Journal of the Operational Research Society

ISSN: 0160-5682 (Print) 1476-9360 (Online) Journal homepage: https://www.tandfonline.com/loi/tjor20

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JOURNAL OF THE OPERATIONAL RESEARCH SOCIETY, 2018
VOL. 69, NO. 4, 571–579
https://doi.org/10.1057/s41274-017-0251-2

Compromise for the complaint: an optimization approach to the ENSC value and the CIS value

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ABSTRACT
The main goal of this paper is to introduce a new solution concept: the optimal compromise value. We propose two kinds of complaint criteria based on which the optimistic complaint and the pessimistic complaint are defined. Two optimal compromise values are obtained by lexicographically minimizing the optimistic maximal complaint and the pessimistic maximal complaint, respectively. Interestingly, these two optimal compromise values coincide with the ENSC value and the CIS value, respectively. Moreover, these values are characterized in terms of equal maximal complaint property and efficiency. As an adjunct, we reveal the coincidence of the Nucleolus and the ENSC value of 1-convex games.

ARTICLE HISTORY
Received 11 July 2016
Accepted 9 May 2017

KEYWORDS
Cooperative game; optimal compromise value; ENSC value; CIS value; equal maximal complaint property

JEL CLASSIFICATIONS
91A12; C71

1. Introduction

A central question in Game Theory is how to share the joint surplus fairly and reasonably among players when they cooperate. For games in characteristic form where the worth of a coalition depends only on the composition of this coalition, Gillies (1953) uses efficiency and group rationality to characterize the Core. The Core of a cooperative game (N, v) is a setvalued solution concept being proposed as a division of $v(N)$ among the players under which no coalition has a worth greater than the sum of its members' payoffs. Therefore, no coalition has incentive to leave the grand coalition and receive a larger payoff. Such a payoff imputation has an inherent stability. The development of the theory in this paper is based on the complaint related to the upper bound and the lower bound of the Core. The involved upper bound of the Core is given in terms of the worth of the player set itself and the $(n-1)$ person coalitions called marginal contribution, and the lower bound of the Core is described as the individual worth vector.

For the cooperative game (N, v) of which the Core is not empty, from the Core's point of view, the marginal contribution is the player's ideal payoff, but in fact, the overall worth $v(N)$ is weakly insufficient to cover all these marginal contributions. Thus, at least one player $i, i \in N$, cannot obtain his marginal contribution, yielding the complaint of player i. On the other, any coalition S is totally satisfied if the outsiders in $N \backslash S$ only obtain their individual worths, by observing that the individual worth vector is the lower bound of the Core. However, for the non-essential cooperate game, at least one player can get more than his individual worth, which will cause the complaint of other players.

The excess proposed by Davis and Maschler (2010) is one criterion to characterize the complaint with respect to the payoff vector. A nonnegative (non-positive, respectively) excess of coalition S at the payoff vector x in the game v represents the gain (loss) to the coalition S if its members withdraw from the payoff vector x in order to form their own coalition. The idea of excess forms the basis of several solution concepts for cooperative games in characteristic function form, such as the (pre-)Kernel, the (pre-)Nucleolus and the τ value (Branzei, Dimitrov, & Tijs, 2005). Beyond the excess criterion, there are other criteria to characterize the complaint with respect to the payoff vector. In our paper, for every payoff vector x, $x \in R^n$, two types of new criteria are proposed to measure the complaint of coalition S, $S \subseteq N$, with respect to the payoff vector x, by observing the fact that the marginal contribution vector and the individual worth vector are the upper bound and the lower bound of the Core, respectively. One is proposed from the perspective of coalitional insiders and to measure the amount (the size of the inequity) by which coalition falls short of its potential marginal contribution, while the other is based on the surplus of the complementary coalition, and the complaint is

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measured by the gap between the real payoff and the least potential payoff of the complementary coalition.

Instead of applying a general axiomatization of fairness to a value function defined on the set of all characteristic functions, we look at a fixed characteristic function v and try to find a payoff vector x that minimizes the maximal complaint under certain complaint criteria. Precisely, we look first at those coalitions S whose complaint, for a fixed payoff vector x, is the largest. Then we adjust x, if possible, to make this largest complaint smaller. When the largest complaint has been made as small as possible, we concentrate on the next largest complaint and adjust x to make it as small as possible, and so on.

At the first sight, the internal relationship between the ENSC value, the CIS value and the (pre)Nucleolus is not obvious. Driessen and Funaki (1991) revealed the coincidence of the ENSC value and the (pre-)Nucleolus in 1991. In this paper, by minimizing the maximal complaint under different complaint criteria paralleling the characterizing of the (pre)Nucleolus, we obtain various significant solutions of the game, respectively. In the case coalition S's complaint is measured by the gap of the sum of the marginal contributions and the payoff x involved, i.e., $\sum_{i\in S}(v(N)-v(N\backslash\{i\}))-\sum_{i\in S}x_i$, and the ENSC value, proposed by Moulin (1985), is achieved by minimizing the optimistic maximal complaint. Alternatively, the coalition S may take the surplus of his complementary coalition $N\backslash S$ into consideration, which yields the other complaint criterion. To be precise, from a coalition point of view, since the players outside S do notjoin S, it is assumed that any player i in $N\backslash S$ does not cooperate and obtains his least potential payoff $v(i)$, which generates a gap between the proposed allocation x_j and the individual value $v(i)$. When coalition S's complaint is measured by the gap of his complementary coalition $N\backslash S$'s least potential payoff and the payoff $\sum_{i\in N\backslash S}x_i$, i.e., $\sum_{i\in N\backslash S}x_i-\sum_{k\in N\backslash S}v(\{k\})$, by minimizing the pessimistic maximal complaint, we achieve the CIS value, which is proposed by Driessen and Funaki (1991). In this way, our result could also be regarded as a kind of coincidence between the (pre)Nucleolus and the single-valued solution. Moreover, inspired from the notion of the (pre)Kernel, we introduce two new properties called equal optimistic maximal complaint property and equal pessimistic maximal complaint property, which can be used to characterize the optimal compromise values involved. On the domain of special games, the ENSC value and the CIS value can be represented by the same manner as in the (pre-)Kernel.

The 1-convex game is initiated by Driessen and Tijs (1983). For the 1-convex game, its Core is non-empty and the Nucleolus is the center of the Core. Moreover, the intersection of the Core with the Kernel contains only one point (Driessen, 1985). The first practical example of a 1-convex game, and the 1-convex complementary unanimity basis for the entire space of cooperative games are introduced in Driessen, Khmelnitskaya, and Jordi (2010). In this paper we show that the Nucleolus and the ENSC value coincide with each other in the 1-convex games.

The paper is organized as follows: Section 2 treats the relevant game theoretic notions and solution concepts, e.g., the ENSC value and the CIS value, in Section 3, we determine the ENSC value and the CIS value by lexicographically minimizing the optimistic maximal complaint and the pessimistic maximal complaint, Section 4 presents another representation of the ENSC value and the CIS value in terms of equal maximal complaint property, and the paper concludes with a brief summary and discussion of further research.

2. Notions and solution concepts

A cooperative game on player set N is a characteristic function v: $\mathcal{P}(N)\to R$ defined on $\mathcal{P}(N)$ satisfying $v(\emptyset)=0$. Here $\mathcal{P}(N)$ denotes the power set of the finite player set N, given by $\mathcal{P}(N)=\{S|S\subseteq N\}$, and in short called a game v on N. Denote all cooperative games in player set N by Γ. With the characteristic function v and supposing that some type of understanding is arrived at by the players, they have to divide the grand value $v(N)$. A distribution of the amount $v(N)$ among the n players is represented by an n-tuple vector $x=(x_1,x_2,\ldots,x_n)$, which are real numbers satisfying the efficiency principle, i.e., $\sum_{i\in N}x_i=v(N)$. For notation convenience, throughout this article, denote $\Sigma_{i\in S}x_i$ by $x(S)$, $S\subseteq N$.

Given the distribution x_i to any player i is not less than his individual worth, i.e., $x_i\geq v(\{i\})$ for all $i\in N$, then the payoff vector x satisfies the individual rationality. Moreover, the group rationality is defined as $x(S)\geq v(S)$, for all $S\subseteq N$. Based on these principles, we review three set solutions of n person game v, the pre-imputation set $I^*(v)$, the imputation set $I(v)$ and the Core $C(v)$.

$$I^*(v):\{x\in R^n|x(N)=v(N)\},$$
$$I(v):\{x\in R^n|x(N)=v(N),\ x_i\geq v(i)\text{ for all }i\in N\},$$
$$C(v)=\{x|x\in I^*(v),\ x(S)\geq v(S)\text{ for all }S\subseteq N\}.$$

It is left to the reader to verify that

$$I(v)\neq\emptyset\text{ if and only if }\sum_{j\in N}v(\{j\})\leq v(N).$$

In this paper, we denote the class of game (N,v) of which $I(v)\neq\emptyset$ by Γ_2.

The Core of an n-person game was first accurately defined by Gillies through group rationality and efficiency (Gillies, 1953). The Core gives rise only to nonpositive excesses. Here the excess of coalition S with respect to the payoff vector x of the game v is defined to be $e^v(S)=v(S)-x(S)$. The idea of excess forms the basis of several solution concepts for cooperative games such as the Core, the (pre)-Nucleolus and the (pre-)Kernel. The excess vector can be used to measure the complaint of players about the payoff vector x. The larger the excess is, the more unsatisfied the players in coalition would feel.

The purpose of this paper is to illustrate other two kinds of complaint criteria, based on which the ENSC value and the CIS value are obtained by lexicographically minimizing the optimistic maximal complaint and the pessimistic maximal complaint, respectively. The CIS value is a solution that concerns about the worths of the individuals and the grand coalition. It first gives every player its individual worth and then distributes the remaining worth of the grand coalition equally among all players, i.e., for any game v,

$$CIS_i(v) = v(\{i\}) + \frac{1}{n}\left[v(N) - \sum_{j\in N} v(\{j\})\right], \text{ for all } i \in N.$$

As an anti-dual value of the CIS value, the ENSC value is a single value based on the separable contributions. For k-coalitional n-person game, Shapley value can be written as a combination of the ENSC value and the CIS value (Driessen & Funaki, 1991). A standard principle requires that the payoff to any player $i \in N$ in the game v is at most the marginal contribution $b_i^v = v(N) - v(N\backslash i)$ of player i with respect to the formation of the grand coalition N. The ENSC value is characterized by the fact that the remaining non-separable contribution $NSC(v) = v(N) - \sum_{j\in N} b_j^v$ is equally charged by the n players in the game v. That is,

$$ENSC_i(v) = b_i^v + \frac{1}{n}\left[v(N) - \sum_{j\in N} b_j^v\right], \text{ for all } i \in N.$$

Thus, the ENSC value can be regarded as a well-known separable contributions remaining benefits method which is a widely used payoff in the water resources field (Young, Okada, & Hashimoto, 1982).

3. The determination of the optimal compromise value

The Nucleolus introduced by Schmeidler (1969) consists of imputations that minimize the complaint of excess in the lexicographic order over the non-empty compact convex imputation set. In this view of point, the Nucleolus is a one- point compromise value of which the corresponding complaint vector is the smallest under the criterion of excess. In this section we introduce one class of optimal compromise value and determine the optimistic optimal compromise value and the pessimistic optimal compromise value under two different complaint criteria, respectively.

For any n-tuple $x \in R^n$, let $\theta(x)$ be the 2^n-tuple whose components are the complaints of coalition S, $S \subseteq N$, arranged in non-increasing order. Thus,

$$\theta_i(x) \geq \theta_j(x), \quad \text{if } 1 \leq i \leq j \leq 2^n.$$

For $x, y \in R^n$ we have $x \leq_L y$, i.e., x is lexicographically smaller than (or equal to) y, if $x = y$ or if there exists an $s \in \{1, 2, \dots, 2^n - 1\}$ such that $x_k = y_k$ for all $k \in \{1, 2, \dots, s-1\}$ and $x_s < y_s$.

Definition 3.1 The optimal compromise value of a balanced game (N, v) is the unique pre-imputation vector y of which the corresponding complaint vector $\theta(y)$ satisfies the lexicographic order $\theta(y) \leq_L \theta(x)$ for any pre-imputation vector x.

Obviously, under the complaint criterion of excess $v(S) - x(S)$, the optimal compromise value and the Nucleolus of the balanced game coincide with each other. The question is: Are there any other kinds of complaint criteria? Under different kinds of complaint criteria, what are the presentations of the optimal compromise values? By answering these questions, we consider the bounds of the Core.

To begin with, we focus on the balanced game (N, v), for which game the Core is not empty and the elementary fact holds that for all $x \in Core(v)$, $v(\{i\}) \leq x_i \leq v(N) - v(N\backslash i)$. Based on this, we can conclude that for the balanced game, the marginal contribution vector and the individual worth vector are upper bound and lower bound of the Core, respectively. The so-called optimal compromise value is based on the idea of minimizing the maximal complaint vector, while the optimistic optimal compromise value and the pessimistic optimal compromise value are related to the complaint vectors with respect to $(b_i^v)_{i\in N}$ and $(v(\{i\}))_{i\in N}$.

Definition 3.2 For each balanced game (N, v), given player i's ideal payoff and the least potential payoff are b_i^v and $v(\{i\})$, respectively, *the optimistic complaint* and *the pessimistic complaint* of coalition S are given by

$$e(S, x, b^v) = b^v(S) - x(S), S \subseteq N, \text{ where } b^v(S) = \sum_{i\in S} b_i^v, \tag{3.1}$$

$$e(S, x, v(.)) = x(N\backslash S) - \sum_{j\in N\backslash S} v(\{j\}), \emptyset \neq S \subseteq N. \tag{3.2}$$

In this paper, we define these two complaint criteria by "optimistic" and "pessimistic", because in the optimistic terms, players always take the surplus of themselves into consideration, while the pessimist always departs from the other players' extra payoff. The optimistic complaint $e(S, x, b^v)$ of coalition S at the payoff vector x with respect to the marginal contribution vector b^v of the balanced game (N, v) represents the amount of the complaint about the payoff distribution x, assuming the member i in coalition S is willing to obtain the monetary utility of which is measured by his marginal contribution $b_i^v = v(N) - v(N\backslash\{i\})$. In the case coalition S, $S \subseteq N$, complains his complementary coalition $N\backslash S$'s payoff by observing that the least potential payoff of player $j \in N\backslash S$ is his individual worth $v(\{j\})$. Thus, the difference of the sum of the $N\backslash S$'s payoff and the sum of the individual worth, i.e., $x(N\backslash S) - \sum_{j\in N\backslash S} v(\{j\})$, is an alternative way to measure the complaint of coalition S. The pessimistic complaint $e(S, x, v(\cdot))$ represents the complaint of a non-empty coalition S at the payoff vector x with respect to the least potential payoff of his complementary coalition in the game v, assuming coalition S only wants the outsiders get their individual worths.

The Nucleolus minimizes the maximal complaint under the complaint criterion of excess $v(S) - x(S)$. One may wonder which payoff vector can minimize the complaint under other complaint criteria. In next subsections, we will determine the optimistic optimal compromise value and the pessimistic optimal compromise value by the same manner as in the Maschler and Peleg (1966).

3.1. The determination of the optimistic optimal compromise value

Definition 3.3 The optimistic optimal compromise value of a balanced game (N, v) is the unique pre-imputation vector x of which the corresponding upper complaint vector $\theta(x)$ satisfies the lexicographic order $\theta(x) \leq_L \theta(y)$ for any preimputation vector y.

Theorem 3.4 *Let (N, v) be a balanced game. The optimistic optimal compromise value x is bounded by the marginal contribution vector b^v*, **i.e.,** $x_i \leq b_i^v, i \in N$.

Proof Suppose $x_i \leq b_i^v$ does not hold for all $i \in N$.

Case 1 If $x_i \geq b_i^v$ holds for all $i \in N$ and there exists at least one player $j, j \in N$ such that $x_j > b_j^v$, then

$$\sum_{i\in N} x_i > \sum_{i\in N} b_i^v \geq v(N).$$

The last inequality holds because the marginal vector b^v is the upper bound of the Core. Thus, $x(N) > v(N)$, which contradicts with the efficiency of the upper optimal compromise value.

Case 2 If there exist at least two players, $i, j \in N$, such that $x_i > b_i^v, x_j < b_j^v$. Denote $\Delta = \min\{b_j^v - x_j, x_i - b_i^v\}$, then $\Delta > 0$ and for the new payoff x^*, where

$$x_k^* = \begin{cases} x_k \text{ for all} & k \neq i, j \\ x_j + \Delta & k = j, \\ x_i - \Delta & k = i. \end{cases} \tag{3.3}$$

It holds $\theta(x) > \theta(x^*)$ by observing that the first coordinates of $\theta(x)$ and $\theta(x^*)$ are $\sum_{b_j^v > x_j, j\in N}(b_j^v - x_j)$ and $\sum_{b_j^v > x_j, j\in N}(b_j^v - x_j) - \Delta$, respectively, contradicting with the claim that x is the optimistic optimal compromise value. Therefore, $x_i \leq b_i^v, i \in N$. □

The critical inequality in the proof of Theorem 3.4 gives rise to look at the class of n-person game (N, v) of which the sum of the marginal contributions cannot be covered by $v(N)$, i.e., $v(N) \leq b^v(N)$. Let Γ_1 denote this class of games. Interestingly, on the domain of this class of games, the conclusion in Theorem 3.4 still holds.

Remark 3.5 Let $(N, v) \in \Gamma_1$. Then the optimistic optimal compromise value x is upper bounded by the marginal contribution vector b^v, i.e., $x_i \leq b_i^v, i \in N$.

The proof is immediate which is similar to that of Theorem 3.4.

Lemma 3.6 *Given x is the optimistic optimal compromise value of a game $(N, v) \in \Gamma_1$, denote $W = \{T | e(T, x, b^v) > b^v(N\backslash m) - x(N\backslash m)\}$. Then for any player m such that $b_l^v - x_l > b_m^v - x_m \geq 0$, $l, m \in N$, it holds that: for any $T \in W$, $m, l \in T$.*

Proof Suppose $m \notin T$, thus, $T \subseteq N\backslash\{m\}$ yielding $e(T, x, b^v) = b^v(T) - x(T) \leq b^v(N\backslash\{m\}) - x(N\backslash\{m\})$, since $b_i^v \geq x_i$ by Remark 3.5. Therefore, $T \notin W$ which contradicts with $T \in W$. Thus, $m \in T$. For any $T \in W$, suppose $l \notin T$, on the one hand, it holds that $T \subseteq N\backslash\{l\}$ which yields $b^v(T) - x(T) < b^v(N\backslash\{l\}) - x(N\backslash\{l\})$ On the other, $b^v(T) - x(T) > b^v(N\backslash m) - x(N\backslash m)$ which holds by the definition of W and Theorem 3.4. Thus, $b^v(N\backslash\{l\}) - x(N\backslash\{l\}) > b^v(N\backslash m) - x(N\backslash m)$ which is equivalent to $b_m^v - x_m > b_l^v - x_l$, and this contradicts with the assumption $b_m^v - x_m < b_l^v - x_l$. Thus, $T \ni l$. □

Lemma 3.7 *Given optimistic complaint criterion, the payoff vector x of the game $(N, v) \in \Gamma_1$ and any player m such that $b_l^v - x_l > b_m^v - x_m \geq 0$, $l, m \in N$, where $x \in I^*(v)$, denote $\Delta = \frac{b_l^v - x_l - (b_m^v - x_m)}{2}$. Then for the new payoff vector x^* constructed in the form of* (3.3), *the following five statements hold.*

(i) *for any $S, S \subseteq N, S \not\ni m, l, e(S, x^*, b^v) = e(S, x, b^v)$.*
(ii) *for any $S, S \subseteq N, S \ni m, l, e(S, x^*, b^v) = e(S, x, b^v)$.*
(iii) *for any S, $S \subseteq N$, $S \not\ni m$, $S \ni l$, $e(S, x^*, b^v) < e(S, x, b^v)$.*
(iv) *for and S, $S \subseteq N, S \ni m, S \not\ni$ l, given $e(S \cup \{l\}\backslash\{m\}, x^*, b^v) \neq e(S \cup \{l\}\backslash\{m\}, x, b^v)$, it is redundant to compare* $e(S, x, b^v)$ and $e(S, x^*, b^v)$.
(v) $\theta(x) > \theta(x^*)$.

Proof

(i) It is trivial that for any S, $S \subseteq N$, $S \not\ni m$, l, $e(S, x, b^v) = e(S, x^*, b^v)$ holds.
(ii) for any $S, S \subseteq N, S \ni m, l,$

$$e(S, x^*, b^v) = b^v(S) - x^*(S)$$

(iii) for any $S, S \subseteq N, S \not\ni m, S \ni l,$

$$\begin{aligned} e(S, x^*, b^v) &= b^v(S) - x^*(S) = b^v(s\backslash\{l\}) - x^*(s\backslash\{l\}) \\ &+ (b_l^v - x_l^*) = b^v(S\backslash\{l\}) - x(S\backslash\{l\}) \\ &+ [b_l^v - (x_l + \Delta)] = b^v(S) - x(S) \\ &- \Delta = e(S, x, b^v) - \Delta < e(S, x, b^v) \end{aligned}$$

The last inequality holds because $\Delta > 0$.

(iv) for any $S, S \subseteq N, S \ni m$, $S \not\ni$ l,

$$e(S, x, b^v) = b^v(S) - x(S)$$

Therefore, for any $S \ni m, S \not\ni 1$, there always exists coalition $S\cup\{l\}\backslash\{m\}$, such that $e(S\cup\{l\}\backslash\{m\},x,b^v) > e(S,x,b^v)$. On the other, note that $b^v_l - x^*_l = b^v_m - x^*_m$, thus,

$$\begin{aligned} e(S,x^*,b^v) &= b^v(S) - x^*(S) = b^v(s\backslash\{m\}) - x^*(s\backslash\{m\}) \\ &+ (b^v_m - x^*_m) = b^v(S\backslash\{m\}) - x(S\backslash\{m\}) \\ &+ (b^v_l - x^*_l) = b^v(S\cup\{l\}\backslash\{m\}) - x^*(S\cup\{l\}\backslash\{m\}) \\ &= e(S\cup\{l\}\backslash\{m\},x^*,b^v) \end{aligned}$$

Because $e(S\cup\{l\}\backslash\{m\},x^*,b^v) < e(S\cup\{l\}\backslash\{m\},x,b^v)$ by (iii), it is redundant to compare $e(S\cup\{l\}\backslash\{m\},x^*,b^v)$ and $e(S\cup\{l\}\backslash\{m\},x,b^v)$ given they do not equal to each other. Therefore, (iv) holds.

(iv) It is trivial (v) holds by (i), (ii), (iii) and (iv). □

Theorem 3.8 *Given x is the optimistic optimal complaint value of the game $(N,v) \in \Gamma_1$, then it holds:*

(i) $b^v_i - x_i, i \in N$, is constant, i.e., $b^v_i - x_i = b^v_m - x_m$, $\forall i, m \in N$.
(ii) $x_i = ENSC_i(v)$, for all $i \in N$.

Proof

(i) Suppose that $b^v_i - x_i, i \in N$, is not constant, then there exist l, m such that $b^v_l - x_l \neq b^v_m - x_m$ Without loss of generality, let $b^v_l - x_l > b^v_m - x_m$. By Lemma 3.7, there exists x^* such that $\theta(x) > \theta(x^*)$, which is in contradiction with that x is the optimistic optimal compromise value. Therefore, $b^v_i - x_i$, $i \in N$, is constant.
(ii) It is easy to obtain that $x_i = ENSC_i(v) = b^v_i + \frac{1}{n}[v(N) - \sum_{j\in N} b_j(v)]$ by (i) together with efficiency. □

3.2. The determination of the pessimistic optimal compromise value

In Section 3.1, we represent the ENSC value by minimizing the maximal complaint under the optimistic complaint criterion. In this section, we will deal with the CIS value which is anti-dual to the ENSC value.

Definition 3.9 The pessimistic optimal compromise value of a balanced game (N,v) is the unique pre-imputation vector x of which the corresponding pessimistic complaint vector $\theta(x)$ satisfies the lexicographic order $\theta(x) \leq_L \theta(y)$ for any pre-imputation vector y.

Theorem 3.10 *Let (N,v) be a balanced game. The pessimistic optimal compromise value x is bounded by individual vector, i.e., $x_i \geq v(\{i\}), i \in N$. That is, the pessimistic optimal compromise value belongs to the imputation set.*

Proof Suppose $x_i \geq v(\{i\})$ does not hold for all $i \in N$.

Case 1 If $x_i \leq v(\{i\})$ holds for all $i \in N$ and there exists at least one player j such that $x_j < v(\{j\})$, then

$$x(N) < \sum_{i\in N} v(\{i\}) \leq v(N).$$

The last inequality holds because individual worth is the lower bound of the Core. Thus, $x(N) < v(N)$, which contradicts with the efficiency of the pessimistic optimal compromise value.

Case 2 If there exist at least two players, $i, j \in N$, such that $x_i < v(\{i\})$, $x_j > v(\{j\})$. Denote $\Delta = \min\{x_j - v(\{j\}), v(\{i\}) - x_i\}$, then $\Delta > 0$ and for the new payoff x^* constructed in the form of (3.3), it holds $\theta(x) > \theta(x^*)$, because the first coordinates of $\theta(x)$ and $\theta(x^*)$ are $\theta_1(x) = \sum_{x_j > v(\{j\}), j\in N}(x_j - v(\{j\}))$ and $\theta_1(x^*) = \sum_{x_j > v(\{j\}), j\in N}(x_j - v(\{j\})) - \Delta$, respectively, yielding $\theta_1(x) > \theta_1(x^*)$. This contradicts with the claim that x is the pessimistic optimal compromise value. Therefore, $x_i \geq v(\{i\}), i \in N$. □

The key inequality in the proof of Theorem 3.10 inspires us to pay attention to the class of n-person game (N,v) of which the sum of the individual worths can be covered by $v(N)$, i.e., $v(N) \geq \sum_{k\in N} v(k)$ Let Γ_2 denote this class of games. Interestingly, on the domain of this class of games, the conclusion in Theorem 3.10 is also justified.

Remark 3.11 Let (N,v) be a game in Γ_2. Then the pessimistic optimal compromise value is lower bounded by the individual worth vector.

The proof is immediate which is similar to that of Theorem 3.10.

Lemma 3.12 *Given x is the pessimistic optimal compromise value of a game $(N,v) \in \Gamma_2$, denote $W = T|e(T,x,v(\cdot)) > x(N\backslash m) - \sum_{i\in N\backslash m} v(\{i\})\}$. Then for any player m such that $x_l - v(\{l\}) > x_m - v(\{m\})$, $l, m \in N$, It holds that for any $T \in W$, $m, l \notin T$.*

Lemma 3.13 *Given pessimistic complaint criterion, the imputation payoff vector x of the game $(N,v) \in \Gamma_2$ and any player m such that $x_l - v(\{l\}) > x_m - v(\{m\})$, $l, m \in N$, denote $\Delta = \frac{x_l - v(\{l\}) - (x_m - v(\{m\}))}{2}$, then for the new payoff vector x^* constructed in the form of* (3.3), *it holds that $\theta(x) > \theta(x^*)$.*

Theorem 3.14 *If x is the pessimistic optimal compromise value of the game $(N,v) \in \Gamma_2$, then the following statements hold.*

(i) $x_i - v(\{i\})$, $i \in N$ is constant, i.e., $x_i - v(\{i\}) = x_m - v(\{m\}), \forall i, m \in N$.
(ii) $x_j = CIS_i(v) = v(\{i\}) + \frac{1}{n}[v(N) - \sum_{j\in N} v(\{j\})]$ for all $i \in N$.

The proof of Lemmas 3.12, 3.13 and Theorem 3.14 is similar to that of Lemmas 3.6, 3.7 and Theorem 3.8, respectively, based on the fact that the CIS value and the ENSC value are anti-dual to each other (Oishi, Nakayama, Hokari, & Funaki, 2016). From this observation, one may explain why these conclusions about the CIS value are similar with that of the ENSC value. The proof is left to the reader to check.

The validity of Theorem 3.14 can also be shown by introducing the anti-dual of cooperative game, which is a useful tool to analyze the relationship between the properties and the solutions (Oishi et al., 2016). Next we explore the alternative proof of Theorem 3.14 by Oishi's approach.

Definition 3.15 For any cooperative game (N, v), the anti-dual of v, denoted by $v^\star$, is defined by

$$v^\star = -[v(N) - v(N\backslash S \text{ for all } S \subseteq N.$$

The value $-v^\star$ indicates the amount that the complement coalition $N\backslash S$ cannot prevent S from obtaining in (N, v). By applying this "anti" operator, we reveal the relationship between the optimistic optimal compromise value and the pessimistic optimal compromise value.

Theorem 3.16 *If x is the pessimistic optimal compromise value of the game $(N, v) \in \Gamma_2$, then*

(i) *$x^\star$ is the optimistic optimal compromise value of the anti-dual game $(N, v^\star)$, where $x^\star = -x$.*
(ii) *$x_i - v(\{i\})$, $i \in N$ is constant, and $x_i = CIS_i(v) = v(\{i\}) + \frac{1}{n}[v(N) - \sum_{j\in N} v(\{j\})], i \in N$.*

Proof

(i) Since x is the pessimistic optimal compromise value of the game $(N, v) \in \Gamma_2$, i.e., $\theta(x, v) \leq_L \theta(y, v)$, for any $y \in I^*(v)$. Since $v^\star(S) = -[v(N) - v(N\backslash S)]$ then by $\sum_{i\in N} v(i) \leq v(N)$ and $v(i) = -b_i^{v^\star}$, it holds $\sum_{i\in N} b_i^{v^\star} = v^\star(N)$. On the other, note that

$$e(N\backslash S, x^\star, b^{v^\star}) = b^{v^\star}(N\backslash S) - x^\star(N\backslash S)$$
$$= x(N\backslash S) - \sum_{i\in N\backslash S} v(i) = e(S, x, v(\cdot)).$$

Therefore, $\theta(x^\star, v^\star) \leq_L \theta(y^\star, v^\star)$ for all $y^\star = -y \in I^\star(v^\star)$, i.e., $x^\star$ is the optimistic optimal compromise value for game $(N, v^\star)$.

(ii) As shown in (i), $x^\star$ is the optimistic optimal compromise value for game $(N, v^\star)$. Hence, by Theorem 3.8(i), $b_i^{v^\star} - x_i^\star$ is constant, and $x_i^\star = b_i^{v^\star} + \frac{1}{n}[v(N) - \sum_{k\in N} b_k^{v^\star}]$. On the other, notice $v(i) = -b_i^{v^\star}$ and $x^\star = -x$. Hence, $b_i^{v^\star} - x_i^\star = x_i - v(\{i\})$, $i \in N$ is constant. And $x_i = -x_i^\star = v(\{i\}) + \frac{1}{n}[v(N) \quad - \sum_{j\in N} v(\{j\})]$, which holds by replacing $b_i^{v^\star}$ and $v^\star(N)$ with $-v(\{i\})$ and $-v(N)$, respectively. □

Theorem 3.16 indicates the intimate connection between the optimal values of the game and its dual game and provide an alternative proof of Theorem 3.8, thanks to the work conducting on the duality and anti-duality in cooperative games from Oishi et al. (2016).

4. A characterization of the optimal compromise value

The pre-kernel was introduced in Maschler, Peleg and Shapley in 1972 based on the ideas of excess and maximum surplus (Maschler, Peleg, & Shapley, 1971). Analogously to this notion, we introduce equal optimistic maximal complaint property and equal pessimistic maximal complaint property, by which we characterize the ENSC value and the CIS value.

4.1. A characterization of the optimistic optimal compromise value

Before we characterize the optimistic optimal compromise value, we define the optimistic maximal complaint of player i over j, $i, j \in N$.

Definition 4.1 Given optimistic complaint criterion, the *optimistic maximal complaint* $m_{ij}^v(x)$ of player $i \in N$ over another player $j \in N$ at payoff vector x of any game $(N, v) \in \Gamma_1$ is given by the maximal complaint among coalitions containing player i, but not containing player j. That is,

$$m_{ij}^v(b^v, x) = \max\, [e(S, b^v, x) | S \subseteq N,\ i \in S, j \notin S]. \tag{4.1}$$

Definition 4.2 A payoff vector x of a game $(N, v) \in \Gamma_1$ is said to satisfy efficiency and equal optimistic maximal complaint property if

(i) efficiency: $\sum_{i\in N} x_i(v) = v(N)$.
(ii) equal optimistic maximal complaint property: for any $i, j \in N$, $m_{ij}^v(b^v, x) = m_{ji}^v(b^v, x)$.

Theorem 4.3 *The optimistic optimal compromise value x of the game $(N, v) \in \Gamma_1$ possesses (i) efficiency and (ii) equal optimistic maximal complaint property.*

Proof

(i) It is trivial that the optimal compromise value is efficient.
(ii) By Remark 3.5, $x_i \leq b_i^v$, $i \in N$; thus,

$$m_{ij}^v(b^v, x) = b^v(N\backslash\{j\}) - x(N\backslash\{j\}) = \tfrac{n-1}{n}[b^v(N) - v(N)],$$
$$m_{ji}^v(b^v, x) = b^v(N\backslash\{i\}) - x(N\backslash\{i\}) = \tfrac{n-1}{n}[b^v(N) - v(N)].$$

Therefore, $m_{ij}^v(b^v, x) = m_{ji}^v(b^v, x)$. □

Lemma 4.4 *Given optimistic complaint criterion, if x is the value of the game $(N, v) \in \Gamma_1$ satisfying efficiency and equal optimistic maximal complaint property, it holds that $b_i^v \geq x_i$, $i \in N$.*

Proof We prove the lemma by proposing the following two claims do not hold.

Claim 1 $b_i^v < x_i$, for all $i \in N$.

Since $b_i^v < x_i$, $i \in N$,

$$\sum_{i\in N} b_i^v < \sum_{i\in N} x_i = v(N). \tag{4.2}$$

The last equation holds by efficiency. On the other hand, for $(N, v) \in \Gamma_1$, $b^v(N) \geq v(N)$, which contradicts with (4.2). Therefore, Claim 1 does not hold.

Claim 2 There exist two players $i, j \in N$, such that $b_i^v < x_i$ and $b_j^v > x_j$, $i, j \in N$.

By (4.1),

$$\begin{aligned} m_{ij}^v(b^v, x) &= \max[e(S, b^v, x) | \emptyset \neq S \subseteq N, i \in S, j \notin S] \\ &= \sum_{b_k^v \geq x_k, k \in N \setminus \{i,j\}} (b_k^v - x_k) + b_i^v - x_i, \\ m_{ji}^v(b^v, x) &= \max[e(S, b^v, x) | S \subseteq N, j \in S,\ i \notin S] \\ &= \sum_{b_k^v \geq x_k, k \in N \setminus \{i,j\}} (b_k^v - x_k) + b_j^v - x_j. \end{aligned}$$

Since $b_j^v > x_j$ and $b_i^v < x_i$, it holds that $m_{ij}^v(b^v, x) < m_{ji}^v(b^v, x)$ contradicting with the equal maximal complaint property. Therefore, Claim 2 does not hold.

By the above claims, $b_i^v \geq x_i$, $i \in N$. □

Theorem 4.5 *The optimistic optimal compromise value is the unique value with the following two properties*: (*i*) *efficiency and* (*ii*) *equal optimistic maximal complaint property*.

Proof It remains to prove the uniqueness part. Suppose that x is a value with the two mentioned properties under optimistic complaint criterion. For all $i \in N$, it holds that $x_i \leq b_i^v$ by Lemma 4.4; thus,

$$m_{ij}^v(b^v, x) = b^v(N \setminus \{j\}) - x(N \setminus \{j\}).$$

Similarly,

$$m_{ji}^v(b^v, x(v)) = b^v(N \setminus \{i\}) - x(N \setminus \{i\}).$$

By property (ii), it holds $b^v(N \setminus \{i\}) - x(N \setminus \{i\}) = b^v(N \setminus \{j\}) - x(N \setminus \{j\})$ yielding the significant equation $b_i^v - x_i = b_j^v - x_j$ for any $i, j \in N$. Therefore, it is easy to obtain $x_i = b_i^v - \frac{b^v(N) - v(N)}{n}$ by applying efficiency to the systems.

An n-person game (N, v) is called 1-convex if it satisfies $b^v(N) \geq v(N)$ and $v(S) \leq v(N) - b^v(N \setminus S)$, $S \subseteq N$, $S \neq \emptyset$. The 1-convex condition states that the amount which remains in the game v for a coalition S, when the total amount $v(N)$ is distributed in such a way that all the players outside S obtain their marginal contributions, is at least its worth $v(S)$. For the 1-convex game, the Core is not empty, i.e., the 1-convex game is balanced. The Nucleolus of an 1-convex n-person game has already been determined by Driessen (1985) as $Nu_i^v = b_i^v - \frac{b^v(N) - v(N)}{n}$. Hence, it is easy to obtain the following remark. □

Remark 4.6 Let (N, v) be a 1-convex game. Then the optimistic optimal compromise value x coincides with the Nucleolus. That is,

$$Nu_i(N, v) = ENSC_i = b_i^v - \frac{b^v(N) - v(N)}{n}, i \in N.$$

4.2. A characterization of the pessimistic optimal compromise value

The equal maximum surplus property is proposed to characterize the pre-Kernel in Machler and Peleg. Inspired by the equal maximum complaint property, we introduce equal pessimistic maximum complaint property and give another representation of the CIS value.

Definition 4.7 Given pessimistic complaint criterion, the *pessimistic maximal complaint* $m_{ij}^v(x)$ of player $i \in N$ over another player $j \in N$ at payoff vector x of any game $(N, v) \in \Gamma_2$ is given by the maximal complaint among coalitions containing player i, but not containing player j. That is,

$$m_{ij}^v(v(\cdot), x) = \max[e(S, v(\{\cdot\}), x) | S \subseteq N, i \in S, j \notin S]. \tag{4.3}$$

Definition 4.8 A payoff vector x of the game $(N, v) \in \Gamma_2$ is said to satisfy efficiency and equal pessimistic maximal complaint property if

(i) efficiency: $\sum_{i \in N} x_i(v) = v(N)$.
(ii) equal pessimistic maximal complaint property:

$$m_{ij}^v(v(\{\cdot\}), x) = m_{ji}^v(v(\{\cdot\}), x).$$

Theorem 4.9 *The pessimistic optimal compromise value x of the game $(N, v) \in \Gamma_2$ possesses* (*i*) *efficiency and* (*ii*) *equal pessimistic maximal complaint property*.

Proof

(i) It is trivial that the pessimistic optimal compromise value is efficient.
(ii) By Theorem 3.10, $x_i \geq v(\{i\})$, $i \in N$; thus,

$$\begin{aligned} m_{ij}^v(v(\{\cdot\}, x) &= x(N \setminus \{j\}) - \sum_{k \in N \setminus \{j\}} v(\{k\}) \\ &= \tfrac{n-1}{n} \left[v(N) - \sum_{k \in N} v(\{k\}) \right], \\ m_{ji}^v(v(\{\cdot\}, x) &= x(N \setminus \{i\}) - \sum_{k \in N \setminus \{i\}} v(\{k\}) \\ &= \tfrac{n-1}{n} \left[v(N) - \sum_{k \in N} v(\{k\}) \right]. \end{aligned}$$

Therefore, $m_{ij}^v(b^v, x) = m_{ji}^v(b^v, x)$. □

Lemma 4.10 *Given pessimistic complaint criterion, if x is the value of the game $(N, v) \in \Gamma_2$ satisfying efficiency and equal pessimistic maximal complaint property, it holds that $x_i \geq v(\{i\})$, $i \in N$.*

Proof We prove the lemma by proposing the following two claims do not hold.

Claim 1 $x_i < v(\{i\})$, for all $i \in N$.

Since $x_i < v(\{i\})$, $i \in N$,

$$\sum_{i\in N} x_i < \sum_{i\in N} v(\{i\}). \tag{4.4}$$

By efficiency, we conclude that $v(N) = \sum_{i\in N} x_i < \sum_{i\in N} v(\{i\})$. On the other hand, for $(N, v) \in \Gamma_2, \sum_{i\in N} v(\{i\}) \le v(N)$, which contradicts with (4.4). Therefore, Claim 1 does not hold.

Claim 2 There exist two players $i, j \in N$, such that $x_i < v(\{i\})$ and $x_j > v(\{j\})$, $i, j \in N$.

By (4.3),

$$m^v_{ij}(v(\{\cdot\}), x) = \max[e(S, v(\{\cdot\}), x)|S \subseteq N,\ i \in S, j \notin S] = \sum_{x_k\ge v(\{k\}), k\in N\setminus\{i,j\}} (x_k - v(\{k\})) + x_i - v(\{i\}),$$

$$m^v_{ji}(v(\{\cdot\}), x) = \max[e(S, v(\{\cdot\}), x)|S \subseteq N, j \in S, i \notin S] = \sum_{x_k\ge v(\{k\}), k\in N\setminus\{i,j\}} (x_k - v(\{k\})) + x_j - v(\{j\}).$$

Since $x_j > v(\{j\})$ and $x_i < v(\{i\})$, it holds that $m^v_{ij}(v(\{\cdot\}), x) < m^v_{ji}(v(\{\cdot\}), x)$ contradicting with property (ii). Therefore, Claim 2 does not hold.

By the above claims, $x_i \ge v(\{i\})$, $i \in N$. □

Theorem 4.11 *The pessimistic optimal compromise value is the unique value with the following two properties: (i) efficiency and (ii) equal pessimistic maximal complaint property.*

Proof It remains to prove the uniqueness part. Suppose that x is a value with the two mentioned properties.

For any players $i \in N$, $x_i \ge v(\{i\})$ by Lemma 4.10; thus,

$$m^v_{ij}(v(\{\cdot\}), x) = x(N\setminus\{j\}) - \sum_{k\in N\setminus\{j\}} v(\{k\}).$$

Similarly,

$$m^v_{ji}(v(\{\cdot\}), x) = x(N\setminus\{i\}) - \sum_{k\in N\setminus\{i\}} v(\{k\}).$$

By property (ii), it holds $x(N\setminus\{i\}) - \sum_{k\in N\setminus\{i\}} v(\{k\}) = x(N\setminus\{j\}) - \sum_{k\in N\setminus\{j\}} v(\{k\})$ yielding the significant equation $x_i - v(\{i\}) = x_j - v(\{j\})$ for any $i, j \in N$. Therefore, it is easily to obtain $x_i = v(\{i\}) - \frac{v(N)-\sum_{k\in N} v(\{k\})}{n}$ by applying efficiency to the systems. □

Remark 4.12 The uniqueness part of Theorem 4.11 can also be proved by constructing the anti-dual game $v^\star$.

Alternative Proof or Theorem 4.11 It remains to prove the uniqueness part. Given φ is a value for game (N, v) satisfying efficiency and the equal pessimistic maximal complaint property. Let $v^\star$ be the anti-dual of game v and construct $\phi = -\varphi$, then it is easy to obtain $\phi(N) = -\varphi(N) = -v(N) = v^*(N)$, i.e., ϕ satisfies the efficiency for game $(N, v^\star)$. Next we prove that for $(N, v^\star)$, the equal optimistic maximal complaint property holds for ϕ.

Clearly, by equal pessimistic maximal complaint property, $m^v_{ij}(v(\{\cdot\}), \varphi) = m^v_{ji}(v(\{\cdot\}), \varphi)$. Therefore,

$$\begin{aligned}
&\max[e(S, v(\{\cdot\}), \varphi)|S \subseteq N, i \in S, j \notin S]\\
&= \max[e(S, v(\{\cdot\}), \varphi)|S \subseteq N, j \in S, i \notin S]\\
&\Leftrightarrow \max[\varphi(N\setminus S) - \textstyle\sum_{k\in N\setminus s} v(k)|i \in S, j \notin S]\\
&= \max[\varphi(N\setminus S) - \textstyle\sum_{l\in N\setminus s} v(l)|j \in S, i \notin S]\\
&\Leftrightarrow \max[-\phi/(N\setminus S) + \textstyle\sum_{k\in N\setminus s} b^{v^\star}_k |i \in S, j \notin S]\\
&= \max[-\phi(N\setminus S) + \textstyle\sum_{l\in N\setminus s} b^{v^\star}_l |j \in S, i \notin S]\\
&\Leftrightarrow m^{v^\star}_{ij}(b^{v^\star}, \phi) = m^{v^\star}_{ji}(b^{v^\star}, \phi),
\end{aligned}$$

which yields ϕ satisfies equal optimistic maximal complaint property. Hence, by Theorem 4.5, ϕ is the optimistic optimal compromise value for game $(N, v^\star)$. That is, $\phi_i = ENSC_i(N, v^\star) = b^{v^\star}_i + \frac{1}{n}[v^\star(N) - \sum_{k\in N} b^{v^\star}_k]$. Note that $\phi_i = -\varphi_i, v^\star(N) = -v(N)$ and $b^{v^\star}_i = -v(\{i\})$; thus, φ is determined and $\varphi_i = v(\{i\}) + \frac{1}{n}[v(N) - \sum_{k\in N} v(\{k\})]$, $i \in N$. □

As a generalization of the conclusions in this paper, we can represent the other optimal compromise values, i.e., the α-ENSC value and the α-CIS value, by the same manner as in the (pre)Kernel. To do that, the α-optimistic complaint and α-pessimistic complaint are proposed such that the coalition S's complaints are given by $e(S, x, \alpha b^v) = \alpha b^v(S) - x(S)$, $S \subseteq N$ and $e(S, x, \alpha v = x(N\setminus S) - \alpha \sum_{j\in N\setminus S} v(\{j\})$ respectively. And the corresponding α-optimistic and α-pessimistic maximal complaints are defined by respectively.

$m^v_{ij}(\alpha b^v, x) = \max[\alpha b^v(S) - x(S) \mid S \subseteq N,\ i \in S, j \notin S]$ and
$m^v_{ij}(\alpha v(\cdot), x) = \max[x(N\setminus S) - \alpha \sum_{k\in N\setminus S} v(k) \mid S \subseteq N,\ \ i \in S, j \notin S]$

Remark 4.13

(i) The α-ENSC value is the unique value with the following two properties: efficiency and equal α-optimistic maximal complaint property.
(ii) The α-CIS value is the unique value with the following two properties: efficiency and equal α-pessimistic maximal complaint property.

The proof is similar with that of Theorems 4.5 and 4.11. Here we omit it.

5. Concluding remarks

In this paper, the optimal compromise value and two kinds of complaint criteria are introduced. The compromise value can be regarded as the solution of one optimization problem. Under the optimistic complaint criterion, the optimal compromise value coincides with the ENSC value, while the optimal compromise value is the CIS value given the pessimistic complaint criterion. Moreover, a characterization of the value is given with the aid of equal maximal complaint property and efficiency. As an adjunct, we reveal the coincidence of the Nucleolus and the ENSC value in 1-convex games. The determination of this optimal compromise value under different complaint criterion is still going on.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The first author acknowledges financial support by NationaI Science Foundation of China (NSFC) through Grant Nos. 71601156, 71671140, 71571143 as well as Shaanxi Province Science and Technology Research and Development Program (No. 2016JQ7008).

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