Non-cooperative queueing games on a Jackson network*

Corine M. Laan[†‡§¶] Judith Timmer[†‖] Richard J. Boucherie[†**]

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Abstract

In this paper we introduce non-cooperative games on a Jackson network. A player has a set of routes available and has to decide which routes to use for its customers. Each player's goal is to minimize the expected sojourn time of its customers.

We consider two cases. First, each player is allowed to divide his customers over multiple routes. This results in a game for which it can be shown that a unique pure-strategy Nash equilibrium exists. This Nash equilibrium can be found by using a best-response algorithm.

Second, each player may only select a single route for its customers. This results in a game with finite strategy spaces. In general, such games need not have a pure-strategy Nash equilibrium, as shown by an example. We show the existence of pure-strategy Nash equilibria for four subclasses of games on a Jackson network: (i) N-player games with equal arrival rates for the players, (ii) 2-player games with identical service rates for all nodes, (iii) 2-player games on a 2×2-grid, and (iv) 2-player games on an $A \times B$-grid with small differences in the service rates.

1 Introduction

In this paper, we introduce and analyze a new type of queueing games: non-cooperative games on a Jackson network. In this game multiple players compete to minimize the expected sojourn time of their customers. All players decide on a routing strategy through the network where they have to send their customers from a given source node to a given sink node. First, we analyze the class of games with continuous strategy spaces. Here

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†Stochastic Operations Research, Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands.

‡TNO, P.O. Box 96864, 2509 JG The Hague, The Netherlands.

§Netherlands Defence Academy, Faculty of Military Sciences, Het Nieuwe Diep 8, 1781 AC Den Helder, The Netherlands.

¶c.m.laan@hva.nl

‖Corresponding author. j.b.timmer@utwente.nl

**r.j.boucherie@utwente.nl

players can split their customers over several routes. Second, we consider the class of games in which each player is only allowed to pick a single route from the source to sink node. Such a game is a special variant of a weighted congestion game [15]. This implies in particular that such a game need not always have a Nash equilibrium in pure strategies. We present several variants of the game that do allow for the existence of a pure-strategy Nash equilibrium.

In the literature, several papers combine models from game theory and queueing theory. For example, in communication networks, game theory is used to determine optimal routing strategies [2], and in security, queueing theory is used to model stochastic elements in interdiction games [10, 20]. A broad overview of models of rational behavior in queueing systems is provided in [7], and [5] discusses a queueing game where Braess' paradox also occurs on a queueing network.

In this paper, we discuss a non-cooperative game on a Jackson network. The cooperative variant of this game is introduced in [17,18]. That cooperative game is played on a network of M/M/1 queues and each server in this network is considered to be a player. In [17] the players are allowed to cooperate by redistributing their service rates over the nodes in order to minimize the long run expected queue length, whereas in [18] the players cooperate by redistributing the arrival rate of the costumers.

The remainder of this paper is organized as follows. In Section 2 we introduce the non-cooperative game on a Jackson network. In Section 3 we consider the game with a continuous strategy space in which each player is allowed to divide his customers over a set of routes. We prove the existence of a unique pure-strategy Nash equilibrium and discuss how to find this. In Section 4, we discuss the game with a discrete strategy space. Now the players have to select a single route for their customers. We show that a pure-strategy Nash equilibrium need not always exist and present four subclasses of games that do allow for a pure-strategy Nash equilibrium. Section 5 concludes the paper.

2 Model

In this section, we introduce our non-cooperative game on a Jackson network.

Consider an open Jackson network with nodes $\mathcal{C} = \{1, \ldots, C\}$, in which agents or players $\mathcal{N} = \{1, \ldots, N\}$ select routes for their customers. Each node i is an M/M/1 queue with service rate μ_i, $i \in \mathcal{C}$. The arrival rate λ_i of node i is determined by the routes chosen by the players, $i \in \mathcal{C}$.

Player $j \in \mathcal{N}$ has to decide how to divide its customers over a given set of routes $R^{(j)}$, where a route $r \in R^{(j)}$ is represented by a set of nodes, i.e., $r \subset \mathcal{C}$. The goal of each player is to minimize the mean sojourn time of its customers via a suitable partition of its customers over the available routes. The total arrival rate of customers for player j is $\lambda^{(j)}$. A strategy of player j is a vector $p^{(j)}$, where component $p_r^{(j)}$ is the fraction of its customers the player sends along route $r \in R^{(j)}$, so that the arrival rate of customers from player j to route r is $p_r^{(j)}\lambda^{(j)}$.

Let $p = (p^{(1)}, ..., p^{(N)})$ denote the strategy profile of all players and $p_{-j} = (p^{(1)}, ..., p^{(j-1)},$

$p^{(j+1)}, ..., p^{(N)})$ the strategies without player j's strategy. Under strategy profile p, the total arrival rate for node i is

$$\lambda_i = \sum_{j=1}^{N} \sum_{\{r: i \in r, r \in R^{(j)}\}} p_r^{(j)} \lambda^{(j)}.$$

Strategy profile $p = (p^{(1)}, ..., p^{(N)})$ is feasible if $\lambda_i < \mu_i$ for all $i \in \mathcal{C}$. This is the stability condition for the Jackson network under which the equilibrium distribution for the number of customers at the nodes exists [8]. Let

$$\mathcal{P} = \left\{ p \,\middle|\, \sum_{\{r: r \in R^{(j)}\}} p_r^{(j)} = 1, p^{(j)} \geq 0, j \in \mathcal{N}, \sum_{j=1}^{N} \sum_{\{r: i \in r, r \in R^{(j)}\}} p_r^{(j)} \lambda^{(j)} < \mu_i, \ i \in \mathcal{C} \right\}$$

be the set of feasible strategy profiles. For a feasible strategy profile $p \in \mathcal{P}$ the mean sojourn time of node i is $1/(\mu_i - \lambda_i)$, and the mean sojourn time of the customers of player j is

$$f^{(j)}(p) = \sum_{i=1}^{C} \frac{\sum_{\{r: i \in r, r \in R^{(j)}\}} p_r^{(j)}}{\mu_i - \lambda_i}, \tag{1}$$

where $\tilde{p}_i^{(j)} := \sum_{\{r: i \in r, r \in R^{(j)}\}} p_r^{(j)}$ is the fraction of customers from player j that routes via node i.

The N-player non-cooperative game on the Jackson network is defined by the tuple $(\mathcal{N}, \{S^{(j)}, f^{(j)}\}_{j \in \mathcal{N}})$. The player set is the set of players $\mathcal{N}$. The strategy space $S^{(j)}$ of player j contains all feasible probability vectors $p^{(j)}$ and its payoff function is $f^{(j)}$ given in (1). The players choose their strategies simultaneously. In this game, the objective of each player is to minimize its payoff, the mean sojourn time of its customers.

A strategy profile $p^{(j)}$ is called dominant if it results in the lowest expected sojourn time for player j no matter the strategies chosen by the other players: $f^{(j)}(p^{(j)}, p_{-j}) \leq f^{(j)}(\bar{p})$ for all p_{-j}, $\bar{p}$. The best-response of player j to the other players' strategies p_{-j} is represented by the set $B^{(j)}(p_{-j}) = \{\hat{p}^{(j)} \in S_j | f^{(j)}(\hat{p}^{(j)}, p_{-j}) \leq f^{(j)}(\bar{p}^{(j)}, p_{-j}) \ \forall \bar{p}^{(j)}\}$. A strategy profile p is a pure-strategy Nash equilibrium if no player can decrease its expected sojourn time by unilateral deviation: for any player j $f^{(j)}(p^{(j)}, p_{-j}) \leq f^{(j)}(\bar{p}^{(j)}, p_{-j})$ for all $\bar{p}^{(j)}$. In particular, p is a pure-strategy Nash equilibrium if each player's strategy is a best-response to the other players' strategies: $p^{(j)} \in B^{(j)}(p_{-j})$ for all j.

In the following sections, we consider continuous and discrete strategy spaces, respectively.

3 Continuous strategy space

This section considers the game with a continuous strategy space, in which each player is allowed to divide its customers over its routes. We show that there exists a unique pure-strategy Nash equilibrium for this game and discuss approaches to find this Nash equilibrium.

The objective of each player is to minimize the mean sojourn time of its customers. Consider player j. Given the strategy p_{-j} of the other players, the optimal strategy for player j can be found by solving the following mathematical program:

$$\min_{p^{(j)}} \quad \sum_{i=1}^{C} \frac{\sum_{\{r:i\in r, r\in R^{(j)}\}} p_r^{(j)}}{\mu_i - \lambda_i} \tag{2}$$

$$\text{s.t.} \quad \lambda_i = \sum_{j=1}^{N} \sum_{\{r:i\in r, r\in R^{(j)}\}} p_r^{(j)} \lambda^{(j)}, \quad i \in \mathcal{C}, \tag{3}$$

$$\lambda_i < \mu_i, \qquad i \in \mathcal{C}, \tag{4}$$

$$\sum_{\{r:r\in R^{(j)}\}} p_r^{(j)} = 1, \tag{5}$$

$$p^{(j)} \geq 0. \tag{6}$$

Solving (2)–(6) simultaneously for all players j gives the optimal strategy profile p. This can be done via the method of Lagrange multipliers. As the example below shows, this is tractable only for small networks.

Example 1 *Consider a network with 3 nodes and two players with routes $R^{(1)} = \{\{1\},\{2\}\}$ and $R^{(2)} = \{\{2\},\{3\}\}$. Assume that $\mu_1 > \lambda^{(1)}$, $\mu_3 > \lambda^{(2)}$ and $\mu_2 > \lambda^{(1)} + \lambda^{(2)}$. The payoff functions for players 1 and 2 are*

$$f^{(1)}(p) = \frac{p_1^{(1)}}{\mu_1 - p_1^{(1)}\lambda^{(1)}} + \frac{1 - p_1^{(1)}}{\mu_2 - (1-p_1^{(1)})\lambda^{(1)} - (1-p_3^{(2)})\lambda^{(2)}} \quad \text{and}$$

$$f^{(2)}(p) = \frac{p_3^{(2)}}{\mu_3 - p_3^{(2)}\lambda^{(2)}} + \frac{1 - p_3^{(2)}}{\mu_2 - (1-p_1^{(1)})\lambda^{(1)} - (1-p_3^{(2)})\lambda^{(2)}},$$

where $p_i^{(j)}$ is the fraction of customers that player j sends along the route through node i, and $1 - p_i^{(j)}$ the fraction that this player sends along the other route.

The Lagrangians for the players are

$$L_1(p^{(1)}, \alpha_1, \alpha_2) = f^{(1)}(p) + \alpha_1 p_1^{(1)} + \alpha_2(1 - p_1^{(1)}),$$

$$L_2(p^{(2)}, \beta_1, \beta_2) = f^{(2)}(p) + \beta_1 p_3^{(2)} + \beta_2(1 - p_3^{(2)}).$$

The optimal solution for $p_1^{(1)}$ and $p_3^{(2)}$ can be found from the Karush-Kuhn-Tucker condi-

tions:

$$\begin{aligned}
\frac{\partial L_1}{\partial p_1^{(1)}} &= \frac{\mu_1}{(\mu_1 - p_1^{(1)}\lambda^{(1)})^2} - \frac{\mu_2 - (1-p_3^{(2)})\lambda^{(2)}}{(\mu_2 - (1-p_1^{(1)})\lambda^{(1)} - (1-p_3^{(2)})\lambda^{(2)})^2} + \alpha_1 - \alpha_2 = 0,\\
\frac{\partial L_2}{\partial p_3^{(2)}} &= \frac{\mu_3}{(\mu_3 - p_3^{(2)}\lambda^{(2)})^2} - \frac{\mu_2 - (1-p_1^{(1)})\lambda^{(1)}}{(\mu_2 - (1-p_1^{(1)})\lambda^{(1)} - (1-p_3^{(2)})\lambda^{(2)})^2} + \beta_1 - \beta_2 = 0,\\
\alpha_1 p_1^{(1)} &= 0,\\
\alpha_2(1-p_1^{(1)}) &= 0,\\
\beta_1 p_3^{(2)} &= 0,\\
\beta_2(1-p_3^{(2)}) &= 0,\\
0 \le p_1^{(1)} &\le 1,\\
0 \le p_3^{(2)} &\le 1.
\end{aligned}$$

Solving this system gives the optimal solution. For $\mu_1 = \mu_3 = 3,\ \mu_2 = 4,$ and $\lambda^{(1)} = \lambda^{(2)} = 1$, we find $p_1^{(1)} = p_3^{(2)} = 0.39$.

We may use the method of Lagrange multipliers and the Karush-Kuhn-Tucker conditions to find a Nash equilibrium for the general non-cooperative queueing game (2)–(6). However, as illustrated in Example 1, this will be intractable for larger networks. A common approach to find Nash equilibria is a best-response approach where all players iteratively implement their best-response strategy; see Algorithm 1. As is shown below, if $\mathcal{P}$ is non-empty, this approach yields a pure-strategy Nash equilibrium.

Algorithm 1 Best-response algorithm

1: Initialize: construct a feasible solution for $p^{(j)}$, $j \in \mathcal{N}$.
2: Select a player $j \in \mathcal{N}$ such that $p^{(j)}$ is not a best response to p_{-j}. Update $p^{(j)}$ to a best response of j to p_{-j}.
3: If p is a Nash equilibrium then stop. Else, go to Step 2 with p.

For the general case (2)–(6), we now show that there exists a unique pure-strategy Nash equilibrium and that the best-response algorithm converges to this equilibrium. We start by proving convexity of the payoff function of a player given fixed strategies of the other players.

Lemma 1 *The payoff function $f^{(j)}(p)$ of player j is strictly convex in $p^{(j)}$, $j \in \mathcal{N}$, $p \in \mathcal{P}$.*

Proof. We rearrange $f^{(j)}(p)$ first. Recall that $\tilde{p}_i^{(j)} := \sum_{\{r: i \in r, r \in R^{(j)}\}} p_r^{(j)}$ is the fraction of customers from player j that routes via node i. As p_{-j} is fixed, the service rate that remains available at node i for customers routed by player j is $\tilde{\mu}_i^{(j)} = \mu_i - \sum_{k=1,\ k \neq j}^{N} \tilde{p}_i^{(k)} \lambda^{(k)}$.

Then

$$f^{(j)}(p) = \sum_{i=1}^{C} \frac{\tilde{p}_i^{(j)}}{\tilde{\mu}_i^{(j)} - \tilde{p}_i^{(j)} \lambda^{(j)}},$$

which may be viewed as a function of $\tilde{p}_i^{(j)}$, $i \in \mathcal{C}$. Observe that

$$\begin{aligned}
\frac{\partial f^{(j)}(p)}{\partial \tilde{p}_i^{(j)}} &= \frac{\tilde{\mu}_i^{(j)}}{(\tilde{\mu}_i^{(j)} - \tilde{p}_i^{(j)} \lambda^{(j)})^2}, & i \in \mathcal{C},\\
\frac{\partial^2 f^{(j)}(p)}{(\partial \tilde{p}_i^{(j)})^2} &= \frac{2\tilde{\mu}_i^{(j)} \lambda^{(j)}}{(\tilde{\mu}_i^{(j)} - \tilde{p}_i^{(j)} \lambda^{(j)})^3}, & i \in \mathcal{C},\\
\frac{\partial^2 f^{(j)}(p)}{\partial \tilde{p}_i^{(j)} \partial \tilde{p}_k^{(j)}} &= 0, & i, k \in \mathcal{C}.\\
\frac{\partial \tilde{p}_i^{(j)}}{\partial p_r^{(j)}} &= \mathbb{1}(i \in r), & i \in \mathcal{C},
\end{aligned}$$

with indicator function $\mathbb{1}$. The Hessian of $f^{(j)}$ has the following entries, for $r, s \in R^{(j)}$:

$$\frac{\partial^2 f^{(j)}(p)}{\partial p_r^{(j)} \partial p_s^{(j)}} = \sum_{\{i : i \in r \cap s\}} \frac{\partial^2 f^{(j)}(p)}{(\partial \tilde{p}_i^{(j)})^2}.$$

We readily obtain that the Hessian is positive-definite, so that $f^{(j)}(p)$ is strictly convex in $p^{(j)}$. ■

We can now show the existence of a unique pure-strategy Nash equilibrium for the non-cooperative game on a Jackson network, and convergence of the best response algorithm.

Theorem 1 *If $\mathcal{P}$ is non-empty, the non-cooperative game on a Jackson network has a unique pure-strategy Nash equilibrium. Moreover, Algorithm 1 converges to a pure-strategy Nash equilibrium.*

Proof. We readily obtain that the strategy space $\mathcal{P}$ is bounded since $\sum_{\{r : r \in R^{(j)}\}} p_r^{(j)} = 1$, and $p^{(j)} \geq 0$. Moreover, this set is convex since it is constructed with linear (in)equalities. Lemma 1 implies that the payoff function is convex for each player. Since $\frac{q}{\mu - q\lambda}$ is continuous for each q where $\mu - q\lambda > 0$, the payoff function is continuous on $\mathcal{P}$. Theorem 1 in [14] now implies that the game posseses a unique pure-strategy Nash equilibrium.

The best-response algorithm converges to this equilibrium, see [13] and [16]. ■

4 Discrete strategy space

This section considers games in which each player is only allowed to send all its customers over a single route, resulting in a finite set of strategies for each player.

If a player is only allowed to select a single route, given the strategy p_{-j} of the other players, the optimal strategy for player j can be found by solving the following mathematical program:

$$\min_{p^{(j)}} \quad \sum_{i=1}^{C} \frac{\sum_{\{r:i\in r, r\in R^{(j)}\}} p_r^{(j)}}{\mu_i - \lambda_i} \tag{7}$$

$$\text{s.t.} \quad \lambda_i = \sum_{j=1}^{N} \sum_{\{r:i\in r, r\in R^{(j)}\}} p_r^{(j)} \lambda^{(j)}, \quad i \in \mathcal{C}, \tag{8}$$

$$\lambda_i < \mu_i, \qquad i \in \mathcal{C}, \tag{9}$$

$$\sum_{\{r:r\in R^{(j)}\}} p_r^{(j)} = 1, \tag{10}$$

$$p_r^{(j)} \in \{0,1\}, \qquad r \in R^{(j)}, \tag{11}$$

which is obtained from (2)–(6) by replacing (6) by $p_r^{(j)} \in \{0,1\}$. To obtain the optimal strategy profile p, we solve these problems simultaneously for all players.

This game has a Nash equilibrium in mixed strategies [12]. We are interested in a pure-strategy Nash equilibrium, where the players do not randomize over the possible routes. First, we show that this game translates to a congestion game. Then, we discuss the existence of a pure-strategy Nash equilibrium for several subclasses of the N-player non-cooperative game on the Jackson network.

In a congestion game multiple resources are available and each player selects a subset of these resources to minimize its own cost. The cost of each resource depends on the number of players selecting that resource. In a traditional congestion game, all players are equivalent and have the same influence on the cost of a single resource. For these games, there exists a Nash equilibrium in pure strategies [15]. In a weighted congestion game each player has a weight and the cost for each resource depends on the weighted sum over the players that pick that resource. In general, weighted congestion games do not always possess a Nash equilibrium in pure strategies (cf. [6, 11]).

There are several subclasses of weighted congestion games for which pure-strategy Nash equilibria do exist, for example, for games in which the underlying matrix is a matroid [1], for games with affine or exponential cost functions [6], and for games in which players can split their total weight (using integer-values only) over the resources and the cost functions are convex and monotonically increasing [19]. For Shapley network congestion games where the players split the cost of a shared edge pure-strategy Nash equilibria exist when at most two players can share an edge or when all players have the same source and sink node [3, 4, 9].

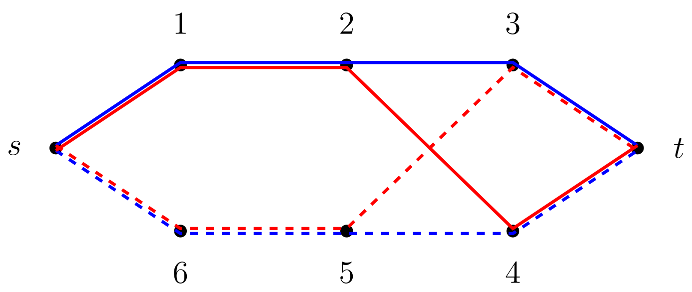

Figure 1: An example of a game on a six-node Jackson network with discrete strategy space without a pure-strategy Nash equilibrium.

For an N-player non-cooperative game on a Jackson network with a discrete strategy space existence of a pure-strategy Nash equilibrium is not guaranteed.

Example 2 (Game without a pure-strategy Nash equilibrium) *Inspired by [6], consider a 2-player non-cooperative game on a Jackson network with 6 nodes and service rates* $\mu_1 = \mu_2 = \mu_5 = \mu_6 = 6$ *and* $\mu_3 = \mu_4 = 4.95$*. Both player 1 (blue) and player 2 (red) have two possible strategies, see Figure 1: the strategies of player 1 are to select route* $r_1^{(1)} = \{1, 2, 3\}$ *or* $r_2^{(1)} = \{4, 5, 6\}$*, and the strategies of player 2 are to select route* $r_1^{(2)} = \{1, 2, 4\}$ *or* $r_2^{(2)} = \{3, 5, 6\}$*. The arrival rates are* $\lambda^{(1)} = 1$ *and* $\lambda^{(2)} = 2$*. A bimatrix representing the mean sojourn times of the customers of both players for this game is:*

		Player 2	
		$r_1^{(2)}$	$r_2^{(2)}$
Player 1	$r_1^{(1)}$	(0.920,1.006)	(0.913,1.013)
	$r_2^{(1)}$	(0.913,1.013)	(0.920,1.006)

For each pair of strategies $(r_r^{(1)}, r_s^{(2)})$*,* $r, s = 1, 2$*, one of the players has the incentive to switch to another strategy. Therefore, this game does not have a pure-strategy Nash equilibrium.*

There are several subclasses of games that allow for a pure-strategy Nash equilibrium. If the arrival rates of all players are identical, this game translates to a traditional congestion game that has a pure-strategy Nash equilibrium.

Theorem 2 *The N-player non-cooperative game on a Jackson network with equal arrival rates* $\lambda^{(j)} = \lambda$ *for all players* j*,* $j \in \mathcal{N}$*, has a pure-strategy Nash equilibrium.*

Proof. We will show that the N-player non-cooperative game on a Jackson network with equal arrival rates $\lambda^{(j)} = \lambda$ for all players j is a congestion game. The result then follows from [15], since a congestion game has a pure-strategy Nash equilibrium.

Consider a strategy profile p for all players. Let $a_i^{(j)}(p)$ equal 1 if node i is used by player j in profile p, and 0 otherwise. Then $x_i(p) := \sum_{j=1}^{N} a_i^{(j)}(p) = \#(\text{players using node } i)$. Under strategy profile p, the sojourn time (1) of player j is

$$f^{(j)}(p) = \sum_{i=1}^{C} \frac{\sum_{\{r:i\in r, r\in R^{(j)}\}} p_r^{(j)}}{\mu_i - \lambda_i} = \sum_{i=1}^{C} \frac{a_i^{(j)}(p)}{\mu_i - \lambda_i},$$

and

$$\lambda_i = \sum_{j=1}^{N} \sum_{\{r:i\in r, r\in R^{(j)}\}} p_r^{(j)} \lambda^{(j)} = \sum_{j=1}^{N} a_i^{(j)}(p)\lambda^{(j)} = x_i(p)\lambda.$$

Let the delay function $d_i : \mathbb{N} \to \mathbb{R}$ be defined as

$$d_i(x) = \frac{1}{\mu_i - x\lambda}.$$

Then

$$f^{(j)}(p) = \sum_{i=1}^{C} a_i^{(j)}(p) d_i(x_i(p)).$$

In its domain $\{x | x < \mu_i/\lambda\}$, $d_i(x)$ is positive and monotone increasing. Note that $x_i(p) < \mu_i/\lambda$ is required for strategy profile p to be feasible. This implies that the N-player non-cooperative game on a Jackson network with equal arrival rates is a congestion game. ■

If the service rates at all nodes are identical in a 2-player non-cooperative game on a Jackson network, then the game has a pure-strategy Nash equilibrium.

Theorem 3 *A 2-player non-cooperative game on a Jackson network with identical service rates μ_i for all nodes $i \in \mathcal{C}$ has a pure-strategy Nash equilibrium.*

Proof. Theorem 3.12 in [6] states that there exists a pure-strategy Nash equilibrium for a two-player game in which the cost function for each resource (node) can be written as $am(x) + b$, where $m(x)$ is a monotone function and $a, b \in \mathbb{R}$.

In our 2-player game we have $\mu_i = \mu$, $i \in \mathcal{C}$. Let $m(x) = \frac{1}{\mu - x}$, which is a monotone increasing function for $x < \mu$. For each node the cost function equals $m(\lambda_i)$ and can therefore be written as $am(x) + b$ with $a = b = 1$. Thus, this game has a pure-strategy Nash equilibrium. ■

The following two theorems discuss 2-player games on a Jackson network on an $A \times B$ grid. The network has AB nodes that may be represented by their coordinates (x, y), $x = 1, \ldots, A,\ y = 1, \ldots, B$. Customers may only route to the neighboring nodes $(x-1, y)$,

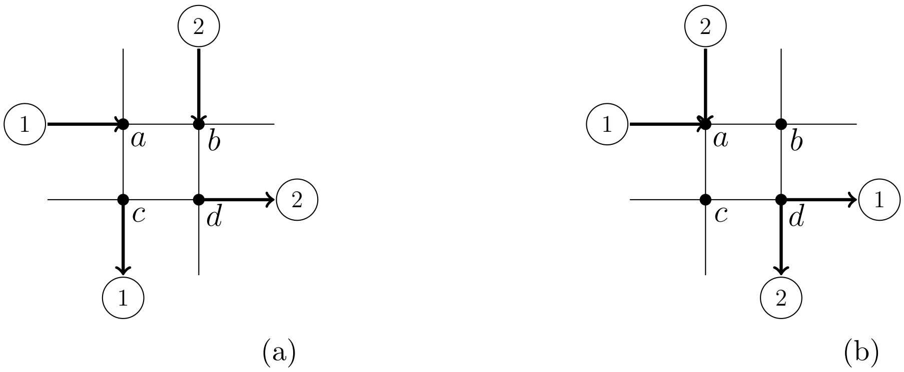

Figure 2: This figure shows two cases of a two-player game on a 2×2-grid. The nodes are named a till d. The source nodes are shown by inward arrows, with the player's name besides the arrow; outward arrows start from the sink nodes. In subfigure (a) the players 1 and 2 have different source and sink nodes. In subfigure (b) the players share their source and sink nodes.

$(x+1, y)$, $(x, y-1)$ and $(x, y+1)$ provided these nodes are part of the grid. Let μ_i be the service rate of node i, $i = (x, y)$, and let $\lambda^{(j)}$ be the arrival rate of customers for player j, $j = 1, 2$. All routes of the players start at a node at the side of the grid and end at a node on the side of the grid. Furthermore, we only allow routes of minimal length, i.e., containing the minimum number of nodes required to move from source node to sink node so that the length of the possible routes for player j is fixed, say $C^{(j)}$ for player j, $j = 1, 2$. Once again, each player's goal is to minimize its expected sojourn time.

First we discuss the subclass of 2-player games on a 2×2-grid. A trivial situation is one in which a player's source and sink nodes are neighbours on the grid. Then the unique optimal route for that player is to go directly from the source node to the sink node. We will not consider such situations.

Two interesting situations are shown in Figure 2. In these situations each player needs to route through three nodes and up front it is not clear which route will be preferred.

Theorem 4 *A 2-player game on a Jackson network on a 2×2-grid has a pure-strategy Nash equilibrium.*

Proof. Consider a 2×2-grid as shown in Figure 2. First, assume the players have different source and sink nodes as depicted on the left in the figure. Player 1 has to decide between routes

$$r_1^{(1)} = \{a, b, d\} \text{ and } r_2^{(1)} = \{a, c, d\}$$

for its arrivals with rate $\lambda^{(1)}$. Player 2 considers routes

$$r_1^{(2)} = \{b, d, c\} \text{ and } r_2^{(2)} = \{b, a, c\}$$

for its arrivals with rate $\lambda^{(2)}$. Let $\Lambda = \lambda^{(1)} + \lambda^{(2)}$ be the total arrival rate and μ_i the service rate of node i. For stability we need $\Lambda < \min\{\mu_a, \mu_b, \mu_c, \mu_d\}$. With slight abuse of notation, let $f^{(i)}(s_1, s_2)$ denote player i's payoff function if player j selects route s_j, $j = 1, 2$. Denote by $B^{(i)}(s_j)$ player i's best-response correspondence to strategy s_j of player $j \neq i$.

From

$$
\begin{aligned}
f^{(1)}(r_1^{(1)}, r_1^{(2)}) &= \frac{1}{\mu_a - \lambda^{(1)}} + \frac{1}{\mu_b - \Lambda} + \frac{1}{\mu_d - \Lambda} \text{ and} \\
f^{(1)}(r_2^{(1)}, r_1^{(2)}) &= \frac{1}{\mu_a - \lambda^{(1)}} + \frac{1}{\mu_c - \Lambda} + \frac{1}{\mu_d - \Lambda}
\end{aligned}
$$

it follows that player 1's best response to route $r_1^{(2)}$ is

$$
B^{(1)}(r_1^{(2)}) = \begin{cases} r_1^{(1)}, & \mu_b \geq \mu_c, \\ r_2^{(1)}, & \mu_c \geq \mu_b. \end{cases}
$$

Similarly, it follows that $B^{(1)}(r_2^{(2)}) = B^{(1)}(r_1^{(2)})$. Player 2's best-responses are

$$
B^{(2)}(r_1^{(1)}) = B^{(2)}(r_2^{(1)}) = \begin{cases} r_1^{(2)}, & \mu_d \geq \mu_a, \\ r_2^{(2)}, & \mu_a \geq \mu_d. \end{cases}
$$

Summarizing, this situation's pure-strategy Nash equilibria are presented in the following table.

	$\mu_d \geq \mu_a$	$\mu_a \geq \mu_d$
$\mu_b \geq \mu_c$	$(r_1^{(1)}, r_1^{(2)})$	$(r_1^{(1)}, r_2^{(2)})$
$\mu_c \geq \mu_b$	$(r_2^{(1)}, r_1^{(2)})$	$(r_2^{(1)}, r_2^{(2)})$

Besides, these results show that if $\mu_b \geq \mu_c$ then route $r_1^{(1)}$ is a *dominant strategy* for player 1, and else route $r_2^{(1)}$ is a dominant strategy. Route $r_1^{(2)}$ is player 2's dominant strategy if $\mu_d \geq \mu_a$, and else route $r_2^{(2)}$ is a dominant strategy.

Second, assume the players share their source and sink nodes as depicted on the right in the figure. Player j's routes are $r_1^{(j)} = \{a, b, d\}$ and $r_2^{(j)} = \{a, c, d\}$.

From

$$
\begin{aligned}
f^{(1)}(r_1^{(1)}, r_1^{(2)}) &= \frac{1}{\mu_a - \Lambda} + \frac{1}{\mu_b - \Lambda} + \frac{1}{\mu_d - \Lambda} \text{ and} \\
f^{(1)}(r_2^{(1)}, r_1^{(2)}) &= \frac{1}{\mu_a - \Lambda} + \frac{1}{\mu_c - \lambda^{(1)}} + \frac{1}{\mu_d - \Lambda}
\end{aligned}
$$

it follows that player 1's best response to route $r_1^{(2)}$ is

$$
B^{(1)}(r_1^{(2)}) = \begin{cases} r_1^{(1)}, & \mu_b - \lambda^{(2)} \geq \mu_c, \\ r_2^{(1)}, & \mu_b - \lambda^{(2)} < \mu_c. \end{cases}
$$

Similar reasoning leads to the best-response correspondences

$$B^{(j)}(r_1^{(3-j)}) = \begin{cases} r_1^{(j)}, & \mu_b - \lambda^{(3-j)} \geq \mu_c, \\ r_2^{(j)}, & \mu_b - \lambda^{(3-j)} \leq \mu_c. \end{cases}$$

$$B^{(j)}(r_2^{(3-j)}) = \begin{cases} r_1^{(j)}, & \mu_b + \lambda^{(3-j)} \geq \mu_c, \\ r_2^{(j)}, & \mu_b + \lambda^{(3-j)} \leq \mu_c. \end{cases}$$

for player $j = 1, 2$. For ease of exposition assume that $\lambda^{(1)} \geq \lambda^{(2)}$. We distinguish five cases.

Case I: $\mu_c \leq \mu_b - \lambda^{(1)}$. In this case route $r_1^{(j)} = \{a, b, d\}$ is a dominant strategy for each player. Thus $(r_1^{(1)}, r_1^{(2)})$ is a pure-strategy Nash equilibrium.

Case II: $\mu_b - \lambda^{(1)} \leq \mu_c \leq \mu_b - \lambda^{(2)}$. Route $r_1^{(1)} = \{a, b, d\}$ is a dominant strategy for player 1. This implies that in this case $(r_1^{(1)}, r_2^{(2)})$ is a pure-strategy Nash equilibrium.

Case III: $\mu_b - \lambda^{(2)} \leq \mu_c \leq \mu_b + \lambda^{(2)}$. From the best-response correspondences two pure-strategy Nash equilibria emerge: $(r_1^{(1)}, r_2^{(2)})$ and $(r_2^{(1)}, r_1^{(2)})$.

Case IV: $\mu_b + \lambda^{(2)} \leq \mu_c \leq \mu_b + \lambda^{(1)}$. Route $r_2^{(1)} = \{a, c, d\}$ is a dominant strategy for player 1. This implies that in this case $(r_2^{(1)}, r_1^{(2)})$ is a pure-strategy Nash equilibrium.

Case V: $\mu_b + \lambda^{(1)} \leq \mu_c$. Now route $r_2^{(j)} = \{a, c, d\}$ is a dominant strategy for each player j. This implies that in this case $(r_2^{(1)}, r_2^{(2)})$ is a pure-strategy Nash equilibrium.

We conclude that there exists a pure-strategy Nash equilibrium for two-player games on a 2×2-grid. ■

Second we consider games on a general $A \times B$ grid. If the source and sink nodes of the players are such that their routes $r^{(j)}$ for player j, $j = 1, 2$, intersect in at least one node, i.e., $r^{(1)} \cap r^{(2)} \notin \emptyset$, then a strategy profile p is feasible if $\lambda^{(1)} + \lambda^{(2)} < \mu$. The mean sojourn times of the customers of the players are

$$\begin{aligned} f^{(1)}(p) &= \sum_{\{i \in r^{(1)} \setminus r^{(2)}\}} \frac{1}{\mu_i - \lambda^{(1)}} + \sum_{\{i \in r^{(1)} \cap r^{(2)}\}} \frac{1}{\mu_i - (\lambda^{(1)} + \lambda^{(2)})}, \\ f^{(2)}(p) &= \sum_{\{i \in r^{(2)} \setminus r^{(1)}\}} \frac{1}{\mu_i - \lambda^{(2)}} + \sum_{\{i \in r^{(1)} \cap r^{(2)}\}} \frac{1}{\mu_i - (\lambda^{(1)} + \lambda^{(2)})}. \end{aligned}$$

For the case $\mu_i = \mu$, $i \in \mathcal{C}$, if $2\lambda < \mu$ according to Theorem 3, the game has a pure-strategy Nash equilibrium. For this Nash equilibrium the intersection of the routes will be in a single node: $r^{(1)} \cap r^{(2)} = \{j\}$, since sharing multiple nodes clearly increases the mean sojourn time of the customers of both players. The following theorem shows that the game has a pure-strategy Nash equilibrium when we vary the service rates $\mu_i \in \{\mu, k\mu\}$ for k sufficiently small to guarantee that routes will intersect in a single point.

Theorem 5 *The 2-player game on a Jackson network on a grid with $\mu_i \in \{\mu, k\mu\}$, for $k > 1$, $i \in \mathcal{C}$, and $\lambda^{(1)} = m\lambda$, $\lambda^{(2)} = \lambda$, $m < 1$, such that $(m+1)\lambda < \mu$ and $k\mu < \mu + m\lambda$,* *has a pure-strategy Nash equilibrium.*

Proof. The condition $k\mu < \mu + m\lambda$ guarantees that routes will intersect in at most 1 node, because then

$$\frac{1}{k\mu - (m+1)\lambda} > \frac{1}{\mu - \lambda} > \frac{1}{\mu - m\lambda}.$$

Let player 1 select a route $r^{(1)}$ that minimizes the sojourn time of its customers, and let player 2 select a route $r^{(2)}$ that is a best response to the route of player 1. If these routes do not intersect, then the strategy profile selecting these routes is a pure-strategy Nash equilibrium.

Now assume these routes intersect in node $i \in r^{(1)} \cap r^{(2)}$. There are two possible values for the service rate at the intersection of the routes: (1) $\mu_i = \mu$ and (2) $\mu_i = k\mu$. For each case we will consider player 1's best-response $r'^{(1)}$ to the route of player 2, which will also intersects with $r^{(2)}$ in one node. Let that be node $i' \in r'^{(1)} \cap r^{(2)}$.

Case (1): $\mu_i = \mu$. If $i = i'$ then the original route profile $(r^{(1)}, r^{(2)})$ is a pure-strategy Nash equilibrium. If $i \neq i'$ assume first that $\mu_{i'} = \mu$. Note that player 2's payoff is unaffected. Because the original route selected by player 1 was optimal without considering the influence of customers of player 2, the part of the route of player 1 that does not intersect with the route of player 2 cannot have a smaller sojourn time. Hence, the payoff of player 1 using route $r'^{(1)}$ cannot be below that of using $r^{(1)}$ and it will also not be larger because it is a best-response. Hence, the profile $(r'^{(1)}, r^{(2)})$ is a pure-strategy Nash equilibrium.

Second, assume $\mu_{i'} = k\mu$. This contradicts the optimality of $r^{(1)}$: if such a node i' is part of player 1's best response, then it should also have been part of the original route. So, this cannot happen.

Case (2): $\mu_i = k\mu$. We will show that under the conditions of the theorem these routes yield a pure-strategy Nash equilibrium.

If $i = i'$ then the original route profile $(r^{(1)}, r^{(2)})$ is a pure-strategy Nash equilibrium. If $i \neq i'$ then assume first that $\mu_{i'} = k\mu$. This will not affect the sojourn time of player 2. Following similar arguments as for case (1), the part of the route of player 1 that does not intersect with the route of player 2 cannot have a smaller sojourn time. Hence, the route profile $(r'^{(1)}, r^{(2)})$ is a pure-strategy Nash equilibrium.

Second, assume $\mu_{i'} = \mu$. For player 1, the new route $r'^{(1)}$ only results in a reduced sojourn time if the other nodes compensate for the increased sojourn time in node i'. Observe that the original route of player 1 was optimal. Therefore, the new route for player 1, that intersects in node i' with service rate $\mu_{i'} = \mu$, cannot contain more nodes with high service rate $k\mu$ than the original route. If the new route has one node with low service rate more than the original route, than the sojourn time of player 1 should have increased. This contradicts our assumption of $r'^{(1)}$ being a best-response. If the new route has the same number of nodes with high service rate, and therefore $r'^{(1)} \setminus r^{(2)}$ contains one extra high service rate node, the difference in sojourn times for player 1 ($r'^{(1)} - r^{(1)}$) is

$$\Delta f^{(1)} = \left(\frac{1}{\mu - (m+1)\lambda} + \frac{1}{k\mu - m\lambda}\right) - \left(\frac{1}{k\mu - (m+1)\lambda} + \frac{1}{\mu - m\lambda}\right),$$

which can readily be seen to be positive for $k > 1$, and $(m+1)\lambda < \mu$. But a positive difference means an increase in sojourn time. This contradicts the assumption that the new route is a best response to player 2's route. Hence, this situation cannot occur. ■

Observe that the assumption $\lambda^{(1)} < \lambda^{(2)} = \lambda$ is without loss of generality. The restriction $\mu \in \{\mu, k\mu\}$ in Theorem 5 is a sufficient condition to guarantee the existence of a pure-strategy Nash equilibrium. Experimental results show that pure-strategy Nash equilibrium exists in most of the random networks we constructed.

Finally, consider an N-player game on a Jackson network on a grid with $\mu_i = \mu$ for all i and $\lambda^{(j)} = \lambda$ for all j. For each pair of players, either their routes must intersect in at least one node, or the players may select routes that do not intersect. Clearly, the players may select their routes such that these routes intersect in at most one node. A strategy p that attains the minimal number of intersections required in the routes of all players is a pure-strategy Nash equilibrium, since deviating from this strategy either does not increase the number of intersection and thus does not influence the mean sojourn time, or increases the number of intersections and therefore increases the mean sojourn time by at least $\frac{1}{\mu - 2\lambda} - \frac{1}{\mu - \lambda}$. The pure-strategy Nash equilibrium also minimizes the total mean sojourn time for all players.

5 Conclusions

In this article we have considered a new type of games: non-cooperative games on a Jackson network, where multiple players route through a network while they are all minimizing their own sojourn time.

In case of continuous strategy spaces, each player is allowed to distribute his arrival rate over multiple fixed routes. This results in a game with continuous strategy space for which it can be proven that a pure-strategy Nash equilibrium exists. This Nash equilibrium can be found by using a best-response algorithm.

In case of discrete strategy spaces, all players are only allowed to select one single route resulting in a game with finite strategy spaces. In general, such games need not have a pure-strategy Nash equilibrium, as shown by an example. We have shown the existence of pure-strategy Nash equilibria for four subclasses of games on a Jackson network: (i) N-player games with equal arrival rates for the players, (ii) 2-player games with identical service rates for all nodes, (iii) 2-player games on a 2×2-grid, and (iv) 2-player games on an $A \times B$-grid with small differences in the service rates.

References

[1] H. Ackermann, H. Röglin, and B. Vöcking. Pure Nash equilibria in player-specific and weighted congestion games. *Theoretical Computer Science*, 410(17):1552–1563, 2009.

[2] E. Altman, T. Boulogne, R. El-Azouzi, T. Jiménez, and L. Wynter. A survey on networking games in telecommunications. *Computers & Operations Research*, 33(2):286–311, 2006.

[3] E. Anshelevich, A. Dasgupta, J. Kleinberg, E. Tardos, T. Wexler, and T. Roughgarden. The price of stability for network design with fair cost allocation. *SIAM Journal on Computing*, 38(4):1602–1623, 2008.

[4] H.L. Chen and T. Roughgarden. Network design with weighted players. *Theory of Computing Systems*, 45(2):302, 2009.

[5] J.E. Cohen and F.P. Kelly. A paradox of congestion in a queuing network. *Journal of Applied Probability*, 27(3):730–734, 1990.

[6] T. Harks and M. Klimm. On the existence of pure Nash equilibria in weighted congestion games. *Mathematics of Operations Research*, 37(3):419–436, 2012.

[7] R. Hassin. *Rational queueing.* Chapman and Hall/CRC, 2016.

[8] F.P. Kelly. *Reversibility and stochastic networks.* Wiley Chichester, 1979.

[9] K. Kollias and T. Roughgarden. Restoring pure equilibria to weighted congestion games. In L. Aceto, M. Henzinger, and J. Sgall, editors, *Automata, Languages, and Programming. ICALP 2011*, volume 6756 of *Lecture Notes in Computer Science*, pages 539–551. Springer, 2011.

[10] C.M. Laan, T. van der Mijden, A.I. Barros, R.J. Boucherie, and H. Monsuur. An interdiction game on a queueing network with multiple intruders. *European Journal of Operational Research*, 260(3):1069–1080, 2017.

[11] I. Milchtaich. Congestion games with player-specific payoff functions. *Games and Economic Behavior*, 13(1):111–124, 1996.

[12] J. Nash. Non-cooperative games. *Annals of Mathematics*, 54(2):286–295, 1951.

[13] J.S. Pang, G. Scutari, D.P. Palomar, and F. Facchinei. Design of cognitive radio systems under temperature-interference constraints: A variational inequality approach. *IEEE Transactions on Signal Processing*, 58(6):3251–3271, 2010.

[14] J.B. Rosen. Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica: Journal of the Econometric Society*, 33(3):520–534, 1965.

[15] R.W. Rosenthal. A class of games possessing pure-strategy Nash equilibria. *International Journal of Game Theory*, 2(1):65–67, 1973.

[16] G. Scutari, D.P. Palomar, F. Facchinei, and J.S. Pang. Convex optimization, game theory, and variational inequality theory. *IEEE Signal Processing Magazine*, 27(3):35–49, 2010.

[17] J. Timmer and W. Scheinhardt. Cost sharing of cooperating queues in a Jackson network. *Queueing Systems*, 75(1):1–17, 2013.

[18] J. Timmer and W. Scheinhardt. Customer and cost sharing in a Jackson network. *International Game Theory Review*, 20(03):1850002, 2018.

[19] L. Tran-Thanh, M. Polukarov, A. Chapman, A. Rogers, and N.R. Jennings. On the existence of pure strategy Nash equilibria in integer–splittable weighted congestion games. In G. Persiano, editor, *Algorithmic Game Theory. SAGT 2011*, volume 6982 of *Lecture Notes in Computer Science*, pages 236–253. Springer, 2011.

[20] L.M. Wein and M.P. Atkinson. The last line of defense: designing radiation detection-interdiction systems to protect cities from a nuclear terrorist attack. *IEEE Transactions on Nuclear Science*, 54(3):654–669, 2007.