isomorphic" classifications.

In the above mentioned example the irreducible semi-unitary multiplier representations have been classified, both up to projective equivalence and up to gauge equivalence. The results of the latter approach turned out to be more satisfactory from a physical point of view[1]. So gauge equivalence seems to be a useful notion for group representations in Hilbert spaces. As a Hilbert space is in fact a degenerate Hilbert bundle, a generalized notion of gauge equivalence might be introduced for group representations in Hilbert bundles.

2. PROJECTIVE BUNDLE REPRESENTATIONS

Hilbert bundles appear in quantum mechanics already on a very elementary level. The solutions of the wave equation for a particle in an external field form a bundle B over a base space F of fields f. Its fibres are (pre)Hilbert spaces H_f of wave functions. The quantum states correspond to rays rather than to vectors, so in fact the state space is a bundle $\hat{B}$ of ray (or: projective) spaces $\hat{H}_f$. The symmetry group $\hat{G}$ operates on the state space by a homomorphism $\hat{U}$ into the group AUT($\hat{B}$) of automorphisms (= ray-product-preserving bundle operators) in $\hat{B}$. This symmetry operation can be lifted from the ray bundle $\hat{B}$ to the Hilbert bundle B, as illustrated by the following diagram[2].

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & \hat{G} & \longrightarrow & 1 \\
 & & & & {\scriptstyle U}\downarrow & & {\scriptstyle \hat{U}}\downarrow & & \\
1 & \longrightarrow & U^{(1)}(B) & \longrightarrow & U/A(B) & \longrightarrow & AUT(\hat{B}) & \longrightarrow & 1
\end{array}
$$

The first line means that G is a covering group of $\hat{G}$, and the second line is the famous theorem of Wigner, here generalized to bundles. The group U/A(B) contains all bundle operators in B that act semi-unitarily between the fibres. Its subgroup $U^{(1)}(B)$ contains all such operators that act as a phase factor in each fibre. The homomorphism $\hat{U}$ is called a projective bundle representation and the almost-homomorphism U is called a multiplier bundle representation. The multiplier, i.e. the defect in the product rule of the operators U(g), is now an f-dependent phase factor, viz. an element of $U^{(1)}(B)$.

The projective equivalence relation (1) can be generalized to bundle representations, by considering all quantities as acting on a whole bundle. Let U and U' be two multiplier bundle representations in bundles B and B' with base spaces F and F' and with fibres H_f and $H'_{f'}$. Then U and U' are called projectively equivalent if equation (1) holds, where $S : B \to B'$ is a bundle transformation acting semi-unitarily between the fibres, and $\nu(g)$ belongs to $U^{(1)}(B')$.

Each bundle transformation induces a base space transformation (denoted by a small letter) and a set of fibre transformations (labeled by a subindex). So $S : B \to B'$ induces $s : F \to F'$ and $S_f : H_f \to H'_{sf}$. Analogously U(g) and U'(g) induce $u(g) : F \to F$, $u'(g) : F' \to F'$, $U_f(g) : H_f \to H_{u(g)f}$ and $U'_{f'}(g) : H'_{f'} \to H'_{u'(g)f'}$. It follows from (1) that $u'(g)s = su(g)$ and

$$U'_{sf}(g) = \nu_{su(g)f}(g)\ S_{u(g)f}\ U_f(g)\ S_f^{-1} \tag{2}$$

where $\nu_{su(g)f}(g)$ is a phase factor. These formulae manifestly display the two ingredients in the equivalence transformations. Firstly there is a "coordinate transformation on the base space" given by the bijection $s : F \to F'$. Secondly there is a "coordinate transformation on the fibres" given by the semi-unitary transformations $S_f : H_f \to H'_{sf}$ acting between Hilbert spaces of wave functions.

The considerations in §1 suggest a refinement of this projective equivalence relation for muliplier bundle representations. The extra condition is that each semi-unitary S shall be a gauge transformation, possibly combined with complex conjugation. At least in elementary quantum mechanics this gauge equivalence may be expected to correspond to physical equivalence.

RFERENCES:

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2. -----------, Memorandum 250 (Dept. of Appl. Math., Twente Univ. of Technology).