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Pure Nash equilibria in restricted budget games

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Abstract In budget games, players compete over resources with finite budgets. For every resource, a player has a specific demand and as a strategy, he chooses a subset of resources. If the total demand on a resource does not exceed its budget, the utility of each player who chose that resource equals his demand. Otherwise, the budget is shared proportionally. In the general case, pure Nash equilibria (NE) do not exist for such games. In this paper, we consider the natural classes of singleton and matroid budget games with additional constraints and show that for each, pure NE can be guaranteed. In addition, we introduce a lexicographical potential function to prove that every matroid budget game has an approximate pure NE which depends on the largest ratio between the different demands of each individual player.

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1 Introduction

Resource allocation problems are widely considered in theory and practice. In computing centers, for example, resources such as processing power and available data rate have to be divided such that the overall performance is optimized. In our paper, we consider the problem that service providers often cannot satisfy the needs of all clients. Here, the total payoff obtainable from a system is often independent of the number of its participants. For example, the computational capacity of a server is usually fixed and does not grow with the number of requests. In a different use case, the overall size of connections between a service provider and all clients may be limited by the amount of data the provider can process. In our model, this is reflected by a limited budget for each resource.

Now, different clients may have different agreed target uses with a provider, which we model by different weights, also called demands throughout the paper. In case a provider cannot fulfill the requirements of all clients, the available resource needs to be split, resulting in clients not being supplied with their full demand. In video streaming, for example, this may lead to a lower quality stream for certain clients. Additionally, we allow part of a resource to be reserved by some external party, which we model as offsets in our setting.

We consider this model in a game theoretic setting called budget games. Here, we are interested in the effects of rational decision making by individuals. In our context, the clients act as the players, who compete over resources with a finite budget. We assume that clients can choose freely among different strategies, with each available strategy being a subset of resources. A player has a specific demand on every resource. For example, in cloud computing, we view each strategy as a distribution of the necessary computing power on different computing centers. Now, each player strives to maximize the overall amount of resource capacities that is supplied to him.

Our main interest lies in states in which no client wants to deviate from his current strategy, as this would yield no or only a marginal benefit for him in the given situation. These states are called pure Nash equilibria, or approximate pure Nash equilibria, respectively. Instead of a global instance enforcing such stable states, they occur as the result of player-induced dynamics. At every point in time, exactly one player changes his strategy such that the amount of received demand is maximized, assuming the strategies of the other players are fixed. It is known that in general, pure Nash equilibria do not exist in budget games. In our earlier research, we considered pure Nash equilibria in ordered budget games (Drees et al. 2014), where the order of the players arriving at a resource influences the distribution of its budget. In Drees et al. (2015), we further discussed approximate pure Nash equilibria in standard budget games, where the resource is distributed proportionally between the players based on their demands. However, the question whether there are pure Nash equilibria for certain restricted instances of standard budget games remained open. In this paper, we focus on budget games with restrictions on the strategies of the players and show that there are indeed certain properties under which pure Nash equilibria always exist. Matroid budget games capture the natural assumption that for any player, the value of a resource is independent of which other resources he has chosen. A special case are singleton budget games in which each player can only choose one resource at a time.

1.1 Our contribution

For matroid budget games, we show that under the restriction of fixed demands per player, they possess the finite improvement property. This implies that the player-induced dynamic mentioned above always leads to a pure Nash equilibrium. On the other hand, we also show that even under this restriction, the matroid property is still required for the existence of pure Nash equilibria. Without any extra conditions on the demands, we can guarantee approximate pure Nash equilibria with a small approximation ratio depending on the maximum ratio between the demands of a single player. By further limiting the structure of the strategies to singleton, we can loosen the restriction on the demands and still obtain positive results regarding equilibria. In some cases, singleton budget games are weakly acyclic, i. e., there is an improving path from each initial state to a pure Nash equilibrium. For the additional class of offset budget games we can guarantee the existence of pure Nash equilibria under some additional restrictions.

1.2 Related work

Budget Games share many properties with congestion games. Although the specific structure of the utility functions makes budget games a special case, the fact that the demand of a player can vary between resources also qualifies them as a more general model for representing different impacts of players on resources. In congestion games, players choose among subsets of resources while trying to minimize personal costs. In the initial (unweighted) version (Rosenthal 1973), the cost of each resource depends only on the number of players choosing it and it is the same for each player using that resource. They are exact potential games (Monderer and Shapley 1996) and therefore always possess pure Nash equilibria. In the weighted version (Milchtaich 1996), each player has a fixed weight and the cost of a resource depends on the sum of weights. For this larger class of games, pure Nash equilibria can no longer be guaranteed. Ackermann et al. (2009) determined that the structure of the strategy spaces is a crucial property in this matter. While a matroid congestion game always has a pure Nash equilibrium, every non-matroid set system induces a game without it. Harks and Klimm (2010) gave a complete characterization of the class of cost functions for which every weighted congestion game possesses a pure Nash equilibrium. The cost functions have to be affine transformations of each other as well as be affine or exponential. Another extension considers player-specific payoff functions for the resources, which only depend on the number of players using a resource, but are different for each player (Milchtaich 1996). For singleton strategy spaces, these games maintain pure Nash equilibria. Ackermann et al. (2009) showed that, again, every player-specific matroid congestion game has a pure Nash equilibrium, while this is also a maximal property.

In a model similar to ours, each player does not only choose his resources, but also his demand on them (Harks and Klimm 2015). In contrast, the players in our model cannot influence their demands. These games have pure Nash equilibria if the cost functions are either exponential or affine. Mavronicolas et al. (2007) combined the concepts of weighted and player-specific congestion games and gave a detailed overview of the existence of pure Nash equilibria. In these games, the cost function $c_{i,r}$ of player i for resource r consists of a base function c_r, which depends on the weights of all players using r, as well as a constant $k_{i,r}$, both connected by abelian group operations. Later, Gairing and Klimm (2013) characterized the conditions for pure Nash equilibria in general player-specific congestion games with weighted players. Pure Nash equilibria exist, if and only if, the cost functions of the resources are affine transformations of each other as well as affine or exponential. Another generalization of congestion games is given by Byde et al. (2009) and Voice et al. (2009). They introduce the model of games with congestion-averse utility functions. They show under which properties pure Nash equilibria exist and identify the matroid as required property for the existence in most cases. Although they consider more general utility functions than standard congestion games, their model does not consider players' weights or demands.

Instead of assigning the whole cost of a resource to each player using it, it can also be shared between those players, so that everyone only pays a part of it. Such games are known as cost sharing games (Jain and Mahdian 2007). One method to determine the share of each player is proportional cost sharing, in which the share increases with the weight of a player. This is exactly what we are doing with budget games, but with utilities instead of costs. Under proportional cost sharing which corresponds to our utility functions, pure Nash equilibria again do not exist in general (Anshelevich et al. 2008). Kollias and Roughgarden (2015) took a different approach by considering weighted games in which the share of each player is identical to his Shapley value (Shapley 1952). Using this method, every weighted congestion game yields a weighted potential function. However, we do not approach this from a mechanism design angle. Instead, we consider this system and especially the structure of the utility functions as given by the scenarios we introduced. Negative results regarding both existence and complexity of pure Nash equilibria lead to the study of approximate pure Nash equilibria (Chien and Sinclair 2011). Caragiannis et al. (2015) and Hansknecht et al. (2014) showed the existence of approximate pure Nash equilibria for weighted congestion games.

2 Model

A budget game $\mathcal{B}$ is a tuple $(\mathcal{N}, \mathcal{R}, (b_r)_{r\in\mathcal{R}}, (\mathcal{S}_i)_{i\in\mathcal{N}}, (\mathcal{D}_i)_{i\in\mathcal{N}})$ where the set of players is denoted by $\mathcal{N} = \{1, \dots, n\}$, the set of resources by $\mathcal{R} = \{r_1, \dots, r_m\}$, the budget of resource r by $b_r \in \mathbb{R}_{>0}$, the strategy space of player i by $\mathcal{S}_i$ and the demands of player i by $\mathcal{D}_i = (d_i(r_1), \dots, d_i(r_m))$. Each strategy $s_i \subseteq 2^{\mathcal{R}}$ is a subset of resources. We call $d_i(r_j) > 0$ the demand of i on r_j and say that a strategy s_i uses a resource r_j if $r_j \in s_i$. The set of strategy profiles is denoted by $\mathcal{S} := \mathcal{S}_1 \times \cdots \times \mathcal{S}_n$. Each player i has a private utility function $u_i : \mathcal{S} \to \mathbb{R}_{\geq 0}$, which he strives to maximize. For a strategy profile $\mathsf{s} = (s_1, \ldots, s_n)$, let $T_r(\mathsf{s}) := \sum_{i \in \mathcal{N}: r \in s_i} d_i(r)$ be the total demand on resource r. The utility of player i from resource r is denoted by $u_{i,r}(\mathsf{s}) \in \mathbb{R}_{\geq 0}$ and defined as $u_{i,r}(\mathsf{s}) = 0$ if $r \notin s_i$ and $u_{i,r}(\mathsf{s}) := d_i(r) \cdot c_r(\mathsf{s})$ if $r \in s_i$, where $c_r(\mathsf{s}) := \min(1, {}^{b_r}/_{T_r(\mathsf{s})})$ denotes the utility per unit demand. The total utility of i is $u_i(\mathsf{s}) := \sum_{r \in \mathcal{R}} u_{i,r}(\mathsf{s})$. When increasing the demand on a resource by some value d, we write $c_r(\mathsf{s}) \oplus d := \min(1, {}^{b_r}/_{(T_r(\mathsf{s})+d)})$. If $\mathcal{M}_i = (\mathcal{R}, \mathcal{I}_i)$ is a matroid with $\mathcal{I}_i = \{x \subseteq s \mid s \in \mathcal{S}_i\}$ for every player i, we call $\mathcal{B}$ a matroid budget game. A matroid budget game is called a singleton budget game if every strategy uses exactly one resource.

Let $\mathsf{s} \in \mathcal{S}$ and $i \in \mathcal{N}$. We denote with $\mathsf{s}_{-i} := (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ the strategy profile excluding i. For any $s_i' \in \mathcal{S}_i$, we can extend this to $(\mathsf{s}_{-i}, s_i) := (s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n) \in \mathcal{S}$. The best-response of i to s_{-i} is denoted by $s_i^b \in \mathcal{S}_i$, i.e. $u_i(\mathsf{s}_{-i}, s_i^b) \geq u_i(\mathsf{s}_{-i}, s_i)$ for all $s_i \in \mathcal{S}_i$. We call the switch from s_i to s_i' with $u_i(\mathsf{s}_{-i}, s_i) < u_i(\mathsf{s}_{-i}, s_i')$ an improving move for player i. Sequential execution of best-response improving moves creates a best-response dynamic. A strategy profile s is called an α-approximate pure Nash equilibrium if $\alpha \cdot u_i(\mathsf{s}) \geq u_i(\mathsf{s}_{-i}, s_i')$ for every $i \in \mathcal{N}$ and $s_i' \in \mathcal{S}_i$. For $\alpha = 1$, s is simply called a pure Nash equilibrium. For the rest of this paper, we mostly omit the prefix *pure*. If from any initial strategy profile, there is a path of improving moves which reaches an (α-approximate) Nash equilibrium, then the game is said to be weakly acyclic. If from any initial strategy profile, each path of improving moves reaches an (α-approximate) Nash equilibrium, then the game possesses the finite improvement property. For a strategy profile s, the lexicographical potential function $\phi : \mathcal{S} \to \mathbb{R}^m_{>0}$ is defined as $\phi(\mathsf{s}) := \big(c_{r_1}(\mathsf{s}), \ldots, c_{r_m}(\mathsf{s})\big)$ with the entries $c_{r_k}(\mathsf{s})$ being sorted in ascending order. The augmented lexicographical potential function $\phi^* : \mathcal{S} \to \mathbb{R}^{m+1}_{>0}$ extends this definition with $\phi^*(\mathsf{s}) := \big(T(\mathsf{s}), c_{r_1}(\mathsf{s}), \ldots, c_{r_m}(\mathsf{s})\big)$, whereas $T(\mathsf{s}) := \sum_{i \in \mathcal{N}} \sum_{r \in s_i} d_i(r)$ is the total demand by all players under s.

3 Matroid budget games

By definition, all strategies of player i in a matroid budget game have the same size m_i. This results from the fact that each strategy space consists of bases of a matroid over the resources and any two bases of the same matroid have the same size. In addition, in any strategy profile s, a strategy change of player i from s_i to s_i' can be decomposed into a sequence $s_i = s_i^0, s_i^1, \ldots, s_i^{m_i} = s_i'$ of *lazy* moves which satisfy $s_i^k \in \mathcal{S}_i$, $|s_i^k \setminus s_i^{k+1}| = 1$ and $u_i(\mathsf{s}_{-i}, s_i^k) < u_i(\mathsf{s}_{-i}, s_i^{k+1})$ for $0 \leq k \leq m_i$ (see Ackermann et al. 2009). In other words, a lazy move is a valid improving strategy change which exchanges exactly one resource for another. In this context, valid means that the resulting strategy is actually part of the corresponding player's strategy space. Any non-lazy move can be decomposed into a sequence of lazy ones.

We start by analyzing matroid budget games in which the demands of each player are fixed, i.e. there exists a constant $d_i \in \mathbb{R}_{>0}$ for every player i such that $d_i(r) = d_i$ for all $r \in \mathcal{R}$.

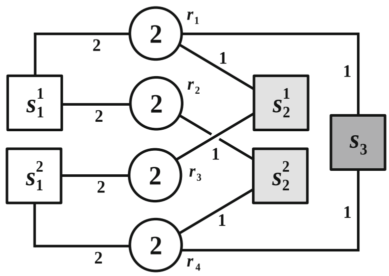

Fig. 1 The budget game $\mathcal{B}_0$ with fixed demands and no pure Nash equilibrium

Theorem 1 *A matroid budget game with fixed demands reaches a pure Nash equilibrium after a finite number of improving moves.*

Proof We show that a single lazy move already increases the lexicographical potential function ϕ. Let player i perform a lazy move in strategy profile s, switching resource r_1 for r_2. Let s' be the resulting strategy profile. We get $u_{i,r_1}(\mathsf{s}) = d_i \cdot c_{r_1}(\mathsf{s}) < d_i \cdot c_{r_2}(\mathsf{s}') = u_{i,r_2}(\mathsf{s}')$ or simply $c_{r_1}(\mathsf{s}) < c_{r_2}(\mathsf{s}')$. Since $c_{r_1}(\mathsf{s}) < c_{r_1}(\mathsf{s}')$ also holds due to $T_{r_1}(\mathsf{s}') = T_{r_1}(\mathsf{s}) - d_i$, we get $\phi(\mathsf{s}) <_{\text{lex}} \phi(\mathsf{s}')$ and see that ϕ is strictly increasing regarding the lexicographical order for every improving lazy move. Since the number of different values of ϕ is finite, the best-response dynamic eventually reaches a strategy profile without any further improving move. By definition, this is a pure Nash equilibrium. □

For this result, the structure of the strategy spaces is a crucial property. Consider the budget game $\mathcal{B}_0$ shown in Fig. 1 which is defined as follows: $\mathcal{N} = \{1, 2, 3\}$, $\mathcal{R} = \{r_1, r_2, r_3, r_4\}$, $b_r = 2$ for all r, $\mathcal{S}_1 = \{s_1^1 = \{r_1, r_2\}, s_1^2 = \{r_3, r_4\}\}$, $\mathcal{S}_2 = \{s_2^1 = \{r_1, r_3\}, s_2^2 = \{r_2, r_4\}\}$, $\mathcal{S}_3 = \{s_3 = \{r_1, r_4\}\}$ and $d_1 = 2$, $d_2 = d_3 = 1$. Note that $\mathcal{B}_0$ is a matroid budget game with fixed demands. Its existence leads to the following result.

Theorem 2 *There is a budget game with fixed demands which is not a matroid budget game and does not have a pure Nash equilibrium.*

Proof We analyze the game $\mathcal{B}_0$. Player 3 has only one strategy, so we focus only on the four different strategy profiles which result from the strategy choices of player 1 and 2. The resulting utilities are stated in Table 1. We see that in each strategy profile, one of the two players is able to increase his utility through a unilateral strategy change. Therefore, no pure Nash equilibrium exists. □

When considering singleton budget games with fixed demands, a pure Nash equilibrium can also be computed efficiently. In order to prove this, we the following technical result.

Lemma 1 *Let $d_1, d_2 \in \mathbb{R}_{>0}$ with $d_1 \le d_2$ and $b_r, T_r(\mathsf{s}) \in \mathbb{R}_{\ge 0}$ with $T_r(\mathsf{s}) + d_1 \ge b_r$. Then $d_1 \cdot \min\left(1, \frac{b_r}{T_r(\mathsf{s})+d_1}\right) \le d_2 \cdot \min\left(1, \frac{b_r}{T_r(\mathsf{s})+d_2}\right)$.*

Proof by case distinction. Due to $d_1 \le d_2$, we only need to consider three cases:

- for $\min\left(1, \frac{b_r}{T_r(\mathsf{s})+d_1}\right) = \min\left(1, \frac{b_r}{T_r(\mathsf{s})+d_2}\right) = 1$, the statement becomes trivial.

Table 1 Overview of the different strategy profiles and the corresponding utilities of the budget game $\mathcal{B}_0$

Players	Strategy profiles			
	(s_1^1, s_2^1)	(s_1^1, s_2^2)	(s_1^2, s_2^1)	(s_1^2, s_2^2)
1	$2 + 1 = 3$	$\frac{4}{3} + \frac{2}{3} = 2$	$\frac{4}{3} + \frac{2}{3} = 2$	$2 + 1 = 3$
2	$1 + \frac{1}{2} = \frac{3}{2}$	$\frac{2}{3} + 1 = \frac{5}{3}$	$\frac{2}{3} + 1 = \frac{5}{3}$	$1 + \frac{1}{2} = \frac{3}{2}$

Since player 3 has only one strategy, we abuse notation and restrict the strategy profiles to the strategies of the two players 1 and 2

– for $\min\left(1, \frac{b_r}{T_r(\mathsf{s})+d_1}\right) = 1$ and $\min\left(1, \frac{b_r}{T_r(\mathsf{s})+d_2}\right) = \frac{b_r}{T_r(\mathsf{s})+d_2}$,

  we get $d_1 \leq b_r - T_r(\mathsf{s})$ while $\frac{b_r - T_r(\mathsf{s})}{d_2} \leq \frac{b_r - T_r(\mathsf{s}) + T_r(\mathsf{s})}{d_2 + T_r(\mathsf{s})} = \frac{b_r}{d_2 + T_r(\mathsf{s})}$, which can be transformed to $b_r - T_r(\mathsf{s}) < d_2 \cdot \frac{b_r}{d_2 + T_r(\mathsf{s})}$.

– for $\min\left(1, \frac{b_r}{T_r(\mathsf{s})+d_1}\right) = \frac{b_r}{T_r(\mathsf{s})+d_1}$ and $\min\left(1, \frac{b_r}{T_r(\mathsf{s})+d_2}\right) = \frac{b_r}{T_r(\mathsf{s})+d_2}$,

  we get $\frac{d_1}{d_2} \leq \frac{d_1 + T_r(\mathsf{s})}{d_2 + T_r(\mathsf{s})}$ (since $\frac{d_1}{d_2} \leq 1$) and therefore $\frac{d_1}{T_r(\mathsf{s})+d_1} \leq \frac{d_2}{T_r(\mathsf{s})+d_2}$.

□

In the context of budget games, this lemma implies that a player with a higher demand always receives a higher utility from a resource than a player with lower demand would from the same resource. In other words, if choosing r is an attractive option for player i, then so it is for any player j with $d_i(r) \leq d_j(r)$ (not taking the current utilities of the players into account).

Theorem 3 *For a singleton budget game with fixed demands, pure Nash equilibria can be computed in time $\mathcal{O}(n)$.*

Proof We start with an *empty* strategy profile where $s_i = \emptyset$ for every player i. The players then choose their actual strategy sequentially in ascending order of their demands. We show that a strategy choice made by player j does not change the best-response of any player i with $d_i \leq d_j$. Let s_j be the strategy profile the moment before j chooses his strategy. If j picks the same resource r as i, then $d_j \cdot (c_r(\mathsf{s}_j) \oplus d_j) \geq d_j \cdot (c_{r'}(\mathsf{s}_j) \oplus d_j) \geq d_i \cdot (c_{r'}(\mathsf{s}_j) \oplus d_i)$ for any $r' \in \mathcal{R}$ due to Lemma 1, meaning that r is still a best-response for i. □

We now turn our attention to approximate Nash equilibria in matroid budget games without fixed demands. In order to give an upper bound on their existence, we again use the potential function ϕ. Starting with an arbitrary strategy profile s_0, we only allow improving moves which also strictly increase ϕ. For player i, let $d_i^{\max} := \max\{d_i(r) \mid r \in \mathcal{R}\}$ and $d_i^{\min} := \min\{d_i(r) \mid r \in \mathcal{R}\}$. We give an upper bound on α depending on the ratio between these two values.

Theorem 4 *A matroid budget game has an α-approximate pure Nash equilibrium for $\alpha = \max\left\{d_i^{max}/d_i^{min} \mid i \in \mathcal{N}\right\}$.*

Proof Let s be a strategy profile of a matroid budget game $\mathcal{B}$ in which player i can switch resource r_1 for r_2 to increase his utility. We restrict the best-response dynamic such that we only allow this lazy move if it also satisfies $d_i(r_1) \cdot c_{r_1}(\mathsf{s}) < d_i(r_1) \cdot (c_{r_2}(\mathsf{s}) \oplus d_i(r_1))$. If this condition holds, player i would still profit from the lazy move if his demands on both r_1 and r_2 were the same. Such a lazy move would also increase ϕ as shown in the proof of Theorem 1. Therefore, the number of such improving moves is finite and this restricted best-response dynamic arrives at a strategy profile s^α. Let s be a strategy profile which originates from s^α through a unilateral improving move by player i to s_i and let $\Delta_\alpha = |s_i \setminus s_i^\alpha|$. We assign an index k to every $r_k^\alpha \in s_i^\alpha$ and every $r_k \in s_i$. If a resource r is used by both s_i^α and s_i, then it has the same index ℓ for both strategies, where $\ell \geq \Delta_\alpha$. The improving move from s_i^α to s_i consists only of lazy moves with $d_i(r_k^\alpha) < d_i(r_k)$ and $c_{r_k^\alpha}(\mathsf{s}^\alpha) \geq (c_{r_k}(\mathsf{s}^\alpha) \oplus d_i(r_k))$. Since $\frac{d_i(r_k)}{d_i(r_k^\alpha)} \leq \frac{d_i^{\max}}{d_i^{\min}}$ holds for all resources, we get

$$
\begin{aligned}
u_i(\mathsf{s}) = \sum_{r \in s_i} u_{i,r}(\mathsf{s}) &= \sum_{k=1}^{\Delta_\alpha} d_i(r_k) \cdot (c_{r_k}(\mathsf{s}^\alpha) \oplus d_i(r_k)) + \sum_{k=\Delta_\alpha+1}^{m_i} d_i(r_k^\alpha) \cdot c_{r_k^\alpha}(\mathsf{s}^\alpha) \\
&\leq \sum_{k=1}^{\Delta_\alpha} \frac{d_i^{\max}}{d_i^{\min}} \cdot d_i(r_k^\alpha) \cdot c_{r_k^\alpha}(\mathsf{s}^\alpha) + \sum_{k=\Delta_\alpha+1}^{m_i} \frac{d_i^{\max}}{d_i^{\min}} \cdot d_i(r_k^\alpha) \cdot c_{r_k^\alpha}(\mathsf{s}^\alpha) \\
&= \frac{d_i^{\max}}{d_i^{\min}} \cdot \sum_{k=1}^{m_i} d_i(r_k^\alpha) \cdot c_{r_k^\alpha}(\mathsf{s}^\alpha) = \frac{d_i^{\max}}{d_i^{\min}} \cdot u_i(\mathsf{s}^\alpha)
\end{aligned}
$$

The theorem follows. □

4 Singleton budget games with two demands

We now consider singleton budget games with only two demands, i.e. every demand $d_i(r)$ of any player i on any resource r satisfies $d_i(r) \in \{d^-, d^+\}$, with both d^- and d^+ being constant values. We assume $d^- < d^+$. Also, all budgets are uniform, i.e. $b_r = b_{r'}$ for all resources r, r'. Finally, every resource r is available to every player i, i.e. there is a strategy $s_i \in \mathcal{S}_i$ using r. This variation models situations in which each player partitions the resources into two sets such that he prefers the resources from the first set over those in the second and he regards all resources from the same set as equally good. In our model, a more preferred resource is identified by a higher demand. Note that the preferences of two different players do not have to be the same. We show that Algorithm 1 always computes a Nash equilibrium by using the best-response dynamic, which proves Theorem 5 stating that such games are weakly acyclic. The algorithm utilizes the best-response dynamic and only controls the order of the improving moves. Since we are only considering singleton games, a player can be associated with his current demand, i.e. the demand he has on the resource which he is currently using. For the following discussion, we separate the improving moves into different *types*. The type depends on the demand of the corresponding player before and after his strategy change. Since we consider only two demands,

Algorithm 1 ComputeNE

$\mathsf{s} \leftarrow$ arbitrary initial strategy profile
Phase 1:
while there is a player in s with best-resp. improving move of type $d^+ \rightarrow d^-$ **do**
  perform best-response improving move of type $d^+ \rightarrow d^-$
  $\mathsf{s} \leftarrow$ resulting strategy profile
Phase 2:
while current strategy profile s is not a pure Nash equilibrium **do**
  if there is a player with best-resp. improving move of type $d^+ \rightarrow d^-$ **then**
    perform best-response improving move of type $d^+ \rightarrow d^-$
  else if there is a player i with b.-r. improving move of type $d^+ \rightarrow d^+$ **then**
    $\mathcal{N}' \leftarrow \{j \in \mathcal{N} \mid j$ has best-response improving move of type $d^+ \rightarrow d^+\}$
    choose $i \in \mathcal{N}'$ such that $T_{s_i}(\mathsf{s}) \geq T_{s_j}(\mathsf{s})$ for all $j \in \mathcal{N}'$
    perform best-response improving move of i
  else
    perform any best-response improving move ▷ $d^- \rightarrow d^-$ or $d^- \rightarrow d^+$
  $\mathsf{s} \leftarrow$ resulting strategy profile
return s ▷ s is pure Nash equilibrium

there are only four different types: $d^+ \rightarrow d^+$, $d^+ \rightarrow d^-$, $d^- \rightarrow d^+$ and $d^- \rightarrow d^-$. Looking at Algorithm 1, we immediately see that in the intermediate strategy profile right after Phase 1 of the algorithm, no improving move of type $d^+ \rightarrow d^-$ exists. In addition, we now introduce the concepts of pushing and pulling strategy changes. Let $\mathcal{B}$ be a singleton budget game with, among others, players i, j, resources r_1, r_2 and strategy profile s. In s, let $s_i = \{r_1\}$ and $s_j = \{r_2\}$ with $u_i(\mathsf{s}) < u_i(\mathsf{s}_{-i}, r_2)$ and $u_j(\mathsf{s}) \geq u_j(\mathsf{s}_{-j}, r)$ for all $r \in \mathcal{R}$. Denote $\mathsf{s}' = (\mathsf{s}_{-i}, r_2)$. If $u_j(\mathsf{s}') < u_j(\mathsf{s}'_{-j}, r_3)$ for some $r_3 \in \mathcal{R}$, then the strategy change by i from r_1 to r_2 is called a *pushing strategy change* for j. Player j is being pushed away from r_2 by player i. In the same scenario, let $u_i(\mathsf{s}) \geq u_i(\mathsf{s}_{-i}, r)$ for all $r \in \mathcal{R}$ and $u_j(\mathsf{s}) < u_j(\mathsf{s}_{-j}, r_3)$ for some $r_3 \in \mathcal{R}$. Denote $\mathsf{s}^* = (\mathsf{s}_{-j}, r_3)$. If $u_i(\mathsf{s}^*) < u_i(\mathsf{s}^*_{-i}, r_2)$, then the strategy change by j from r_2 to r_3 is called a *pulling strategy change* for i. Player i is being pulled towards r_2 by the strategy change of player j. Both strategy changes are illustrated in Fig. 2. If a strategy change is both pushing and pulling for the same player, we always regard it as the former.

Based on these characterizations, we analyze the effects of different strategy changes between the players and formulate our results in three lemmas.

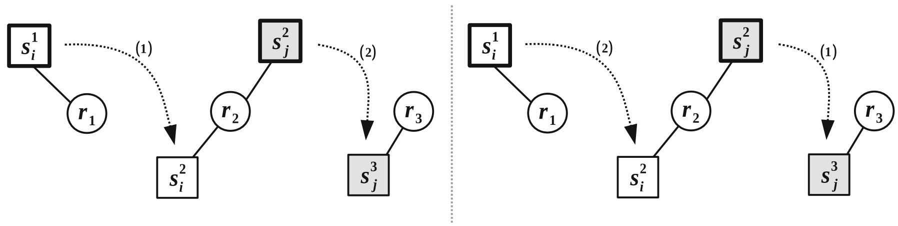

Fig. 2 Examples for pushing (left) and pulling (right) strategy changes

Lemma 2 *Let* s *be a strategy profile during Phase 2 of Algorithm 1 in which no best-response improving move of type* $d^+ \rightarrow d^-$ *exists. In* s*, no best-response improving move of type* $d^+ \rightarrow d^-$ *is created by a pushing strategy change.*

Proof Let s be a strategy profile with $s_i = \{r_1\}$ and $s_j = \{r_2\}$ and in which player i can increase his utility by moving to resource r_2:

$$u_i(\mathsf{s}) = d_i(r_1) \cdot c_{r_1}(\mathsf{s}) < d_i(r_2) \cdot (c_{r_2}(\mathsf{s}) \oplus d_i(r_2)) = u_i(\mathsf{s}_{-i}, r_2)$$

Set $\mathsf{s}' = (s_{-i}, r_2)$. Now assume that this pushes player j by creating an improving move of type $d^+ \rightarrow d^-$:

$$u_j(\mathsf{s}') = d^+ \cdot (c_{r_2}(\mathsf{s}) \oplus d_i(r_2)) < d^- \cdot (c_{r_3}(\mathsf{s}) \oplus d^-) = u_j(\mathsf{s}'_{-j}, r_3).$$

This would imply

$$\begin{aligned} d_i(r_1) \cdot c_{r_1}(\mathsf{s}) &< d_i(r_2) \cdot (c_{r_2}(\mathsf{s}) \oplus d_i(r_2)) \leq d^+ \cdot (c_{r_2}(\mathsf{s}) \oplus d_i(r_2)) \\ &< d^- \cdot (c_{r_3}(\mathsf{s}) \oplus d^-) \leq d_i(r_3) \cdot (c_{r_3}(\mathsf{s}) \oplus d_i(r_3)). \end{aligned}$$

or simply

$$u_i(\mathsf{s}_{-i}, r_2) < d_i(r_3) \cdot (c_{r_3}(\mathsf{s}) \oplus d_i(r_3)) = u_i(\mathsf{s}_{-i}, r_3).$$

Therefore, player i would have chosen resource r_3 instead of r_2. If $d_i(r_3) = d^-$, then a strategy change of type $d^+ \rightarrow d^-$ would have already existed in s. Note that

$$d_i(r_3) \cdot (c_{r_3}(\mathsf{s}) \oplus d_i(r_3)) < d_i(r_3) \cdot c_{r_3}(\mathsf{s}),$$

which covers the case $r_1 = r_3$. □

Lemma 3 *Let* s *be a strategy profile during Phase 2 of Algorithm 1. In* s*, no best-response improving move of type* $d^+ \rightarrow d^-$ *is created by a pulling strategy change of type* $d^- \rightarrow d^-$*.*

Proof by contradiction. Let s be a strategy profile with $s_i = \{r_2\}$ and $s_j = \{r_1\}$, $d_i(r_2) = d_i(r_3) = d^-$, $d_j(r_1) = d^+$, $d_j(r_2) = d^-$ and both $u_i(\mathsf{s}) < u_i(\mathsf{s}_{-i}, r_3)$ and $u_j(\mathsf{s}') < u_j(\mathsf{s}'_{-j}, r_2)$ for $\mathsf{s}' = (\mathsf{s}_{-i}, r_3)$. According to the structure of the algorithm, the current strategy profile s is an equilibrium for j if i is allowed to change his strategy. From

$$u_i(\mathsf{s}) = d^- \cdot c_{r_2}(\mathsf{s}) < d^- \cdot (c_{r_3}(\mathsf{s}) \oplus d^-) = u_i(\mathsf{s}_{-i}, r_3)$$

and

$$u_j(\mathsf{s}) = d^+ \cdot c_{r_1}(\mathsf{s}) < d^- \cdot (c_{r_2}(\mathsf{s}) \oplus (d^- - d^-)) = d^- \cdot c_{r_2}(\mathsf{s}) = u_i(\mathsf{s})$$

we can conclude that

$$u_j(\mathsf{s}) = d^+ \cdot c_{r_1}(\mathsf{s}) < d^- \cdot (c_{r_3}(\mathsf{s}) \oplus d^-) \leq u_j(\mathsf{s}_{-j}, r_3).$$

If r_2 is the best-response of j after the strategy change of i, then r_3 had to be his best-response before it, which contradicts our assumption that j is in an equilibrium. For both possible values of $d_j(r_3)$, j would have performed this strategy change before i. □

Lemma 4 *Let* s *be a strategy profile during phase 2 of Algorithm 1. In* s*, no best-response improving move of type* $d^+ \to d^-$ *is created by a pulling strategy change of type* $d^+ \to d^+$.

Proof by contradiction. Let s be a strategy profile with $s_i = \{r_2\}$ and $s_j = \{r_1\}$, $d_i(r_2) = d_i(r_3) = d^+$, $d_j(r_1) = d^+$, $d_j(r_2) = d^-$ and both $u_i(\mathsf{s}) < u_i(\mathsf{s}_{-i}, r_3)$ and $u_j(\mathsf{s}') < u_j(\mathsf{s}'_{-j}, r_2)$ for $\mathsf{s}' = (\mathsf{s}_{-i}, r_3)$. According to the structure of the algorithm, $u_j(\mathsf{s}) \geq u_j(\mathsf{s}_{-j}, r_2)$ has to hold. From

$$u_i(\mathsf{s}) = d^+ \cdot c_{r_2}(\mathsf{s}) < d^+ \cdot (c_{r_3}(\mathsf{s}) \oplus d^+) = u_i(\mathsf{s}_{-i}, r_3)$$

we get $c_{r_2}(\mathsf{s}) < c_{r_3}(\mathsf{s}) \oplus d^+$, which can be written as

$$\min\left(1, \frac{b}{T_{r_2}(\mathsf{s})}\right) < \min\left(1, \frac{b}{T_{r_3}(\mathsf{s}) + d^+}\right)$$

and finally gives us $T_{r_3}(\mathsf{s}) < T_{r_2}(\mathsf{s}) - d^+$. Also

$$u_j(s) = u_j(\mathsf{s}') = d^+ \cdot c_{r_1}(\mathsf{s}) < d^- \cdot (c_{r_2}(\mathsf{s}) \oplus (d^+ - d^-)) = u_j(\mathsf{s}'_{-j}, r_2).$$

The rest of the proof is done via case distinction.

- $d_j(r_3) = d^-$

  First we show that

$$u_j(\mathsf{s}) = d^+ \cdot c_{r_1}(\mathsf{s}) \geq d^- \cdot (c_{r_3}(\mathsf{s}) \oplus d^-) = u_j(\mathsf{s}_{-j}, r_3). \quad (1)$$

  If this would be false, then there has to be some resource r_4 with $d_j(r_4) = d^+$ and

$$u_j(\mathsf{s}_{-j}, r_3) < u_j(\mathsf{s}_{-j}, r_4) = d^+ \cdot (c_{r_4}(\mathsf{s}) \oplus d^+),$$

  otherwise the algorithm would choose j instead of i to perform an improving move. Since both players would perform the same type of best-response improving move ($d^+ \to d^+$) in s but i is chosen over j, the relation $T_{r_1}(\mathsf{s}) \leq T_{r_2}(\mathsf{s})$ has to hold.

However, this leads to a contradiction with $u_j(\mathsf{s}') < u_j(\mathsf{s}'_{-j}, r_2)$, which can be written as

$$d^+ \cdot \min\left(1, \frac{b}{T_{r_1}(\mathsf{s}}\right) < d^- \cdot \min\left(1, \frac{b}{T_{r_2}(\mathsf{s}) + d^- - d^+}\right) \\ \leq d^+ \cdot \min\left(1, \frac{b}{T_{r_2}(\mathsf{s})}\right) \Rightarrow T_{r_2}(\mathsf{s}) < T_{r_1}(\mathsf{s}).$$

For this, also note that $c_{r_1}(\mathsf{s}) = c_{r_1}(\mathsf{s}')$. So Eq. 1 has to hold and we get

$$u_j(\mathsf{s}'_{-j}, r_2) > u_j(\mathsf{s}') = u_j(\mathsf{s}) \geq u_j(\mathsf{s}_{-j}, r_3)$$

which in combination with $T_{r_3}(\mathsf{s}) < T_{r_2}(\mathsf{s}) - d^+$ leads to

$$d^- \cdot (c_{r_2}(\mathsf{s}) \oplus (d^- - d^+)) > d^- \cdot (c_{r_3}(\mathsf{s}) \oplus d^-) \\ = d^- \cdot \min\left(1, \frac{b}{T_{r_3}(\mathsf{s}) + d^-}\right) \geq d^- \cdot \min\left(1, \frac{b}{T_{r_2}(\mathsf{s}) - d^+ + d^-}\right) \\ = d^- \cdot (c_{r_2}(\mathsf{s}) \oplus (d^- - d^+)),$$

the final contradiction.

– $d_j(r_3) = d^+$

We can write $u_j(\mathsf{s}') < u_j(\mathsf{s}'_{-j}, r_2)$ as

$$d^+ \cdot \min\left(1, \frac{b}{T_{r_1}(\mathsf{s})}\right) < d^- \cdot \min\left(1, \frac{b}{T_{r_2}(\mathsf{s}) - d^+ + d^-}\right) \\ < d^+ \cdot \min\left(1, \frac{b}{T_{r_2}(\mathsf{s}) - d^+ + d^+}\right) = d^+ \cdot \min\left(1, \frac{b}{T_{r_2}(\mathsf{s})}\right),$$

which implies both

$$\min\left(1, \frac{b}{T_{r_1}(\mathsf{s})}\right) < \min\left(1, \frac{b}{T_{r_2}(\mathsf{s})}\right) \Rightarrow T_{r_1}(\mathsf{s}) > T_{r_2}(\mathsf{s})$$

and (together with $T_{r_3}(\mathsf{s}) < T_{r_2}(\mathsf{s}) - d^+$)

$$u_j(\mathsf{s}) = d^+ \cdot \min\left(1, \frac{b}{T_{r_1}(\mathsf{s})}\right) < d^+ \cdot \min\left(1, \frac{b}{T_{r_2}(\mathsf{s})}\right) \\ < d^+ \cdot \min\left(1, \frac{b}{T_{r_3}(\mathsf{s}) + d^+}\right) = u_j(\mathsf{s}_{-j}, r_3).$$

If r_2 is a best-response of j in s', after the strategy change of i, then r_3 would have been a best-response of j before, in s. According to how the algorithm works,

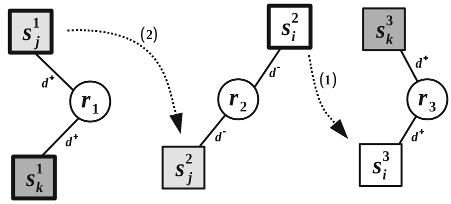

Fig. 3 Example for a macro strategy change. This sequence of strategy changes is equivalent to the strategy change of a virtual player k from r_1 to r_3

both players have the same priority when choosing the next improving move, but since $T_{r_1}(\mathsf{s}) > T_{r_2}(\mathsf{s})$, j would have been chosen before i.

This concludes our proof. □

In order to show that the best-response dynamic controlled by our algorithm actually results in a pure Nash equilibrium, we use the augmented lexicographical potential function ϕ^* Together with the three lemmas above, we see that during Phase 2, any best-response improving move of type $d^+ \to d^-$ has to be created by a pulling strategy change of type $d^- \to d^+$. We can combine these two strategy changes into a single one, a *macro strategy change*. In this combination, both individual strategy changes are executed right after another. In a macro strategy change, two players i, j change their resources, with r being both the old resource of i and the new resource of j and $d_i(r) = d_j(r)$. As a result, the total demand on r does not change during a macro strategy change. An example can be seen in Fig. 3. Although not associated with an actual player, we say that a macro strategy change is performed by a virtual player. The following lemma shows that this virtual player would actually benefit from his strategy change.

Lemma 5 *Let* s *be a strategy profile in which a macro strategy change of type* $d^+ \to d^+$ *from* r_1 *to* r_3 *is executed. Then* $d^+ \cdot c_{r_1}(\mathsf{s}) < d^+ \cdot (c_{r_3}(\mathsf{s}) \oplus d^+)$.

Proof Due to the underlying strategy changes and Lemma 1, we get

$$
\begin{aligned}
d^+ \cdot c_{r_1}(\mathsf{s}) &< d^- \cdot (c_{r_2}(\mathsf{s}) \oplus (d^- - d^+)) < d^+ \cdot (c_{r_2}(\mathsf{s}) \oplus (d^+ - d^+)) \\
&< d^- \cdot (c_{r_3}(\mathsf{s}) \oplus d^-) < d^+ \cdot (c_{r_3} \oplus d^+)
\end{aligned}
$$

□

With this lemma, we conclude that a macro strategy change strictly increases ϕ^*. Its type is $d^+ \to d^+$ and from the results in the previous section, we know that such a strategy change already strictly increases ϕ. Since the total demand of all players does not change, this holds for ϕ^* as well.

Theorem 5 *A singleton budget game with two demands and uniform budgets is weakly acyclic.*

Proof By construction, the output of Algorithm 1 is a pure Nash equilibrium. It remains to show that the algorithm actually terminates at some point. The number of improving moves in the first phase is at most n, as every player changes his strategy at most once.

For the second phase, we use the augmented lexicographical potential function ϕ^*. This function is strictly increasing regarding $<_{\text{lex}}$ for all strategy changes of type $d^- \to d^-$ and $d^+ \to d^+$, since ϕ is strictly increasing for these types and the total demand of all players does not change. For strategy changes of type $d^- \to d^+$, ϕ^* is also strictly increasing because the total demand is always the first entry in $\phi^*(\mathtt{s})$ and it increases with strategy changes of this type. The value of ϕ^* can only decrease as the result of improving moves of type $d^+ \to d^-$.

Let $\mathtt{s}_1$ be the strategy profile right after Phase 1 has terminated. Then $\mathtt{s}_1$ contains no best-response improving moves of type $d^+ \to d^-$. According to Lemmas 2, 3 and 4, such moves can only appear as the result of a pulling strategy change of type $d^- \to d^+$. In this case, both the resulting strategy change of type $d^+ \to d^-$ as well as its creator can be regarded as a single macro strategy change of type $d^+ \to d^+$. Because of Lemma 5 and the fact that such a macro strategy change does not change the total demand, ϕ^* strictly increases due to such a macro strategy change. If a pulling strategy change creates multiple best-response improving moves of type $d^+ \to d^-$ to a resource r, then the algorithm executes one of them, chosen by some arbitrary tie-breaker. Afterwards, the total demand on r is the same as it was before the pulling strategy change. Hence, the other best-response moves of type $d^+ \to d^-$ cease to exist.

ϕ^* strictly increases after at least every second strategy change, so our algorithm has to terminate at some point. The resulting strategy profile is a Nash equilibrium. During its execution, the algorithm performs only best-response improving moves and only controls the order in which they are executed. Therefore, the game is weakly acyclic. □

At this point, we do not know if this result carries over to matroid budget games under the same restrictions. Regarding budget games with two demands and uniform budgets, but an arbitrary structure on the strategy spaces, we already know that Nash equilibria do not exist in general (Drees et al. 2015).

5 Singleton offset budget games

In this section, we introduce a new variant of budget games in which we allow a fixed offset to the total demand on a resource. As already mentioned in the introduction, this enables us to model reserved instances for specific users in our games. An offset $\sigma_r \in \mathbb{R}_{\geq 0}$ for resource $r \in \mathcal{R}$ changes the utility of any player i from r in strategy profile $\mathtt{s}$ to $u_{i,r}(\mathtt{s}) = d_i(r) \cdot \min\left(1, {}^{b_r}/_{(T_r(\mathtt{s})+\sigma_r)}\right)$.

It is easy to see that by setting $\sigma_r = 0$ for every $r \in \mathcal{R}$, an offset budget game becomes a regular budget game. We now start by considering budget games with two additional restrictions: a total order on the players based on their demands and increasing demand ratios. Let i, j be players with $d_i(r) \leq d_j(r)$ for some resource r. Our first restriction states that although the demands of an individual player can differ between resources, the order between the demands of all players is the same for every resource. In other words: $d_i(r') \leq d_j(r')$ for all resources r'. This is a natural assumption, as bigger players (like global companies) normally have a higher demand than smaller ones on all resources.

The second restriction requires larger players to have larger deviations between their demands, i.e. $d_i(r')/d_i(r) \leq d_j(r')/d_j(r)$ for $d_i(r) \leq d_i(r')$. Again, this assumption is only natural, as larger players (e. g. jobs on servers) offer more room for optimization and are more influenced by their choice of resource (e. g. servers with better support for certain kinds of operations) than smaller ones, which are already quite compact. For the class of offset budget games which satisfy these two restrictions and only allow singleton strategies, we can guarantee the existence of pure Nash equilibria.

Theorem 6 *Singleton offset budget games with ordered players and increasing demand ratios always have a pure Nash equilibrium.*

Proof by induction over the number of players. For a game with n players, we denote the offset of resource r by $\sigma_n(r)$. For $n = 2$, the statement becomes trivial. For $n > 2$, we assume without loss of generality that $d_n(r_1) \geq d_i(r_1)$ for all $i \in \mathcal{N}$ and $d_n(r_1) \geq d_n(r)$ for all $r \in \mathcal{R}$. Fix the strategy of n to $\{r_1\}$. The resulting game is identical to one with $n-1$ players and $\sigma_{r_1}^{n-1} = \sigma_{r_1}^n + d_n(r_1)$ and $\sigma_r^{n-1} = \sigma_r^n$ for all $r \in \mathcal{R} \setminus \{r_1\}$. By induction hypothesis, this sub-game has a Nash equilibrium s'. Let $\mathsf{s} := (\mathsf{s}', r_1)$ and assume that s is not already a Nash equilibrium for n. Then

$$u_n(\mathsf{s}) = d_n(r_1) \cdot (c_{r_1}(\mathsf{s}) \oplus \sigma_{r_1}^n) < d_n(r_2) \cdot (c_{r_2}(\mathsf{s}) \oplus (\sigma_{r_2}^n + d_n(r_2)) = u_n(\mathsf{s}_{-n}, r_2)$$

for some $r_2 \in R$. Let i be another player on r_1, i.e. $s_i = \{r_1\}$. Since r_1 is the best-response of i, we get

$$u_i(\mathsf{s}) = d_i(r_1) \cdot (c_{r_1}(\mathsf{s}) \oplus \sigma_{r_1}^n) \geq d_i(r_2) \cdot (c_{r_2}(\mathsf{s}) \oplus (\sigma_{r_2}^n + d_i(r_2)) = u_i(\mathsf{s}_{-i}, r_2).$$

Combining these two inequalities, we get

$$\frac{d_i(r_2)}{d_i(r_1)} \cdot (c_{r_2}(\mathsf{s}) \oplus (\sigma_{r_2}^n + d_i(r_2)) < \frac{d_n(r_2)}{d_n(r_1)} \cdot (c_{r_2}(\mathsf{s}) \oplus (\sigma_{r_2}^n + d_n(r_2)). \tag{2}$$

Since the players are ordered, we know that $d_n(r_2) \geq d_i(r_2)$ and therefore $(c_{r_2}(\mathsf{s}) \oplus (\sigma_{r_2}^n + d_i(r_2)) \geq (c_{r_2}(\mathsf{s}) \oplus (\sigma_{r_2}^n + d_n(r_2))$. Equation 2 thus implies $\frac{d_i(r_1)}{d_i(r_2)} > \frac{d_n(r_1)}{d_n(r_2)}$. This contradicts our restriction that the demand ratios are increasing, hence player i cannot exist. So n is the only player on resource r_1 and since this is his preferred resource (the one where he has the highest demand), it also has to be his best-response. We conclude that s has to be a Nash equilibrium for all n players. □

As mentioned before, this result also holds for regular budget games in particular by setting every offset to 0.

Corollary 1 *Singleton budget games with ordered players and increasing demand ratios always have a pure Nash equilibrium.*

For singleton (offset) budget games with only two resources, this result can be improved even more. In this case, we can drop both restrictions regarding ordered players and increasing demand ratios. In addition, besides always having a pure equilibrium, such games are also weakly acyclic.

Lemma 6 *Every singleton offset budget game with two resources is weakly acyclic.*

Proof The proof is similar to the one of Theorem 6 and uses induction over the number of players. For a game with n players, we denote the offset of resource r by $\sigma_n(r)$. For $n = 2$, the statement becomes trivial. For $n > 2$, we assume without loss of generality that $d_n(r_1) \geq d_i(r_1)$ for all $i \in \mathcal{N}$ and $d_n(r_1) \geq d_n(r_2)$. Fix the strategy of n to $\{r_1\}$. The resulting game is identical to one with $n-1$ players and $\sigma_{r_1}^{n-1} = \sigma_{r_1}^n + d_n(r_1)$ and $\sigma_{r_2}^{n-1} = \sigma_{r_2}^n$. By induction hypothesis, this game is weakly acyclic and the remaining players can reach a pure Nash equilibrium s^1 after a finite number of improving moves. By the same arguments, the game has a pure Nash equilibrium s^2 in which we fix the strategy of player n to $\{r_2\}$. If at least one of these two strategy profiles, s^1 or s^2, is an actual equilibrium (in which the strategy of n is not fixed), then the lemma has been proven. So now we assume that this is not the case, i.e. $u_n(\mathsf{s}^1) < u_n(\mathsf{s}^1_{-n}, r_2)$ and $u_n(\mathsf{s}^2) < u_n(\mathsf{s}^2_{-n}, r_1)$. From each of these two assumptions, we obtain one inequality:

$$\begin{aligned} d_n(r_1) \cdot (c_{r_1}(\mathsf{s}^1) \oplus \sigma_{r_1}^n) &< d_n(r_2) \cdot (c_{r_2}(\mathsf{s}^1) \oplus (\sigma_{r_2}^n + d_n(r_2))) \\ d_n(r_2) \cdot (c_{r_2}(\mathsf{s}^2) \oplus \sigma_{r_2}^n) &< d_n(r_1) \cdot (c_{r_1}(\mathsf{s}^2) \oplus (\sigma_{r_1}^n + d_n(r_1))) \end{aligned} \tag{3}$$

which can be combined to

$$\frac{c_{r_1}(\mathsf{s}^1) \oplus \sigma_{r_1}^n}{c_{r_1}(\mathsf{s}^2) \oplus (\sigma_{r_1}^n + d_n(r_1))} < \frac{c_{r_2}(\mathsf{s}^1) \oplus (\sigma_{r_2}^n + d_n(r_2))}{c_{r_2}(\mathsf{s}^2) \oplus \sigma_{r_2}^n}$$

We now make a crucial observation: there has to be at least one common player i who is located on the same resource as player n in both s^1 and s^2, i.e. r_1 in s^1 and r_2 in s^2. To see this, first consider the case that n could be the only player on resource r_1 in s^1, with all other players on r_2. Equation 3 states that n is still willing to switch over to r_2, contradicting $d_n(r_1) \geq d_n(r_2)$. So there has to be at least one additional player i on r_1 in s^1. For any strategy profile s, let $n_r(\mathsf{s})$ be the set of all players located on resource r. If $n_{r_1}(\mathsf{s}^1) \cap n_{r_2}(\mathsf{s}^2) = \{n\}$, then $n_{r_2}(\mathsf{s}^2) \setminus \{n\} \subseteq n_{r_2}(\mathsf{s}^1)$, which contradicts n's preference of r_2 over r_1 in (s^1_{-n}, r_2) but not in s^2: the set of competing players on r_2 is bigger in s^1. So at least one common player i shares the same resource as n in both s^1 and s^2.

By definition, player i cannot improve his utility in neither s^1 nor s^2, so

$$\begin{aligned} d_i(r_1) \cdot (c_{r_1}(\mathsf{s}^1) \oplus \sigma_{r_1}^n) &\geq d_i(r_2) \cdot (c_{r_2}(\mathsf{s}^1) \oplus (\sigma_{r_2}^n + d_i(r_2))) \\ \Rightarrow \quad \frac{c_{r_1}(\mathsf{s}^1) \oplus \sigma_{r_1}^n}{(c_{r_2}(\mathsf{s}^1) \oplus (\sigma_{r_2}^n + d_i(r_2)))} &\geq \frac{d_i(r_2)}{d_i(r_1)} \end{aligned}$$

and

$$\begin{aligned} d_i(r_2) \cdot (c_{r_2}(\mathsf{s}^2) \oplus \sigma_{r_2}^n) &\geq d_i(r_1) \cdot (c_{r_1}(\mathsf{s}^2) \oplus (\sigma_{r_1}^n + d_i(r_1))) \\ \Rightarrow \quad \frac{(c_{r_1}(\mathsf{s}^2) \oplus (\sigma_{r_1}^n + d_i(r_1)))}{c_{r_2}(\mathsf{s}^2) \oplus \sigma_{r_2}^n} &\leq \frac{d_i(r_2)}{d_i(r_1)}. \end{aligned}$$

Combining these two inequalities with those above yields the final contradiction that $d_i(r_2)/d_i(r_1) < d_i(r_2)/d_i(r_1)$. Therefore, at least one of the two strategy profiles, s^1 or s^2, has to be a pure Nash equilibrium for all n players. □

Note that the proof above indeed shows that such budget games are weakly acyclic. Starting from an arbitrary initial strategy profile, (recursively) create either s^1 or s^2 via improving moves of the first $n-1$ players, depending on whether player n is located on r_1 or r_2. If the result is not already a pure Nash equilibrium, let player n switch his resource, which increases his utility. Afterwards, from the opposite strategy profile (e. g. s^2 instead of s^1). Because at least one of the two has to be a pure Nash equilibrium, this method produces the desired state.

Since every budget game is also an offset budget game, we can immediately draw the following conclusion.

Corollary 2 *Every singleton budget game with two resources is weakly acyclic.*

This last result also holds for matroid budget games with only two resources. Let i be a player whose strategy space does not consist solely of singleton strategies. By definition, he then can only have a single strategy which uses both resources and his existence simply introduces a fixed offset to them. This holds for all players with non-singleton strategy spaces and their demands can be summed up to a single offset value for each resource. According to Lemma 6, the remaining players who only possess singleton strategies can create a pure Nash equilibrium through improving moves.

Corollary 3 *Every matroid (offset) budget game with two resources is weakly acyclic.*

6 Conclusion

The model of budget games enables us to analyze different effects which appear specifically in, but are not limited to, cloud computing. In emerging markets with shared resources, the question of resource allocation gets more and more important. In our current work, we focus on a specific method of distributing resources among the market participants.

This article gives a first insight into the existence of pure Nash equilibria in restricted instances of budget games. In our previous research, we already considered approximate pure Nash equilibria in general budget games (Drees et al. 2015) and gave both upper and lower bounds for the approximation factor as well as an algorithm to compute such equilibria. These bounds depend on the relative size of the demands, i.e. how much of a resource's budget can be claimed by a single strategy. The higher this value, the larger the approximation factor of the resulting equilibria. Our new result from Theorem 4 uses a different approach, with the approximation factor depending on a different criteria: the ratio between the different demands of each individual player. In a matroid budget game where the various demands of each player are close to each other, this new approach can yield a better solution than our old method, especially if at least one demand is relatively high compared to the budget of the corresponding resource (e. g. $d_i(r) = b_r$).

While we managed to characterize some classes of budget games with pure Nash equilibria, our initial question is still left unanswered: what is the fundamental property which guarantees the existence of pure Nash equilibria. If we look at existing work in the field of congestion games (e. g. Ackermann et al. 2009), we see that the matroid structure of the strategy spaces can be a promising candidate. In fact, it is our believe that any matroid budget game features at least one equilibria. For future works, it would be very interesting to either prove or refute this proposition.

Finally, we propose two modifications of our model. On the one hand, strategies can extended by (fixed) prices, which makes the choice of each player even more interesting. In that case, he is no longer interested just in his incoming utilities, but his total revenue (= utilities − prices). This extension would bring the model even closer to the setting of cloud computing, for example. On the other hand, the current model allocates the budget of a resource in a proportional fashion (based on the demands and only if the the total demand exceeds the budget). It would be interesting to analyze other allocation mechanisms. We already did this in Drees et al. (2014) and produced completely different results regarding existence and complexity of Nash equilibria. It therefore seems promising that these problems can also be approached from a mechanism design angle instead of restricting the structure of the strategy spaces.

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