Implementation of Shear Deformable Thin-Walled Beam Element for Flexible Multibody Dynamics

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Abstract

This paper presents a geometrically nonlinear beam finite element that captures non-uniform torsion and flexural-torsional coupling of shear deformable thin-walled beams with an open unsymmetrical cross-section. The beam model is based on the generalized strain beam formulation. In this formulation, a set of independent deformation modes is defined as generalized strains in a co-rotational frame which are related to dual stress resultants using Timoshenko-Reissner's beam formulation. Cross-sectional warping is accounted for by a single warping function. Geometric nonlinearities are accounted for by additional second order terms in the expressions for the deformation modes. Some benchmark examples illustrate the accuracy of the new beam element.

Keywords: beam element, deformation modes, shear deformable, thin-walled beam, warping, geometrically nonlinear.

1. Introduction

Thin-walled members found in multibody systems are often modelled as thin-walled beams. Beam models are computationally efficient and convenient for parametric analysis and shape optimization of complex multibody geometries such as large stroke flexure hinges and compliant mechanisms [31]. Due to their geometric characteristics thin-walled beams with open cross-sections exhibit complex structural behaviour including cross-sectional warping and lateral-torsional buckling. An analysis of these phenomena must include, at least the warping of the beam cross-section and a second order approximation of the beam strain-displacement expressions for determining the buckling loads. Furthermore, the effect of shear deformation can gain importance in the buckling behaviour of thin-walled beams, especially when materials with relatively low shear modulus are used.

The majority of beam models used in flexible multibody analysis are based on geometrically exact theories such as those proposed by Reissner [34] and used as a basis for a computational approach by Simo and Vu-Quoc [35, 36]. These models are frame independent and capable of accounting for finite rotations and for arbitrarily large deformations including transverse shear and torsion warping of the cross-section; however the applied constitutive laws are only valid for small strains. Several authors developed nonlinear geometrically exact beam finite elements incorporating torsion warping effects [37, 11, 9] and constitutive equations for composite material [13, 39] to mention just a few. Since the dominant nonlinearity of flexible multibody systems is due to large rigid-body motions, the co-rotational formulation where the rigid-body motions are separated from deformations (e.g. see references ([3], [6], [1]) could be attractive for modelling of flexible multibody systems. However, when modelling flexible multibody systems with elastic and rigid bodies, conventional co-rotational formulations treat rigid bodies in the same way as elastic bodies with large stiffness. Thus they are not able of to model rigid-body dynamics exactly, yielding a less efficient formulation in terms of computational time and eliminating high frequency modes of deformation. For this reason conventional co-rotational formulations are still rarely used in flexible multibody dynamic analyses.

Co-rotational beam formulations bear much resemblance to the generalized strain formulation proposed by Besseling [4]. This formulation refers to the use of deformation modes, which define elastic deformations as well as implicitly rigid-body motion of the element and serves as the basis for the development of a finite element based formulation for rigid–flexible multibody system analysis [16, 18]. For each element, a fixed number of independent deformation modes is defined which are invariant under arbitrary rigid-body motions of the element. The deformation modes are characterized by generalized strains, which are expressed as analytical functions of the nodal coordinates referred to the global coordinate system. The deformation modes include the specification of rigid-body motions as displacements and rotations for which the deformations are zero, van der Werff [44]. Flexible elements are modelled by allowing non-zero deformations. If the deformations remain sufficiently small, they can be described in a single co-rotational frame and related to the dual stress-resultants using existing linear beam models at different levels of sophistication ranging from elementary beam theory to relatively advanced formulations for shear deformable and composite thin-walled beams with an open unsymmetrical cross-section [20, 38, 29, 30]. Discrete interpolation of finite rotations is avoided, leading to an intrinsic objective description. Furthermore the nonlinear strain-displacement relationships can be approximated

through its Taylor series expansion without noticeable inaccuracy. This formulation combines the advantages of the co-rotational formulation with the consistency of the inertial frame approach, viz derivation of the inertia forces in terms of absolute nodal velocities and accelerations. Recently some papers [32, 24] have been published which are based on a similar formulation.

In this paper, the generalized strain beam formulation, presented in [18] is extended to thin-walled beams with an open unsymmetrical cross-section. A first- and second order stiffness formulation is derived. The first order stiffness formulation includes the derivation of the element stiffness matrix for a thin-walled beam element with distinct shear centre and centroid. The kinematics of the cross-sectional deformations is based on Timoshenko's bending theory and Reissner's torsion theory [33]. The stiffness matrix is derived by interpolating both flexure displacements and twisting rotations by means of modified Hermitian polynomials [29], using parameters accounting for the influence of shear deformations. Coupling effects between shear deformations due to the shear forces and the non-uniform torsion in two orthogonal principal planes are included using a procedure outlined in Kollar and Springer [21, 22] and Kim at al. [20]. The second order stiffness formulation deals with the modelling of geometric nonlinearities arising from change in configuration involving finite deflections and pre and post-buckling. These geometric nonlinearities are accounted for by additional second order terms in the expressions for the deformation modes. For two-dimensional beams, Jennings [14] and Visser and Besseling [41] derived a second order approximation for the axial shortening due to bending. Meijaard [27] used a third order Taylor expansion of the strain energy to derive a set of modified deformation modes which include additional quadratic terms that account for the geometrically nonlinear couplings among bending and torsion of three dimensional beams. However the derivation of these second order terms is not easy for beams with unsymmetrical cross-section since the elastic rotations are defined about different points [19]. Bending rotations are defined about the (principal) axes through the centroid while twisting rotations are defined about the shear centre axis. In order to circumvent these difficulties a new approach is proposed in this work which consists of two steps:

1. The coupling among bending and torsion due to a non-coinciding centroid and shear centre is incorporated at the level of the first order stiffness formulation using a transformation matrix which accounts for the eccentric location of the shear centre axis with respect to the line of centroids.

2. On the basis of the elastic line (line of centroids) a nonlinear displacement field is defined, in which each material point is associated with a cross-section, whose orientation is specified by an attendant elastic rotation matrix resulting from a 3-2-1 rotation sequence for three modified Euler angles. A Taylor series expansion is used to expand the nonlinear curvature and strain-displacement equations into a polynomial form of second order. The second order displacement and rotation fields are interpolated in the same way as in the linear case by means of modified Hermitian polynomials. Integrating the interpolated second order curvature and strain-displacement equations over the length of the beam using the second moment-area theorem [10], yields a set of modified deformation modes which includes additional quadratic terms that account for the geometrically nonlinear couplings such as extension-torsion, bending-torsion and bending-extension.

The outline of the paper is as follows: The basic assumptions and the deformation modes for the thin-walled beam element with the dual stress resultants and the equilibrium equations are presented in Section 2. First and second order stiffness formulations are presented in Section 3. The (linearized) equations of motion for the second order beam element are derived in Section 4. Finally numerical examples illustrating the performance of the present beam element are presented in Section 5.

2. Generalized Strain Beam Formulation

We consider a spatial beam element based on the generalized strain beam formulation. In the original description developed for stability and post-buckling analysis of structures [4], Euler angles were used to parameterize global nodal rotations. Van der Werff and Jonker [45] introduced a description including Euler parameters which is more appropriate for computations in multibody system codes and made possible an implementation in the program SPACAR [15].

2.1. Description of Thin-Walled Beam Model

The kinematic model is based on the following assumptions:

1. The beam is prismatic and slender, i.e. the dimensions of the cross-sections are small with respect to the beam length.

2. The cross-section remains plane and keeps its shape, but is subjected to an additional warping displacement perpendicular to the displaced plane.

Figure 1 shows a thin-walled beam element which is represented by an elastic line which coincides with the centroid of the cross-sections of the beam. The configuration of the element is described by the position vectors r^p and r^q of
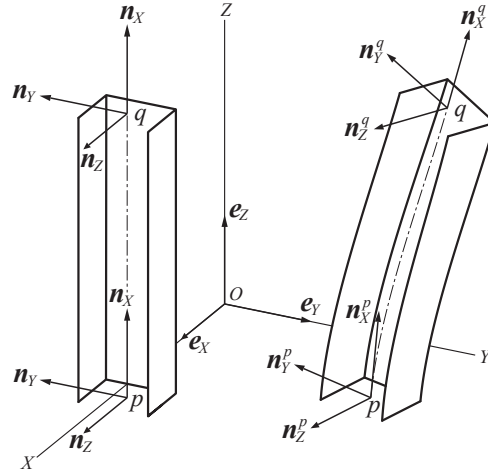
Figure 1: spatial beam element, reference and deformed state.

the centroid at the end (section) nodes p and q respectively, in the global inertial coordinate system $(OXYZ)$. The orientations are described by orthogonal triads of unit vectors $(\boldsymbol{n}_X^p, \boldsymbol{n}_Y^p, \boldsymbol{n}_Z^p)$ and $(\boldsymbol{n}_X^q, \boldsymbol{n}_Y^q, \boldsymbol{n}_Z^q)$ which are rigidly attached to the nodes. In the undeflected reference configuration the triads coincide and are given by $(\boldsymbol{n}_X, \boldsymbol{n}_Y, \boldsymbol{n}_Z)$; the vector $\boldsymbol{n}_X$ points in the direction of the axis pq and $\boldsymbol{n}_Y$ and $\boldsymbol{n}_Z$ are in the directions of the principal axes of the cross-section. The rotations of the triads are described by rotation matrices $\boldsymbol{R}^p$ and $\boldsymbol{R}^q$, so

$$
\begin{aligned}
\boldsymbol{n}_X^p &= \boldsymbol{R}^p \boldsymbol{n}_X, & \boldsymbol{n}_X^q &= \boldsymbol{R}^q \boldsymbol{n}_X, \\
\boldsymbol{n}_Y^p &= \boldsymbol{R}^p \boldsymbol{n}_Y, & \boldsymbol{n}_Y^q &= \boldsymbol{R}^q \boldsymbol{n}_Y, \\
\boldsymbol{n}_Z^p &= \boldsymbol{R}^p \boldsymbol{n}_Z, & \boldsymbol{n}_Z^q &= \boldsymbol{R}^q \boldsymbol{n}_Z.
\end{aligned}
\tag{1}
$$

If the beam is rigid then the rotation matrices $\boldsymbol{R}^p$ and $\boldsymbol{R}^q$ are identical and in the initial undeformed state they are equal to the identity matrix. In the present description Euler parameters are used to parametrize the rotation matrices, but the formulation can easily be transformed if a different choice is made. If the Euler parameters are denoted by $(\lambda_0, \boldsymbol{\lambda})$ with the scalar part λ_0 and the vector part $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)^{\mathrm{T}}$, a rotation matrix can be expressed as [23]

$$
\boldsymbol{R}(\lambda_0, \boldsymbol{\lambda}) = (\lambda_0^2 - \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{\lambda})\mathbf{I} + 2\lambda_0 \tilde{\boldsymbol{\lambda}} + 2\boldsymbol{\lambda}\boldsymbol{\lambda}^{\mathrm{T}},
\tag{2}
$$

where $\mathbf{I}$ is a 3×3 unitary matrix. Furthermore use has been made of the tilde notation to denote the skew symmetric matrix associated with a vector,

$$
\tilde{\boldsymbol{\lambda}} = \begin{bmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{bmatrix}.
\tag{3}
$$

By definition, the Euler parameters must satisfy the constraint equation

$$
\lambda_0^2 + \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{\lambda} = 1.
\tag{4}
$$

2.2. Deformation Modes

The nodal coordinates of the spatial beam element are the six Cartesian coordinates representing the position vectors $\boldsymbol{r}^p$ and $\boldsymbol{r}^q$, two sets of four Euler parameters $(\lambda_0^p, \boldsymbol{\lambda}^p)$ and $(\lambda_0^q, \boldsymbol{\lambda}^q)$ and two warping coordinates α^p and α^q, describing the change of twist at the nodes p and q. If a redundant parametrization for the rotations is used, only three of them are independent. Therefore, as the beam has six degrees of freedom as a rigid-body, eight independent deformation modes,

specified by a set of generalized strains ε_i, can be expressed as analytical functions of the nodal coordinates X_k, referred to the fixed global coordinate system

$$\varepsilon_i = D_i(X_k); \ i = 1, \dots, 8; \ k = 1, \dots, 16.$$

or $\hspace{12cm}$ (5)

$$\boldsymbol{\varepsilon} = \boldsymbol{D}(\boldsymbol{X}),$$

where

$$\boldsymbol{X} = \left[\boldsymbol{r}^{p\text{T}}, (\lambda_0, \boldsymbol{\lambda}^{p\text{T}}), \alpha^p, \boldsymbol{r}^{q\text{T}}, (\lambda_0, \boldsymbol{\lambda}^{q\text{T}}), \alpha^q\right]^\text{T}. \tag{6}$$

A suitable choice for the deformation functions is [18]

$$\varepsilon_1 = \ l - l_0, \hspace{3cm} \text{(axial} - \text{elongation)} \tag{7a}$$

$$\varepsilon_2 = \ l_0\left(\boldsymbol{n}_Z^{p\text{T}}\boldsymbol{n}_Y^q - \boldsymbol{n}_Y^{p\text{T}}\boldsymbol{n}_Z^q\right)/2, \hspace{1.2cm} \text{(torsion)} \tag{7b}$$

$$\left.\begin{aligned} \varepsilon_3 &= -l_0\,\boldsymbol{n}_l^\text{T}\boldsymbol{n}_Z^p, & \varepsilon_4 &= \ l_0\,\boldsymbol{n}_l^\text{T}\boldsymbol{n}_Z^q, \\ \varepsilon_5 &= \ l_0\,\boldsymbol{n}_l^\text{T}\boldsymbol{n}_Y^p, & \varepsilon_6 &= -l_0\,\boldsymbol{n}_l^\text{T}\boldsymbol{n}_Y^q, \end{aligned}\right\} \text{(bending)} \tag{7c}$$

$$\varepsilon_7 = \ l_0^2\,\alpha^p, \hspace{1cm} \varepsilon_8 = \ l_0^2\,\alpha^q, \hspace{0.8cm} \text{(warping)} \tag{7d}$$

where l_0 is the reference length of the element, $\boldsymbol{l} = \boldsymbol{r}^q - \boldsymbol{r}^p$ is the vector from node p to node q, $l = \|\boldsymbol{l}\|$ is the distance between the nodes and $\boldsymbol{n}_l = (\boldsymbol{r}^q - \boldsymbol{r}^p)/l$ is the unit vector directed from node p to node q. The deformation modes are frame-invariant, which means that they do not change if the element undergoes an arbitrary Euclidean displacement. The first deformation mode ε_1 describes the axial elongation, the second mode ε_2 describes the torsional deformation, $\varepsilon_3 - \varepsilon_6$ represent bending modes in the xz and xy-planes respectively and $\varepsilon_7, \varepsilon_8$ represent the warping modes. The bending modes are defined in terms of inner products of unit vector $\boldsymbol{n}_l$ and the global unit vectors $\boldsymbol{n}_Y$, $\boldsymbol{n}_Z$ attached at the element nodes. Note that $\varepsilon_3 - \varepsilon_6$ do not change if the beam undergoes an axial elongation with fixed orientations of the nodes and fixed direction $\boldsymbol{n}_l$. The physical dimension of all generalized strains is length. The generalized strains can be constrained by imposing conditions on ε_i; in particular, the element can be made rigid by imposing eight conditions $\varepsilon_i = 0 \ (i = 1, ..., 8)$. The bending modes are visualized in Figure 2.
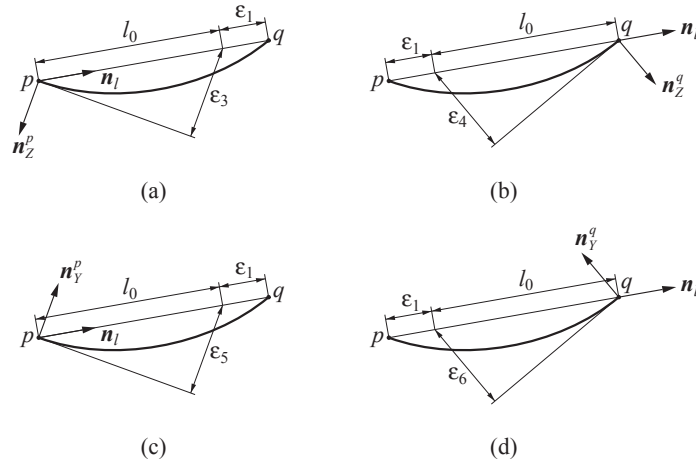

(a) $\hspace{6cm}$ (b)

(c) $\hspace{6cm}$ (d)

Figure 2: Visualization of bending modes: $\varepsilon_3, \varepsilon_4$ and $\varepsilon_5, \varepsilon_6$.

2.3. Generalized Stress Resultants and Equilibrium Equations

Let us consider an equilibrium force system defined by the forces $\boldsymbol{F}^p$, $\boldsymbol{F}^q$, the moments $\boldsymbol{T}^p$, $\boldsymbol{T}^q$ and the bi-moments B^p, B^q applied at the nodal points p and q of the free beam element, which are placed in a vector of element nodal forces

$$\boldsymbol{F} = \left[\boldsymbol{F}^{p\text{T}}, \boldsymbol{T}^{p\text{T}}, B^p, \boldsymbol{F}^{q\text{T}}, \boldsymbol{T}^{q\text{T}}, B^q\right]^\text{T}. \tag{8}$$

The bi-moments have the physical dimension of moment multiplied by length [42]. Furthermore, we consider virtual

variations of the nodal positions δr^p, δr^q, rotations $\delta\varphi^p$, $\delta\varphi^q$ and warping coordinates $\delta\alpha^p$, $\delta\alpha^q$ which are collected in a vector of virtual nodal displacements

$$\delta u = \left[\begin{array}{c} \delta u^p \\ \delta u^q \end{array} \right] = [\delta r^{p\mathrm{T}}, \delta\varphi^{p\mathrm{T}}, \delta\alpha^p, \delta r^{q\mathrm{T}}, \delta\varphi^{q\mathrm{T}}, \delta\alpha^q]^{\mathrm{T}}. \tag{9}$$

The variations $\delta\varphi^p$ and $\delta\varphi^q$ define infinitesimally small rotations from a general configuration with components along the axes of the inertial coordinate system. These are related to the variations of the Euler parameters by a 3×4 transformation matrix $\Lambda(\lambda_0, \lambda)$, as

$$\delta\varphi = 2\Lambda \left[\begin{array}{c} \delta\lambda_0 \\ \delta\lambda \end{array} \right] = 2[-\lambda, \ \lambda_0 I + \tilde{\lambda}] \left[\begin{array}{c} \delta\lambda_0 \\ \delta\lambda \end{array} \right]. \tag{10}$$

Because $\delta R^p = \delta\tilde{\varphi}^p R^p$ and $\delta R^q = \delta\tilde{\varphi}^q R^q$, the variations of the unit vectors in Eq.(1) can be expressed as

$$\begin{array}{ll}
\delta n_X^p = \delta\varphi^p \times n_X^p, & \delta n_X^q = \delta\varphi^q \times n_X^q, \\
\delta n_Y^p = \delta\varphi^p \times n_Y^p, & \delta n_Y^q = \delta\varphi^q \times n_Y^q, \\
\delta n_Z^p = \delta\varphi^p \times n_Z^p, & \delta n_Z^q = \delta\varphi^q \times n_Z^q.
\end{array} \tag{11}$$

The virtual variations of the generalized strains $\delta\varepsilon$ are related to the virtual displacements δu by the relationship

$$\delta\varepsilon = D_{,u}\,\delta u, \tag{12}$$

where the components of matrix $D_{,u}$ are derived using relations (11). For the generalized strains defined in Eq.(7) we obtain

$$
\begin{array}{ccccccc}
(\delta r^{p\mathrm{T}}) & (\delta\varphi^{p\mathrm{T}}) & (\delta\alpha^p) & (\delta r^{q\mathrm{T}}) & (\delta\varphi^{q\mathrm{T}}) & (\delta\alpha^q)
\end{array}
$$

$$
D_{,u} = \begin{bmatrix}
-n_l^{\mathrm{T}} & 0^{\mathrm{T}} & 0 & n_l^{\mathrm{T}} & 0^{\mathrm{T}} & 0 \\[6pt]
0^{\mathrm{T}} & \frac{l_0}{2}\left[n_Z^p \times n_Y^q - n_Y^p \times n_Z^q\right]^{\mathrm{T}} & 0 & 0^{\mathrm{T}} & \frac{l_0}{2}\left[n_Y^q \times n_Z^p - n_Z^q \times n_Y^p\right]^{\mathrm{T}} & 0 \\[6pt]
\frac{l_0}{l}\left[n_Z^p - (n_l^{\mathrm{T}} n_Z^p)n_l\right]^{\mathrm{T}} & -l_0(n_Z^p \times n_l)^{\mathrm{T}} & 0 & -\frac{l_0}{l}\left[n_Z^p - (n_l^{\mathrm{T}} n_Z^p)n_l\right]^{\mathrm{T}} & 0^{\mathrm{T}} & 0 \\[6pt]
-\frac{l_0}{l}\left[n_Z^q - (n_l^{\mathrm{T}} n_Z^q)n_l\right]^{\mathrm{T}} & 0^{\mathrm{T}} & 0 & \frac{l_0}{l}\left[n_Z^q - (n_l^{\mathrm{T}} n_Z^q)n_l\right]^{\mathrm{T}} & l_0(n_Z^q \times n_l)^{\mathrm{T}} & 0 \\[6pt]
-\frac{l_0}{l}\left[n_Y^p - (n_l^{\mathrm{T}} n_Y^p)n_l\right]^{\mathrm{T}} & l_0(n_Y^p \times n_l)^{\mathrm{T}} & 0 & \frac{l_0}{l}\left[n_Y^p - (n_l^{\mathrm{T}} n_Y^p)n_l\right]^{\mathrm{T}} & 0^{\mathrm{T}} & 0 \\[6pt]
\frac{l_0}{l}\left[n_Y^q - (n_l^{\mathrm{T}} n_Y^q)n_l\right]^{\mathrm{T}} & 0^{\mathrm{T}} & 0 & -\frac{l_0}{l}\left[n_Y^q - (n_l^{\mathrm{T}} n_Y^q)n_l\right]^{\mathrm{T}} & -l_0(n_Y^q \times n_l)^{\mathrm{T}} & 0 \\[6pt]
0 & 0 & l_0^2 & 0 & 0 & 0 \\[6pt]
0 & 0 & 0 & 0 & 0 & l_0^2
\end{bmatrix} \tag{13}
$$

Generalized stress resultants σ_i are defined to be energetically dual to the generalized strains ε_i, that is, the virtual work supplied by the generalized stress resultants is $-\sigma^{\mathrm{T}}\delta\varepsilon$. According to the principle of virtual work, the element will be in a state of equilibrium if

$$F^{\mathrm{T}}\delta u - \sigma^{\mathrm{T}}\delta\varepsilon = 0, \tag{14}$$

holds for all δu and $\delta\varepsilon$ depending on δu by the relationship (12). Substitution Eq.(12) in Eq.(14) yields with the transpose matrix $D_{,u}^{\mathrm{T}}$

$$D_{,u}^{\mathrm{T}}\sigma = F. \tag{15}$$

These are the equilibrium equations formulated in the deformed configuration of the beam element. From these equations the equilibrium nodal force systems, expressed in terms of the generalized stress-resultant components σ_i, are calculated and visualized in Fig.3(b)-(g). In all cases, perfect equilibrium is obtained for arbitrary large rigid-body displacements of the element. This is a direct consequence of the invariance of the generalized strains under rigid-body displacements, as the generalized stress resultants have no contribution to the virtual work in Eq.(14) for virtual rigid-body displacements, so rigid-body displacements leaving the deformations unchanged can be described.

In order to identify the generalized stress resultant components of a beam element we consider the undeformed configuration of the beam element, in which the unit vectors (n_X^p, n_Y^p, n_Z^p) and (n_X^q, n_Y^q, n_Z^q) coincide with the global coordinate axes X, Y, Z as shown in Fig.3(a). Subsequently, the beam is loaded by an equilibrium force system F defined in Eq.(8). The generalized stress resultants σ_i corresponding to the nodal point forces and moments can then be recognized as
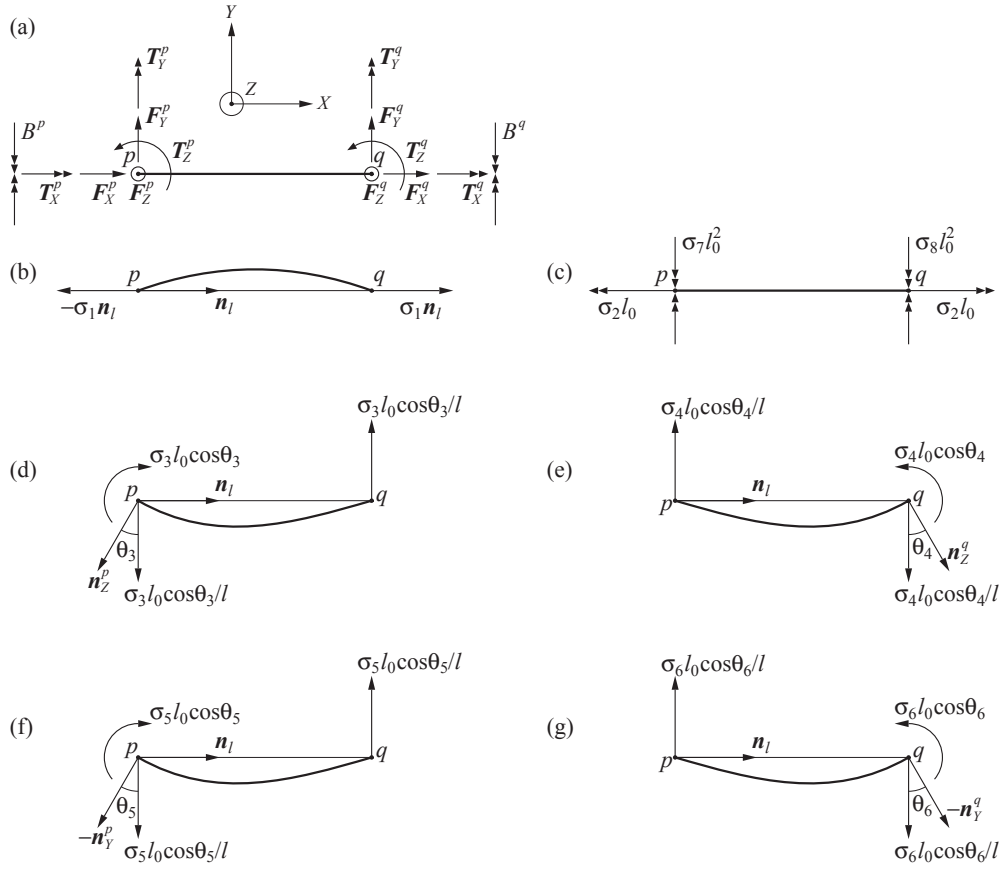
Figure 3: (a) Equilibrium nodal force system $\boldsymbol{F}$ of nodal forces, moments and bi-moments defined by Eq.(8) and (b)-(g) generalized generalized stress resultants $\sigma_1 - \sigma_8$ corresponding to Eq.(15), where $\cos\theta_i = \arcsin(\varepsilon_i/l)$.

$$
\begin{aligned}
\sigma_1 &= -F_x^p = F_x^q, && \text{(normal force)} \\
l_0\sigma_2 &= -T_x^p = T_x^q, && \text{(twisting moment)} \\
l_0\,\sigma_3 &= -T_y^p, & l_0\,\sigma_4 &= T_y^q, \\
l_0\,\sigma_5 &= -T_z^p, & l_0\,\sigma_6 &= T_z^q,
\end{aligned}
\Biggr\} \text{(bending moments)} \tag{16}
$$
$$
l_0^2\,\sigma_7 = B^p, \qquad l_0^2\,\sigma_8 = B^q. \quad \text{(bi-moments)}
$$

3. Stiffness Properties

Flexible elements are modelled by allowing non-zero deformations and specifying constitutive equations relating generalized strains and dual stress-resultants. In the present finite-element formulation, the generalized strains are described in local frames which move with the element as it is translated and rotated. If the deformations remain sufficiently small, they can be described in a single co-rotational frame and related to the dual stress-resultants using a linear thin-walled beam model. In this section a first and second order stiffness formulation is derived. The first order stiffness formulation includes the derivation of the stiffness matrix based on Timoshenko-Reissner's thin-walled beam model with distinct shear centre and centroid. The second order stiffness formulation deals with the derivation of modified definitions for the deformation modes in which geometric nonlinearities arising from change in configuration are accounted for by additional quadratic terms.
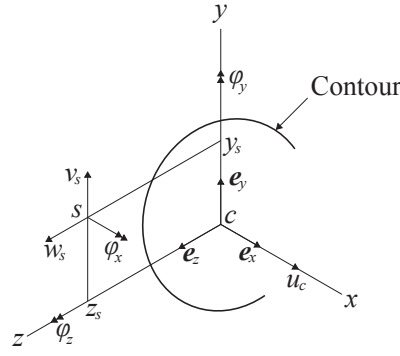
Figure 4: Local coordinate system and displacements.

3.1. First Order Analytical Formulation

We consider a typical cross-section of a thin-walled beam in a Cartesian co-rotational frame (x,y,z) with base vectors, $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_z$ pointing along the respective positive directions of the coordinate axes as shown in Fig. 4. A plane normal to the x-axis, cuts the middle surface in a line called contour of the cross-section. The origin coincides with centroid c and the x-axis coincides with the centroid of the cross-sections. The y-axis and z-axis are the principal axes; y_s and z_s are the coordinates of the shear centre s; u_c, v_s and w_s are rigid-body translations of the cross-section in the x-direction at the centroid c and the y and z-directions at the shear centre s respectively; φ_x is a rigid-body rotation about the shear centre axis and φ_y, φ_z are rigid-body rotations about the principal axes y and z, respectively. According to the second assumption of in-plane rigid cross-sections, see Section 2.1, the beam is deformed in a way such that the shape of the cross-section remains plane and keeps its shape. Under this assumption the cross-section translates and rotates as a rigid-body, i.e. three translations, two bending rotations and a twisting rotation are required to describe the movement of the cross-section. An additional assumption is made that the deformations remain sufficiently small, so that Timoshenko-Reissner beam kinematics is applicable. Let $\boldsymbol{r}_0^P(x,y,z)$ denote the position vector of an arbitrary point P on the cross-section, in the initial undeformed configuration, i.e.

$$\boldsymbol{r}_0^P = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z. \tag{17}$$

Reissner's beam theory considers the warping displacements due to torsion and Timoshenko's beam theory includes flexural shear deformation. Hence, the position of point P in the deformed configuration becomes

$$\boldsymbol{r}^P = (x+u_c)\boldsymbol{e}_x + v_s\boldsymbol{e}_y + w_s\boldsymbol{e}_z + (y\mathbf{R} - y_s\mathbf{A})\boldsymbol{e}_y + (z\mathbf{R} - z_s\mathbf{A})\boldsymbol{e}_z + \omega\,\alpha\,\boldsymbol{e}_x, \tag{18}$$

where $\mathbf{R}(x)$ and $\mathbf{A}(x)$ are small rotation matrices which are defined by [40]

$$\mathbf{R} = \begin{bmatrix} 1 & -\varphi_z & \varphi_y \\ \varphi_z & 1 & -\varphi_x \\ -\varphi_y & \varphi_x & 1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\varphi_x \\ 0 & \varphi_x & 0 \end{bmatrix}, \tag{19}$$

$\omega(y,z)$ is a normalized warping function with respect to the shear centre and $\alpha(x)$ is the change of twist per unit of length taken as an additional degree of freedom [12]. Note that $\alpha(x)$ is not equal to the derivative $\varphi_{x,x}$ owing to the inclusion of shear deformation in warping. Since the y and z-axes are principal axes of inertia of the cross-section, we have

$$\int_A \omega\,dA = \int_A \omega y\,dA = \int_A \omega z\,dA = 0. \tag{20}$$

Substitution of the rotation matrices $\mathbf{R}$ and $\mathbf{A}$ into Eq.(18), the local displacement vector $\boldsymbol{u} = (u,v,w)^T = \boldsymbol{r}^P - \boldsymbol{r}_0^P$ can be written as [7]

$$\begin{aligned} u &= u_c - y\varphi_z + z\varphi_y + \omega\alpha, \\ v &= v_s - (z - z_s)\varphi_x, \\ w &= w_s + (y - y_s)\varphi_x. \end{aligned} \tag{21}$$

From Eq.(21) it can be observed that the axial displacement u is expressed as the sum of the axial displacement of the centroid, the cross-section displacement due to Timoshenko's beam theory and by the axial warping due to Reissner's thin-walled beam theory. The transverse displacements v, w are given by the transverse displacements of the shear centre
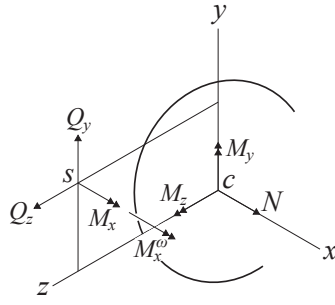
Figure 5: Stress-resultants.

and by the angle of twist φ_x of the whole cross-section about the shear centre. The non-vanishing strain components are

$$\varepsilon_{xx} = u_{,x} = u_{c,x} - y\,\varphi_{z,x} + z\,\varphi_{y,x} + \omega\,\alpha_{,x}\,,$$

$$\gamma_{xy} = v_{,x} + u_{,y} = v_{s,x} - \varphi_z - (z - z_s)\varphi_{x,x} + \omega_{,y}\,\alpha\,, \tag{22}$$

$$\gamma_{xz} = w_{,x} + u_{,z} = w_{s,x} + \varphi_y + (y - y_s)\varphi_{x,x} + \omega_{,z}\,\alpha\,.$$

In these relations, a comma followed by an index denotes differentiation with respect to the corresponding variable. Based on equilibrium considerations the stress resultants on a cross-section are, see Fig. 5

$$N = \int_A \sigma_x \, dA, \qquad M_y = \int_A \sigma_x z \, dA, \qquad M_z = -\int_A \sigma_x y \, dA, \qquad B = \int_A \sigma_x \omega \, dA, \tag{23a}$$

$$Q_y = \int_A \tau_{yx} \, dA, \qquad Q_z = \int_A \tau_{zx} \, dA, \qquad M_x^\omega = \int_A \left(\tau_{zx} \frac{\partial \omega}{\partial z} + \tau_{yx} \frac{\partial \omega}{\partial y} \right) dA, \tag{23b}$$

$$M_x = \int_A \left[\tau_{zx}(y - y_s) - \tau_{yx}(z - z_s) \right] dA, \qquad T_x = M_x + M_x^\omega, \tag{23c}$$

where A represents the cross-sectional area, N, M_y, M_z and B are the axial force, bending moments about the y-axis and z-axis and the bi-moment; Q_y, Q_z and M_x^ω are the shear force components acting along the principal axes at the shear centre and the twisting moment due to non-uniform torsion (warping torque) and finally M_x, T_x are De Saint Venant twisting moment and the total twisting moment acting on the cross-section, respectively. For a linear elastic and isotropic material with elastic modulus E and and shear modulus G, the components of the stresses σ_x, τ_{xy} and τ_{xz} can be expressed as follows

$$\sigma_x = E\varepsilon_{xx}, \qquad \tau_{xy} = G\gamma_{xy}, \qquad \tau_{xz} = G\gamma_{xz}. \tag{24}$$

Substituting Eqs.(22) and(24) into expressions (23a) and making use of Eq.(20) we obtain

$$\begin{bmatrix} N \\ M_y \\ M_z \\ B \end{bmatrix} = E \begin{bmatrix} A & & & \\ & I_y & & \\ & & I_z & \\ & & & I_\omega \end{bmatrix} \begin{bmatrix} u_{c,x} \\ \varphi_{y,x} \\ \varphi_{z,x} \\ \alpha_{,x} \end{bmatrix} \tag{25a}$$

and

$$M_x = GI_t\,\varphi_{x,x} \tag{25b}$$

where

$$I_y = \int_A z^2 \, dA, \qquad I_z = \int_A y^2 \, dA, \qquad I_\omega = \int_A \omega^2 \, dA, \tag{26a}$$

are the second area and sectorial moments respectively. The torsional constant I_t is defined as [25]

$$I_t = \int_A \left[(z - z_s + \frac{\partial \omega}{\partial y})^2 + (y - y_s - \frac{\partial \omega}{\partial z})^2 \right] dA\,. \tag{26b}$$

Substituting Eqs.(22) and (24) into the expressions (23b), the following equations relating shear forces Q_y, Q_z and twisting moment M_x^ω with the shear deformations are obtained [21, 20]

$$
\begin{bmatrix} Q_y \\ Q_z \\ M_x^\omega \end{bmatrix} = G \begin{bmatrix} D_y & D_{yz} & D_{y\omega} \\ & D_z & D_{z\omega} \\ \text{sym} & & D_\omega \end{bmatrix} \begin{bmatrix} v_{s,x} - \varphi_z \\ w_{s,x} + \varphi_y \\ \varphi_{x,x} + \alpha \end{bmatrix}. \tag{27}
$$

The off-diagonal shear rigidities represent the coupling effects between shear resultants into the principal planes (D_{yz}) and between shear and non-uniform torsion $(D_{y\omega}, D_{z\omega})$. Thus in a shear-deformable beam with unsymmetrical cross-section, a shear force acting along the principal axis passing through the shear centre generates an orthogonal shear centre displacement component and a twisting rotation. Alternatively, a twisting moment will produce shear centre displacements into the principal planes. The off-diagonal terms vanish for double-symmetric cross-sections. A pre-processing procedure [22, 20] is applied to evaluate the shear rigidities in Eq.(27) for simple cross-sectional shapes. Finally, the linear elastic strain energy for a beam of length l can be written in the form

$$
W^e = \frac{1}{2} \int_0^l \left[N u_{c,x} + M_x \varphi_{x,x} + M_y \varphi_{y,x} + M_z \varphi_{z,x} + B \alpha_{,x} + Q_y(v_{s,x} - \varphi_z) + Q_z(w_{s,x} + \varphi_y) + M_x^\omega(\varphi_{x,x} + \alpha) \right] dx. \tag{28}
$$

Substituting Eqs.(25) and (27) gives

$$
\begin{aligned}
W^e \quad = \quad & \frac{1}{2} \int_0^l E \left[A u_{c,x}^2 + I_y \varphi_{y,x}^2 + I_z \varphi_{z,x}^2 + I_\omega \alpha_{,x}^2 \right] dx \\
& + \frac{1}{2} \int_0^l G \left[I_t \varphi_{x,x}^2 + D_y(v_{s,x} - \varphi_z)^2 + D_z(w_{s,x} + \varphi_y)^2 + D_\omega(\varphi_{x,x} + \alpha)^2 \right. \\
& \left. + 2 D_{yz}(v_{s,x} - \varphi_z)(w_{s,x} + \varphi_y) + 2 D_{y\omega}(v_{s,x} - \varphi_z)(\varphi_{x,x} + \alpha) + 2 D_{z\omega}(w_{s,x} + \varphi_y)(\varphi_{x,x} + \alpha) \right] dx.
\end{aligned} \tag{29}
$$

The last three terms refer to coupling between the two shear resultants and between each shear resultant and the non-uniform torsion component.

3.2. First Order Discrete Formulation

If the deformations remain sufficiently small the generalized strains $(\varepsilon_2, ..., \varepsilon_6)$ specified in Eqs.(7) can be described in a single co-rotational frame (x, y, z) with the origin at node p and base vectors $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_z$ pointing along the respective positive directions of the coordinate axes as shown in Fig. 6. The centroidal axis of the beam is taken as the x-axis which coincides with the line connecting the nodal points p and q and the y-axis and z-axis are principal axes of the cross-section. By replacing the global unit vectors $\boldsymbol{n}_Y^p$, $\boldsymbol{n}_Z^p$ and $\boldsymbol{n}_Y^q$, $\boldsymbol{n}_Z^q$ in Eqs.(7) by the respective local unit vectors $\boldsymbol{e}_y^p$, $\boldsymbol{e}_z^p$ and $\boldsymbol{e}_y^q$, $\boldsymbol{e}_z^q$ we obtain

$$
\begin{aligned}
\varepsilon_2 &= \ l_0 \left(\boldsymbol{e}_z^{pT} \boldsymbol{e}_y^q - \boldsymbol{e}_y^{pT} \boldsymbol{e}_z^q \right)/2, && \text{(torsion)} \\
\varepsilon_3 &= -l_0 \, \boldsymbol{e}_x^T \boldsymbol{e}_z^p, & \varepsilon_4 &= \ l_0 \, \boldsymbol{e}_x^T \boldsymbol{e}_z^q, & \Big\} \ \text{(bending)} \\
\varepsilon_5 &= \ l_0 \, \boldsymbol{e}_x^T \boldsymbol{e}_y^p, & \varepsilon_6 &= -l_0 \, \boldsymbol{e}_x^T \boldsymbol{e}_y^q,
\end{aligned} \tag{30}
$$
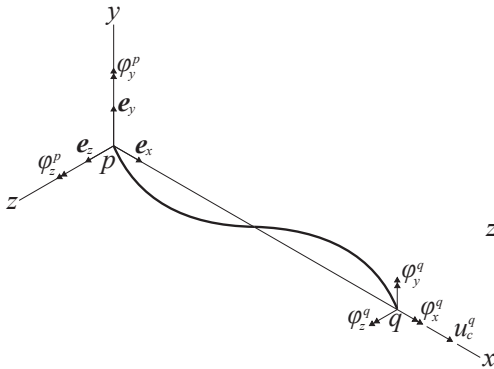
Figure 6: Co-rotational frame (x,y,z).   Figure 7: Bending deformations ε_3^s/l_0 and ε_4^s/l_0.
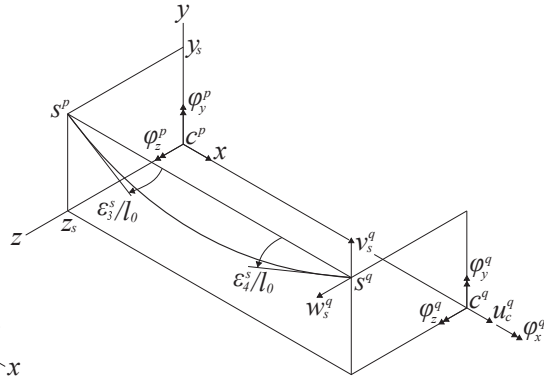
where

$$e_y^p = \mathbf{R}^p e_y, \qquad e_y^q = \mathbf{R}^q e_y,$$
$$e_z^p = \mathbf{R}^p e_z, \qquad e_z^q = \mathbf{R}^q e_z. \tag{31}$$

The rotation matrices $\mathbf{R}^p$ and $\mathbf{R}^q$ are obtained by evaluating the small rotation matrix $\mathbf{R}$ from Eq.(19) at the nodes p and q. In accordance with the definition of the co-rotational frame, the boundary conditions are

$$
\begin{aligned}
u_c(0) &= 0, & u_c(l_0) &= u_c^q, & \varphi_x(0) &= \varphi_x^p = 0, & \varphi_x(l_0) &= \varphi_x^q, \\
v_c(0) &= 0, & v_c(l_0) &= 0, & \varphi_y(0) &= \varphi_y^p, & \varphi_y(l_0) &= \varphi_y^q, \\
w_c(0) &= 0, & w_c(l_0) &= 0, & \varphi_z(0) &= \varphi_z^p, & \varphi_z(l_0) &= \varphi_z^q, \\
\alpha(0) &= \alpha^p, & \alpha(l_0) &= \alpha^q,
\end{aligned}
\tag{32}
$$

where α^p and α^q are the nodal warping coordinates in Eq.(6). Substitution of Eq.(31) into Eq.(30), using boundary conditions (32) and disregarding second and higher order terms yields the following first order approximations for the generalized strains.

$$\varepsilon_1 = u_c^q, \qquad\qquad \text{(axial elongation)} \tag{33a}$$

$$\varepsilon_2 = l_0 \varphi_x^q, \qquad\qquad \text{(torsion)} \tag{33b}$$

$$\left.\begin{aligned}
\varepsilon_3 &= -l_0 \varphi_y^p, & \varepsilon_4 &= l_0 \varphi_y^q, \\
\varepsilon_5 &= -l_0 \varphi_z^p, & \varepsilon_6 &= l_0 \varphi_z^q,
\end{aligned}\right\} \text{(bending)} \tag{33c}$$

$$\varepsilon_7 = l_0^2 \alpha^p, \qquad \varepsilon_8 = l_0^2 \alpha^q. \quad \text{(warping)} \tag{33d}$$

As seen in Eqs.(21), the transverse displacements are referred to the shear centre s, while the axial displacement is referred to the centroid of the cross-section. In order to derive a stiffness matrix for a beam element with an unsymmetrical cross-section, it is necessary that the bending deformations are defined with respect to the shear axis as is illustrated for ε_3^s/l_0 and ε_4^s/l_0 in x-z plane, see Fig. 7. Let v_s^q and w_s^q be the transverse displacements of the cross-section in the y and z-directions at the shear centre of the end-section q, respectively. Since $\varphi_x^p = 0$ and $v_s^p = w_s^p = 0$, we obtain

$$
\begin{aligned}
\varepsilon_3^s &= -w_s^q - l_0 \varphi_y^p, & \varepsilon_4^s &= w_s^q + l_0 \varphi_y^q, \\
\varepsilon_5^s &= v_s^q - l_0 \varphi_z^p, & \varepsilon_6^s &= -v_s^q + l_0 \varphi_z^q,
\end{aligned}
\tag{34}
$$

where

$$w_s^q = y_s \varphi_x^q, \qquad v_s^q = -z_s \varphi_x^q. \tag{35}$$

Substituting Eqs.(35) into (34) yields with Eq.(33b)

$$
\begin{aligned}
\varepsilon_3^s &= \varepsilon_3 - y_s\, \varepsilon_2/l_0, & \varepsilon_4^s &= \varepsilon_4 + y_s\, \varepsilon_2/l_0, \\
\varepsilon_5^s &= \varepsilon_5 - z_s\, \varepsilon_2/l_0, & \varepsilon_6^s &= \varepsilon_6 + z_s\, \varepsilon_2/l_0.
\end{aligned}
\tag{36}
$$

From Eqs.(36) it can be observed that the transformation of bending deformations from the centroid coordinate system to the system of parallel axes passing through the shear centre depends on the twist rotation angle $\varphi_x^q = \varepsilon_2/l_0$. The remaining deformations will not change by this transformation. Consequently, the transformation for the generalized strains can be expressed in the following matrix form

$$
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3^s \\
\varepsilon_4^s \\
\varepsilon_5^s \\
\varepsilon_6^s \\
\varepsilon_7 \\
\varepsilon_8
\end{bmatrix}
=
\begin{bmatrix}
1 & & & & & & & \\
& 1 & & & & & & \\
-y_s/l_0 & & 1 & & & & & \\
y_s/l_0 & & & 1 & & & & \\
-z_s/l_0 & & & & 1 & & & \\
z_s/l_0 & & & & & 1 & & \\
& & & & & & 1 & \\
& & & & & & & 1
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6 \\
\varepsilon_7 \\
\varepsilon_8
\end{bmatrix},
\tag{37a}
$$

or

$$\boldsymbol{\varepsilon}^s = \mathbf{E}\boldsymbol{\varepsilon}, \tag{37b}$$

where $\mathbf{E}$ is a transformation matrix which accounts for the eccentricity of the shear centre. The derivation of the stiffness matrix is based on assumed displacement fields. We employ a linear displacement field for the axial displacement and so-called modified Hermitian shape functions [29] for the twisting angle and bending displacements which allow to

take the influence of shear deformations into account. The displacement and rotation fields can be approximated by the following matrix form

$$
\begin{bmatrix} u_c(\xi) \\ l_0\varphi_x(\xi) \\ w_s(\xi) \\ l_0\varphi_y(\xi) \\ v_s(\xi) \\ l_0\varphi_z(\xi) \\ l_0\alpha(\xi) \end{bmatrix} = \begin{bmatrix} N_1 & & & & & & \\ & N_{13}^\omega & & & & l_0 N_{12}^\omega & -l_0 N_{14}^\omega \\ & & N_{12}^z & N_{14}^z & & & \\ & & N_{22}^z & N_{24}^z & & & \\ & & & & N_{12}^y & N_{14}^y & \\ & & & & N_{22}^y & N_{24}^y & \\ & N_{23}^\omega & & & & -l_0 N_{22}^\omega & l_0 N_{24}^\omega \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3^s \\ \varepsilon_4^s \\ \varepsilon_5^s \\ \varepsilon_6^s \\ \varepsilon_7 \\ \varepsilon_8 \end{bmatrix}.
$$
(38a)

The expressions for the interpolation functions are given by

$$
\begin{aligned}
N_1 &= \xi \\
N_{13}^i &= \xi^2(3-2\xi)+\Phi_i, & N_{23}^i &= 6\xi(1-\xi), \\
N_{12}^i &= \frac{1}{1+\Phi_i}\xi\left[(1-\xi)^2+\frac{\Phi_i}{2}(1-\xi)\right], & N_{14}^i &= \frac{1}{1+\Phi_i}\xi\left[-\xi^2+\xi+\frac{\Phi_i}{2}(1-\xi)\right], \\
N_{22}^i &= \frac{1}{1+\Phi_i}\left[-3\xi^2+4\xi-1+\Phi_i(\xi-1)\right], & N_{24}^i &= \frac{1}{1+\Phi_i}\xi\left[3\xi-2+\Phi_i\right].
\end{aligned}
$$
(38b)

In these expressions $\xi = x/l_0$ and coefficients $\Phi_i(i=y,z,\omega)$ take the expressions

$$
\Phi_y = \frac{12EI_z}{GD_y l^2} \quad \text{(bending in x-y plane)},
$$

$$
\Phi_z = \frac{12EI_y}{GD_z l^2} \quad \text{(bending in x-z plane)},
$$
(38c)

$$
\Phi_\omega = \frac{12EI_\omega}{GD_\omega l^2} \quad \text{(torsion/warping)}.
$$

Note that the interpolation functions N_{14}^ω and N_{22}^ω for torsion/warping in Eq.(38a) take minus signs. By substituting Eq.(38a) into expression (29) of the elastic strain energy and applying the principle of minimum potential energy, a set of stiffness equations is obtained, which are expressed with respect to the coordinate axes parallel to the principal axes trough the shear centre

$$
\boldsymbol{\sigma}^s = \boldsymbol{S}^s \boldsymbol{\varepsilon}^s.
$$
(39)

The components of matrix $\boldsymbol{S}^s$ can be found in Appendix A. When shear deformations tend to vanish, i.e. $\Phi_i = 0$ $(i=y,z,\omega)$ and consequently polynomials N_{1j}^i and $N_{2j}^i(i=2,..,4)$ reduce to the classical Hermitian polynomials. Correspondingly, equations (39) reduce to the stiffness equations of the Euler-Bernoulli-Vlasov beam model [2]. The torsional stress resultant σ_2 and the warping stress resultants σ_7, σ_8 are then calculated by the equations

$$
\begin{bmatrix} \sigma_2 \\ \sigma_7 \\ \sigma_8 \end{bmatrix} = \frac{GI_t}{30l_0^3}\begin{bmatrix} 36 & -3 & -3 \\ -3 & 4 & -1 \\ -3 & -1 & 4 \end{bmatrix}\begin{bmatrix} \varepsilon_2 \\ \varepsilon_7 \\ \varepsilon_8 \end{bmatrix} + \frac{EI_\omega}{l_0^5}\begin{bmatrix} 12 & -6 & -6 \\ -6 & 4 & 2 \\ -6 & 2 & 4 \end{bmatrix}\begin{bmatrix} \varepsilon_2 \\ \varepsilon_7 \\ \varepsilon_8 \end{bmatrix}.
$$
(40a)

Note that if $I_\omega = 0$, we obtain $\varepsilon_7 = \varepsilon_8 = \varepsilon_2$, yielding $\sigma_2 = GI_t/30l_0^3$ and $\sigma_7 = \sigma_8 = 0$.
For bending along the local shear principal y_s and z_s-axes, the generalized stress resultants are calculated by

$$
\begin{bmatrix} \sigma_3^s \\ \sigma_4^s \end{bmatrix} = \frac{EI_y}{l_0^3}\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} \varepsilon_3^s \\ \varepsilon_4^s \end{bmatrix}, \quad \begin{bmatrix} \sigma_5^s \\ \sigma_6^s \end{bmatrix} = \frac{EI_z}{l_0^3}\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} \varepsilon_5^s \\ \varepsilon_6^s \end{bmatrix}.
$$
(40b)

The vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^s$ are related by

$$
\boldsymbol{\sigma} = \mathbf{E}^{\mathrm{T}}\boldsymbol{\sigma}^s.
$$
(41)

By substituting Eqs.(39) with (37b) into Eq.(41) the element stiffness equations can be expressed in terms of centroidal coordinates as

$$
\boldsymbol{\sigma} = \boldsymbol{S}\boldsymbol{\varepsilon},
$$
(42a)

where

$$S = \mathbf{E}^{\mathrm{T}} S^s \mathbf{E}, \tag{42b}$$

in which $\mathbf{E}$ is the transformation matrix defined in Eq.(37a). With the stiffness matrix S the configuration of the elastic line of a beam with unsymmetrical cross-section can be determined in the centroidal coordinate system.

3.3. Second Order Analytical Formulation

Once the configuration of the elastic line has been determined in the centroidal coordinate system it is straightforward to derive a second order stiffness formulation. We again consider the two-node beam element in the local co-rotational frame (x,y,z) with origin at node p and base vectors $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ pointing along the respective positive directions of the coordinate axes as shown in Fig. 8. The x-axis coincides with the line connecting the nodal points p and q in the current configuration. The y-axis and z-axis coincide with the principal axes of the cross-section. An orientation is attached to it, describing the rotation of the cross-section independent of the tangent to the elastic line. The total displacement of the cross-section can be considered as the result of two successive motions: first, a rigid displacement and rotation of each cross-section due to bending and warping free torsion; next, a warping displacement perpendicular to the displaced cross-section. Let $\mathbf{r}_0^{\mathrm{P}}(x,y,z)$ denote the position vector of an arbitrary point P on the cross-section in the initial undeformed configuration and let $\mathbf{r}^{\mathrm{P}}(x,y,z)$ denote the position vector of point P in the current configuration. These two vectors are given by

$$\mathbf{r}_0^{\mathrm{P}} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z, \tag{43a}$$

$$\mathbf{r}^{\mathrm{P}} = (x + u_c)\mathbf{e}_x + v_c\mathbf{e}_y + w_c\mathbf{e}_z + y\mathbf{R}\mathbf{e}_y + z\mathbf{R}\mathbf{e}_z + \omega(y,z)\,\alpha\,\mathbf{R}\mathbf{e}_x, \tag{43b}$$

where u_c, v_c and w_c are the components of the displacement vector $\mathbf{u}_c$, $\omega(y,z)$ is a normalized warping function and $\alpha(x)$ is the change of twist per unit of length taken as an additional degree of freedom. The elastic rotation matrix $\mathbf{R}$ describes the rotation of a cross-section relative to the local frame due to flexural and torsional deformations of the beam. Following Besseling [4], the matrix $\mathbf{R}(x)$ is defined as the product of three successive rotations about the rotated coordinate axes parametrized by modified Euler angles $(\varphi_x, \varphi_y, \varphi_z)$ as

$$\mathbf{R} = \mathbf{R}(\varphi_z)\mathbf{R}(\varphi_y)\mathbf{R}(\varphi_x), \tag{44}$$

where

$$\mathbf{R}(\varphi_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_x & -\sin\varphi_x \\ 0 & \sin\varphi_x & \cos\varphi_x \end{bmatrix}, \quad \mathbf{R}(\varphi_y) = \begin{bmatrix} \cos\varphi_y & 0 & \sin\varphi_y \\ 0 & 1 & 0 \\ -\sin\varphi_y & 0 & \cos\varphi_y \end{bmatrix}, \quad \mathbf{R}(\varphi_z) = \begin{bmatrix} \cos\varphi_z & -\sin\varphi_z & 0 \\ \sin\varphi_z & \cos\varphi_z & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In contrast with Euler angles, modified Euler angles avoid singularity problems for small rotations. Another reason for taking modified Euler angles is that for small rotations they may be uniquely defined as components of a rotation pseudovector. Expanding matrix $\mathbf{R}$ around the identity matrix up to second order terms yields

$$\mathbf{R} = \begin{bmatrix} 1 - \frac{1}{2}\varphi_y^2 - \frac{1}{2}\varphi_z^2 & -\varphi_z + \varphi_x\varphi_y & \varphi_y + \varphi_x\varphi_z \\ \varphi_z & 1 - \frac{1}{2}\varphi_x^2 - \frac{1}{2}\varphi_z^2 & -\varphi_x + \varphi_y\varphi_z \\ -\varphi_y & \varphi_x & 1 - \frac{1}{2}\varphi_x^2 - \frac{1}{2}\varphi_y^2 \end{bmatrix}. \tag{45}$$
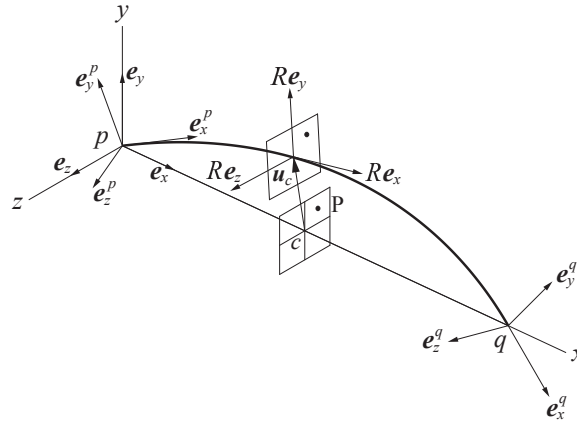

Figure 8: Configuration of the elastic line in a local, co-rotational frame (x,y,z).

With the above kinematic description, six independent strains can be defined for each cross-section, one for the axial elongation and warping deformation, two for the transverse shear deformations, one torsion and two bending curvatures. A second order approximation of the torsion and bending curvatures of the elastic line is defined by three independent components of the derivative of matrix $\mathbf{R}$ with respect to x [34]

$$\tilde{\boldsymbol{\kappa}}(x) = \mathbf{R}^{\mathrm{T}} \mathbf{R}_{,x} \tag{46a}$$

where

$$\tilde{\boldsymbol{\kappa}} = \begin{bmatrix} 0 & -\kappa_z & \kappa_y \\ \kappa_z & 0 & -\kappa_x \\ -\kappa_y & \kappa_x & 0 \end{bmatrix}. \tag{46b}$$

Written out, the components of $\tilde{\kappa}$ are

$$\kappa_x = \varphi_{x,x} - \varphi_y \, \varphi_{z,x} \tag{47a}$$

$$\begin{aligned} \kappa_y &= \varphi_{y,x} + \varphi_x \, \varphi_{z,x} \\ \kappa_z &= \varphi_{z,x} - \varphi_x \, \varphi_{y,x} \end{aligned} \tag{47b}$$

where κ_x is the torsional curvature and κ_y, κ_z are the bending curvatures about the y-axis and z-axis respectively. The Green-Lagrange strain formulation is adopted to obtain a second order approximation for the axial and transverse shear strains including warping. Substituting the second order rotation matrix of Eq.(45) into Eq.(43b), the local displacement vector $\boldsymbol{u} = (u, v, w)^{\mathrm{T}} = \boldsymbol{r}^{\mathrm{P}} - \boldsymbol{r}_0^{\mathrm{P}}$ of point P, can be evaluated as

$$\begin{aligned} u &= u_c + y\left(-\varphi_z + \varphi_x \varphi_y\right) + z\left(\varphi_y + \varphi_x \varphi_z\right) + \omega\, \alpha \left(1 - \tfrac{1}{2}\varphi_y^2 - \tfrac{1}{2}\varphi_z^2\right), \\ v &= v_c - y\left(\tfrac{1}{2}\varphi_x^2 + \tfrac{1}{2}\varphi_z^2\right) + z\left(-\varphi_x + \varphi_y \varphi_z\right) + \omega\, \alpha\, y\, \varphi_z, \\ w &= w_c + y\, \varphi_x - z\left(\tfrac{1}{2}\varphi_x^2 + \tfrac{1}{2}\varphi_y^2\right) - \omega\, \alpha\, z\, \varphi_y. \end{aligned} \tag{48}$$

Assuming that the axial strain ε_{xx} and the shear strains γ_{xy}, γ_{xz} are small, we can neglect the second order terms involving $(u_{,x}), (v_{,x}v_{,y}), (w_{,x}w_{,y})$ and $(w_{,x}w_{,z}), (v_{,x}v_{,z})$ yielding the following expressions for the nonlinear Green-Lagrange tensor

$$\begin{aligned} \varepsilon_{xx} &= u_{,x} + \tfrac{1}{2}\left(v_{,x}^2 + w_{,x}^2\right), \\ \gamma_{xy} &= u_{,y} + v_{,x}(1 + v_{,y}) + w_{,x}w_{,y} = u_{,y} + v_{,x}, \\ \gamma_{xz} &= u_{,z} + w_{,x}(1 + w_{,z}) + v_{,x}v_{,z} = u_{,z} + w_{,x}. \end{aligned} \tag{49}$$

After substituting Eq.(48) into (49), a second order approximation of the Green-Lagrange strains is obtained

$$\varepsilon_{xx} = u_{c,x} + \tfrac{1}{2}(v_{c,x}^2 + w_{c,x}^2) - y\,\kappa_z + z\,\kappa_y + \tfrac{1}{2}(y^2 + z^2)\,\varphi_{x,x}^2 + \omega\,\alpha_{,x}, \tag{50a}$$

$$\begin{aligned} \gamma_{xy} &= v_{c,x} - \varphi_z - z\,\varphi_{x,x} + \omega_{,y}\,\alpha, \\ \gamma_{xz} &= w_{c,x} + \varphi_y + y\,\varphi_{x,x} + \omega_{,z}\,\alpha. \end{aligned} \tag{50b}$$

Since the x and y-axes are principal axes of inertia, we have [11]

$$\int_A \omega_{,y}\, dA = \int_A \omega_{,z}\, dA = 0. \tag{51}$$

By integrating Eq.(50a) over the cross-section area and applying Eq.(20) it follows that

$$\varepsilon_x = \frac{1}{A}\int_A \varepsilon_{xx}\, dA = u_{c,x} + \tfrac{1}{2}(v_{c,x}^2 + w_{c,x}^2) + \tfrac{1}{2}\frac{I_p}{A}\varphi_{x,x}^2, \tag{52}$$

where ε_x is the axial stretch and I_p is the polar moment of area about the centroid of the cross-section. The second and third part on the right-hand side of Eq.(52), refers to the additional axial stretch due to bending and torsion respectively. The latter is denoted as Wagner effect [43]. Integration of Eqs.(50b) over the cross-section area yields with (51)

$$\gamma_y = \frac{1}{A}\int_A \gamma_{xy}\, dA = v_{c,x} - \varphi_z, \qquad \gamma_z = \frac{1}{A}\int_A \gamma_{xz}\, dA = w_{c,x} + \varphi_y. \tag{53}$$

In the above equations, γ_y and γ_z represent the average transverse shear strains due to flexure.

3.4. Second Order Discrete Formulation

Substitution of Eqs.(31) with the second order rotation matrix of Eq.(45) into Eqs.(30), using boundary conditions (32) and disregarding third and higher order terms we obtain the second order approximations for the generalized strains

$$\varepsilon_2 = l_0\varphi_x^q + \frac{1}{2l_0}(\varepsilon_3 - \varepsilon_4)(\varepsilon_5 + \varepsilon_6) \quad \text{(torsion)} \tag{54a}$$

$$\left.\begin{aligned}
\varepsilon_3 &= -l_0\varphi_y^p \\
\varepsilon_4 &= l_0\varphi_y^q + \tfrac{1}{l_0}\varepsilon_2\,\varepsilon_6 \\
\varepsilon_5 &= -l_0\varphi_z^p \\
\varepsilon_6 &= l_0\varphi_z^q - \tfrac{1}{l_0}\varepsilon_2\,\varepsilon_4
\end{aligned}\right\} \quad \text{(bending)} \tag{54b}$$

The quadratic terms in the above equations originate from the additional nonlinearity due to the relative rotations of the unit vectors e_y^p, e_z^p and e_y^q, e_z^q at the nodal points p and q as shown in Fig. 8. The elastic displacement vector $u_c(\xi)$ and rotation vector $l_0\boldsymbol{\varphi}(\xi)$ of an arbitrary point on the elastic line with coordinate $x = \xi l_0$ can be expressed as a vectorial sum of the axial and lateral displacements, i.e. in the form

$$u_c(\xi) = u_c(\xi)e_x + v_c(\xi)e_y + w_c(\xi)e_z,$$

$$l_0\boldsymbol{\varphi}(\xi) = l_0\left[\varphi_x(\xi)e_x + \varphi_y(\xi)e_y + \varphi_z(\xi)e_z\right].$$

The displacements and rotations are interpolated in the same way as in the linear case, i.e. by the modified Hermitian shape functions of Eqs.(38) for the case zero eccentricity ($y_s = z_s = 0$) resulting in

$$u_c(\xi) = u_c^q N_1(\xi)e_x + l_0\left[-\varphi_z^p N_{12}^y(\xi) + \varphi_z^q N_{14}^y(\xi)\right]e_y + l_0\left[-\varphi_y^p N_{12}^z(\xi) + \varphi_y^q N_{14}^z(\xi)\right]e_z, \tag{56a}$$

$$l_0\boldsymbol{\varphi}(\xi) = l_0\left[\varphi_x^q N_{13}^\omega(\xi) + l_0\varepsilon_7 N_{12}^\omega(\xi) - l_0\varepsilon_8 N_{14}^\omega(\xi)\right]e_x$$

$$\quad + l_0\left[-\varphi_y^p N_{22}^z(\xi) + \varphi_y^q N_{24}^z(\xi)\right]e_y + l_0\left[-\varphi_z^p N_{22}^y(\xi) + \varphi_z^q N_{24}^y(\xi)\right]e_z. \tag{56b}$$

Substituting for $l_0\varphi_x^q, l_0\varphi_y^p, l_0\varphi_y^q$ and $l_0\varphi_z^p, \varphi_z^q$ from Eqs.(54) into Eqs.(56) yields the following second order approximation of the displacement and rotation fields

$$u_c(\xi) = u_c^q N_1(\xi)e_x + \left[\varepsilon_5 N_{12}^y(\xi) + (\varepsilon_6 + \tfrac{1}{l_0}\varepsilon_2\,\varepsilon_4)N_{14}^y(\xi)\right]e_y + \left[\varepsilon_3 N_{12}^z(\xi) + (\varepsilon_4 - \tfrac{1}{l_0}\varepsilon_2\,\varepsilon_6)N_{14}^z(\xi)\right]e_z, \tag{57a}$$

$$l_0\boldsymbol{\varphi}(\xi) = \left[(\varepsilon_2 - \tfrac{1}{2l_0}(\varepsilon_3 - \varepsilon_4)(\varepsilon_5 + \varepsilon_6))N_{13}^\omega(\xi) + l_0\varepsilon_7 N_{12}^\omega(\xi) - l_0\varepsilon_8 N_{14}^\omega(\xi)\right]e_x$$

$$\quad + \left[\varepsilon_3 N_{22}^z(\xi) + (\varepsilon_4 - \tfrac{1}{l_0}\varepsilon_2\,\varepsilon_6)N_{24}^z(\xi)\right]e_y + \left[\varepsilon_5 N_{22}^y(\xi) + (\varepsilon_6 + \tfrac{1}{l_0}\varepsilon_2\,\varepsilon_4)N_{24}^y(\xi)\right]e_z. \tag{57b}$$

Integrating the second part of Eq.(52) over the length of the beam using the derivatives of the polynomial components of Eq.(57a) yields

$$\frac{1}{2}\int_0^{l_0}(v_{c,x}^2 + w_{c,x}^2)\,dx = \frac{1}{2l_0}\int_0^1(v_{c,\xi}^2 + w_{c,\xi}^2)\,d\xi = \frac{1}{30l_0}(2\varepsilon_3^2 + \varepsilon_3\varepsilon_4 + 2\varepsilon_4^2 + 2\varepsilon_5^2 + \varepsilon_5\varepsilon_6 + 2\varepsilon_6^2). \tag{58}$$

The quadratic terms in the above equation account for the additional axial shortening of the beam centroid axis due to bending. The second order shortening due to a twisting rotation φ_x^q of node q is calculated through integration of the third term (Wagner term) of Eq.(52) over the length of the beam

$$\frac{1}{2}\frac{I_p}{A}\int_0^{l_0}\varphi_{x,x}^2\,dx = \frac{1}{2l_0}\frac{I_p}{A}\int_0^1\varphi_{x,\xi}^2\,d\xi. \tag{59}$$

However, since the shear centre is the pole of the twisting rotation, there will also be a second order elongation of the beam axis associated with the twisting rotation, if the shear centre does not coincide with the centroid of the cross-section. This will be explained on the basis of figure 9. The figure shows the elastic line of the beam element in the corotational frame (x,y,z) with the origin at node c^p. The x-axis coincides with the centroid of the cross-sections and the y-axis and z-axis are principal axes; y_s and z_s are the coordinates of the shear centre with respect to the centroid. We
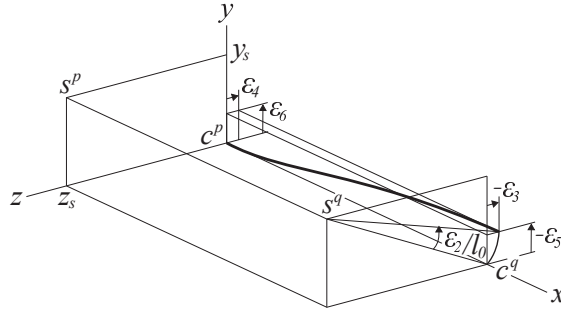
Figure 9: Axial elongation due to rotation about shear axis.

consider the end-sections at the nodes c^p and c^q normal to the shear centre axis. The cross-section at node c^q rotates through an angle $\varphi_x^q = \varepsilon_2/l_0$ about the shear axis. As a result of this rotation, the elastic line becomes helical after rotation as shown in figure 9 . The associated generalized strains are

$$-\varepsilon_3 = y_s \frac{\varepsilon_2}{l_0}, \quad \varepsilon_4 = y_s \frac{\varepsilon_2}{l_0}, \quad -\varepsilon_5 = z_s \frac{\varepsilon_2}{l_0}, \quad \varepsilon_6 = z_s \frac{\varepsilon_2}{l_0}. \tag{60}$$

Substitution of Eqs.(60) into Eq.(58) yields an expression which describes the second order elongation of the elastic line due to the twisting rotation ε_2/l_0, about the shear centre axis

$$\frac{1}{30l_0}\left(2\varepsilon_3^2 + \varepsilon_3\varepsilon_4 + 2\varepsilon_4^2 + 2\varepsilon_5^2 + \varepsilon_5\varepsilon_6 + 2\varepsilon_6^2\right) = \frac{\left(y_s^2 + z_s^2\right)}{10l_0}\frac{\varepsilon_2^2}{l_0^2}. \tag{61}$$

3.5. Modified Deformation Modes

As a final step we derive a set of modified generalized strains $\overline{\varepsilon}_i$ through integration of the second order analytical expressions for the axial stretch and the torsion and bending curvatures, over the length of the beam, using the second order displacement and rotation fields of Eqs.(57). Integrating Eq.(52) over the length of the beam and including the second order elongation term of Eq.(61) we obtain

$$\overline{\varepsilon}_1 = \int_0^{l_0} \varepsilon_x \mathrm{d}x = \int_0^1 u_{c,\xi}\,\mathrm{d}\xi + \frac{1}{2l_0}\int_0^1 \left(v_{c,\xi}^2 + w_{c,\xi}^2 + \frac{I_p}{A}\varphi_{x,\xi}^2\right)\mathrm{d}\xi - \frac{\left(y_s^2 + z_s^2\right)}{10l_0}\frac{\varepsilon_2^2}{l_0^2}. \tag{62}$$

In a similar way, integration of the torsion curvature of Eq.(47a) over the length of the beam yields

$$\overline{\varepsilon}_2 = l_0 \int_0^{l_0} \kappa_x\,\mathrm{d}x = l_0 \int_0^1 (\varphi_{x,\xi} - \varphi_y\,\varphi_{z,\xi})\,\mathrm{d}\xi. \tag{63}$$

Expressions for $(\overline{\varepsilon}_3,...,\overline{\varepsilon}_6)$ are obtained using the moment-area theorem [10] including the effects of transverse shear deformation. Using the nonlinear bending curvature and average transverse shear strain relations of Eqs.(47b) and (53) respectively, we obtain

$$\overline{\varepsilon}_3 = \int_0^{l_0} \left[\kappa_y(l_0 - x) - \gamma_z\right]\mathrm{d}x = l_0 \int_0^1 \left[(\varphi_{y,\xi} + \varphi_x\,\varphi_{z,\xi})(1 - \xi) - (\frac{1}{l_0}w_{c,\xi} + \varphi_y)\right]\mathrm{d}\xi,$$

$$\overline{\varepsilon}_4 = \int_0^{l_0} \left[\kappa_y x + \gamma_z\right]\mathrm{d}x = l_0 \int_0^1 \left[(\varphi_{y,\xi} + \varphi_x\,\varphi_{z,\xi})\xi + \frac{1}{l_0}w_{c,\xi} + \varphi_y\right]\mathrm{d}\xi,$$

$$\overline{\varepsilon}_5 = \int_0^{l_0} \left[\kappa_z(l_0 - x) + \gamma_y\right]\mathrm{d}x = l_0 \int_0^1 \left[(\varphi_{z,\xi} - \varphi_x\,\varphi_{y,\xi})(1 - \xi) + \frac{1}{l_0}v_{c,\xi} - \varphi_z\right]\mathrm{d}\xi, \tag{64}$$

$$\overline{\varepsilon}_6 = \int_0^{l_0} \left[\kappa_z x - \gamma_y\right]\mathrm{d}x = l_0 \int_0^1 \left[(\varphi_{z,\xi} - \varphi_x\,\varphi_{y,\xi})\xi - (\frac{1}{l_0}v_{c,\xi} - \varphi_z)\right]\mathrm{d}\xi.$$

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Substituting the components of $\boldsymbol{\varphi}(\xi)$ and the derivatives of the components $\boldsymbol{u}_c(\xi)$ and $\boldsymbol{\varphi}(\xi)$ of Eqs.(57)into Eqs.(62)-(64) and neglecting third and higher order terms, we obtain a set of modified generalized strains including second order coupling terms

$$\overline{\boldsymbol{\varepsilon}} = \boldsymbol{E}(\boldsymbol{\varepsilon}),\tag{65}$$

where

$$\overline{\varepsilon}_1 = \varepsilon_1 + \frac{1}{30l_0}\left[2\varepsilon_3^2 + \varepsilon_3\varepsilon_4 + 2\varepsilon_4^2 + 2\varepsilon_5^2 + \varepsilon_5\varepsilon_6 + 2\varepsilon_6^2\right]$$
$$+ \frac{1}{30l_0^3}\frac{I_p}{A}\left[3\varepsilon_2\left(6\varepsilon_2 - \varepsilon_7 - \varepsilon_8\right) + 2\varepsilon_7^2 - \varepsilon_7\varepsilon_8 + 2\varepsilon_8^2\right] - \frac{1}{10l_0^3}\left(y_s^2 + z_s^2\right)\varepsilon_2^2 \qquad \Big\}\ \text{(elongation)}$$

$$\overline{\varepsilon}_2 = \varepsilon_2 + \frac{1}{l_0}(-\varepsilon_3\varepsilon_6 + \varepsilon_4\varepsilon_5),\qquad\qquad\qquad\qquad\qquad\qquad \Big\}\ \text{(torsion)}$$

$$\overline{\varepsilon}_3 = \varepsilon_3 + \frac{1}{10l_0}\varepsilon_2(\varepsilon_5 + 2\varepsilon_6) + \frac{1}{30l_0}\left[\ 3\varepsilon_5\varepsilon_7 - \varepsilon_8(\varepsilon_5 + \varepsilon_6)\right],$$

$$\overline{\varepsilon}_4 = \varepsilon_4 - \frac{1}{10l_0}\varepsilon_2(2\varepsilon_5 + \varepsilon_6) + \frac{1}{30l_0}\left[-3\varepsilon_6\varepsilon_8 + \varepsilon_7(\varepsilon_5 + \varepsilon_6)\right],$$

$$\overline{\varepsilon}_5 = \varepsilon_5 - \frac{1}{10l_0}\varepsilon_2(\varepsilon_3 + 2\varepsilon_4) + \frac{1}{30l_0}\left[-3\varepsilon_3\varepsilon_7 + \varepsilon_8(\varepsilon_3 + \varepsilon_4)\right],$$

$$\overline{\varepsilon}_6 = \varepsilon_6 + \frac{1}{10l_0}\varepsilon_2(2\varepsilon_3 + \varepsilon_4) + \frac{1}{30l_0}\left[\ 3\varepsilon_4\varepsilon_8 - \varepsilon_7(\varepsilon_3 + \varepsilon_4)\right],$$

$$\overline{\varepsilon}_7 = \varepsilon_7,$$

$$\overline{\varepsilon}_8 = \varepsilon_8.$$

The linear terms ε_i represent the first order approximations of the generalized strains as they are defined in Eqs.(33). The quadratic terms in the equation for $\overline{\varepsilon}_1$ take into account the additional shortening of the beam axis caused by its bending, torsion and warping. The latter two terms represent the so-called "Wagner-effect" which strongly influence the beam response in cases of flexural/torsional buckling of open thin-walled beams. The quadratic terms in the expression for ε_2 account for the extra torsion of the beam caused by its bending. The quadratic terms in the expressions for ε_3 to ε_6 take into account the effect of unsymmetrical bending caused by torsion and warping deformations of the beam. Because the rigidity against axial elongation is much larger than the flexural and torsional rigidity, the modification of $\overline{\varepsilon}_1$ is the most relevant one and the other modifications are only relevant to more special cases, for instance when large differences in flexural rigidities occur.

Substitution of Eq.(5) in Eq.(65) yields a new set of modified deformation modes, specified by a set of modified generalized strains $\overline{\varepsilon}_i$ expressed as functions of the global nodal coordinates X_i

$$\overline{\boldsymbol{\varepsilon}} = \boldsymbol{E}(\boldsymbol{D}(\boldsymbol{X})) = \overline{\boldsymbol{D}}(\boldsymbol{X})\,.\tag{67}$$

These deformation modes form the basis of a second order beam finite element that captures non-uniform torsion and flexural-torsional coupling of thin-walled beams with unsymmetrical cross-section. In the next section the equations of motion are derived for this second order beam model.

4. Equations of Motion

The equations of motion are derived using d'Alembert's principle, which states that the balance of virtual work

$$\overline{\boldsymbol{\sigma}}^{\mathrm{T}}\delta\overline{\boldsymbol{\varepsilon}} = \boldsymbol{F}^{\mathrm{T}}\delta\boldsymbol{u} - \delta\boldsymbol{X}^{\mathrm{T}}[\boldsymbol{M}\ddot{\boldsymbol{X}} + \boldsymbol{h}],\tag{68}$$

holds for all $\delta\overline{\varepsilon}$ and $\delta\boldsymbol{u}$ depending on the variations of the nodal coordinates $\delta\boldsymbol{X}$, by the relations

$$\delta\overline{\boldsymbol{\varepsilon}} = \overline{\boldsymbol{D}}_{,X}\delta\boldsymbol{X},\quad \delta\boldsymbol{u} = \boldsymbol{A}\,\delta\boldsymbol{X},\tag{69}$$

where

$$\overline{\boldsymbol{D}}_{,X} = \boldsymbol{E}_{,D}\,\boldsymbol{D}_{,X}\,,\qquad \boldsymbol{A} = \mathrm{diag}(\mathbf{I},\ 2\boldsymbol{\Lambda}^p,\ 1,\ \mathbf{I},\ 2\boldsymbol{\Lambda}^q,\ 1).\tag{70}$$

in which $\mathbf{I}$ is a 3×3 unitary matrix and $\boldsymbol{\Lambda}$ is defined in Eq.(10). The last term in Eq.(68) represents the virtual work due to the inertia forces, where $\boldsymbol{M}(\boldsymbol{X})$ is a position dependent mass matrix and $\boldsymbol{h}(\boldsymbol{X},\dot{\boldsymbol{X}})$ a convective inertia term being a quadratic function of nodal velocities. Both quantities are obtained by adding the consistent and lumped inertia parts, i.e. $\boldsymbol{M} = \boldsymbol{M}_c + \boldsymbol{M}_\ell$ and $\boldsymbol{h} = \boldsymbol{h}_c + \boldsymbol{h}_\ell$. Substitution of the compatibility relations (69) in Eq.(68) yields with the transpose matrices $\overline{\boldsymbol{D}}_{,X}^{\mathrm{T}}$ and $\boldsymbol{A}^{\mathrm{T}}$

$$\overline{\boldsymbol{D}}_{,X}^{\mathrm{T}}\overline{\boldsymbol{\sigma}} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{F} - (\boldsymbol{M}\ddot{\boldsymbol{X}} + \boldsymbol{h})\,.\tag{71}$$

These are the equations of motion for the free second order beam element. With the inclusion of the second order terms in the definitions of the modified generalized strains $\overline{\varepsilon}_i$, relationship(42a) between generalized strains and dual stress resultants remains valid, i.e.

$$\overline{\boldsymbol{\sigma}} = \boldsymbol{S}\overline{\boldsymbol{\varepsilon}}, \tag{72}$$

where $\boldsymbol{S}$ is defined by Eq.(42b). The dual stress-resultants $\overline{\sigma}_i$ have now a slightly modified meaning in case of finite deformations. A consistent mass formulation is derived by Jonker [16] and Meijaard [26]. A lumped mass matrix $\boldsymbol{M}_\ell$ and corresponding convective term $\boldsymbol{h}_\ell$ is obtained by applying one-half of the total twist, rotary and warping inertia's to both end points of the beam [16]

$$\boldsymbol{M}_\ell = \tfrac{1}{2}\rho l_0 \operatorname{diag}\left[A\boldsymbol{I}\,,\, 4\overline{\boldsymbol{\Lambda}}^{p\mathrm{T}}\boldsymbol{J}\,\overline{\boldsymbol{\Lambda}}^p,\, I_\omega,\, A\boldsymbol{I}\,,\, 4\overline{\boldsymbol{\Lambda}}^{q\mathrm{T}}\boldsymbol{J}\,\overline{\boldsymbol{\Lambda}}^q\,,\, I_\omega\right], \tag{73}$$

$$\boldsymbol{h}_\ell = \tfrac{1}{2}\rho l_0 \left[\boldsymbol{0}\,,\, 8\overline{\boldsymbol{\Lambda}}^{pT}\boldsymbol{J}\,\overline{\boldsymbol{\Lambda}}^p \begin{bmatrix} \dot{\lambda}_0^p \\ \dot{\boldsymbol{\lambda}}^p \end{bmatrix},\, 0,\, \boldsymbol{0},\, 8\overline{\boldsymbol{\Lambda}}^{q\mathrm{T}}\boldsymbol{J}\,\overline{\boldsymbol{\Lambda}}^q \begin{bmatrix} \dot{\lambda}_0^q \\ \dot{\boldsymbol{\lambda}}^q \end{bmatrix},\, 0 \right]^{\mathrm{T}},$$

in which $\boldsymbol{I}$ and $\boldsymbol{0}$ are a 3×3 unitary and zero matrix respectively, I_ω is the sectorial moment defined in Eq.(23a) and matrices $\overline{\boldsymbol{\Lambda}}^{(i)}$, $(i = p, q)$ and $\boldsymbol{J}$ are defined by

$$\overline{\boldsymbol{\Lambda}}^{(i)} = (-\boldsymbol{\lambda}^{(i)}, \lambda_0^{(i)}\boldsymbol{I} - \tilde{\boldsymbol{\lambda}}^{(i)}), \qquad \boldsymbol{J} = \operatorname{diag}(I_\mathrm{p},\, I_x,\, I_y). \tag{74}$$

In order to study the vibration properties and the elastic stability we consider small disturbances from an equilibrium configuration $\sigma_i^0 \overline{D}_{i,k}^0 = A_{lk}^0 F_l^0$. Substitution of (72) into (71) and expanding the equations in terms of disturbances $\Delta \boldsymbol{x}$, $\Delta \ddot{\boldsymbol{x}}$ and disregarding second and higher order terms, yields the linearized equations of motion [17, 26]

$$\left[\overline{D}_{i,k}^0 S_{ij} \overline{D}_{j,l}^0 + \sigma_i^0 \overline{D}_{i,kl}^0 - (A_{mk}F_m)_{,l}^0\right]\Delta x_l + M_{kl}^0 \Delta \ddot{x}_l = 0, \tag{75}$$

or in matrix form

$$(\boldsymbol{K}^0 + \boldsymbol{G}^0 + \boldsymbol{N}^0)\Delta \boldsymbol{x} + \boldsymbol{M}^0 \Delta \ddot{\boldsymbol{x}} = \boldsymbol{0}, \tag{76}$$

where $(\boldsymbol{K}^0 + \boldsymbol{G}^0 + \boldsymbol{N}^0)$ is the tangent stiffness matrix, with $\boldsymbol{K}^0$ the constitutive stiffness matrix and $\boldsymbol{G}^0$ the geometric stiffness matrix due to the reference load $\boldsymbol{F}^0$ giving rise to reference stress resultants $\boldsymbol{\sigma}^0$. The matrix $\boldsymbol{N}^0$ is the dynamic stiffness matrix containing the terms $(A_{mk}F_m)_{,l}^0$ which need not be symmetric. For static problems the tangent stiffness matrix $(\boldsymbol{K}^0 + \boldsymbol{G}^0)$ is always symmetric.

5. Numerical Simulations

The presented second order beam-warping element has been implemented in the computer programme SPACAR [15] under the name BEAMW-element. This programme can make computations for mechanical systems with interconnected rigid and flexible elements. Specifically, the motion can be simulated for given initial conditions, the equations can be linearized about an arbitrary state of motion, stationary solutions can be determined and with the linearized equations, eigen frequencies and corresponding frequency modes as well as the elastic stability can be determined. An incremental-iterative method based on the Newton-Raphson method combined with constant arc length of the incremental displacement vector is employed for the solution of the quasi-static equilibrium equations [28].

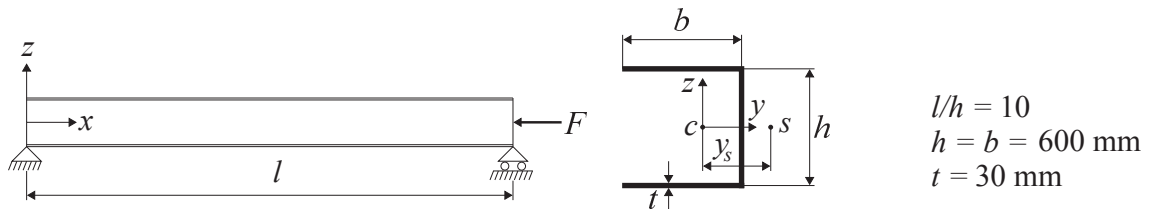### 5.1. Flexural-Torsional Buckling of a Simply-Supported C-shaped Beam

Figure 10: Simply supported C-shaped beam with geometrical properties.

In this example we determine the global buckling load of a simply supported C-shaped beam subjected to a centroidal axial force F. See Fig. 10. The cross-sectional properties of the beam are presented in table 1. In this example the high ratio E/G between elastic and the transverse shear elastic moduli requires a beam model (e.g.Timoshenko-Reissner) that accounts for the shear strain effects on both non-uniform torsion and bending. Because of symmetry only one half of the beam need to be modelled. The half-beam is divided into 2 equal beam warping elements.

Table 1: Cross-sectional properties simply supported c-shaped beam

Elastic modulus	E	$= 1.44 \times 10^2\,\text{kN/mm}^2$
Shear modulus	G	$= 4.14\,\text{kN/mm}^2$
cross-sectional area	A	$= 5.4 \times 10^4\,\text{mm}^2$
Position of shear centre	y_s	$= 457.1\,\text{mm}$
Moment of inertia	I_y	$= 3.78 \times 10^9\,\text{mm}^4$
Moment of inertia	I_z	$= 2.16 \times 10^9\,\text{mm}^4$
Torsion constant	I_t	$= 1.62 \times 10^7\,\text{mm}^4$
Warping constant	I_ω	$= 1.3886 \times 10^{14}\,\text{mm}^6$
Wagner term	I_p/A	$= 1.1 \times 10^5\,\text{mm}^2$
Shear stiffness	D_y	$= 2.79 \times 10^4\,\text{mm}^2$
Shear stiffness	D_z	$= 1.58 \times 10^4\,\text{mm}^2$
Warping shear stiffness	D_ω	$= 3.65 \times 10^6\,\text{mm}^4$
Warping shear stiffness	$D_{z\omega}$	$= -3.7 \times 10^6\,\text{mm}^3$

In Table 2, the buckling loads are presented for a simply supported beam with $l/h = 10$, including shear deformation or not. When two finite elements are adopted, the error with respect to the analytic solution of Cortinez and Pivian [8] is negligible for the Timoshenko-Reissner beam model and less than 4% for the Bernoulli-Vlasov beam model. It is quite evident that ignoring shear deformations implies a strong overestimate of the buckling load (35% for l/h = 10). Furthermore the eigen frequencies of modes 1-6 and of mode 12 and 13 of the simply-supported beam have been computed and compared with the analytical solutions values of Cortinez and Piovan [8].The beam is divided into 30 BEAMW elements. The results are shown in Table 3. Eigenfrequencies are evaluated including effects of shear flexibility or not. There is a good agreement with the analytic results. It can be observed that neglecting shear strain effects leads to significant overestimation of natural frequencies.

Table 2: Buckling loads of simply supported C-shaped beam with $l/h = 10$. Comparison between present analysis using SPACAR and the analytical solution proposed by Cortinez and Piovan [8]

Beam model	Cortinez and Piovan [8] (analytic solution)	SPACAR (2 EL)	SPACAR (4 EL)
Timoshenko-Reissner	$11.94 .10^3\,\text{kN}$	$12.86 .10^3\,\text{kN}$	$11.90 .10^3\,\text{kN}$
Bernoulli-Euler-Vlasov	$16.12 .10^3\,\text{kN}$	$18.34 .10^3\,\text{kN}$	$16.67 .10^3\,\text{kN}$

Table 3: Flexural-torsional frequencies f_n (Hz) of simply supported beam with C-shaped cross-section.

Mode nr.	Cortinez and Piovan	SPACAR	Cortinez and Piovan	SPACAR
	(a) Including shear flexibility		(b) Excluding shear flexibility	
1	33.23	33.18	38.63	38.61
2	67.17	67.30	88.85	88.85
3	99.03	98.65	153.86	153.86
4	102.79	103.01	207.98	207.38
5	169.05	168.65	345.90	344.37
6	178.01	178.41	355.42	355.43
12			1421.67	1421.8
13	395.60	397.28		

5.2. Rotating C-Shaped Cantilever Beam Subjected to Eccentric Tip Force

A uniform C-shaped beam of length L is attached to a rigid hub of radius $R = 0$, rotating at constant angular speed Ω. The geometry of the beam and its cross-section are shown in Fig. 11. With these data the cross-sectional input has been calculated and presented in table 4. The beam is subjected to a vertical tip force F at the free end applied at the points D or E of the cross-section respectively. Its lateral-torsional post-buckling behaviour is investigated for different rotation speeds. The beam is divided into 10 equal BEAMW elements. In Fig. 12 the load-deflection curves $(-w, F)$ of point D are depicted for the case where $\Omega = 0$ (static case) and load F is applied at point D or E, respectively. This case was recently analysed by Bourihane et al [5], using ABAQUS. The comparison of these curves with those computed using 120 shell elements of type S8R in ABAQUS shows a good agreement. It appeared that the Vlasov beam model provides satisfactory results for this nonlinear beam problem. In addition the effect of the second order terms that account for geometric non-linearities appears to be negligibly small for this problem. Fig. 13 shows the load-deflection curves $(-w, F)$ of point D for four different angular velocities, where load F is applied at point D.
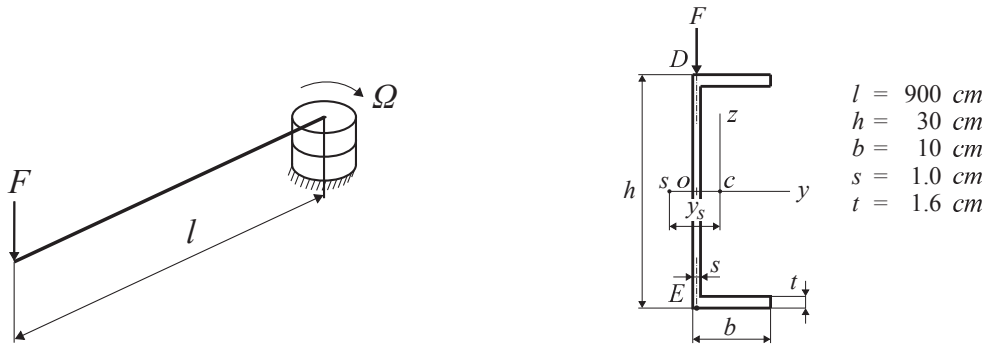
Figure 11: Rotating C-shaped cantilever beam with geometrical properties.

Table 4: Cross-sectional properties rotating C-shaped beam

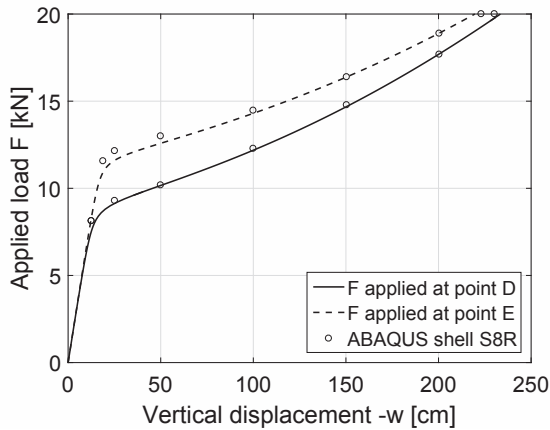|		
Elastic modulus	E	$= 21000$ kN/cm^2
Shear modulus	G	$= 8077$ kN/cm^2
cross-sectional area	A	$= 58.8$ cm^2
Position	y_o	$= -2.456$ cm
Position of shear centre	y_s	$= -6.0779$ cm
Moment of inertia	I_y	$= 8038.7$ cm^4
Moment of inertia	I_z	$= 559.92$ cm^4
Torsion constant	I_t	$= 35.408$ cm^4
Warping constant	I_ω	$= 78943.0$ cm^6
Mass density	ρ	$= 7.8 \times 10^{-3}$ kg/cm^3

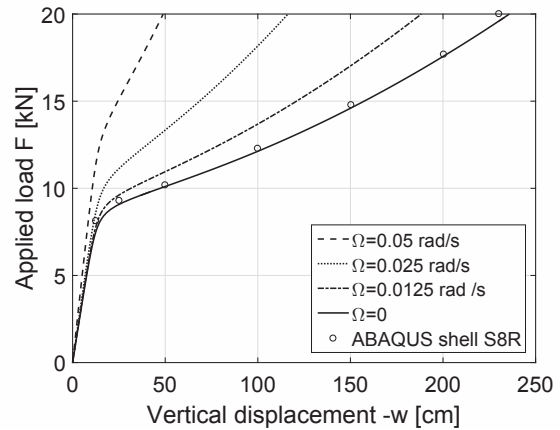Figure 12: Load-deflection curves of C-shaped cantilever beam, static case (Ω=0).

Figure 13: Load-deflection curves of C-shaped cantilever beam rotating at different speeds Ω.

6. Conclusions

The generalized strain beam formulation is used to derive a co-rotational beam element for buckling and post-buckling analysis of thin-walled beams. A first- and second order stiffness formulation is derived. The first order stiffness formulation includes the derivation of the stiffness matrix based on Timoshenko-Reissner's thin-walled beam model. Cross-sectional warping is accounted for by a single warping function. Coupling among bending and torsion due to non-coinciding centroid and shear centre is incorporated using a transformation matrix which accounts for the eccentricity of the shear centre. The second order stiffness formulation deals with the derivation of modified deformation modes which includes quadratic terms that account for geometric nonlinearities, intended to analyse buckling and post-buckling behaviour of thin-walled beams. With the inclusion of additional second order terms, a clear separation of stiffness due to elongation, torsion, warping and bending in two directions is retained, so deformation modes with a large stiffness can be eliminated by constraining them to be zero. Some numerical examples including large torsion and deflections, and pre- and post buckling are presented in order to illustrate the performance of the present beam element and to investigate the influence of second order terms on the accuracy and the rate of convergence. The following conclusions can be drawn:

1. In flexural-torsional buckling and vibration analysis of beams with unsymmetric channel cross-section under axial forces, the present solutions show excellent agreement with the analytical solutions from literature.

2. In the post buckling analysis of thin-walled open beams under eccentric bending loads the use of second order terms does not lead to a reduction in the number of elements needed to perform the analysis with the same accuracy. This is probably due to the use of torsional deformation mode (7b) which is limited to small bending and torsion deformations.

3. The shear strain effects become significant for beams made of materials characterized by large E/G-ratios.

Acknowledgement

Appendix A–Stiffness Matrix of Beam Element

The 8×8 stiffness matrix of the BEAMW element based on the modified Hermitian interpolation, is written in block matrix form as [29, 30]

$$
S^s =
\begin{bmatrix}
[\,S_{11}\,] & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & [\,S_{22}\,] & [\,S_{23} & S_{24}\,] & [\,S_{25} & S_{26}\,] & [\,S_{27} & S_{28}\,] \\
 & & S_{33} & S_{34} & S_{35} & S_{36} & S_{37} & S_{38} \\
 & & S_{43} & S_{44} & S_{45} & S_{46} & S_{47} & S_{48} \\
 & \text{Symm} & & & S_{55} & S_{56} & S_{57} & S_{58} \\
 & & & & S_{65} & S_{66} & S_{67} & S_{68} \\
 & & & & & & S_{77} & S_{78} \\
 & & & & & & S_{87} & S_{88}
\end{bmatrix}
\begin{array}{c}(\varepsilon_1)\ (\varepsilon_2)\ (\varepsilon_3^s)\ (\varepsilon_4^s)\ (\varepsilon_5^s)\ (\varepsilon_6^s)\ (\varepsilon_7)\ (\varepsilon_8)\end{array}
$$

$$
[S_{11}] = \frac{EA}{l_0}, \qquad
[S_{23}\ \ S_{24}] = \frac{2\,GD_{\omega z}\,\Phi_\omega\,\Phi_z}{l_0(1+\Phi_\omega)(1+\Phi_z)}[-1\ \ 1], \qquad
[S_{25}\ \ S_{26}] = \frac{2\,GD_{\omega y}\,\Phi_\omega\,\Phi_y}{l_0(1+\Phi_\omega)(1+\Phi_y)}[1\ \ -1],
$$

$$
\begin{bmatrix} S_{22} & S_{27} & S_{28} \\ & S_{77} & S_{78} \\ & & S_{88} \end{bmatrix}
= \frac{GI_t}{30l_0^3(1+\Phi_\omega)^2}
\begin{bmatrix} 36+P_1 & -3 & -3 \\ & 4+P_2 & -(1+P_2) \\ & & 4+P_2 \end{bmatrix}
+ \frac{EI_\omega}{l_0^5(1+\Phi_\omega)}
\begin{bmatrix} 12 & -6 & -6 \\ & 4+\Phi_\omega & 2-\Phi_\omega \\ & & 4+\Phi_\omega \end{bmatrix},
$$

$$\begin{bmatrix} S_{33} & S_{34} \\ S_{43} & S_{44} \end{bmatrix} = \frac{EI_y}{l_0^3(1+\Phi_z)} \begin{bmatrix} 4+\Phi_z & -2+\Phi_z \\ -2+\Phi_z & 4+\Phi_z \end{bmatrix}, \quad \begin{bmatrix} S_{35} & S_{36} \\ S_{45} & S_{46} \end{bmatrix} = \frac{GD_{yz}\,\Phi_y\,\Phi_z}{4l_0(1+\Phi_y)(1+\Phi_z)} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$\begin{bmatrix} S_{37} & S_{38} \\ S_{47} & S_{48} \end{bmatrix} = \frac{GD_{\omega z}\,\Phi_\omega\,\Phi_z}{4l_0^2(1+\Phi_\omega)(1+\Phi_z)} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},$$

$$\begin{bmatrix} S_{55} & S_{56} \\ S_{65} & S_{66} \end{bmatrix} = \frac{EI_z}{l_0^3(1+\Phi_y)} \begin{bmatrix} 4+\Phi_y & -2+\Phi_y \\ -2+\Phi_y & 4+\Phi_y \end{bmatrix}, \quad \begin{bmatrix} S_{57} & S_{58} \\ S_{67} & S_{68} \end{bmatrix} = \frac{GD_{\omega y}\,\Phi_\omega\,\Phi_y}{4l_0^2(1+\Phi_\omega)(1+\Phi_y)} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.$$

In the above equations, $P_1 = 30\,\Phi_\omega\,(2+\Phi_\omega)$ and $P_2 = 5\,\Phi_\omega\,(1+\Phi_\omega/2)$. The shear coefficients $D_\omega, D_{\omega y}, D_{\omega z}$ and D_{yz} for simple cross-sections can be calculated using a pre-processing procedure [21, 22].

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