Recursive Solution Procedures for Flexible Multibody Systems: Comparing Condensation and Transfer Matrix Methods

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Abstract

The dynamic behavior of flexible multibody systems that consist of a chain of bodies can be studied efficiently by recursive solution procedures. In these, it is common to express the kinematics using the relative coordinates formulation and the dynamics using the floating frame of reference formulation. A comprehensive overview of the developments in this field for open-loop and closed-loop rigid and flexible multibody systems is given in [1].

The purpose of this work is to provide additional insights to these recursive methods. Key concept in most recursive solution procedures is to recursively eliminate a body from the chain, upon which the mass matrix and interface forces of the remaining adjacent body must be updated appropriately. In this work, it is shown that by expressing the motion with respect to an inertial frame, the mass matrix update can directly be related to the effective mass of the eliminated body, giving it a clear physical interpretation. This method is compared with a recursive solution procedure based on transfer matrices. This transfer matrix method may have advantages in the case of closed-loop systems and systems that consist of a chain of identical bodies, which is often the case for solar panel arrays, robotic manipulators, offshore equipment, etcetera.

Consider the equation of motion of a single flexible body using the floating frame of reference formulation:

$$\begin{bmatrix} \mathbf{M}_r & \mathbf{\Gamma}^T \\ \mathbf{\Gamma} & \mathbf{M}_e \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_r \\ \ddot{\mathbf{q}}_e \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_e \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{q}_e \end{bmatrix} = \begin{bmatrix} \mathbf{F}_r \\ \mathbf{F}_e \end{bmatrix}. \tag{1}$$

The rigid body coordinates $\mathbf{q}_r$ describe the configuration of a floating frame with respect to an inertial frame. The elastic coordinates $\mathbf{q}_e$ describe a linear elastic displacement field with respect to the floating frame, using a linear combination of deformation modes. The mass matrix consists of the rigid body mass matrix $\mathbf{M}_r$, the generalized mass matrix due to the elastic modes $\mathbf{M}_e$ and the modal participation factors $\mathbf{\Gamma}$ that couple rigid and elastic motions. $\mathbf{K}_e$ is the generalized stiffness matrix due to the elastic modes. $\mathbf{F}_r$ is the resultant generalized force acting on the body and $\mathbf{F}_e$ is the generalized modal force. Here, quadratic velocity forces are contained in the right hand side generalized forces.

In [2], it is explained how this equation of motion can be used in a recursive order-n-method that is applied for the simulation of flexible space structures. To explain the procedure, Figure 1 highlights the last two bodies, A and B, of an open-loop system. From the equation of motion of the last body B, the second line is solved for $\ddot{\mathbf{q}}_e$ and substituted in the first line. By doing so, it is possible to express the interface force between the two bodies in terms of the acceleration of the interface point and the generalized forces of body B. Next, this is substituted in the right hand side of the equation of motion of second to last body A. The term containing the acceleration of the interface point can be combined with the other inertia forces and forms the mass matrix update of this body. All other terms remain on the right hand side and form the update of the interface force. The procedure is repeated recursively, eliminating one body at a time, until one body remains. This equation of motion is solved and the result is back substituted in the equations of motion of the eliminated bodies subsequently. In case of a closed-loop system, a loop closure constraint is enforced using Lagrange multipliers.

If the elastic deformation is described by six appropriate modes, e.g. static Craig-Bampton modes, it can be shown that for a two-node body, it is always possible to find a coordinate transformation from $\mathbf{q}_r$ and $\mathbf{q}_e$ towards the global motion of the interface points, e.g. $\mathbf{q}_i$ and $\mathbf{q}_j$ for body A. With this, the equation of motion eq. (1) can be rewritten to the following form:

$$\begin{bmatrix} \mathbf{M}_{ii} & \mathbf{M}_{ij} \\ \mathbf{M}_{ji} & \mathbf{M}_{jj} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_i \\ \ddot{\mathbf{q}}_j \end{bmatrix} = \begin{bmatrix} \mathbf{F}_i \\ \mathbf{F}_j \end{bmatrix} + \begin{bmatrix} \mathbf{Q}_i \\ \mathbf{Q}_j \end{bmatrix}. \tag{2}$$

Note that these are nonlinear equations in terms of the (large) global motion of the interface coordinates. $\mathbf{F}_i$ and $\mathbf{F}_j$ are the interface forces of interface points i and j. $\mathbf{Q}_i$ and $\mathbf{Q}_j$ contains all other generalized forces acting on interface points i and j. This form is similar to the co-rotational frame formulation of a finite element, with the difference that in eq. (2) also quadratic velocity forces are included in $\mathbf{Q}$, which are normally neglected in a co-rotational formulation.

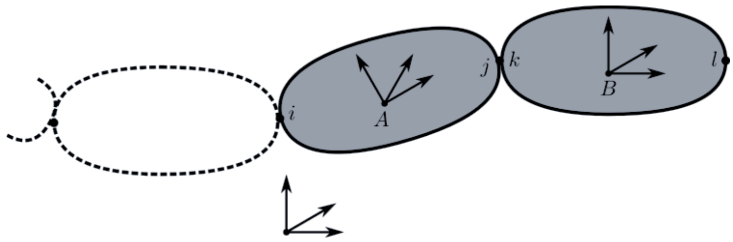

Figure 1. The last two bodies, A and B, of an open-loop flexible multibody system.

Using eq. (2) instead of eq. (1), a recursive solution procedure is set up similarly as explained above. The equation of motion of body B can be manipulated such that the interface force $\mathbf{F}_k$ is expressed as:

$$\mathbf{F}_k = [\mathbf{M}_{kk} - \mathbf{M}_{kl}\mathbf{M}_{ll}^{-1}\mathbf{M}_{lk}]\ddot{\mathbf{q}}_k + \mathbf{M}_{kl}\mathbf{M}_{ll}^{-1}\mathbf{F}_l + \mathbf{Q}_k - \mathbf{M}_{kl}\mathbf{M}_{ll}^{-1}\mathbf{Q}_k. \tag{3}$$

The term in between square brackets can be recognized as the mass matrix condensed on interface point k, which consists of its relevant partition $\mathbf{M}_{kk}$ and the added effective mass of the body $-\mathbf{M}_{kl}\mathbf{M}_{ll}^{-1}\mathbf{M}_{lk}$, similar to linear vibration problems [3]. Without loss of generality, it is assumed that the bodies are connected rigidly, such that the kinematic and dynamic coupling conditions are $\ddot{\mathbf{q}}_j = \ddot{\mathbf{q}}_k$ and $\mathbf{F}_j = -\mathbf{F}_k$. With this, the equation of motion of body A can be expressed as:

$$\begin{bmatrix} \mathbf{M}_{ii} & \mathbf{M}_{ij} \\ \mathbf{M}_{ji} & \mathbf{M}_{jj} + \mathbf{M}_{kk} - \mathbf{M}_{kl}\mathbf{M}_{ll}^{-1}\mathbf{M}_{lk} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_i \\ \ddot{\mathbf{q}}_j \end{bmatrix} = \begin{bmatrix} \mathbf{F}_i \\ \mathbf{M}_{kl}\mathbf{M}_{ll}^{-1}\mathbf{F}_l \end{bmatrix} + \begin{bmatrix} \mathbf{Q}_i \\ \mathbf{Q}_j + \mathbf{Q}_k - \mathbf{M}_{kl}\mathbf{M}_{ll}^{-1}\mathbf{Q}_l \end{bmatrix}. \tag{4}$$

In this form, it is clear that the elimination of body B results in a mass matrix update of body A, but only in the partition directly related to interface point j. This can be physically interpreted as follows: In order to have equivalent motion of the system from which body B is eliminated, the appropriate condensed mass is added to interface point j. Simultaneously its interface force is updated to take into account the interface force $\mathbf{F}_l$ and elastic and quadratic velocity forces of body B. By applying a coordinate transformation back to rigid and elastic coordinates on eq. (4), the same result is obtained as in [2]. With this, it can be understood that in this recursive solution procedure the mass matrix update can be related to a specific condensed mass matrix and a coordinate transformation.

With the equations of motion in the form of eq. (2), an alternative recursive formulation is possible in terms of transfer matrices. To this end, the acceleration and interface force of interface point j are expressed in terms of the acceleration and interface force of interface point i and generalized forces as:

$$\begin{bmatrix} \ddot{\mathbf{q}}_j \\ \mathbf{F}_j \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{ij} & \mathbf{0} \\ \mathbf{M}_{jj} & -\mathbf{1} \end{bmatrix}^{-1} \begin{bmatrix} -\mathbf{M}_{ii} & \mathbf{1} \\ -\mathbf{M}_{ji} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_i \\ \mathbf{F}_i \end{bmatrix} + \begin{bmatrix} \mathbf{M}_{ij} & \mathbf{0} \\ \mathbf{M}_{jj} & -\mathbf{1} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{Q}_i \\ \mathbf{Q}_j \end{bmatrix}. \tag{5}$$

The combination of acceleration and interface force of an interface point is referred to as the state vector of that interface point. In this formulation, the state vector of j is related to the state vector of i by a so-called transfer matrix and an additional vector due to elastic and quadratic velocity terms. With this, it is possible to eliminate all intermediate bodies from a multibody chain by successive multiplications of transfer matrices and to obtain a reduced equation of motion in terms of the first and last states only.

An advantage of this transfer matrix method is that these transfer matrices can be determined without any a priori knowledge of the boundary conditions of the entire structure, i.e. changing the boundary conditions on the first and last interface points does not influence the transfer matrices. As a consequence, no loop-closure constraints are required for closed-loop systems. Using the transfer matrix method becomes increasingly beneficial for systems that consists of a chain of identical bodies. In this case, the reduction of the full size equation of motion consists of multiplications of transfer matrices that are all the same, whereas when using techniques based on condensation and elimination of bodies, the mass matrix update still needs to be executed one body at a time. However, since this mass matrix update can be given the clear physical interpretation presented in this work, it is still an attractive alternative to the more mathematical transfer matrix method.

References

[1] T.M. Wasfy, A.K. Noor. Computational strategies for flexible multibody systems. *Applied Mechanics Reviews,* 56 (6), pages 553-613, 2003.

[2] M.H.M. Ellenbroek. On the fast simulation of flexible space structures. PhD Thesis, University of Twente, Enschede, The Netherlands, 1994.

[3] M. Géradin, D.J. Rixen. *Mechanical Vibrations.* Third Edition. Wiley, Chichester, United Kindom, 2015.