J Optim Theory Appl (2017) 173:336–352
DOI 10.1007/s10957-017-1092-5

Optimization Implementation and Characterization of the Equal Allocation of Nonseparable Costs Value

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Received: 9 August 2016 / Accepted: 28 February 2017 / Published online: 9 March 2017

Abstract This paper devotes to the study of the equal allocation of nonseparable costs value for cooperative games. On the one hand, we show that the equal allocation of nonseparable costs value is the unique optimal solution that minimizes the total complaints for individual players over the pre-imputation set. On the other hand, analogously to the way of determining the Nucleolus, we obtain the equal allocation of nonseparable costs value by applying the lexicographic order over the individual complaints. Moreover, we offer alternative characterizations of the equal allocation of nonseparable costs value by proposing several new properties such as dual nullifying player property, dual dummifying player property and grand marginal contribution monotonicity.

Keywords Cooperative games · The equal allocation of nonseparable costs value · Optimization · Complaint · Characterization

Communicated by Kyriakos G. Vamvoudakis.

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Mathematics Subject Classification 91A12

1 Introduction

Cooperative game theory provides practical mathematical tools to model situations of cooperation, and it is widely used in economics and political science. The main subject of cooperative game theory is that of determining the allocation rule which defines what portion of the societal benefit is to be shared with each participating player. Various allocation rules for cooperative games have been proposed concerning different criteria of a fair payoff.

Several allocation rules for cooperative games have been studied based on the concept of excess which is the gap between the worth of a coalition and what it can obtain from the proposed payoff. The notion of excess usually is taken as a measurement of complaints for coalitions toward a given payoff. The Nucleolus, introduced by Schmeidler [1], is the outcome of a lexicographic minimization procedure over the excess vectors which are associated with allocations. More specifically, the Nucleolus is the only allocation rule that minimizes the maximal complaint among all coalitions over the imputation set, which contains all the payoff vectors that completely distribute the total benefit and assign to every single player at least his individual worth. Instead of pushing down the highest excess, Ruiz et al. [2–4] introduced the so-called least square value as the unique minimizer of the variance of the total excesses for coalitions.

In this paper, we implement the equal allocation of nonseparable costs value (the ENSC value for short), proposed by Moulin [5], as the unique optimal solution to several optimization problems that involve the complaints for individual players instead of the coalitions. Recall that cooperative game theory deals with the mathematical models of cooperative situations in which the grand coalition forms. Thus, every single player plays a significant role in the formation of the grand coalition, suggesting that the ideal payoff for players is the grand marginal contribution. Given any allocation, there always exists a gap between the ideal payoff and the proposed allocation, resulting in the complaints of the players. With the same method applied in the determination of the least square value, we select the unique payoff vector that minimizes the summation of the variance of the total complaints for individual players. It will be shown that the optimal solution coincides with the ENSC value.

Alternatively, inspired by the interpretation of the Shapley value [6] both in [7] and [8], another ideal payoff for individual players is proposed. Suppose that players join the game one by one and every new entrant charges the marginal contribution he creates to the formed coalition as his payoff, then the Shapley value is defined by averaging the marginal vectors over all possible orders. The Shapley value can be treated as an egoistic allocation in the sense that every new entrant takes all his marginal contributions without sharing with others. In this context, we deal with the situation in which players are totally egoistic. That is, every new entrant claims his grand marginal contribution rather than the marginal contribution of the formed coalition and what's left is shared equally among the preceding players. Notice that in the formation of the grand coalition, every player obtains his grand marginal contribution as a new entrant but also has to undertake a portion of gaps generated by his successors, yielding the so-called compromised ideal payoff. In this paper, it will be proved that the ENSC value is the unique pre-imputation that minimizes the summation of the variance of the total compromised complaints for individual players. By analyzing the involved procedure, we could conclude that the ENSC value is a more egoistic value than the Shapley value. In fact, Marcin [8] elaborated that the equal division value is more altruistic than the solidarity value [9] by proposing the so-called "Procedural" value. And the Shapley value is in some sense egoistic compared with the solidarity value, while our work reveals the fact that the ENSC value is more egoistic than the Shapley value. Moreover, analogously to the way of obtaining the Nucleolus, we implement the ENSC value by lexicographically minimizing the individual excess vector over the pre-imputation set. From the perspective of optimization, the conclusion indicates a coincidence between these two values, which was first studied by Driessen and Funaki [10].

In order to explain the rationality and fairness of the ENSC value for cooperative games, it entails characterization of this allocation rule. There exists a lot of literature concentrating on the characterization of the ENSC value. For instance, Xu [11] and Hwang [12] both characterized the ENSC value with the aid of associated consistency, which was inspired by Hamiache's [13] characterization of the Shapley value. The associated consistency of an allocation rule implies that it gives the same payoff to each participating agent in the original game and in the imaginary associated game, which reflects the expectations elaborated by the agents. Ju and Wettstein [14] characterized the ENSC value as the implementation result of a bidding mechanism through noncooperative viewpoint. In this paper, we provide two alternative characterizations of the ENSC value: the first as the unique solution to an optimization problem, the second as the unique satisfier of a particular collection of axioms. Several new properties are proposed including the dual nullifying player property, the dual dummifying player property and grand marginal contribution monotonicity. The dual nullifying player property and dual dummifying player property are inspired by the concepts of the nullifying and dummifying player property, which were introduced by Brink [15] and André [16], respectively. The dual nullifying player property states that any player that brings nullifying influence to coalitions containing him through an indirect way will get nothing. The dual dummifying player property guarantees the grand marginal contribution for players who carry out dummifying influence to coalitions. In addition, grand marginal contribution monotonicity ensures that players with a larger grand marginal contribution can obtain a larger portion of the total benefit.

The paper is organized as follows: in Sect. 2, we recall necessary definitions of cooperative games. Then we present the optimization models that lead to the ENSC value. In Sect. 3, characterizations of the ENSC value with several new properties are presented and we end this paper with a short conclusion in Sect. 4.

2 Optimization Implementation of the Equal Allocation of Nonseparable Costs Value

A cooperative game is a pair (N, v), where N is the finite player set and $v{:}2^N \to \mathbb{R}$ is the characteristic function on the set 2^N of all subsets of N such that $v(\emptyset) = 0$. For any $S \subseteq N$, $v(S)$ is the worth that S can earn by acting alone and the cardinality of S is denoted by s. Denote Γ_n as the class of all cooperative n-person games with player set N. Player i's *grand marginal contribution* $v(N) - v(N \backslash i)$ is denoted by b_i^v. Any $x \in \mathbb{R}^n$ is called a *payoff vector*. A *pre-imputation* [3] is a payoff vector with *efficiency*, i.e., $x(N) = v(N)$, where $x(N) = \sum_{i \in N} x_i$. An *imputation* is a pre-imputation with *individual rationality*, i.e., $x_i \geq v(i)$, $\forall i \in N$. For any $v \in \Gamma_n$, $I^*(v)$ and $I(v)$ denote the pre-imputation set and the imputation set [17], respectively.

A value $\varphi: \Gamma_n \to \mathbb{R}^n$ is a function that maps every $v \in \Gamma_n$ to an n tuple. The ENSC value, introduced by Moulin [5] under the full name "equal allocation of nonseparable costs," allocates to each player his grand marginal contribution and then evenly divides the remaining benefit among all players, i.e.,

$$\mathrm{ENSC}_i(v) := b_i^v + \frac{1}{n}\left[v(N) - \sum_{j \in N} b_j^v\right], \quad \forall i \in N. \tag{1}$$

Ruiz [2] defined the least square value as the unique pre-imputation that minimizes the variance of the excess of coalitions. Tijs [18] explained that the grand marginal contribution $b^v \in \mathbb{R}^n$ is an upper bound for the core of game $v \in \Gamma_n$. The core, introduced by Gillies [19], is the set of all pre-imputations that cannot be improved upon by any coalition, that is, the core is always internally stable. Based on this fact, any player $i \in N$ in a game v will regard b_i^v as his ideal payoff and certainly prefer an allocation which is closest to the vector b^v. Inspired by Ruiz, our aim is to select a payoff vector that has the least Euclidean distance to the marginal contribution vector b^v. The following optimization problem is taken into account.

Problem 1 Minimize $\sum_{i \in N} (b_i^v - x_i)^2$, s.t. $x \in \mathbb{R}^n$ and $\sum_{i \in N} x_i = v(N)$.

Theorem 2.1 *For any $v \in \Gamma_n$, there exists a unique optimal solution x^* of Problem 1, which coincides with the ENSC value, i.e.,*

$$x_i^* = b_i^v + \frac{1}{n}\left[v(N) - \sum_{j \in N} b_j^v\right], \quad \forall i \in N. \tag{2}$$

Proof It is not difficult to verify that the objective function of Problem 1 is strictly convex by calculating the Hessian matrix. Together with the convexity of the feasible set, we conclude that the optimal solution is unique if it exists. Thus, in order to find the optimal solution, it is necessary and sufficient to verify the Lagrange conditions. The Lagrange function of Problem 1 is

$$L(x, \lambda) = \sum_{i \in N} \left(b_i^v - x_i\right)^2 + \lambda \left(\sum_{i \in N} x_i - v(N)\right). \tag{3}$$

The Lagrange conditions are then

$$L_{x_i}(x, \lambda) = 2\left(x_i^* - b_i^v\right) + \lambda = 0, \quad i \in N. \tag{4}$$

Therefore, it holds that

$$x_i^* - b_i^v = x_j^* - b_j^v, \quad \forall i, j \in N. \tag{5}$$

Further, by the constraint equation $\sum_{i \in N} x_i = v(N)$, the solution for Eq. (5) is

$$x_i^* = b_i^v + \frac{1}{n}\left[v(N) - \sum_{j \in N} b_j^v\right] = \mathrm{ENSC}_i(v). \tag{6}$$

□

Remark 2.1 One may reasonably deduce that if every player aspires to get the worth of the grand coalition $v(N)$, then the gap between the worth of the grand coalition and proposed payoff can be regarded as complaints for players. By minimizing $\sum_{i \in N}(v(N) - x_i)^2$, $x \in \mathbb{R}^n$ over the pre-imputation set, one could find that the optimal solution coincides with the equal division value.

As stated in the Introduction, the Shapley value is in some sense an egoistic value which assigns to every new entrant his marginal contribution to the formed coalition. In this paper, we deal with the situation where players are totally egoistic, that is, all the newcomers will charge the grand marginal contributions as their ideal payoff.

Definition 2.1 For any $v \in \Gamma_n$ and $\pi \in \Pi(N)$, where $\Pi(N)$ denotes the set of all permutations on N, a player $i \in N$ is ***totally egoistic*** if he claims C_i^π when he joins the game, where

$$C_i^\pi := \begin{cases} v(i), & \text{if} \quad i = \pi(1), \\ b_i^v, & \text{otherwise}. \end{cases} \tag{7}$$

The gap $v\left(S_\pi^i\right) - v\left(S_\pi^i \backslash i\right) - C_i^\pi$ generated by player i is shared among the players. The portion for player $k \in N$ is G_{ik}^π, i.e.,

$$G_{ik}^\pi := \begin{cases} \frac{v(S_\pi^i) - v(S_\pi^i \backslash i) - C_i^\pi}{\pi(i) - 1}, & \text{if} \quad k \in S_\pi^i \backslash \{i\}, \\ 0, & \text{otherwise}. \end{cases} \tag{8}$$

Here $S_\pi^i := \{j \in N : \pi(j) \leq \pi(i)\}$ denotes the set that consists of player i and all his predecessors under permutation π.

For any dynamic coalitional formation order, a totally egoistic player will claim his grand marginal contribution, while what's left is equally shared among the preceding participants. Particularly, a player who takes up the first position only obtains his individual value since there are no predecessors.

In order to illustrate the above procedure, we consider the following three-person game $v \in \Gamma_n$ under the assumption that all players are totally egoistic and the dynamic coalitional formation order is (3, 1, 2).

Table 1 Payoff for players when 3 joins

Players	Player 1	Player 2	Player 3
Payoff	0	0	$v(3)$

Table 2 Payoff for players when 1 joins

Players	Player 1	Player 2	Player 3
Payoff	b_1^v	0	$v(1,3) - v(3) - b_1^v$

Table 3 Payoff for players when 2 joins

Players	Player 1	Player 2	Player 3
Payoff	$\frac{v(N)-v(1,3)-b_2^v}{2}$	b_2^v	$\frac{v(N)-v(1,3)-b_2^v}{2}$

Table 4 Payoff for players under the formation order (3, 1, 2)

Formation of N	Payoff in every step	Player's final payoff
Player 3	$v(3),\ v(1,3) - v(3) - b_1^v,\ \frac{v(N)-v(1,3)-b_2^v}{2}$	$-b_1^v + \frac{v(N)+v(1,3)-b_2^v}{2}$
Player 1	$b_1^v,\ \frac{v(N)-v(1,3)-b_2^v}{2}$	$b_1^v + \frac{v(N)-v(1,3)-b_2^v}{2}$
Player 2	b_2^v	b_2^v

At the beginning, player 3 joins the game. As the sole present player, player 3 obtains his individual worth $v(3)$, while players 1 and 2 obtain nothing at the moment. The payoff for players in this stage is shown in Table 1.

Then player 1 joins, and he obtains his grand marginal contribution b_1^v. The only preceding player 3 achieves an extra payoff, the amount of which equals to the new entrant's marginal contribution to the preceding player 3 minus his grand marginal contribution, i.e., $v(1,3) - v(3) - b_1^v$. The payoff for players in this stage is shown in Table 2.

Finally player 2 joins the game and obtains b_2^v. The remaining $v(N) - v(1,3) - b_2^v$ is shared equally among the preceding players 1 and 3. The payoff for players in the final stage is shown in Table 3.

By summing the payoffs in these three stages, we get the final payoff as shown in Table 4.

In the formation of the grand coalition, each new entrant first obtains his grand marginal contribution and then bears the gaps generated by his successors, which somehow reflects a compromise between the ideal payoff b^v and the gaps. As shown in the above example, the final payoff for player 1 consists of two parts, his ideal payoff b_1^v and the compromised part $\frac{v(N)-v(1,3)-b_2^v}{2}$ generated by his successor 2.

Definition 2.2 For any $v \in \Gamma_n$ and $\pi \in \Pi(N)$, let all players be totally egoistic. The ***compromised ideal payoff*** for player i under coalitional formation order π is defined as

$$\eta_i^{v\pi} := \begin{cases} v(i) + \sum\limits_{k=\pi^{-1}(i)+1}^{n} \frac{v\left(S_\pi^{\pi(k)}\right) - v\left(S_\pi^{\pi(k)} \backslash \pi(k)\right) - b_{\pi(k)}^v}{k-1}, & \text{if } i = \pi(1), \\ b_i^v + \sum\limits_{k=\pi^{-1}(i)+1}^{n} \frac{v\left(S_\pi^{\pi(k)}\right) - v\left(S_\pi^{\pi(k)} \backslash \pi(k)\right) - b_{\pi(k)}^v}{k-1}, & \text{otherwise.} \end{cases} \tag{9}$$

Here

$$\sum_{k=n+1}^{n} \frac{v\left(S_\pi^{\pi(k)}\right) - v\left(S_\pi^{\pi(k)} \backslash \pi(k)\right) - b_{\pi(k)}^v}{k-1} = 0.$$

Definition 2.3 For any $v \in \Gamma_n$ and $\pi \in \Pi(N)$, let all players be totally egoistic. The ***compromised complaint*** of player i toward the payoff vector $x \in \mathbb{R}^n$ with respect to the compromised ideal payoff $\eta_i^{v\pi}$ is

$$e\left(i, x, \eta^{v\pi}\right) := \eta_i^{v\pi} - x_i, i \in N. \tag{10}$$

The expression $e(i, x, \eta^{v\pi})$ indicates the gap between player's compromised ideal payoff and the proposed payoff. The larger $e(i, x, \eta^{v\pi})$ is, the more dissatisfied player i feels. By minimizing the summation of the variance of the compromised complaints over the pre-imputation set, a unique optimal solution is obtained. This solution can be seen as the least unsatisfied payoff by the fact that it is the vector closest to the compromised ideal payoff vector under Euclidean distance. Formally, the following optimization problem is considered.

Problem 2 Minimize $\sum_{i \in N} \sum_{\pi \in \Pi(N)} (\eta_i^{v\pi} - x_i)^2$, s.t. $x \in \mathbb{R}^n$ and $\sum_{i \in N} x_i = v(N)$.

Theorem 2.2 *For any $v \in \Gamma_n$, there exists a unique optimal solution x^* for Problem 2, which coincides with the ENSC value.*

In order to show the validity of Theorem 2.2, the following lemma is taken into account.

Lemma 2.1 *For any game $v \in \Gamma_n$, the average compromised ideal payoff coincides with the ENSC value, i.e.,*

$$\frac{1}{n!} \sum_{\pi \in \Pi(N)} \eta_i^{v\pi} = \text{ENSC}_i(v), \tag{11}$$

where $\eta_i^{v\pi}$ is defined as in (9).

See the proof of this lemma in "Appendix."

Proof of Theorem 2.2 The objective function and the feasible set of Problem 2 are both convex; hence, there is only one optimal solution if it exists. It is necessary and sufficient to verify the Lagrange conditions so as to find the optimal solution. Formally, the Lagrange function of Problem 2 is

$$L(x,\lambda)=\sum_{i\in N}\sum_{\pi\in\Pi(N)}\left(\eta_i^{v\pi}-x_i\right)^2+\lambda\left(\sum_{i\in N}x_i-v(N)\right). \tag{12}$$

Taking the derivative of this function, the Lagrange conditions are obtained as

$$L_{x_i}(x,\lambda)=-2\sum_{\pi\in\Pi(N)}\left(\eta_i^{v\pi}-x_i\right)+\lambda=0,\quad \forall i\in N. \tag{13}$$

Thus for all $i, j\in N$, we have

$$\sum_{\pi\in\Pi(N)}\left(\eta_i^{v\pi}-x_i\right)=\sum_{\pi\in\Pi(N)}\left(\eta_j^{v\pi}-x_j\right). \tag{14}$$

By Lemma 2.1 and the constraint equation $\sum_{i\in N}x_i=v(N)$, we may straightforward to get that the optimal solution x^* is

$$x_i^*=\frac{1}{n!}\sum_{\pi\in\Pi(N)}\eta_i^{v\pi}=\mathrm{ENSC}_i(v). \tag{15}$$

□

The foregoing optimization problems both characterize the ENSC value as the unique allocation rule that minimizes the total complaints for individual players under the least square criterion. We now turn to explore the optimal solution that pushes down the maximal complaints for individual players under the lexicographic criterion, which is similar to the method of obtaining the Nucleolus. Given $v\in\Gamma_n$ and $x\in\mathbb{R}^n$, the gap between $v(S)$ and $x(S)$ is known as the excess $\mathrm{e}^v(S,x)=v(S)-x(S)$, which in some sense reflects the loss or complaint for coalition S toward the allocation x. Let $\theta(x)$ be a 2^n tuple whose components are the excesses $\mathrm{e}^v(S,x)$ arranged in nonincreasing order.

Definition 2.4 The Nucleolus of a cooperative game $v\in\Gamma_n$ is the set of all imputations $x\in I(v)$ satisfying

$$\theta(x)\leq_L\theta(y),\quad \forall y\in I(v), \tag{16}$$

where $\leq_L$ represents the lexicographic order.

Instead of considering the complaints for coalitions, we select the pre-imputation that minimizes the maximal average compromised complaint for the individual players. For any $x\in I^*(v)$, let $\theta^*(x)$ be the n tuple whose components are the average compromised complaint, i.e., $\theta_i^*(x)=\frac{1}{n!}\sum_{\pi\in\Pi(N)}e(i,x,\eta^{v\pi})$, arranged in nonincreasing order. The next theorem states that the ENSC value is the unique pre-imputation that lexicographically minimizes the average compromised complaint vector $\theta^*(x)$ over $I^*(v)$.

Theorem 2.3 *For any $v \in \Gamma_n$, the ENSC value is the unique pre-imputation $x \in I^*(v)$ satisfying*

$$\theta^*(x) \leq_L \theta^*(y), \quad \forall y \in I^*(v). \tag{17}$$

Proof Given any $v \in \Gamma_n$, let x be the pre-imputation satisfying

$$\theta^*(x) \leq_L \theta^*(y), \quad \forall y \in I^*(v). \tag{18}$$

Now we assert that $\theta_i^*(x) = \theta_j^*(x)$ for all $i, j \in N$. Otherwise, there must exist $i, j \in N$ such that $\theta_i^*(x) \neq \theta_j^*(x)$. Without loss of generality, let $\theta_i^*(x) < \theta_j^*(x)$. Define the n-tuple $\widehat{x}$ by

$$\widehat{x}_k := \begin{cases} x_k, & k \in N \backslash \{i, j\}, \\ x_k - \delta, & k = i, \\ x_k + \delta, & k = j. \end{cases} \tag{19}$$

Here $\delta = \frac{\theta_j^*(x) - \theta_j^*(x)}{2}$. It is obvious that $\widehat{x} \in I^*(v)$. Then, the average compromised complaint vector of $\widehat{x}$ is

$$\theta_k^*(\widehat{x}) = \begin{cases} \theta_k^*(x), & k \in N \backslash \{i, j\}, \\ \theta_k^*(x) + \delta, & k = i, \\ \theta_k^*(x) - \delta, & k = j. \end{cases} \tag{20}$$

Moreover, since $\delta > 0$, we have

$$\theta_j^*(x) > \theta_j^*(\widehat{x}) = \theta_i^*(\widehat{x}) > \theta_i^*(x).$$

This implies $\theta^*(\widehat{x}) <_L \theta^*(x)$, which is contradiction with that x lexicographically minimizes the average compromised complaint vector over the pre-imputation set. Hence, $\theta_i^*(x) = \theta_j^*(x)$ for all $i, j \in N$. By the efficiency of x and Lemma 2.1, it is not difficult to obtain that $x_i = \text{ENSC}_i(v)$ for all $i \in N$. □

We conclude this section with a simple example to illustrate the involved optimization models.

Example 2.1 Given a 3-person game $v \in \Gamma_n$ with $N = \{1, 2, 3\}$. Let the characteristic function v be given by $v(1) = 1, v(2) = 2, v(3) = 3, v(12) = 4, v(23) = 9, v(13) = 9$ and $v(N) = 10$. Then we have $b_1^v = 1, b_2^v = 2$ and $b_3^v = 6$. It is easy to figure out that the ENSC value is $\text{ENSC}(v) = \left(\frac{4}{3}, \frac{7}{3}, \frac{19}{3}\right)$.

As to Problem 1, the following problem is taken into account

$$\text{Minimize} \quad (x_1 - 1)^2 + (x_2 - 2)^2 + (x_3 - 6)^2, \text{s.t.}\, x_1 + x_2 + x_3 = 10.$$

The Lagrange function of this problem is

$$L(x, \lambda) = (x_1 - 1)^2 + (x_2 - 2)^2 + (x_3 - 6)^2 + \lambda(x_1 + x_2 + x_3 - 10),$$

which gives the Lagrange conditions

$$2(x_1-1)+\lambda=0;\quad 2(x_2-2)+\lambda=0;\quad 2(x_3-6)+\lambda=0. \tag{21}$$

Together with $x_1+x_2+x_3=10$, the optimal solution $x^*=\left(\frac{4}{3},\frac{7}{3},\frac{19}{3}\right)$ is exactly the ENSC value, verifying the validity of Theorem 2.1.

As to Problem 2, we first denote the permutations on N by $\pi_1=(1,2,3)$, $\pi_2=(1,3,2)$, $\pi_3=(2,1,3)$, $\pi_4=(2,3,1)$, $\pi_5=(3,1,2)$ and $\pi_6=(3,2,1)$. The compromised ideal payoffs for the players are

$$\begin{aligned}
&\eta_1^{v\pi_1}=\eta_1^{v\pi_2}=2,\quad \eta_1^{v\pi_3}=\eta_1^{v\pi_4}=\eta_1^{v\pi_5}=\eta_1^{v\pi_6}=1,\\
&\eta_2^{v\pi_3}=\eta_2^{v\pi_4}=3,\quad \eta_2^{v\pi_1}=\eta_2^{v\pi_2}=\eta_2^{v\pi_5}=\eta_2^{v\pi_6}=2,\\
&\eta_3^{v\pi_5}=\eta_3^{v\pi_6}=7,\quad \eta_3^{v\pi_1}=\eta_3^{v\pi_2}=\eta_3^{v\pi_3}=\eta_3^{v\pi_4}=6.
\end{aligned}$$

Thus, the explicit formulation of Problem 2 is

$$\begin{aligned}
&\text{Minimize}\quad 2(x_1-2)^2+4(x_1-1)^2+2(x_2-3)^2+4(x_2-2)^2\\
&+2(x_3-7)^2+4(x_3-6)^2,\quad \text{s.t.}\ \sum_{i=1}^{3}x_i=10.
\end{aligned}$$

The Lagrange function is

$$\begin{aligned}
L(x,\lambda)=\;&2(x_1-2)^2+4(x_1-1)^2+2(x_2-3)^2+4(x_2-2)^2+2(x_3-7)^2\\
&+4(x_3-6)^2+\lambda(x_1+x_2+x_3-10),
\end{aligned}$$

which gives the Lagrange conditions

$$\begin{aligned}
&4(x_1-2)+8(x_1-1)+\lambda=0;\quad 4(x_2-3)+8(x_2-2)+\lambda=0;\\
&4(x_3-7)+8(x_3-6)+\lambda=0.
\end{aligned} \tag{22}$$

Together with $x_1+x_2+x_3=10$, the optimal solution $x^*=\left(\frac{4}{3},\frac{7}{3},\frac{19}{3}\right)$ coincides with the ENSC value, which verifies the validity of Theorem 2.2.

3 Characterization of the Equal Allocation of Nonseparable Costs Value

In Sect. 2, we implement the ENSC value as the unique optimal solution to some optimization problems. Alternative characterizations of the ENSC value will be shown in this section by proposing several new properties. Before doing that, we first list some usual principles that have been proposed to reflect the fairness and rationality for values. A value $\varphi:\Gamma_n\to\mathbb{R}^n$ satisfies

- **Additivity**: if $\varphi_i(v+w)=\varphi_i(v)+\varphi_i(w)$ for any $v,w\in\Gamma_n$.
- **Invariance**: if for any $a\in\mathbb{R}$ and $b\in\mathbb{R}^n$, it holds that $\varphi_i(av+b)=a\varphi_i(v)+b$, where $(av+b)\in\Gamma_n$ is given by $(av+b)(S):=av(S)+\sum_{i\in S}b_i, \forall S\subseteq N$.

- **Symmetry**: if $\varphi_i(v) = \varphi_j(v)$, where $i, j \in N$ are symmetric players in game $v \in \Gamma_n$, that is, $v(S \cup i) = v(S \cup j), \forall S \subseteq N \backslash \{i, j\}$.
- **Inessential game property**: if $\varphi_i(v) = v(i), \forall i \in N$ for any inessential game $v \in \Gamma_n$, that is, $v(S) = \sum_{i \in S} v(i), \forall S \subseteq N$.

Brink [15] introduced the nullifying player property to characterize both the equal division value and the center of gravity of the imputation set value [10]. A player is nullifying if the worth of all coalitions containing him equals zero. The nullifying player property implies that the nullifying players will obtain nothing due to their nullifying influence to coalitions. Different from the nullifying players, we consider the players who bring nullifying influence to coalitions in an indirect way.

Definition 3.1 For any $v \in \Gamma_n$, player $i \in N$ is called a ***dual nullifying player*** in v if $v(N) - v(N \backslash S) = 0$ for all $S \subseteq N$ with $i \in S$.

Obviously, if a coalition S contains a dual nullifying player, then the remaining part of the total worth $v(N)$ for it equals 0 whenever $v(N)$ is distributed in such a way that players outside S receive the amount of $v(N \backslash S)$. In this case, an allocation rule should assign nothing to the dual nullifying players. This yields the dual nullifying player property. A value $\varphi : \Gamma_n \to \mathbb{R}^n$ satisfies

- **Dual nullifying player property**: if $\varphi_i(v) = 0$, for any given $v \in \Gamma_n$ such that i is a dual nullifying player in game v.

By replacing the nullifying player property with dual nullifying player property in the characterization of the equal division value proposed by Brink [15], we obtain a new characterization of the equal division value.

Lemma 3.1 *For all $v \in \Gamma_n$, the equal division value is the unique value that satisfies efficiency, additivity, symmetry and the dual nullifying player property.*

Proof Obviously, the equal division value satisfies efficiency, additivity and symmetry. Next we verify the dual nullifying player property for the equal division value. Suppose that player $i \in N$ is dual nullifying in game $v \in \Gamma_n$. By the definition of the dual nullifying player, we have $v(N) - v(N \backslash N) = v(N) = 0$, thus $ED_i(v) = \frac{v(N)}{n} = 0$, which means that the dual nullifying property holds for the equal division value.

Uniqueness will be proved in a similar way as for the Shapley value but using the standard games instead of the unanimity games which are introduced by Harsanyi [20].

For any $T \subseteq N$, $T \neq \emptyset$, the standard game $b^{\mathrm{T}} \in \Gamma_n$ is defined by

$$b^{\mathrm{T}}(S) := \begin{cases} 1, & \text{if} \quad S = T, \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

Now suppose that a value $\varphi : \Gamma_n \to \mathbb{R}^n$ satisfies these four properties. For any $T \subsetneq N$, all players in T are dual nullifying in game b^{T}. According to the dual nullifying player property, $\varphi_i(b^{\mathrm{T}}) = 0$ for all $i \in T$. By efficiency of φ, it holds that $\sum_{i \in N \backslash T} \varphi_i(b^{\mathrm{T}}) = b^{\mathrm{T}}(N) = 0$. Moreover, all players in $N \backslash T$ are symmetric to each other, hence $\varphi_i(b^{\mathrm{T}}) = 0$ for all $i \in N \backslash T$.

For $T = N$, $\varphi_i(b^N) = \frac{b^N}{n} = \frac{1}{n}$ for all $i \in N$ simply by efficiency and symmetry.

Recall that the set $\{b^{\mathrm{T}}\}_{T \subset N, T \neq \emptyset}$ forms a basis of Γ_n, yielding $v = \sum_{T \subseteq N, T \neq \emptyset} v(T) b^{\mathrm{T}}$ for all $v \in \Gamma_n$. Therefore, additivity implies that $\varphi_i(v) = \sum_{T \subseteq N, T \neq \emptyset} v(T) \varphi_i(b^{\mathrm{T}}) = \frac{v(N)}{n}$ for all $i \in N$, which completes the uniqueness part. □

The ENSC value satisfies all the properties except the dual nullifying player property in Lemma 3.1. This property holds for the ENSC value only on the domain of games of which the grand marginal contributions for all players equal 0.

Definition 3.2 A game $v \in \Gamma_n$ is ***zero marginal normalized*** if $b_i^v = 0$ for all $i \in N$.

A value $\varphi : \Gamma_n \to \mathbb{R}^n$ satisfies

- **Zero marginal normalized game property**: if $\varphi_i(v) = 0$ for any zero marginal normalized game $v \in \Gamma_n$ in which i is a dual nullifying player.

Clearly, the zero marginal normalized game property is generated by restricting the dual nullifying player property on zero marginal normalized games. Adding the invariance property and replacing the dual nullifying player property with the zero marginal normalized game property in Lemma 3.1 give a new characterization of the ENSC value.

Theorem 3.1 *For all $v \in \Gamma_n$, the ENSC value is the unique value that satisfies efficiency, additivity, symmetry, invariance and the zero marginal normalized game property.*

Proof It is not difficult to verify that the ENSC value satisfies all above properties. It is sufficient to prove the uniqueness.

Let $\varphi : \Gamma_n \to \mathbb{R}^n$ be a value satisfying these five properties. For any $v \in \Gamma_n$, we define $w \in \Gamma_n$ as $w(S) := v(S) - \sum_{j \in S} b_j^v$, $\forall S \subseteq N$. It is obvious that w is a zero marginal normalized game. Since φ satisfies all properties for zero marginal normalized games in Lemma 3.1, φ coincides with the equal division value for such games. Thus, $\varphi_i(w) = ED_i(v) = \frac{w(N)}{n} = \frac{v(N) - \sum_{j \in N} b_j^v}{n}$, $\forall i \in N$.

Moreover, $v = w + d$, where $d \in \mathbb{R}^n$ with $d_i = b_i^v$ for all $i \in N$. Together with invariance, it holds that $\varphi_i(v) = \varphi_i(w) + d_i = \frac{w(N)}{n} + b_i^v = \frac{v(N) - \sum_{j \in N} b_j^v}{n} + b_i^v = \mathrm{ENSC}_i(v)$ for all $i \in N$, which completes the uniqueness part. □

A dual nullifying player not only neutralizes productiveness of coalitions containing him, but also blocks cooperation within such coalitions. Discarding the neutralization effect gives a new kind of players called the dual dummifying players.

Definition 3.3 For any game $v \in \Gamma_n$, player $i \in N$ is a ***dual dummifying player*** in v if $v(N) - v(N \backslash S) = \sum_{j \in S} b_j^v$ for all $S \subseteq N$ with $i \in S$.

The remaining part of the total worth $v(N)$ for any coalition S that contains a dual dummifying player is exactly the amount of all the grand marginal contributions of its members whenever coalition $N \backslash S$ receives the worth $v(N \backslash S)$. Analogously to the dual nullifying player property, we define the dummifying player property as follows. A value $\varphi:\Gamma_n \to \mathbb{R}^n$ satisfies:

- **Dual dummifying player property**: if $\varphi_i(v) = b_i^v$, for any $v \in \Gamma_n$ such that i is a dual dummifying player in v.

As we showed in Sect. 2, the grand marginal contributions can be regarded as the ideal payoff for players which in general cannot be guaranteed. While the dual dummifying player property requires this outcome for the dual dummifying player.

Theorem 3.2 *For all $v \in \Gamma_n$, the ENSC value is the unique value that satisfies efficiency, additivity, symmetry and the dual dummifying player property.*

Proof It is easy to verify that the ENSC value satisfies efficiency, additivity and symmetry. We show that the ENSC value also has the dual dummifying player property. Let $i \in N$ be a dual dummifying player in game $v \in \Gamma_n$, then $v(N) = v(N) - v(N\backslash N) = \sum_{j\in N} b_j^v$. According to the definition of the ENSC value, we have $\text{ENSC}_i(v) = b_i^v$, $\forall i \in N$, which completes the validity of the dual dummifying player property. Now it remains to prove the uniqueness.

Let $\varphi:\Gamma_n \to \mathbb{R}^n$ be a value that satisfies the mentioned properties. One may notice that the dual dummifying player property implies the dual nullifying player property restricted on zero marginal normalized games. In view of Theorem 3.1, it is sufficient to show that invariance holds. For any $v \in \Gamma_n$ and $b \in \mathbb{R}^n$, it remains to prove $\varphi_i(v+b) = \varphi_i(v) + b_i$, where $b(S) := \sum_{i\in S} b_i$, $\forall S \subseteq N$. By additivity, we have $\varphi(v+b) = \varphi(v) + \varphi(b)$. Moreover, since b is an inessential game, all players are dual dummifying in b. The dual dummifying player property implies $\varphi_i(b) = b(N) - b(N\backslash i) = b_i$, $\forall i \in N$. Thus, the invariance property holds. □

Monotonicity is a quite general standard for reasonable allocation in cooperative games. Young [21] characterized the Shapley value with strong monotonicity. The strong monotonicity states that if a game evolves such that some players' marginal contributions to all coalitions that contain them increase or stay unchanged, then the payoff to these players will not decrease. Instead of discussing a situation which involves game changing, we introduce a new monotonicity named grand marginal contribution monotonicity which reveals the relation between payoffs to players in a given game. A value $\varphi:\Gamma_n \to \mathbb{R}^n$ satisfies

- **Grand marginal contribution monotonicity**: if $\varphi_i(v) \geq \varphi_j(v)$ for any $v \in \Gamma_n$ such that $b_i^v \geq b_j^v$, where $i, j \in N$.

Grand marginal contribution monotonicity expresses the fact that players with larger grand marginal contribution will be assigned with a larger portion of the total benefit. Together with efficiency, additivity and inessential game property, we derive a new characterization of the ENSC value.

Theorem 3.3 *For all $v \in \Gamma_n$, the ENSC value is the unique value that satisfies efficiency, additivity, inessential game property and grand marginal contribution monotonicity.*

Proof It is trivial that the ENSC value satisfies the properties mentioned in the theorem. It remains to prove the uniqueness part.

Given a value $\varphi : \Gamma_n \to \mathbb{R}^n$ that satisfies the four properties. For any $v \in \Gamma_n$, we decompose v into two games, i.e., $v = u + w$, where $u(S) := v(S) - \sum_{j \in S} b_j^v$ and $w(S) := \sum_{j \in S} b_j^v$, $\forall S \subseteq N$. Obviously, w is an inessential game, thus $\varphi_i(w) = w(i) = v(N) - v(N \backslash i) = b_i^v$ by applying inessential game property of φ.

For any $i \in N$, $b_i^u = u(N) - u(N \backslash i) = \sum_{k \in N} b_k^v$, which indicates that all players have the same marginal contribution to the grand coalition in game u. According to grand marginal contribution monotonicity of φ, we have $\varphi_i(u) = \frac{u(N)}{n} = \frac{v(N) - \sum_{j \in N} b_j^v}{n}$, $\forall i \in N$. Finally, additivity implies that $\varphi_i(v) = \varphi_i(u + w) = \varphi_i(u) + \varphi_i(w) = \frac{v(N) - \sum_{j \in N} b_j^v}{n} + b_i^v = \text{ENSC}_i(v)$. □

Remark 3.1 To show the independence of the four axioms in Theorem 3.3, note:

1. The value $\varphi_i^1(v) := v(N) - v(N \backslash i)$ for all $i \in N$ satisfies linearity, inessential game property and grand marginal contribution monotonicity. But it does not satisfy efficiency.
2. The value $\varphi_i^2(v) := \frac{v(N) - v(N \backslash i)}{\sum_{j \in N} (v(N) - v(N \backslash j))} v(N)$ for all $i \in N$ satisfies efficiency, inessential game property and grand marginal contribution monotonicity. But it does not satisfy linearity.
3. The value $\varphi_i^3(v) := \frac{v(N)}{n}$ for all $i \in N$ satisfies efficiency, linearity and grand marginal contribution monotonicity. But it does not satisfy inessential game property.
4. The value $\varphi_i^4(v) := v(i) + \frac{v(N) - \sum_{j \in N} v(j)}{n}$ for all $i \in N$ satisfies efficiency, linearity and inessential game property. But it does not satisfy grand marginal contribution monotonicity.

4 Conclusions

In this paper, we provide two alternative ways to characterize the equal allocation of nonseparable costs value: one is the unique solution to an optimization problem, and the other is the unique satisfier of a particular collection of axioms. We show that the equal allocation of nonseparable costs value is the unique solution that minimizes the variance of the complaints for individual players under the least square criterion. Besides, the equal allocation of nonseparable costs value is the unique pre-imputation that minimizes the maximal complaint for individual players under lexicographic criterion. Several new properties are involved to characterize the equal allocation of nonseparable costs value including the zero marginal normalized game property, dual dummifying player property and grand marginal contribution monotonicity. For further research, the determination of other allocation rules for cooperative games from the perspective of optimization is expected, such as the solidarity value, the center of gravity of the imputation set value [10], the Banzhaf value [22]. The supreme step to achieve this is to figure out the corresponding complaint criteria.

Appendix: The Proof of Lemma 3.1

$$\sum_{\pi \in \Pi(N)} \frac{1}{n!}\eta_i^{v\pi} = \sum_{\pi:\pi(1)=i} \frac{1}{n!}\left[v(i) + \sum_{k=\pi^{-1}(i)+1}^{n} \frac{m_{\pi(k)} - b^v_{\pi(k)}}{k-1}\right]$$

$$+ \sum_{\pi:\pi(1)\neq i} \frac{1}{n!}\left[b_i^v + \sum_{k=\pi^{-1}(i)+1}^{n} \frac{m_{\pi(k)} - b^v_{\pi(k)}}{k-1}\right]$$

$$= \frac{(n-1)!}{n!}v(i) + \sum_{\pi:\pi(1)=i} \frac{1}{n!} \sum_{k=\pi^{-1}(i)+1}^{n} \frac{m_{\pi(k)} - b^v_{\pi(k)}}{k-1}\Bigg]$$

$$+ \frac{(n-1)(n-1)!}{n!}b_i^v + \sum_{\pi:\pi(1)\neq i} \frac{1}{n!} \sum_{k=\pi^{-1}(i)+1}^{n} \frac{m_{\pi(k)} - b^v_{\pi(k)}}{k-1}$$

$$= \frac{1}{n}v(i) + \frac{n-1}{n}b_i^v + \sum_{\pi} \frac{1}{n!} \sum_{k=\pi^{-1}(i)+1}^{n} \frac{m_{\pi(k)} - b^v_{\pi(k)}}{k-1}$$

$$= \frac{1}{n}v(i) + \frac{n-1}{n}b_i^v + \sum_{\pi} \frac{1}{n!} \sum_{l\neq i} \sum_{\pi:S_\pi^l \ni i} \frac{1}{n!} \frac{\left(v\left(S_\pi^l\right) - v\left(S_\pi^l \backslash l\right)\right) - b_l^v}{|S_\pi^l| - 1}$$

$$= \frac{1}{n}v(i) + \frac{n-1}{n}b_i^v$$

$$+ \sum_{l\neq i} \sum_{S\ni i,l} \frac{(v(S) - v(S\backslash l)) - b_l^v}{s-1} \cdot \frac{(s-1)!(n-s)!}{n!}$$

$$= \frac{1}{n}v(i) + \frac{n-1}{n}b_i^v$$

$$+ \sum_{l\neq i} \sum_{S\ni i,l} \left[(v(S) - v(S\backslash l)) - b_l^v\right] \cdot \frac{(s-2)!(n-s)!}{n!}$$

$$= \frac{1}{n}v(i) + \frac{n-1}{n}b_i^v + \sum_{l\neq i} \sum_{S\ni i,l} v(S) \cdot \frac{(s-2)!(n-s)!}{n!}$$

$$- \sum_{l\neq i} \sum_{S\ni i,l} v(S\backslash l) \cdot \frac{(s-2)!(n-s)!}{n!} - \sum_{l\neq i} \sum_{S\ni i,l} b_l^v \cdot \frac{(s-2)!(n-s)!}{n!}$$

$$= \frac{1}{n}v(i) + \frac{n-1}{n}b_i^v + \sum_{S\ni i,|S|\geq 2} \sum_{l\in S\backslash i} v(S) \cdot \frac{(s-2)!(n-s)!}{n!}$$

$$- \sum_{l\neq i} \sum_{T\subseteq N\backslash l, T\ni i} v(T) \cdot \frac{(t+1-2)!(n-(t+1))!}{n!}$$

$$- \sum_{l\neq i} \sum_{S\ni i,l} b_l^v \cdot \frac{(s-2)!(n-s)!}{n!}$$

$$= \frac{1}{n}v(i) + \frac{n-1}{n}b_i^v + \sum_{S\ni i,|S|\geq 2} (s-1)v(S) \cdot \frac{(s-2)!(n-s)!}{n!}$$

$$
\begin{aligned}
&- \sum_{T \ni i, |T| \le n-1} \sum_{l \notin T} v(T) \cdot \frac{(t-1)!(n-t-1)!}{n!} \\
&- \sum_{l \ne i} \sum_{s=2}^{n} b_l^v \cdot \frac{(s-2)!(n-s)!}{n!} \binom{n-2}{s-2} \\
&= \frac{1}{n} v(i) + \frac{n-1}{n} b_i^v + \sum_{S \ni i, |S| \ge 2} v(S) \cdot \frac{(s-1)!(n-s)!}{n!} \\
&- \sum_{T \ni i, |T| \le n-1} (n-t) v(T) \cdot \frac{(t-1)!(n-t-1)!}{n!} - \sum_{l \ne i} \sum_{s=2}^{n} b_l^v \cdot \frac{1}{n(n-1)} \\
&= \frac{1}{n} v(i) + \frac{n-1}{n} b_i^v + \sum_{S \ni i, |S| \ge 2} v(S) \cdot \frac{(s-1)!(n-s)!}{n!} \\
&- \sum_{T \ni i, |T| \le n-1} v(T) \cdot \frac{(t-1)!(n-t)!}{n!} - \sum_{l \ne i} \frac{b_l^v}{n} \\
&= \frac{1}{n} v(i) + \frac{n-1}{n} b_i^v - \frac{1}{n} v(i) + \alpha \frac{v(N)}{n} - \sum_{l \ne i} \frac{b_l^v}{n} \\
&= b_i^v + \frac{v(N) - \sum_{l \in N} b_l^v}{n} \\
&= \mathrm{ENSC}_i(v).
\end{aligned}
$$

Acknowledgements We are very grateful to the associate editor and two anonymous referees. Their valuable comments and suggestions contributed to a substantial improvement of the paper. We are also very grateful to Dr. Harry Aarts from University of Twente and Dr. Artem Sedakov from St. Petersburg State University who help us improve the use of the language. This work is supported by National Science Foundation of China (NSFC) through Grant Nos. 71601156, 71671140 and 71571143 as well as Shaanxi Province Science and Technology Research and Development Program (No. 2016JQ7008).

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