



On a property of random walk polynomials involving Christoffel functions



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ABSTRACT

Discrete-time birth-death processes may or may not have certain properties known as *asymptotic aperiodicity* and the *strong ratio limit property*. In all cases known to us a suitably normalized process having one property also possesses the other, suggesting equivalence of the two properties for a normalized process. We show that equivalence may be translated into a property involving Christoffel functions for a type of orthogonal polynomials known as *random walk polynomials*. The prevalence of this property – and thus the equivalence of asymptotic aperiodicity and the strong ratio limit property for a normalized birth-death process – is proven under mild regularity conditions.

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1. Introduction

In what follows $\mathcal{X} := \{X(n), n = 0, 1, \dots\}$ is a (discrete-time) birth-death process on $\mathcal{N} := \{0, 1, \dots\}$, with tridiagonal matrix of one-step transition probabilities

$$P \equiv (P_{ij})_{i,j \in \mathcal{N}} := \begin{pmatrix} r_0 & p_0 & 0 & 0 & 0 & \dots \\ q_1 & r_1 & p_1 & 0 & 0 & \dots \\ 0 & q_2 & r_2 & p_2 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

We assume throughout that $p_j > 0$, $q_{j+1} > 0$, $r_j \geq 0$, and (save for the last section) that $p_j + q_j + r_j = 1$ for $j \in \mathcal{N}$, where $q_0 := 0$. The polynomials Q_n are defined by the recurrence relation

$$\begin{aligned} xQ_n(x) &= q_n Q_{n-1}(x) + r_n Q_n(x) + p_n Q_{n+1}(x), \quad n > 1, \\ Q_0(x) &= 1, \quad p_0 Q_1(x) = x - r_0, \end{aligned} \tag{1}$$

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so that $Q_n(1) = 1$ for all n . Karlin and McGregor [11] referred to \mathcal{X} as a *random walk* and to $\{Q_n\}$ as a sequence of *random walk polynomials*. Since the latter terminology is rather well established (contrary to the former) we will stick with it. But note that the random walk polynomials in, for example, Askey and Ismail [1] have $r_j = 0$ for all j , so the present setting is more general.

It has been shown in [11] that the n -step transition probabilities

$$P_{ij}(n) := \Pr\{X(n) = j \mid X(0) = i\}, \quad i, j \in \mathcal{N}, \quad n \geq 0,$$

which satisfy $P_{ij}(n) = (P^n)_{ij}$, may also be represented in the form

$$P_{ij}(n) = \pi_j \int_{[-1,1]} x^n Q_i(x) Q_j(x) \psi(dx), \quad i, j \in \mathcal{N}, \quad n \geq 0, \quad (2)$$

where

$$\pi_0 := 1, \quad \pi_j := \frac{p_0 p_1 \cdots p_{j-1}}{q_1 q_2 \cdots q_j}, \quad j \geq 1,$$

and ψ is the (unique) Borel measure on the real axis of total mass 1 with respect to which the polynomials Q_n are orthogonal. Moreover, $\text{supp}(\psi)$, the support of the measure ψ , is infinite and a subset of the interval $[-1, 1]$. Adopting the terminology of [8] we will refer to ψ as a *random walk measure*.

The process \mathcal{X} is said to have the *strong ratio limit property* if the limits

$$\lim_{n \rightarrow \infty} \frac{P_{ij}(n)}{P_{kl}(n)}, \quad i, j, k, l \in \mathcal{N}, \quad (3)$$

exist simultaneously. \mathcal{X} is *asymptotically periodic* if, in the long run, the process evolves cyclically between the even and the odd states, and *asymptotically aperiodic* otherwise. These properties will be discussed in more detail in Section 2. At this point we only remark that in all cases known to us a suitably normalized process having the strong ratio limit property is also asymptotically aperiodic, and vice versa. So we conjecture that for a birth-death process that is normalized (in a sense to be defined in the next section) the two properties are in fact equivalent.

It will be shown in this paper that equivalence of the strong ratio limit property and asymptotic aperiodicity for a normalized birth-death process may be translated into a property of random walk polynomials and the associated measure involving *Christoffel functions*. Concretely, with ρ_n denoting the n th Christoffel function associated with the random walk measure ψ , and η the largest point in the support of ψ , we have equivalence of the two properties for the corresponding normalized birth-death process if and only if

$$\lim_{n \rightarrow \infty} \frac{\int_{[-1,0)} (-x)^n \psi(dx)}{\int_{(0,1]} x^n \psi(dx)} = 0 \iff \lim_{n \rightarrow \infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)} = 0. \quad (4)$$

So our conjecture amounts to validity of (4). But actually we conjecture validity of the stronger property

$$\lim_{n \rightarrow \infty} \frac{\int_{[-1,0)} (-x)^n \psi(dx)}{\int_{(0,1]} x^n \psi(dx)} = \lim_{n \rightarrow \infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)}, \quad (5)$$

if the left-hand limit exists. We will subsequently disclose mild conditions for (5) to prevail, and hence for equivalence of the strong ratio limit property and asymptotic aperiodicity for a normalized birth-death process.

The next section contains a number of preliminary and introductory results. Then, in Section 3, the conjectured property of random walk polynomials is motivated and its relation with the associated birth-death process is discussed. In the Sections 4 and 5 we collect a number of asymptotic results for the quantities featuring in the conjectured property of random walk polynomials. Our main conclusions – sufficient conditions for (5) to be valid – are drawn in Section 6. In the last section the consequences of allowing $p_j + q_j + r_j \leq 1$ will be examined.

2. Preliminaries

This section contains additional information on the strong ratio limit property and on asymptotic aperiodicity of a birth-death process. We also define the normalization of a birth-death process referred to in the introduction, and start off by collecting a number of relevant properties of the random walk polynomials Q_n and the measure ψ with respect to which they are orthogonal.

2.1. Random walk polynomials and measure

By (2) we have

$$r_j \equiv (P)_{jj} = \pi_j \int_{[-1,1]} xQ_j^2(x)\psi(dx), \quad j \in \mathcal{N},$$

so our assumption $r_j \geq 0$ implies

$$\int_{[-1,1]} xQ_n^2(x)\psi(dx) \geq 0, \quad n \geq 0. \tag{6}$$

Whitehurst [21, Theorem 1.6] has shown that, conversely, any Borel measure ψ on the interval $[-1, 1]$, of total mass 1 and with infinite support, is a random walk measure if it satisfies (6) (see also [8, Theorem 1.2]).

Obviously $P_{ij}(0) = \delta_{ij}$ (Kronecker’s delta), so, letting

$$p_n(x) := \sqrt{\pi_n}Q_n(x), \quad n \geq 0, \tag{7}$$

(2) leads to

$$\int_{[-1,1]} p_i(x)p_j(x)\psi(dx) = \delta_{ij}, \quad i, j \geq 0,$$

that is, $\{p_n\}$ constitutes the sequence of *orthonormal* polynomials with respect to the random walk measure ψ . Writing $p_n(x) = \gamma_n x^n + \dots$ we note for future reference that

$$\gamma_n^{-2} = \prod_{i=1}^n p_{i-1}q_i, \quad n \geq 1. \tag{8}$$

The *Christoffel functions* ρ_n associated with ψ are defined by

$$\rho_n(x) := \left\{ \sum_{j=0}^{n-1} p_j^2(x) \right\}^{-1}, \quad n \geq 1. \tag{9}$$

A direct relation between the measure ψ and its Christoffel functions is given by the classic result (Shohat and Tamarkin [16, Corollary 2.6])

$$\lim_{n \rightarrow \infty} \rho_n(x) = \psi(\{x\}), \quad x \in \mathbb{R}. \tag{10}$$

Of particular interest to us is $\eta := \sup \text{supp}(\psi)$, the largest point of the support of the measure ψ , which may also be characterized in terms of the polynomials Q_n by

$$x \geq \eta \iff Q_n(x) > 0 \text{ for all } n \geq 0 \tag{11}$$

(see, for example, Chihara [3, Theorem II.4.1]). Evidently, (6) already implies $\eta > 0$, but it can actually be shown (see, for example, [3, Corollary 2 to Theorem IV.2.1]) that

$$0 \leq r_j < \eta \leq 1, \quad j \in \mathcal{N}. \tag{12}$$

Letting $\zeta := \inf \text{supp}(\psi)$ we also have

$$\inf_j \{r_j + r_{j+1}\} \leq \zeta + \eta \leq \sup_j \{r_j + r_{j+1}\}, \quad j \in \mathcal{N},$$

by [9, Lemma 2.3]. It follows that

$$\zeta \geq -\eta, \tag{13}$$

and hence $\text{supp}(\psi) \subset [-\eta, \eta]$. Moreover, the counterpart of (11) (obtained from (11) by considering, instead of $Q_n(x)$, the polynomials $(-1)^n Q_n(-x)$) gives us

$$x \leq \zeta \iff (-1)^n Q_n(x) > 0 \text{ for all } n \geq 0. \tag{14}$$

The recurrence relations (1) imply the *Christoffel-Darboux* identity

$$p_n \pi_n (Q_n(x)Q_{n+1}(y) - Q_n(y)Q_{n+1}(x)) = (y - x) \sum_{j=0}^n \pi_j Q_j(x)Q_j(y) \tag{15}$$

(see, for example, [3, Theorem I.4.5]), whence, by (11),

$$\eta \leq x < y \implies Q_n(x)Q_{n+1}(y) > Q_n(y)Q_{n+1}(x) > 0 \text{ for all } n \geq 0. \tag{16}$$

Since $Q_n(1) = 1$ for all n this leads in particular to

$$\eta \leq x < 1 \implies 0 < Q_{n+1}(x) < Q_n(x) < Q_0(x) = 1 \text{ for all } n \geq 1. \tag{17}$$

The measure ψ is symmetric about 0 if (and only if) the process \mathcal{X} is *periodic*, that is, if $r_j = 0$ for all j (see [11, p. 69]). Evidently, the process will evolve cyclically between the even and the odd states if it is periodic. The process is *aperiodic* if it is not periodic. Whitehurst [20, Theorem 5.2] has shown that

$$\mathcal{X} \text{ is aperiodic} \implies \int_{[-\eta, \eta]} \frac{\psi(dx)}{\eta + x} < \infty, \tag{18}$$

so that in particular $\psi(\{-\eta\}) = 0$ if \mathcal{X} is aperiodic. It will also be useful to note from (1) that

$$\mathcal{X} \text{ is periodic} \iff (-1)^n Q_n(-x) = Q_n(x), \quad n \geq 0. \tag{19}$$

We now introduce the normalization of the process \mathcal{X} referred to in the Introduction. Namely, letting $\tilde{q}_0 := 0$ and

$$\tilde{p}_j := \frac{Q_{j+1}(\eta)}{Q_j(\eta)} \frac{p_j}{\eta}, \quad \tilde{r}_j := \frac{r_j}{\eta}, \quad \tilde{q}_{j+1} := \frac{Q_j(\eta)}{Q_{j+1}(\eta)} \frac{q_{j+1}}{\eta}, \quad j \in \mathcal{N}, \tag{20}$$

it follows from (1) and (11) that $\tilde{p}_j > 0$, $\tilde{q}_{j+1} > 0$, $\tilde{r}_j \geq 0$, and $\tilde{p}_j + \tilde{q}_j + \tilde{r}_j = 1$, so that the parameters \tilde{p}_j , \tilde{q}_j and \tilde{r}_j may be interpreted as the one-step transition probabilities of a birth-death process $\tilde{\mathcal{X}}$ on \mathcal{N} , the *normalized* version of \mathcal{X} . Note that $\tilde{\mathcal{X}}$ is periodic if and only if \mathcal{X} is periodic. Since $Q_n(1) = 1$ for all n we have $\tilde{\mathcal{X}} = \mathcal{X}$ if (and only if) $\eta = 1$. By [9, Appendix 2] the random walk polynomials \tilde{Q}_n and measure $\tilde{\psi}$ associated with the process $\tilde{\mathcal{X}}$ may be expressed as

$$\tilde{Q}_n(x) = \frac{Q_n(\eta x)}{Q_n(\eta)}, \quad n \geq 0, \tag{21}$$

and

$$\tilde{\psi}([-1, x]) = \psi([- \eta, x\eta]), \quad -1 \leq x \leq 1, \tag{22}$$

respectively. Consequently,

$$\tilde{\zeta} := \inf \text{supp}(\tilde{\psi}) = \frac{\zeta}{\eta} \geq -1 \quad \text{and} \quad \tilde{\eta} := \sup \text{supp}(\tilde{\psi}) = 1.$$

So normalizing \mathcal{X} amounts to stretching the support of the associated measure such that its largest point becomes 1.

We know from [6, Lemma 2.1] that $(-1)^n \tilde{Q}_n(-1)$ is increasing, and strictly increasing for n sufficiently large, if $\tilde{r}_j > 0$ for some $j \in \mathcal{N}$, that is, if $\tilde{\mathcal{X}}$ is aperiodic. It follows that $|Q_n(\eta)/Q_n(-\eta)|$ is decreasing, and strictly decreasing for n sufficiently large, if \mathcal{X} is aperiodic. Since, by (19), $(-1)^n \tilde{Q}_n(-1) = \tilde{Q}_n(1) = 1$ for all n if \mathcal{X} is periodic, we can conclude the following.

Lemma 1. *If \mathcal{X} is periodic then $Q_n^2(\eta)/Q_n^2(-\eta) = 1$ for all n . If \mathcal{X} is aperiodic then $Q_n^2(\eta)/Q_n^2(-\eta)$ is decreasing and tends to a limit satisfying*

$$0 \leq \lim_{n \rightarrow \infty} \frac{Q_n^2(\eta)}{Q_n^2(-\eta)} < 1.$$

In view of (7) this lemma tells us that the ratio $p_n^2(\eta)/p_n^2(-\eta)$ tends to a limit as $n \rightarrow \infty$, while, by (10) and (18),

$$\mathcal{X} \text{ is aperiodic} \implies \lim_{n \rightarrow \infty} \frac{1}{\rho_n(-\eta)} = \sum_{j=0}^{\infty} p_j^2(-\eta) = \infty.$$

Applying the Stolz-Cesàro theorem therefore leads to the conclusion that, as $n \rightarrow \infty$, the ratio $\rho_n(-\eta)/\rho_n(\eta)$ tends to a limit satisfying

$$\lim_{n \rightarrow \infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)} = \lim_{n \rightarrow \infty} \frac{p_n^2(\eta)}{p_n^2(-\eta)} \tag{23}$$

if \mathcal{X} is aperiodic. But (23) is obviously also valid if \mathcal{X} is periodic (both limits then being one), so we have the following result.

Proposition 1. *If \mathcal{X} is periodic then $\rho_n(-\eta)/\rho_n(\eta) = Q_n^2(\eta)/Q_n^2(-\eta) = 1$ for all n . If \mathcal{X} is aperiodic then $\rho_n(-\eta)/\rho_n(\eta)$ tends, as $n \rightarrow \infty$, to a limit satisfying*

$$0 \leq \lim_{n \rightarrow \infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)} = \lim_{n \rightarrow \infty} \frac{Q_n^2(\eta)}{Q_n^2(-\eta)} < 1.$$

With $\tilde{\rho}_n$ denoting the Christoffel functions associated with the normalized process $\tilde{\mathcal{X}}$ it follows readily from (7), (21) and (23) that

$$\lim_{n \rightarrow \infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)} = \lim_{n \rightarrow \infty} \frac{\tilde{\rho}_n(-1)}{\tilde{\rho}_n(1)}, \tag{24}$$

so in studying the asymptotic behaviour of the ratio $\rho_n(-\eta)/\rho_n(\eta)$ it is no restriction to assume $\eta = 1$.

We will see in the next subsections that Proposition 1 enables us to establish a link between the Christoffel functions associated with a sequence of random walk polynomials and probabilistic properties of the normalized version of the corresponding birth-death process.

2.2. Strong ratio limit property

The strong ratio limit property (SRLP) was introduced in the setting of discrete-time Markov chains on a countable state space by Orey [14] and Pruitt [15], but the problem of finding conditions for the limits (3) to exist in the more restricted setting of discrete-time birth-death processes had been considered before in [11]. For more information on the history of the problem we refer to [10] and [12].

A necessary and sufficient condition for the process \mathcal{X} to possess the SRLP is known in terms of the associated random walk measure ψ . Namely, letting

$$C_n(\psi) := \frac{\int_{[-1,0)} (-x)^n \psi(dx)}{\int_{(0,1]} x^n \psi(dx)}, \quad n \geq 0, \tag{25}$$

[10, Theorem 3.1] tells us the following.

Theorem 1. *The process \mathcal{X} has the SRLP if and only if $\lim_{n \rightarrow \infty} C_n(\psi) = 0$, in which case*

$$\lim_{n \rightarrow \infty} \frac{P_{ij}(n)}{P_{kl}(n)} = \frac{\pi_j Q_i(\eta) Q_j(\eta)}{\pi_l Q_k(\eta) Q_l(\eta)}, \quad i, j, k, l \in \mathcal{N}.$$

Note that the denominator in (25) is positive since $\eta > 0$, so that $C_n(\psi)$ exists and is nonnegative for all n . Moreover, in view of (22) we clearly have

$$C_n(\psi) = C_n(\tilde{\psi}), \quad n \geq 0, \tag{26}$$

so normalization does not affect prevalence of the SRLP.

If \mathcal{X} is periodic then $P_{ij}(n) = 0$ if $n + i + j$ is odd, as a consequence of (2) and (19). Hence the limits in (3) do not exist, which is reflected by the fact that $C_n(\psi) = 1$ for all n in this case. So aperiodicity is necessary for \mathcal{X} to have the SRLP. A sufficient condition for \mathcal{X} to have the SRLP is implied by [10, Theorem 3.2], which states that

$$\lim_{n \rightarrow \infty} \left| \frac{Q_n(\eta)}{Q_n(-\eta)} \right| = 0 \quad \Rightarrow \quad \lim_{n \rightarrow \infty} C_n(\psi) = 0. \tag{27}$$

The reverse implication is conjectured in [10] to be valid as well. We can actually establish a result that is stronger than (27).

Lemma 2. *We have*

$$0 \leq \limsup_{n \rightarrow \infty} C_n(\psi) \leq \lim_{n \rightarrow \infty} \frac{Q_n^2(\eta)}{Q_n^2(-\eta)}.$$

Proof. The first inequality is obvious since $C_n(\psi) \geq 0$ for all n . If \mathcal{X} is periodic, then, by (19) and the fact that ψ is symmetric, both sides of the second inequality are one, so in the remainder of this proof we will assume that \mathcal{X} is aperiodic. Let

$$c_1 := \limsup_{n \rightarrow \infty} C_{2n}(\psi), \quad c_2 := \limsup_{n \rightarrow \infty} C_{2n+1}(\psi),$$

and

$$L_n(f, \psi) := \frac{\int_{[-\eta, \eta]} x^n f(x) \psi(dx)}{\int_{[-\eta, \eta]} x^n \psi(dx)}, \quad n \geq 0. \tag{28}$$

In view of the representation formula (2) the denominator in (28) equals $P_{00}(n)$ and is therefore nonnegative for all n . But, \mathcal{X} being aperiodic, we must have $P_{00}(n) > 0$ for n sufficiently large so the denominator is actually positive for n sufficiently large. Choosing a subsequence n_k of the positive integers such that $C_{2n_k}(\psi) \rightarrow c_1$ as $k \rightarrow \infty$, we have, by [10, Lemmas 3.1 and 3.2],

$$\lim_{k \rightarrow \infty} L_{n_k}(Q_j Q_{j+1}, \psi) = \frac{Q_j(\eta)Q_{j+1}(\eta) + c_1 Q_j(-\eta)Q_{j+1}(-\eta)}{1 + c_1}.$$

Since, by the representation formula (2) again, $L_n(Q_j Q_{j+1}, \psi) \geq 0$ for all n , the limit must be nonnegative. Moreover, by (13) and (14) we have $(-1)^n Q_n(-\eta) > 0$ for all $n \geq 0$, so that $Q_j(-\eta)Q_{j+1}(-\eta) < 0$. Hence

$$c_1 \leq -\frac{Q_j(\eta)Q_{j+1}(\eta)}{Q_j(-\eta)Q_{j+1}(-\eta)}, \quad j \geq 0,$$

so that

$$c_1 \leq \lim_{j \rightarrow \infty} -\frac{Q_j(\eta)Q_{j+1}(\eta)}{Q_j(-\eta)Q_{j+1}(-\eta)} = \lim_{n \rightarrow \infty} \frac{Q_n^2(\eta)}{Q_n^2(-\eta)}.$$

Turning to c_2 we first note that $0 \leq c_2 < 1$ by [10, Lemma 3.3]. Next proceeding in a similar way as before, by considering $L_{n_k}(Q_j^2, \psi)$ with n_k a subsequence of the integers such that $C_{2n_k+1}(\psi) \rightarrow c_2$, we obtain

$$\lim_{k \rightarrow \infty} L_{n_k}(Q_j^2, \psi) = \frac{Q_j^2(\eta) - c_2 Q_j^2(-\eta)}{1 - c_2},$$

so that

$$c_2 \leq \lim_{n \rightarrow \infty} \frac{Q_n^2(\eta)}{Q_n^2(-\eta)},$$

which completes the proof. \square

In view of Proposition 1 we can thus state the following.

Theorem 2. *If \mathcal{X} is aperiodic then*

$$0 \leq \limsup_{n \rightarrow \infty} C_n(\psi) \leq \lim_{n \rightarrow \infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)} < 1.$$

It has recently been shown in [6, Lemma 2.1] that

$$\sum_{j=0}^{\infty} \frac{1}{p_j \pi_j} \sum_{k=0}^j r_k \pi_k = \infty \iff \lim_{n \rightarrow \infty} (-1)^n Q_n(-1) = \infty, \quad (29)$$

while it follows from [6, Corollary 3.2 and Lemma 3.3] that

$$\lim_{n \rightarrow \infty} |Q_n(-1)| = \infty \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{Q_n(\eta)}{Q_n(-\eta)} \right| = 0. \quad (30)$$

Hence, by Proposition 1,

$$\sum_{j=0}^{\infty} \frac{1}{p_j \pi_j} \sum_{k=0}^j r_k \pi_k = \infty \Rightarrow \lim_{n \rightarrow \infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)} = 0, \quad (31)$$

which, in view of Theorem 2, gives us a sufficient condition for the SRLP directly in terms of the parameters of the process. The condition is not necessary since [6, Example 4.1] provides a counterexample to the reverse implication in (30).

2.3. Asymptotic aperiodicity

A discrete-time Markov chain on \mathcal{N} may, in the long run, evolve cyclically through a number of sets constituting a partition of \mathcal{N} . The maximum number of sets involved in this cyclic behaviour is called the *asymptotic period* of the chain, and the chain is said to be *asymptotically aperiodic* if such cyclic behaviour does not occur, in which case we also say that the asymptotic period equals one. The *asymptotic period* of a Markov chain may be larger than its *period*. For rigorous definitions and developments we refer to [7], where it is also shown that in the specific setting of a birth-death process the asymptotic period equals either one, or two, or infinity. Precise conditions for these values to prevail are given as well. In particular, [7, Theorem 12] tells us the following.

Theorem 3. *The process \mathcal{X} is asymptotically aperiodic if and only if*

$$\sum_{j=0}^{\infty} \frac{1}{p_j \pi_j} \sum_{k=0}^j r_k \pi_k = \infty. \quad (32)$$

Note that (32) is precisely the sufficient condition for prevalence of the SRLP derived in the previous subsection.

Letting

$$L_n := \sum_{j=0}^n \frac{1}{p_j \pi_j}, \quad 0 \leq n \leq \infty, \quad (33)$$

it follows from Theorem 3 that

$$\mathcal{X} \text{ is aperiodic and } L_\infty = \infty \Rightarrow \mathcal{X} \text{ is asymptotically aperiodic.} \tag{34}$$

So, recalling from [11] that

$$\mathcal{X} \text{ is } \begin{cases} \text{recurrent} & \iff L_\infty = \infty \\ \text{transient} & \iff L_\infty < \infty, \end{cases} \tag{35}$$

and noting the obvious fact that asymptotic aperiodicity implies aperiodicity, we conclude that for a recurrent process *aperiodicity* and *asymptotic aperiodicity* are equivalent. The study of asymptotic aperiodicity is therefore relevant in particular for transient processes.

Another sufficient condition for asymptotic aperiodicity is obtained by observing that

$$\sum_{j=0}^n \frac{1}{p_j \pi_j} \sum_{k=0}^j r_k \pi_k \geq \sum_{j=0}^n \frac{r_j}{p_j}, \tag{36}$$

so that

$$\sum_{j=0}^{\infty} \frac{r_j}{p_j} = \infty \Rightarrow \mathcal{X} \text{ is asymptotically aperiodic.} \tag{37}$$

Now turning to the normalized version $\tilde{\mathcal{X}}$ of \mathcal{X} we observe from the analogues for $\tilde{\mathcal{X}}$ of (29) and Theorem 3 that

$$\tilde{\mathcal{X}} \text{ is asymptotically aperiodic} \iff \lim_{n \rightarrow \infty} (-1)^n \tilde{Q}_n(-1) = \infty,$$

which, by (21) and Proposition 1, may be formulated as

$$\tilde{\mathcal{X}} \text{ is asymptotically aperiodic} \iff \lim_{n \rightarrow \infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)} = 0. \tag{38}$$

With Theorem 3 it now follows that (31) may be translated into

$$\mathcal{X} \text{ is asymptotically aperiodic} \Rightarrow \tilde{\mathcal{X}} \text{ is asymptotically aperiodic,} \tag{39}$$

but we emphasize again that the reverse implication is not valid.

3. Conjecture

In view of (31) and the Theorems 1, 2 and 3 the birth-death process \mathcal{X} has the SRLP if it is asymptotically aperiodic. But, bearing in mind that the reverse implication in (31) does not hold, the two properties are definitely *not* equivalent. However, if, instead of \mathcal{X} , we consider the *normalized* process $\tilde{\mathcal{X}}$, then $|\tilde{Q}_n(\tilde{\eta})/\tilde{Q}_n(-\tilde{\eta})| = |1/\tilde{Q}_n(-1)|$, so that the reverse implication in (30) – and hence in (31) – is trivially true. In all cases known to us a normalized process having the SRLP is asymptotically aperiodic, so we conjecture that $\tilde{\mathcal{X}}$ is in fact asymptotically aperiodic if it has the SRLP, which, by Theorem 1, (26) and (38), amounts to the following.

Conjecture 1. *We have*

$$\lim_{n \rightarrow \infty} C_n(\psi) = 0 \quad \Rightarrow \quad \lim_{n \rightarrow \infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)} = 0. \tag{40}$$

Recall that, by Proposition 1, the limit on the right-hand side exists, and that, by Theorem 2, the right-hand side of (40) implies the left-hand side. Note also that (40) is equivalent to the conjecture already put forward in [10]. Actually, as announced in the introduction, we venture to state the following, stronger conjecture.

Conjecture 2. *If $C_n(\psi)$ tends to a limit as $n \rightarrow \infty$, then*

$$\lim_{n \rightarrow \infty} C_n(\psi) = \lim_{n \rightarrow \infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)}. \tag{41}$$

In what follows we will verify Conjecture 2 – and hence Conjecture 1 – under some mild regularity conditions. But before drawing our conclusions in Section 6, we collect some asymptotic properties of $C_n(\psi)$ in the next section and study the asymptotic behaviour of the ratio $\rho_n(-\eta)/\rho_n(\eta)$ in Section 5.

4. Asymptotic results for $C_n(\psi)$

By definition of $C_n(\psi)$ we obviously have $C_n(\psi) = 0$ for all n if $\zeta \geq 0$. Moreover, if $-\eta < \zeta < 0$ then, for $0 < \epsilon < \eta + \zeta$,

$$C_n(\psi) = \frac{\int_{[\zeta, 0)} (-x)^n \psi(dx)}{\int_{(0, \eta]} x^n \psi(dx)} \leq \frac{(-\zeta)^n}{(\eta - \epsilon)^n \psi([\eta - \epsilon, \eta])} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Finally, if $\zeta = -\eta$ we have, for $0 < \epsilon < \eta$,

$$\frac{\int_{(0, \eta - \epsilon]} x^n \psi(dx)}{\int_{[\eta - \epsilon, \eta]} x^n \psi(dx)} \leq \frac{(\eta - \epsilon)^n}{(\eta - \epsilon/2)^n \psi([\eta - \epsilon/2, \eta])} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

while

$$\frac{\int_{[-\eta + \epsilon, 0)} (-x)^n \psi(dx)}{\int_{[-\eta, -\eta + \epsilon]} (-x)^n \psi(dx)} \leq \frac{(\eta - \epsilon)^n}{(\eta - \epsilon/2)^n \psi([- \eta, -\eta + \epsilon/2])} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

With these results we readily obtain the next proposition, which extends [10, Lemma 3.5].

Proposition 2. *If $\zeta > -\eta$ then $\lim_{n \rightarrow \infty} C_n(\psi) = 0$. If $\zeta = -\eta$ then we have for any $\epsilon \in (0, \eta)$,*

$$\limsup_{n \rightarrow \infty} C_n(\psi) = \limsup_{n \rightarrow \infty} \frac{\int_{[-\eta, -\eta + \epsilon]} (-x)^n \psi(dx)}{\int_{[\eta - \epsilon, \eta]} x^n \psi(dx)}, \tag{42}$$

and a similar result with \limsup replaced by \liminf .

As an aside we note that the first statement of this proposition follows also from Theorem 6 in the next section and Theorem 2.

Corollary 1. *Let $0 < \epsilon < \eta$. Then $\lim_{n \rightarrow \infty} C_n(\psi)$ exists if and only if the ratio of integrals in (42) tends to a limit as $n \rightarrow \infty$, in which case the two limits are equal.*

This corollary and (18) imply in particular that $\lim_{n \rightarrow \infty} C_n(\psi) = 0$ if \mathcal{X} is aperiodic and $\psi(\{\eta\}) > 0$. But this result is encompassed in the next proposition.

Proposition 3. *We have*

$$\liminf_{\epsilon \downarrow 0} \frac{\psi([- \eta, - \eta + \epsilon])}{\psi([\eta - \epsilon, \eta])} \leq \liminf_{n \rightarrow \infty} C_n(\psi) \leq \limsup_{n \rightarrow \infty} C_n(\psi) \leq \limsup_{\epsilon \downarrow 0} \frac{\psi([- \eta, - \eta + \epsilon])}{\psi([\eta - \epsilon, \eta])}. \tag{43}$$

Proof. The result is obviously true if ψ is symmetric about 0 (that is, if \mathcal{X} is periodic) or, by Proposition 2, if $\zeta > -\eta$. Moreover, if \mathcal{X} is aperiodic, $\zeta = -\eta$ and $\psi(\{\eta\}) > 0$ then, by (18) and Corollary 1, all components of the inequalities (43) are zero. In the remainder of the proof we will therefore assume that \mathcal{X} is aperiodic, $\zeta = -\eta$ and $\psi(\{\eta\}) = \psi(\{-\eta\}) = 0$. Now let c be such that

$$c > L := \limsup_{\epsilon \downarrow 0} \frac{\psi([- \eta, - \eta + \epsilon])}{\psi([\eta - \epsilon, \eta])}.$$

Then there exists an ϵ , $0 < \epsilon < \eta$, such that

$$\psi([- \eta, - \eta + x]) \leq c\psi([\eta - x, \eta]), \quad 0 < x \leq \epsilon. \tag{44}$$

Next defining

$$\Psi(x) := \begin{cases} 0 & \text{if } x < -\eta \\ \psi([- \eta, x]) & \text{if } -\eta \leq x \leq \eta \\ 1 & \text{if } x > \eta, \end{cases} \tag{45}$$

integration by parts of the relevant Stieltjes integrals gives us, for all n ,

$$\begin{aligned} \int_{[\eta - \epsilon, \eta]} x^n \psi(dx) &= \int_{[\eta - \epsilon, \eta]} x^n d\Psi(x) \\ &= \eta^n - (\eta - \epsilon)^n \Psi(\eta - \epsilon) - n \int_{\eta - \epsilon}^{\eta} x^{n-1} \Psi(x) dx \\ &= n \int_{\eta - \epsilon}^{\eta} [1 - \Psi(x)] x^{n-1} dx + (\eta - \epsilon)^n [1 - \Psi(\eta - \epsilon)] \\ &= n \int_{\eta - \epsilon}^{\eta} \psi([x, \eta]) x^{n-1} dx + (\eta - \epsilon)^n \psi([\eta - \epsilon, \eta]), \end{aligned}$$

while

$$\begin{aligned} \int_{[- \eta, - \eta + \epsilon]} (-x)^n \psi(dx) &= \int_{[\eta - \epsilon, \eta]} x^n d(1 - \Psi(-x)) \\ &= n \int_{\eta - \epsilon}^{\eta} \Psi(-x) x^{n-1} dx + (\eta - \epsilon)^n \Psi(-\eta + \epsilon) \end{aligned}$$

$$\begin{aligned}
 &= n \int_{\eta-\epsilon}^{\eta} \psi([- \eta, -x])x^{n-1}dx + (\eta - \epsilon)^n \psi([- \eta, -\eta + \epsilon]) \\
 &\leq cn \int_{\eta-\epsilon}^{\eta} \psi([x, \eta])x^{n-1}dx + c(\eta - \epsilon)^n \psi([\eta - \epsilon, \eta]),
 \end{aligned}$$

where we have used (44) in the last step. It follows that

$$\limsup_{n \rightarrow \infty} \frac{\int_{[-\eta, -\eta+\epsilon]} (-x)^n \psi(dx)}{\int_{[\eta-\epsilon, \eta]} x^n \psi(dx)} \leq c,$$

and since c can be chosen arbitrarily close to L , the right-hand inequality in (43) follows by Proposition 2. The left-hand inequality is proven similarly. \square

In combination with Theorem 2 this proposition leads to the following.

Theorem 4. *If \mathcal{X} is aperiodic we have*

$$\lim_{n \rightarrow \infty} C_n(\psi) = \lim_{\epsilon \downarrow 0} \frac{\psi([- \eta, -\eta + \epsilon])}{\psi([\eta - \epsilon, \eta])} < 1,$$

if the second limit exists.

With a view to the analysis in Section 6 we will employ this theorem to obtain a limit result in a more specific situation. Concretely, we consider the condition

(i) Ψ is continuously differentiable on \mathbb{R} and $\Psi'(x) > 0$ for $x \in (-\eta, \eta)$,

where Ψ denotes the function defined in (45). Note that this condition implies $\Psi(-\eta+) = 0$, $\Psi(\eta-) = 1$ and also $\Psi'(-\eta+) = \Psi'(\eta-) = 0$. If condition (i) prevails we let

$$\begin{aligned}
 \alpha &:= \sup\{a : \lim_{x \uparrow \eta} (\eta - x)^{-a} \Psi'(x) = 0\}, \\
 \beta &:= \sup\{b : \lim_{x \downarrow -\eta} (\eta + x)^{-b} \Psi'(x) = 0\},
 \end{aligned} \tag{46}$$

so that α and β are nonnegative (but possibly infinity). A second condition is

(ii) α and β are finite.

Finally, if conditions (i) and (ii) prevail we define

$$w(x) := (\eta - x)^{-\alpha} (\eta + x)^{-\beta} \Psi'(x), \quad -\eta < x < \eta, \tag{47}$$

so that $w(x) > 0$ for $x \in (-\eta, \eta)$. A third condition is

(iii) the limits $w(-\eta+)$ and $w(\eta-)$ exist and are finite, and $w(\eta-) > 0$.

Theorem 5. *If \mathcal{X} is aperiodic and the corresponding measure ψ satisfies the conditions (i), (ii) and (iii) above, then $0 < \alpha \leq \beta$ and*

$$\lim_{n \rightarrow \infty} C_n(\psi) = \begin{cases} 0 & \text{if } \alpha < \beta \\ \frac{w(-\eta+)}{w(\eta-)} & \text{if } \alpha = \beta. \end{cases}$$

Proof. We must have $\alpha > 0$, since $\alpha = 0$ would imply $w(\eta-) = (2\eta)^{-\beta}\Psi'(\eta-) = 0$. Further, since Ψ is continuously differentiable we may apply l'Hôpital's rule to conclude that

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \frac{\psi([- \eta, -\eta + \epsilon])}{\psi([\eta - \epsilon, \eta])} &= \lim_{\epsilon \downarrow 0} \frac{\Psi(-\eta + \epsilon)}{1 - \Psi(\eta - \epsilon)} \\ &= \lim_{\epsilon \downarrow 0} \frac{\Psi'(-\eta + \epsilon)}{\Psi'(\eta - \epsilon)} = \frac{(2\eta)^{\alpha-\beta}}{w(\eta-)} \lim_{\epsilon \downarrow 0} \epsilon^{\beta-\alpha} w(-\eta + \epsilon), \end{aligned} \tag{48}$$

if the limit on the right exists. By definition of β this limit is zero if $\alpha < \beta$, while it obviously equals $w(-\eta+)$ if $\alpha = \beta$. Finally, if $\alpha > \beta$ the right-hand limit in (48) is infinity, which, however, would contradict Theorem 4. So we must have $\alpha \leq \beta$. The result now follows from Theorem 4. \square

Note that $w(-\eta+) = 0$ if $\zeta > -\eta$, so the theorem is consistent with Proposition 2.

5. Asymptotic results for $\rho_n(-\eta)/\rho_n(\eta)$

Formulating (29) and Proposition 1 in terms of the normalized process $\tilde{\mathcal{X}}$ (recall that $\tilde{\eta} = 1$), and translating the results with the help of (20) and (24) in terms of quantities related to the original process \mathcal{X} , leads to the next result.

Lemma 3. *We have*

$$\lim_{n \rightarrow \infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)} = 0 \iff \sum_{j=0}^{\infty} \frac{1}{p_j \pi_j Q_j(\eta) Q_{j+1}(\eta)} \sum_{k=0}^j r_k \pi_k Q_k^2(\eta) = \infty. \tag{49}$$

Defining \tilde{L}_n in analogy with (33) we readily obtain

$$\tilde{L}_\infty = \sum_{j=0}^{\infty} \frac{1}{p_j \pi_j Q_j(\eta) Q_{j+1}(\eta)}.$$

So, in analogy with (34), Lemma 3 yields

$$\mathcal{X} \text{ is aperiodic and } \tilde{L}_\infty = \infty \implies \lim_{n \rightarrow \infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)} = 0. \tag{50}$$

By (17) we have $\tilde{L}_\infty \geq L_\infty$, so the premise in (50) certainly prevails if \mathcal{X} is aperiodic and recurrent. For later use we note that the condition $\tilde{L}_\infty = \infty$ has an interpretation in terms of the measure ψ , namely, by [9, Theorem 3.2],

$$\tilde{L}_\infty = \infty \iff \int_{[-\eta, \eta]} \frac{\psi(dx)}{\eta - x} = \infty, \tag{51}$$

so that in particular $\psi(\{\eta\}) = 0$ if $\tilde{L}_\infty < \infty$.

Another sufficient condition for the left-hand side of (49) is obtained in analogy with (37), namely

$$\sum_{j=0}^{\infty} \frac{r_j Q_j(\eta)}{p_j Q_{j+1}(\eta)} = \infty \implies \lim_{n \rightarrow \infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)} = 0. \tag{52}$$

Note that by (17) we have

$$\sum_{j=0}^{\infty} \frac{r_j Q_j(\eta)}{p_j Q_{j+1}(\eta)} \geq \sum_{j=0}^{\infty} \frac{r_j}{p_j}, \tag{53}$$

so that (52) improves upon the sufficient condition implied by (37), (38) and (39).

The following is a sufficient condition for the left-hand side of (49) in terms of the orthogonalizing measure ψ .

Theorem 6. *We have*

$$\zeta > -\eta \Rightarrow \lim_{n \rightarrow \infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)} = 0.$$

Proof. In view of (52) and (53) it is no restriction to assume in the remainder of this proof that $r_j \rightarrow 0$. Define the polynomials S_n by

$$\begin{aligned} xS_n(x) &= q_n S_{n-1}(x) + p_n S_{n+1}(x), \quad n > 1, \\ S_0(x) &= 1, \quad p_0 S_1(x) = x, \end{aligned}$$

and let ϕ be the measure with respect to which these polynomials are orthogonal. Then ϕ is symmetric about 0. Let $[-\theta, \theta]$ be the smallest interval containing the support of ϕ . By J_ψ and J_ϕ we denote the operators

$$J_\psi a_n = q_n a_{n-1} + r_n a_n + p_n a_{n+1} \quad \text{and} \quad J_\phi a_n = q_n a_{n-1} + p_n a_{n+1}.$$

The spectra of J_ψ and J_ϕ on the space of square summable sequences correspond to $\text{supp}(\psi)$ and $\text{supp}(\phi)$, respectively, and any mass point of ψ (ϕ) is an eigenvalue of J_ψ (J_ϕ) (see, for example, Van Assche [19] for these and subsequent results). Since $r_j \rightarrow 0$ the difference $J_\psi - J_\phi$ is a compact operator, so, by Weyl’s theorem on bounded linear operators, $\text{supp}(\psi)$ and $\text{supp}(\phi)$ differ by at most countably many points, each being a mass point of the corresponding measure. Since $r_j \geq 0$ we also have $\zeta \geq -\theta$ and $\eta \geq \theta$. (This follows also from [3, Theorems III.5.7 and IV.2.1].) If $\eta > \theta$ then η is a mass point of ψ and, by (50) and (51), we are done. On the other hand, if $\eta = \theta$ then $\zeta > -\theta$, so that $-\theta$ is a mass point of ϕ and, by symmetry, also $\theta = \eta$ is a mass point of ϕ . It follows that

$$\int_{[-\eta, \eta]} \frac{\phi(dx)}{\eta - x} = \infty. \tag{54}$$

From [3, Theorem IV.2.1] and (12) we know that the sequence

$$\left\{ \frac{p_{n-1}q_n}{(\eta - r_{n-1})(\eta - r_n)} \right\}_n$$

constitutes a *chain sequence*. Moreover, ψ not being symmetric, we have $r_j > 0$ for some j , while

$$\frac{p_{n-1}q_n}{\eta^2} \leq \frac{p_{n-1}q_n}{(\eta - r_{n-1})(\eta - r_n)},$$

so that $\{p_{n-1}q_n/\eta^2\}_n$ constitutes a chain sequence that does not determine its parameters uniquely. But this contradicts (54), by [18, Theorem 1], so $\eta = \theta$ is not possible. \square

Remark. An alternative proof involving a probabilistic argument is the following. Define the polynomials \tilde{S}_n by

$$\begin{aligned} x\tilde{S}_n(x) &= \tilde{q}_n\tilde{S}_{n-1}(x) + \tilde{p}_n\tilde{S}_{n+1}(x), \quad n > 1, \\ \tilde{S}_0(x) &= 1, \quad \tilde{p}_0\tilde{S}_1(x) = x, \end{aligned}$$

with \tilde{p}_n and \tilde{q}_n as in (20). Since $\tilde{p}_j + \tilde{q}_j = 1 - \tilde{r}_j \leq 1$, the polynomials \tilde{S}_n correspond to a discrete-time birth-death process \mathcal{Y} with an ignored state δ that can be reached with probability \tilde{r}_j from state $j \in \mathcal{N}$ (see [4, Sect. 3]). Since $\tilde{r}_j > 0$ for at least one $j \in \mathcal{N}$, the process \mathcal{Y} is transient and, as a consequence (see [11, p. 70]), the (symmetric) measure $\tilde{\phi}$ associated with \mathcal{Y} satisfies

$$\int_{[-1,1]} \frac{\tilde{\phi}(dx)}{1-x} < \infty. \tag{55}$$

As before, let $[-\tilde{\theta}, \tilde{\theta}]$ be the smallest interval containing the support of $\tilde{\phi}$. Now applying the argument involving Weyl’s theorem in the proof above to the operators $J_{\tilde{\psi}}$ and $J_{\tilde{\phi}}$, the assumption $\tilde{\theta} = \tilde{\eta} (= 1)$ implies $-\tilde{\theta} = -1 < \zeta/\eta = \tilde{\zeta}$, so that -1 , and hence, by symmetry, 1 , is a mass point of $\tilde{\phi}$. This, however, contradicts (55). On the other hand, the assumption $\tilde{\theta} < 1$ implies that 1 is a mass point of $\tilde{\psi}$, and hence η a mass point of ψ , which, by (50) and (51), yields the result. \square

Our next step will be to study the asymptotic behaviour of $\rho_n(-\eta)/\rho_n(\eta)$ in the specific setting of Theorem 5. So we will now assume that the random walk measure ψ satisfies the conditions (i), (ii) and (iii) preceding Theorem 5, so that $\text{supp}(\psi) = [-\eta, \eta]$. In addition we will assume that ψ is regular in the sense of Ullman-Stahl-Totik (see Stahl and Totik [17, Def. 3.1.2]), which amounts to assuming that $\lim_{n \rightarrow \infty} \gamma_n^{1/n} = 2\eta$. (Recall that γ_n is the coefficient of x^n in $p_n(x)$.) Applying Theorem 1.2 of Danka and Totik [5] then leads to the conclusion that

$$\lim_{n \rightarrow \infty} n^{2\alpha+2} \rho_n(\eta) = (2\eta)^{-\alpha-1} w(\eta-) \Gamma(\alpha + 1) \Gamma(\alpha + 2).$$

By considering the measure with respect to which the polynomials $(-1)^n Q_n(-x)$ are orthogonal, one obtains in a similar way

$$\lim_{n \rightarrow \infty} n^{2\beta+2} \rho_n(-\eta) = (2\eta)^{-\beta-1} w(-\eta+) \Gamma(\beta + 1) \Gamma(\beta + 2).$$

From Theorem 5 we know already that $0 < \alpha \leq \beta$, so the preceding limit results lead to the following theorem.

Theorem 7. *If \mathcal{X} is aperiodic, and the corresponding measure ψ is regular and satisfies the conditions (i), (ii) and (iii) preceding Theorem 5, then $0 < \alpha \leq \beta$ and*

$$\lim_{n \rightarrow \infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)} = \begin{cases} 0 & \text{if } \alpha < \beta \\ \frac{w(-\eta+)}{w(\eta-)} & \text{if } \alpha = \beta. \end{cases}$$

We note again that $w(-\eta+) = 0$ if $\zeta > -\eta$, so the result is consistent with Theorem 6.

6. Results

In this section we will verify Conjecture 2 under mild regularity conditions on the one-step transition probabilities of the process \mathcal{X} and the associated random walk measure ψ . Unless stated otherwise we will

assume \mathcal{X} , and hence $\tilde{\mathcal{X}}$, to be aperiodic, that is, $r_j > 0$ for at least one state $j \in \mathcal{N}$. We may further restrict our analysis to the setting in which

$$\sum_{j=0}^{\infty} \frac{1}{p_j \pi_j} \sum_{k=0}^j r_k \pi_k < \infty \quad \text{and} \quad \tilde{L}_{\infty} < \infty,$$

since we know already by (31), (50) and Theorem 2 that the conjecture holds true in the opposite case, both sides of (41) then being equal to zero. In view of (36) we thus have $\sum r_n < \infty$, and hence $r_n \rightarrow 0$ as $n \rightarrow \infty$.

In what follows we denote the smallest and largest *limit* point of $\text{supp}(\psi)$ by σ and τ , respectively. Evidently, $\zeta \leq \sigma \leq \tau \leq \eta$. The next lemma shows that we can draw some useful conclusions on the measure ψ if, besides $\tilde{L}_{\infty} < \infty$ and $r_n \rightarrow 0$, the product $p_{n-1}q_n$ tends to a limit as $n \rightarrow \infty$.

Lemma 4. *Let $\lim_{n \rightarrow \infty} r_n = 0$ and $\tilde{L}_{\infty} < \infty$. If $\lim_{n \rightarrow \infty} p_{n-1}q_n = \beta$, then $\eta = \tau = 2\sqrt{\beta} > 0$ and $\zeta = \sigma = -2\sqrt{\beta}$.*

Proof. The *monic* polynomials $P_n = p_0 \dots p_{n-1}Q_n$ satisfy the recurrence

$$\begin{aligned} P_{n+1}(x) &= (x - r_n)P_n(x) - p_{n-1}q_n P_{n-1}(x), \quad n > 0, \\ P_0(x) &= 1, \quad P_1(x) = x - r_0. \end{aligned}$$

By Blumenthal's theorem (see Chihara [2]) we have $\sigma = -\tau = -2\sqrt{\beta}$ when $r_n \rightarrow 0$ and $p_{n-1}q_n \rightarrow \beta$ as $n \rightarrow \infty$. If $\eta > \tau$ then η must be an isolated point of $\text{supp}(\psi)$, and hence $\psi(\{\eta\}) > 0$. But in view of (51) this would contradict our assumption $\tilde{L}_{\infty} < \infty$, so we must have $\eta = \tau = 2\sqrt{\beta}$ and hence $\beta > 0$, by (12). Finally, by (13), $\zeta \geq -\eta$, but since $\zeta \leq \sigma = -\eta$, we must have $\zeta = \sigma$. \square

Note that, as a consequence of this lemma, Theorem 6 is of no use to us in verifying Conjecture 2 when $p_{n-1}q_n$ tends to limit, for in that case $\zeta > -\eta$ can only occur if $\tilde{L}_{\infty} = \infty$ or $r_n \not\rightarrow 0$.

Regarding the parameters p_j and q_j we will now impose the condition

$$\sum_{j=1}^{\infty} |p_j q_{j+1} - p_{j-1} q_j| < \infty, \tag{56}$$

implying in particular that $p_n q_{n+1}$ tends to a limit. We will further assume

$$\lim_{n \rightarrow \infty} p_n q_{n+1} = \frac{1}{4}, \tag{57}$$

so that, by the previous lemma, $\zeta = \sigma = -1$ and $\eta = \tau = 1$. The latter assumption entails no loss of generality, since, in view of (24) and (26), verifying Conjecture 2 is equivalent to verifying a similar conjecture in terms of $\tilde{\mathcal{X}}$, while by (20) and the previous lemma,

$$\tilde{p}_n \tilde{q}_{n+1} = \frac{p_n q_{n+1}}{\eta^2} \rightarrow \frac{\beta}{\eta^2} = \frac{1}{4} \quad \text{as } n \rightarrow \infty.$$

Letting Ψ as in (45) we can now invoke a theorem of Máté and Nevai [13] stating that Ψ is continuously differentiable in $(-1, 1)$ and $\Psi'(x) > 0$ for $x \in (-1, 1)$, so that $\text{supp}(\psi) = [-1, 1]$. In view of (8) and (57) we also have $\lim_{n \rightarrow \infty} \gamma_n^{1/n} = 2$, so that ψ is regular in the sense of Ullman-Stahl-Totik.

In what follows we will assume that the limits $\Psi'(-1+)$ and $\Psi'(1-)$ exist. Recalling our earlier assumptions that \mathcal{X} is aperiodic and $\tilde{L}_{\infty} < \infty$, we now have, by (18) and (51), not only $\Psi(-1+) = \Psi(-1) = 0$

and $\Psi(1-) = \Psi(1) = 1$ (implying the continuity of Ψ), but also $\Psi'(-1+) = \Psi'(1-) = 0$, which implies the continuity of Ψ' on \mathbb{R} . Next defining α, β and w as in (46) and (47), the Theorems 5 and 7 lead to the conclusion that, under the preceding conditions and if $0 < w(1-) < \infty$, we have $0 < \alpha \leq \beta$ and

$$\lim_{n \rightarrow \infty} C_n(\psi) = \lim_{n \rightarrow \infty} \frac{\rho_n(-1)}{\rho_n(1)} = \begin{cases} 0 & \text{if } \alpha < \beta \\ \frac{w(-1+)}{w(1-)} & \text{if } \alpha = \beta. \end{cases} \tag{58}$$

Collecting all our results we can now establish the following theorem, which amounts to validity of Conjecture 2 under mild regularity conditions.

Theorem 8. *Let \mathcal{X} be a birth-death process with corresponding random walk measure ψ , and let Ψ, α, β and w be defined as in (45), (46) and (47).*

(i) *If \mathcal{X} is periodic, then*

$$\lim_{n \rightarrow \infty} C_n(\psi) = \lim_{n \rightarrow \infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)} = 1.$$

(ii) *If \mathcal{X} is aperiodic and*

$$\sum_{j=0}^{\infty} \frac{1}{p_j \pi_j} \sum_{k=0}^j r_k \pi_k = \infty \quad \text{or} \quad \sum_{j=0}^{\infty} \frac{1}{p_j \pi_j Q_j(\eta) Q_{j+1}(\eta)} = \infty, \tag{59}$$

then

$$\lim_{n \rightarrow \infty} C_n(\psi) = \lim_{n \rightarrow \infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)} = 0.$$

(iii) *If \mathcal{X} is aperiodic, (59) does not hold (so that $r_n \rightarrow 0$), and in addition,*

- (a) *the one-step transition probabilities of \mathcal{X} satisfy $\sum_{j=1}^{\infty} |p_j q_{j+1} - p_{j-1} q_j| < \infty$,*
- (b) *the limits $\Psi'(-\eta+)$ and $\Psi'(\eta-)$ exist,*
- (c) *the quantities α and β are finite,*
- (d) *the limits $w(-\eta+)$ and $w(\eta-)$ exist and are finite, and $w(\eta-) > 0$, then $0 < \alpha \leq \beta$ and*

$$\lim_{n \rightarrow \infty} C_n(\psi) = \lim_{n \rightarrow \infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)} = \begin{cases} 0 & \text{if } \alpha < \beta \\ \frac{w(-\eta+)}{w(\eta-)} & \text{if } \alpha = \beta. \end{cases} \tag{60}$$

Proof. The first statement is implied by the fact that ψ is symmetric if \mathcal{X} is periodic, while the second statement follows from (31), (50) and Theorem 2. To prove the third statement we apply to the normalized version $\tilde{\mathcal{X}}$ of \mathcal{X} the argument preceding this theorem. Obviously, $\tilde{\Psi}'(x) = \eta \Psi'(\eta x)$ and $\tilde{w}(x) = \eta^{\alpha+\beta+1} w(\eta x)$, so subsequently rephrasing, with the help of (24) and (26), conclusion (58) and the conditions preceding it in terms of the original process \mathcal{X} , gives us (60). \square

7. Concluding remarks

The previous analysis remains largely valid if we allow $p_j + q_j + r_j \leq 1$ and interpret $\kappa_j := 1 - p_j - q_j - r_j$ as the *killing probability* of \mathcal{X} in state j , that is, the probability of absorption into an (ignored) cemetery state ∂ , say. Karlin and McGregor’s representation formula (2) still holds in this more general setting, but if $\kappa_j > 0$ for at least one state $j \in S$ (so that ∂ is accessible from \mathcal{N}) we have to make some adjustments to the preceding analysis.

First, asymptotic aperiodicity is not defined for \mathcal{X} in this case, but since the normalization (20) results in a process $\tilde{\mathcal{X}}$ which, as before, satisfies $\tilde{p}_j + \tilde{q}_j + \tilde{r}_j = 1$ for all $j \in \mathcal{N}$, the content of Subsection 2.3 remains relevant if \mathcal{X} is replaced by $\tilde{\mathcal{X}}$ (which will be different from \mathcal{X} , also if $\eta = 1$.) Then, from [4, Eq. (25)] we know that

$$Q_{n+1}(1) = 1 + \sum_{j=0}^n \frac{1}{p_j \pi_j} \sum_{k=0}^j \kappa_k \pi_k Q_k(1), \quad n \geq 0,$$

so that $Q_{n+1}(1) \geq Q_n(1)$ with strict inequality for n sufficiently large. So we no longer have $Q_n(1) = 1$ and therefore cannot assume the validity of (17) and its consequence (53). Note that

$$\lim_{n \rightarrow \infty} Q_n(1) = \infty \iff \sum_{j=0}^{\infty} \frac{1}{p_j \pi_j} \sum_{k=0}^j \kappa_k \pi_k = \infty, \quad (61)$$

while [4, Theorem 5] tells us that τ_j , the probability of eventual absorption at ∂ from state j , is given by

$$\tau_j = 1 - \frac{Q_j(1)}{Q_{\infty}(1)}, \quad j \in \mathcal{N}.$$

So eventual absorption at ∂ is certain if and only if $\lim_{n \rightarrow \infty} Q_n(1) = \infty$.

It is easily seen that [6, Lemma 2.1], and hence (29), remain valid in the more general setting at hand, but that is not so obvious for (30). In fact, it may be shown that (30) should be replaced by

$$\lim_{n \rightarrow \infty} \left| \frac{Q_n(1)}{Q_n(-1)} \right| = 0 \implies \lim_{n \rightarrow \infty} \left| \frac{Q_n(\eta)}{Q_n(-\eta)} \right| = 0, \quad (62)$$

and so the conclusion (31) cannot be maintained. However, in view of (61), we may replace (31) by

$$\sum_{j=0}^{\infty} \frac{1}{p_j \pi_j} \sum_{k=0}^j r_k \pi_k = \infty \text{ and } \sum_{j=0}^{\infty} \frac{1}{p_j \pi_j} \sum_{k=0}^j \kappa_k \pi_k < \infty \implies \lim_{n \rightarrow \infty} \frac{\rho_n(-\eta)}{\rho_n(\eta)} = 0. \quad (63)$$

In other words, (31) remains valid if we add the condition that absorption at ∂ is *not* certain. This has consequences for Theorem 8, where the first condition in (59) should be replaced by the two conditions in (63).

All other results remain valid.

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References

- [1] R.A. Askey, M.E.H. Ismail, Recurrence Relations, Continued Fractions and Orthogonal Polynomials, Memoirs of the American Mathematical Society, vol. 300, American Mathematical Society, Providence, R.I., 1984.
- [2] T.S. Chihara, Orthogonal polynomials whose zeros are dense in intervals, J. Math. Anal. Appl. 24 (1968) 362–371.
- [3] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.
- [4] P. Coolen-Schrijner, E.A. van Doorn, Quasistationary distributions for a class of discrete-time Markov chains, Methodol. Comput. Appl. Probab. 8 (2006) 449–465.
- [5] T. Danka, V. Totik, Christoffel functions with power type weights, J. Eur. Math. Soc. (JEMS) 20 (2018) 747–796.
- [6] E.A. van Doorn, On the strong ratio limit property for discrete-time birth-death processes, SIGMA Symmetry Integrability Geom. Methods Appl. 14 (2018) 047.

- [7] E.A. van Doorn, Asymptotic period of an aperiodic Markov chain, *Markov Process. Related Fields* 24 (2018) 759–778.
- [8] E.A. van Doorn, P. Schrijner, Random walk polynomials and random walk measures, *J. Comput. Appl. Math.* 49 (1993) 289–296.
- [9] E.A. van Doorn, P. Schrijner, Geometric ergodicity and quasi-stationarity in discrete-time birth-death processes, *J. Aust. Math. Soc. B* 37 (1995) 121–144.
- [10] E.A. van Doorn, P. Schrijner, Ratio limits and limiting conditional distributions for discrete-time birth-death processes, *J. Math. Anal. Appl.* 190 (1995) 263–284.
- [11] S. Karlin, J.L. McGregor, Random walks, *Illinois J. Math.* 3 (1959) 66–81.
- [12] H. Kesten, A ratio limit theorem for (sub) Markov chains on $\{1, 2, \dots\}$ with bounded jumps, *Adv. Appl. Probab.* 27 (1995) 652–691.
- [13] A. Máté, P. Nevai, Orthogonal polynomials and absolutely continuous measures, in: C.K. Chui, et al. (Eds.), *Approximation Theory IV*, Academic Press, New York, 1983, pp. 611–617.
- [14] S. Orey, Strong ratio limit property, *Bull. Amer. Math. Soc.* 67 (1961) 571–574.
- [15] W.E. Pruitt, Strong ratio limit property for R -recurrent Markov chains, *Proc. Amer. Math. Soc.* 16 (1965) 196–200.
- [16] J.A. Shohat, J.D. Tamarkin, *The Problem of Moments*. *Mathematical Surveys I*, rev. ed., American Mathematical Society, Providence, R.I., 1963.
- [17] H. Stahl, V. Totik, *General Orthogonal Polynomials*, *Encyclopedia of Mathematics and its Applications*, vol. 43, Cambridge University Press, Cambridge, 1992.
- [18] R. Szwarc, Chain sequences and compact perturbations of orthogonal polynomials, *Math. Z.* 217 (1994) 57–71.
- [19] W. Van Assche, Compact Jacobi matrices: from Stieltjes to Krein and $M(a, b)$, *Ann. Fac. Sci. Toulouse Math.* (6) S5 (1996) 195–215.
- [20] T.A. Whitehurst, *On Random Walks and Orthogonal Polynomials*, Ph.D. Thesis, Indiana University, Bloomington, 1978.
- [21] T.A. Whitehurst, An application of orthogonal polynomials to random walks, *Pacific J. Math.* 99 (1982) 205–213.