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Solvability conditions and design for H_∞ & H_2 almost state synchronization of homogeneous multi-agent systems

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ABSTRACT

This paper studies the H_∞ and H_2 almost state synchronization problem for homogeneous multi-agent systems with general linear agents affected by external disturbances and with a directed communication topology. Agents are connected via diffusive full-state coupling or diffusive partial-state coupling. A necessary and sufficient condition is developed for the solvability of the H_∞ and H_2 almost state synchronization problem. Moreover, a family of protocols based on either an algebraic Riccati equation (ARE) method or a directed eigen structure assignment method are developed such that the impact of disturbances on the network disagreement dynamics, expressed in terms of the H_∞ and H_2 norm of the corresponding closed-loop transfer function, is reduced to any arbitrarily small value. The protocol for full-state coupling is static, while for partial-state coupling it is dynamic.

1. Introduction

Over the past decade, the synchronization problem of multi-agent system (MAS) has received substantial attention because of its potential applications in cooperative control of autonomous vehicles, distributed sensor network, swarming and flocking and others. The objective of synchronization is to secure an asymptotic agreement on a common state or output trajectory through decentralized protocols (see [1,8,16,30] and references therein).

State synchronization inherently requires homogeneous MAS (i.e. agents have identical dynamics). Most works have focused on state synchronization based on diffusive full-state coupling, where the agent dynamics progress from single- and double-integrator dynamics (e.g. [9,10,13–15]) to more general dynamics (e.g. [22,27,29,31]). State synchronization based on diffusive partial-state coupling has also been considered (e.g. [3,4,22–25,28]).

Most research has focused on the idealized case where the agents are not affected by external disturbances. In the literature where external disturbances are considered, γ-suboptimal H_∞ design is developed for MAS to achieve H_∞ norm from an external disturbance to the synchronization error among agents less than an, a priori given, bound γ. In particular, [4,34] considered the H_∞ norm from an external disturbance to the output error among agents. [21] considered the H_∞ norm from an external disturbance to the state error among agents. These papers do not present an explicit methodology for designing protocols. Refs. [5] and [6] try to obtain an H_∞ norm from a disturbance to the average of the states in a network of single or double integrators.

By contrast, Peymani et al. [11] introduced the notion of H_∞ *almost synchronization* for homogeneous MAS, where the goal is to reduce the H_∞ norm from an external disturbance to the synchronization *error*, to any arbitrary desired level. But it requires an additional layer of communication among distributed controllers, which is completely dispensed in this paper. This work is extended later in Refs. [12,32], and [33]. Ref. [33], where heterogeneous MAS are considered, provides a solution for the case of right-invertible agents with the addional objective beyond output synchronization that the agents track a regulated signal given to some or all of the agents. Although homogeneous MAS, which are considered in this paper, are a subset of heterogeneous MAS, the results of Zhang et al. [33] cannot be directly applied to the case of full-state coupling since the agents are not right-invertible. Secondly, the results for synchronization without regulation cannot be obtained from results obtained for regulated synchronization. Thirdly, we consider state synchronization instead of output synchronization in both full- and partial-state coupling. Finally, by restricting to homogeneous networks more explicit designs can be obtained under weaker conditions.

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In this paper, we will study H_∞ *almost state synchronization* for a MAS with full-state coupling or partial-state coupling. We will also study H_2 almost state synchronization, since it is closely related to the problems of H_∞ almost state synchronization. In H_∞ we look at the worst case disturbance with the only constraints being the power, while in H_2 we only consider white noise disturbances which is a more restrictive class. In both cases, disturbances or noises are restricted in the process, not in the measurement. Our contribution in this paper is three-fold.

- We obtain necessary and sufficient conditions for H_∞ and H_2 almost state synchronization for a MAS in the presence of external disturbances
- We develop a protocol design for H_∞ and H_2 almost state synchronization based on an algebraic Riccati equation (ARE) method
- We develop a protocol design for H_∞ and H_2 almost state synchronization based on an asymptotic time-scale eigenstructure assignment (ATEA) method for the full-state coupling case, and on the direct eigenstructure assignment method for the partial-state coupling case.

It is worth noting that our solvability conditions and protocol designs are developed for a MAS associated with a set of network graphs. Specifically, only rough information of a network graph is utilized.

1.1. Notations and definitions

Given a matrix $A \in \mathbb{C}^{m\times n}$, A' denotes its conjugate transpose, $\|A\|$ is the induced 2-norm. A square matrix A is said to be Hurwitz stable if all its eigenvalues are in the open left half complex plane. $A \otimes B$ depicts the Kronecker product between A and B. I_n denotes the n-dimensional identity matrix and 0_n denotes $n \times n$ zero matrix; sometimes we drop the subscript if the dimension is clear from the context. Given a complex number λ, $\mathrm{Re}(\lambda)$ is the real part of λ and $\mathrm{Im}(\lambda)$ is the imaginary part of λ.

A *weighted directed graph* $\mathcal{G}$ is defined by a triple $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, \dots, N\}$ is a node set, $\mathcal{E}$ is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the weighting matrix, and $a_{ij} > 0$ iff $(i, j) \in \mathcal{E}$ which denotes an *edge* from node j to node i. In our case, we have $a_{ii} = 0$. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1, \dots, i_k\}$ such that $(i_{j+1}, i_j) \in \mathcal{E}$ for $j = 1, \dots, k-1$. A *directed tree* is a subgraph (subset of nodes and edges) in which every node has exactly one parent node except for one node, called the *root*, which has no parent node. In this case, the root has a directed path to every other node in the tree. A *directed spanning tree* is a directed tree containing all the nodes of the graph. For a weighted graph $\mathcal{G}$, a matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^N a_{ik}, & i = j, \\ -a_{ij}, & i \neq j, \end{cases}$$

is called the *Laplacian matrix* associated with the graph $\mathcal{G}$. The Laplacian L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector **1**. A specific class of graphs needed in this paper is presented below:

Definition 1. For any given $\alpha \geq \beta > 0$, let $\mathbb{G}^N_{\alpha,\beta}$ denote the set of directed graphs with N nodes that contain a directed spanning tree and for which the corresponding Laplacian matrix L satisfies $\|L\| < \alpha$ while its nonzero eigenvalues have a real part larger than or equal to β.

2. Problem formulation

Consider a MAS composed of N identical linear time-invariant agents of the form,

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i + E\omega_i, \\ y_i &= Cx_i, \end{aligned} \qquad (i = 1, \dots, N) \tag{1}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are respectively the state, input, and output vectors of agent i, and $\omega_i \in \mathbb{R}^\omega$ is the external disturbances.

The communication network provides each agent with a linear combination of its own outputs relative to that of other neighboring agents. In particular, each agent $i \in \{1, \dots, N\}$ has access to the quantity,

$$\zeta_i = \sum_{j=1}^N a_{ij}(y_i - y_j), \tag{2}$$

where $a_{ij} \geq 0$ and $a_{ii} = 0$ indicate the communication among agents. This communication topology of the network can be described by a weighted and directed graph $\mathcal{G}$ with nodes corresponding to the agents in the network and the weight of edges given by the coefficient a_{ij}. In terms of the coefficients of the Laplacian matrix L, ζ_i can be rewritten as

$$\zeta_i = \sum_{j=1}^N \ell_{ij} y_j. \tag{3}$$

We refer to this case as *partial-state coupling*. Note that if C has full column rank then, without loss of generality, we can assume that $C = I$, and the quantity ζ_i becomes

$$\zeta_i = \sum_{j=1}^N a_{ij}(x_i - x_j) = \sum_{j=1}^N \ell_{ij} x_j. \tag{4}$$

We refer to this case as *full-state coupling*.

If the graph $\mathcal{G}$ describing the communication topology of the network contains a directed spanning tree, then it follows from [15, Lemma 3.3] that the Laplacian matrix L has a simple eigenvalue at the origin, with the corresponding right eigenvector **1** and all the other eigenvalues are in the open right-half complex plane. Let $\lambda_1, \dots, \lambda_N$ denote the eigenvalues of L such that $\lambda_1 = 0$ and $\mathrm{Re}(\lambda_i) > 0$, $i = 2, \dots, N$.

Let N be any agent and define $\bar{x}_i = x_N - x_i$ and

$$\bar{x} = \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_{N-1} \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_N \end{pmatrix}.$$

Obviously, synchronization is achieved if $\bar{x} = 0$. That is

$$\lim_{t\to\infty} (x_i(t) - x_N(t)) = 0, \quad \forall i, \in \{1, \dots, N-1\}. \tag{5}$$

We denote by $T_{\omega\bar{x}}$, the transfer function from ω to $\bar{x}$

Remark 1. Agent N is not necessarily a root agent. Obviously, (5) is equivalent to the condition that

$$\lim_{t\to\infty} (x_i(t) - x_j(t)) = 0, \quad \forall i, j \in \{1, \dots, N\}.$$

We formulate below four almost state synchronization problems for a network with either H_2 or H_∞ almost synchronization.

Problem 1. Consider a MAS described by (1) and (4). Let $\mathbf{G}$ be a given set of graphs such that $\mathbf{G} \subseteq \mathbb{G}^N$. The ***$H_\infty$* almost state synchronization problem via full-state coupling (in short H_∞-ASSFS)** with a set of network graphs $\mathbf{G}$ is to find, if possible, a linear static protocol parameterized in terms of a parameter ε, of the form,

$$u_i = F(\varepsilon)\zeta_i, \tag{6}$$

such that, for any given real number $r>0$, there exists an ε^* such that for any $\varepsilon\in(0,\varepsilon^*]$ and for any graph $\mathcal{G}\in\mathbf{G}$, (5) is satisfied for all initial conditions in the absence of disturbances and the closed loop transfer matrix $T_{\omega\bar{x}}$ satisfies

$$\|T_{\omega\bar{x}}\|_\infty < r. \tag{7}$$

Problem 2. Consider a MAS described by (1) and (3). Let $\mathbf{G}$ be a given set of graphs such that $\mathbf{G}\subseteq\mathbb{G}^N$. The ***$H_\infty$*** almost state synchronization problem via partial-state coupling (in short **H_∞-ASSPS)** with a set of network graphs $\mathbf{G}$ is to find, if possible, a linear time-invariant dynamic protocol parameterized in terms of a parameter ε, of the form,

$$\begin{aligned}\dot{\chi}_i &= A_c(\varepsilon)\chi_i + B_c(\varepsilon)\zeta_i,\\ u_i &= C_c(\varepsilon)\chi_i + D_c(\varepsilon)\zeta_i,\end{aligned} \tag{8}$$

where $\chi_i\in\mathbb{R}^{n_c}$, such that, for any given real number $r>0$, there exists an ε^* such that for any $\varepsilon\in(0,\varepsilon^*]$ and for any graph $\mathcal{G}\in\mathbf{G}$, (5) is satisfied for all initial conditions in the absence of disturbances and the closed loop transfer matrix $T_{\omega\bar{x}}$ satisfies (7).

Problem 3. Consider a MAS described by (1) and (4). Let $\mathbf{G}$ be a given set of graphs such that $\mathbf{G}\subseteq\mathbb{G}^N$. The ***$H_2$*** almost state synchronization problem via full-state coupling (in short **H_2-ASSFS)** with a set of network graphs $\mathbf{G}$ is to find, if possible, a linear static protocol parameterized in terms of a parameter ε, of the form (6) such that, for any given real number $r>0$, there exists an ε^* such that for any $\varepsilon\in(0,\varepsilon^*]$ and for any graph $\mathcal{G}\in\mathbf{G}$, (5) is satisfied for all initial conditions in the absence of disturbances and the closed loop transfer matrix $T_{\omega\bar{x}}$ satisfies

$$\|T_{\omega\bar{x}}\|_2 < r. \tag{9}$$

Problem 4. Consider a MAS described by (1) and (3). Let $\mathbf{G}$ be a given set of graphs such that $\mathbf{G}\subseteq\mathbb{G}^N$. The ***$H_2$*** almost state synchronization problem via partial-state coupling (in short **H_2-ASSPS)** with a set of network graphs $\mathbf{G}$ is to find, if possible, a linear time-invariant dynamic protocol parameterized in terms of a parameter ε, of the form (8) such that, for any given real number $r>0$, there exists an ε^* such that for any $\varepsilon\in(0,\varepsilon^*]$ and for any graph $\mathcal{G}\in\mathbf{G}$, (5) is satisfied for all initial conditions in the absence of disturbances and the closed loop transfer matrix $T_{\omega\bar{x}}$ satisfies (9).

Note that the problems of H_∞ almost state synchronization and H_2 almost state synchronization are closely related. Roughly speaking, H_2 almost synchronization is easier to achieve than H_∞ almost synchronization. This is related to the fact that in H_∞ we look at the worst case disturbance with the only constraints being the power:

$$\limsup_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\omega_i'(t)\omega_i(t)dt<\infty.$$

while in H_2 we only consider white noise disturbances which is a more restrictive class.

3. MAS with full-state coupling

In this section, we establish a connection between the almost state synchronization among agents in the network and a robust H_∞ or H_2 almost disturbance decoupling problem via state feedback with internal stability (in short H_∞ or H_2-ADDPSS) (see [20]). Then, we use this connection to derive the necessary and sufficient condition and design appropriate protocols.

3.1. Necessary and sufficient condition for H_∞-ASSFS

The MAS system described by (1) and (4) after implementing the linear static protocol (6) is described by

$$\dot{x}_i = Ax_i + BF(\varepsilon)\zeta_i + E\omega_i,$$

for $i=1,\dots,N$. Let

$$x=\begin{pmatrix}x_1\\ \vdots\\ x_N\end{pmatrix},\quad \omega=\begin{pmatrix}\omega_1\\ \vdots\\ \omega_N\end{pmatrix}.$$

Then, the overall dynamics of the N agents can be written as

$$\dot{x} = (I_N\otimes A + L\otimes BF(\varepsilon))x + (I_N\otimes E)\omega. \tag{10}$$

We define the robust H_∞-ADDPSS with bounded input as follows. Given $\Lambda\subset\mathbb{C}$, there should exist $M>0$ such that for any given real number $r>0$, we can find a parameterized controller

$$u = F(\varepsilon)x \tag{11}$$

for the following subsystem,

$$\dot{x} = Ax + \lambda Bu + B\omega, \tag{12}$$

such that for any $\lambda\in\Lambda$ the following hold:

1. The interconnection of the systems (12) and (11) is internally stable;
2. The resulting closed-loop transfer function $T_{\omega x}$ from ω to x has an H_∞ norm less than r.
3. The resulting closed-loop transfer function $T_{\omega u}$ from ω to u has an H_∞ norm less than M.

In the above, Λ denotes all possible locations for the nonzero eigenvalues of the Laplacian matrix L when the graph varies over the set $\mathbf{G}$. It is also important to note that M is independent of the choice for r.

In the following lemma we give a necessary condition for the H_∞-ASSFS. Moreover, for sufficiency, we connect the H_∞-ASSFS problem to the robust H_∞-ADDPSS with bounded input problem which we will address later.

Lemma 1. *Let $\mathbf{G}$ be a set of graphs such that the associated Laplacian matrices are uniformly bounded and let Λ consist of all possible nonzero eigenvalues of Laplacian matrices associated with graphs in $\mathbf{G}$.*

*(*Necessity*) The H_∞-ASSFS for the MAS described by (1) and (4) given $\mathbf{G}$ is solvable by a parameterized protocol $u_i=F(\varepsilon)\zeta_i$ only if*

$$\operatorname{im} E\subset \operatorname{im} B. \tag{13}$$

*(*Sufficiency*) The H_∞-ASSFS for the MAS described by (1) and (4) given $\mathbf{G}$ is solved by a parameterized protocol $u_i=F(\varepsilon)\zeta_i$ if the robust H_∞-ADDPSS with bounded input for the system (12) with $\lambda\in\Lambda$ is solved by the parameterized controller $u=F(\varepsilon)x$.*

Proof. Note that L has eigenvalue 0 with associated right eigenvector $\mathbf{1}$. Let

$$L = TS_LT^{-1}, \tag{14}$$

with T unitary and S_L the upper-triangular Schur form associated to the Laplacian matrix L such that $S_L(1,1)=0$. Let

$$\eta := (T^{-1}\otimes I_n)x = \begin{pmatrix}\eta_1\\ \vdots\\ \eta_N\end{pmatrix},\qquad \bar{\omega} = (T^{-1}\otimes I)\omega = \begin{pmatrix}\bar{\omega}_1\\ \vdots\\ \bar{\omega}_N\end{pmatrix}$$

where $\eta_i\in\mathbb{C}^n$ and $\bar{\omega}_i\in\mathbb{C}^q$. In the new coordinates, the dynamics of η can be written as

$$\dot{\eta}(t) = (I_N\otimes A + S_L\otimes BF(\varepsilon))\eta + (T^{-1}\otimes E)\omega, \tag{15}$$

which is rewritten as

$$\dot{\eta}_1 = A\eta_1 + \sum_{j=2}^{N} s_{1j}BF(\varepsilon)\eta_j + E\bar{\omega}_1,$$

$$\dot{\eta}_i = (A+\lambda_i BF(\varepsilon))\eta_i + \sum_{j=i+1}^{N} s_{ij}BF(\varepsilon)\eta_j + E\bar{\omega}_i,$$
$$\dot{\eta}_N = (A+\lambda_N BF(\varepsilon))\eta_N + E\bar{\omega}_N, \tag{16}$$

for $i\in\{2,\dots,N-1\}$ where $S_L=[s_{ij}]$. The first column of T is an eigenvector of L associated to eigenvalue 0 with length 1, i.e. it is equal to $\pm\mathbf{1}/\sqrt{N}$. Using this we obtain:

$$\bar{x} = \begin{pmatrix} x_N-x_1 \\ x_N-x_2 \\ \vdots \\ x_N-x_{N-1}\end{pmatrix} = \left(\begin{pmatrix} -1 & 0 & \cdots & 0 & 1 \\ 0 & -1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & -1 & 1\end{pmatrix}\otimes I_n\right)(T\otimes I_n)\eta = \left(\begin{pmatrix}0 & V\end{pmatrix}\otimes I_n\right)\eta,$$

for some suitably chosen matrix V. Therefore we have

$$\bar{x} = (V\otimes I_n)\begin{pmatrix}\eta_2\\ \vdots \\ \eta_N\end{pmatrix}, \tag{17}$$

Note that since T is unitary, also the matrix T^{-1} is unitary and the matrix V is uniformly bounded. Therefore the H_∞ norm of the transfer matrix from ω to $\bar{x}$ can be made arbitrarily small if and only if the H_∞ norm of the transfer matrix from $\bar{\omega}$ to η can be made arbitrarily small.

In order for the H_∞ norm from $\bar{\omega}$ to η to be arbitrarily small we need the H_∞ norm from $\bar{\omega}_N$ to η_N to be arbitrarily small. From classical results (see [18,26]) on H_∞ almost disturbance decoupling we find that this is only possible if (13) is satisfied.

Conversely, suppose $u=F(\varepsilon)x$ solves the robust H_∞-ADDPSS with bounded input for (12) and assume (13) is satisfied. We show next that $u_i=F(\varepsilon)\zeta_i$ solves the H_∞-ASSFS for the MAS described by (1) and (4). Let X be such that $E=BX$.

The fact that $u=F(\varepsilon)x$ solves the robust H_∞-ADDPSS with bounded input for (12) implies that for small ε we have that $A+\lambda BF(\varepsilon)$ is asymptotically stable for all $\lambda\in\Lambda$. In particular, $A+\lambda_i BF(\varepsilon)$ is asymptotically stable for $i=2,\dots,N$ which guarantees that $\eta_i\to 0$ for $i=2,\dots,N$ for zero disturbances and all initial conditions. Therefore we have state synchronization.

Next, we are going to show that for any $\bar{r}>0$, we can choose ε sufficiently small such that the transfer matrix from $\bar{\omega}$ to η_i is less than $\bar{r}$ for $i=2,\dots,N$. This guarantees that we can achieve (7) for any $r>0$. We have that

$$T^{\lambda}_{\omega x}(s) = (sI-A-\lambda BF(\varepsilon))^{-1}B,$$
$$T^{\lambda}_{\omega u}(s) = F(\varepsilon)(sI-A-\lambda BF(\varepsilon))^{-1}B.$$

For a given M and parameter ε, the following is satisfied

$$\|T^{\lambda}_{\omega x}\|_\infty < \tilde{r}_\varepsilon, \qquad \|T^{\lambda}_{\omega u}\|_\infty < M$$

for all $\lambda\in\Lambda$ where $\tilde{r}_\varepsilon$ is a parameter depending on ε with the property that $\lim_{\varepsilon\downarrow 0}\tilde{r}_\varepsilon=0$. Denote $\nu_i=F(\varepsilon)\eta_i$.

When $i=N$, it is easy to find that,

$$T_{\bar{\omega}\eta_N} = T^{\lambda_N}_{\omega x}\begin{pmatrix}0 & \cdots & 0 & X\end{pmatrix},$$
$$T_{\bar{\omega}\nu_N} = T^{\lambda_N}_{\omega u}\begin{pmatrix}0 & \cdots & 0 & X\end{pmatrix}$$

and hence

$$\|T_{\bar{\omega}\eta_N}\|_\infty < \bar{r}, \qquad \|T_{\bar{\omega}\nu_N}\|_\infty < \bar{M}_N$$

provided

$$\|X\|\tilde{r}_\varepsilon < \bar{r}, \qquad \|X\|M < \bar{M}_N. \tag{18}$$

Recall that we can make $\tilde{r}_\varepsilon$ arbitrarily small by reducing ε without affecting the bound M. Assume

$$\|T_{\bar{\omega}\eta_j}\|_\infty < \bar{r}, \qquad \|T_{\bar{\omega}\nu_j}\|_\infty < \bar{M}_j$$

holds for $j=i+1,\dots,N$. We have:

$$T_{\bar{\omega}\eta_i}(s) = T^{\lambda_i}_{\omega x}(s)\left[e_i\otimes X + \sum_{j=i+1}^{N} s_{ij}T_{\bar{\omega}\nu_j}(s)\right]$$
$$T_{\bar{\omega}\nu_i}(s) = T^{\lambda_i}_{\omega u}(s)\left[e_i\otimes X + \sum_{j=i+1}^{N} s_{ij}T_{\bar{\omega}\nu_j}(s)\right]$$

where e_i is a row vector of dimension N with elements equal to zero except for the ith component which is equal to 1. Since

$$\left\| e_i\otimes X + \sum_{j=i+1}^{N} s_{ij}T_{\bar{\omega}\nu_j}\right\|_\infty < \|X\| + \sum_{j=i+1}^{N}|s_{ij}|\bar{M}_j$$

we find:

$$\|T_{\bar{\omega}\eta_i}\|_\infty < \bar{r}, \qquad \|T_{\bar{\omega}\nu_i}\|_\infty < \bar{M}_i \tag{19}$$

provided:

$$\left(\|X\| + \sum_{j=i+1}^{N}|s_{ij}|\bar{M}_j\right)\tilde{r}_\varepsilon < \bar{r}, \quad \left(\|X\| + \sum_{j=i+1}^{N}|s_{ij}|\bar{M}_j\right)\tilde{M} < \bar{M}_i. \tag{20}$$

Note that s_{ij} depends on the graph in $\mathbf{G}$ but since the Laplacian matrices associated to graphs in $\mathbf{G}$ are uniformly bounded we find that also the s_{ij} are uniformly bounded. In this way for any arbitrary $\bar{r}$, we can recursively obtain the bounds in (19) for $i=2,\dots,N$ provided we choose ε sufficiently small such that the corresponding $\tilde{r}_\varepsilon$ satisfies (18) and (20) for $i=2,\dots,N-1$. Hence, we can choose ε sufficiently small such that the transfer matrix from $\bar{\omega}$ to η_i is less than $\bar{r}$ for $i=2,\dots,N$. As noted before this guarantees that we can achieve (7) for any $r>0$. □

For the case where the set of graphs $\mathbf{G}$ equals $\mathbb{G}^N_{\alpha,\beta}$ for some given α, $\beta>0$, we develop necessary and sufficient conditions for the solvability of the H_∞-ASSFS for MAS as follows:

Theorem 1. *Consider a MAS described by (1) and (4) with an associated graph from the set $\mathbf{G}=\mathbb{G}^N_{\alpha,\beta}$ for some α, $\beta>0$.*

Then, the H_∞-ASSFS is solvable if and only if (13) is satisfied and (A, B) is stabilizable.

Proof. From Lemma 1, we note that (13) is actually a necessary condition for H_∞-ASSFS. Clearly, also (A, B) stabilizable is a necessary condition. Sufficiency is a direct result of Theorems 3 or5 for H_∞-ASSFS. □

3.2. Necessary and sufficient conditions for H_2*-ASSFS*

We define the robust H_2-ADDPSS with bounded input as follows. Given $\Lambda\subset\mathbb{C}$, there should exist $M>0$ such that for any given real number $r>0$, we can find a parameterized controller (11) for the system, (12) such that the following holds for any $\lambda\in\Lambda$:

1. The interconnection of the systems (11) and (12) is internally stable;
2. The resulting closed-loop transfer function $T_{\omega x}$ from ω to x has an H_2 norm less than r.
3. The resulting closed-loop transfer function $T_{\omega u}$ from ω to u has an H_∞ norm less than M.

In the above, Λ denotes all possible locations for the nonzero eigenvalues of the Laplacian matrix L when the graph varies over the set $\mathbf{G}$. It is also important to note that M is independent of the choice for r. Note that we need to consider two aspects in our controller H_2 disturbance rejection and robust stabilization (because of a set of network graphs $\mathbb{G}^N_{\alpha,\beta}$). The latter translates in the H_∞ norm constraint from ω to u.

Lemma 2. *Let* **G** *be a set of graphs such that the associated Laplacian matrices are uniformly bounded and let Λ consist of all possible nonzero eigenvalues of Laplacian matrices associated with graphs in* **G**.

*(*Necessity*) The H_2-ASSFS for the MAS described by (1) and (4) given* **G** *is solvable by a parameterized protocol $u_i = F(\varepsilon)\zeta_i$ only if (13) is satisfied.*

*(*Sufficiency*)The H_2-ASSFS for the MAS described by (1) and (4) given* **G** *is solvable by a parameterized protocol $u_i = F(\varepsilon)\zeta_i$ if the robust H_2-ADDPSS with bounded input for the system (12) with $\lambda \in \Lambda$ is solved by the parameterized controller $u = F(\varepsilon)x$.*

Proof. The proof is similar to the proof of Lemma 1. This time we need the H_2 norm from $\bar{\omega}_N$ to η_N to be arbitrarily small and also H_2 almost disturbance decoupling then immediately yields that we need that (13) is satisfied.

The rest of the proof follows the same lines except that we require the H_2 norm from $\bar{\omega}$ to η_j arbitrarily small while we keep the H_∞ norm from $\bar{\omega}$ to ν_j bounded. Recall that for any two stable, strictly proper transfer matrices T_1 and T_2 we have:

$$\|T_1T_2\|_2 \le \|T_1\|_2\|T_2\|_\infty$$

which we need in the modifications of the proof of Lemma 1. □

For the case with a set of graph $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ (with given α, $\beta > 0$), we develop necessary and sufficient conditions for the solvability of the H_2-ASSFS for MAS as follows:

Theorem 2. *Consider a MAS described by (1) and (4) with an associated graph from the set $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ for some α, $\beta > 0$.*

Then, the H_2-ASSFS is solvable if and only if (13) is satisfied and (A, B) is stabilizable.

Proof. We have already noted before that (13) is actually a necessary condition for H_2-ASSFS. Clearly, also (A, B) being stabilizable is a necessary condition. Sufficiency for H_2-ASSFS, is a direct result of either Theorems 4 or6. □

3.3. Protocol design for H_∞-ASSFS and H_2-ASSFS

We present below two protocol design methods for both H_∞-ASSFS and H_2-ASSFS problems. One relies on an algebraic Riccati equation (ARE), and dummyTXdummy- the other is based on an asymptotic time-scale eigenstructure assignment (ATEA) method.

3.3.1. ARE-based method

Using an algebraic Riccati equation, we can design a suitable protocol provided (A, B) is stabilizable. We consider the protocol,

$$u_i = \rho F\zeta_i, \tag{21}$$

where $\rho = \frac{1}{\varepsilon}$ and $F = -B'P$ with P being the unique solution of the continuous-time algebraic Riccati equation

$$A'P + PA - 2\beta PBB'P + I = 0, \tag{22}$$

where β is a lower bound for the real part of the non-zero eigenavlues of all Laplacian matrices associated with a graph in $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$.

The main result regarding H_∞-ASSFS is stated as follows.

Theorem 3. *Consider a MAS described by (1) and (4) such that (13) is satisfied. Let any real numbers α, $\beta > 0$ and a positive integer N be given, and hence a set of network graphs $\mathbb{G}^N_{\alpha,\beta}$ be defined.*

If (A, B) is stabilizable then the H_∞-ASSFS stated in Problem 1 with $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ is solvable. In particular, for any given real number $r > 0$, there exists an ε^*, such that for any $\varepsilon \in (0, \varepsilon^*)$, the protocol (21) achieves state synchronization and the resulting system from ω to $x_i - x_j$ has an H_∞ norm less than r for any $i, j \in 1, \dots, N$ and for any graph $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$.

Proof. Using Lemma 1, we know that we only need to verify that $u = \rho Fx$ solves the robust H_∞-ADDPSS with bounded input for the system (12) with $\lambda \in \Lambda$. Given $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$, we know that $\lambda \in \Lambda$ implies $\operatorname{Re}\lambda \ge \beta$. Clearly, the Laplacian matrices are uniformly bounded since $\|L\| \le \alpha$.

Consider the interconnection of (12) and $u = \rho Fx$. We define

$$V(x) = x'Px$$

and we obtain:

$$\begin{aligned}
\dot{V} &= x'(A - \rho\lambda BB'P)'Px + \omega'B'Px + x'P(A - \rho\lambda BB'P)x + x'PB\omega \\
&= x'PBB'Px - x'x - 2\rho\beta x'PBB'Px + 2x'PB\omega \\
&\le (1 - \frac{\beta}{\varepsilon})x'PBB'Px - x'x + \frac{\varepsilon}{\beta}\omega'\omega \\
&\le -\frac{\beta}{2}\varepsilon u'u - x'x + \frac{\varepsilon}{\beta}\omega'\omega
\end{aligned}$$

which implies that the system is asymptotically stable and the H_∞ norm of the transfer function from ω to x is less that ε/β while the H_∞ norm of the transfer function from ω to u is less that $2/\beta^2$. Therefore, $u = \rho Fx$ solves the robust H_∞-ADDPSS with bounded input for the system (12) as required. □

For H_2-ASSFS we have the following classical result:

Lemma 3. *Consider an asymptotically stable system:*

$$\dot{p} = A_1 p + B_1\omega$$

The H_2 norm from ω to p is less than δ if there exists a matrix Q such that:

$$A_1Q + QA_1' + B_1B_1' \le 0, \qquad Q < \delta I$$

The main result regarding H_2-ASSFS is stated as follows.

Theorem 4. *Consider a MAS described by (1) and (4) such that (13) is satisfied. Let any real numbers α, $\beta > 0$ and a positive integer N be given, and hence a set of network graphs $\mathbb{G}^N_{\alpha,\beta}$ be defined.*

If (A, B) is stabilizable then the H_2-ASSFS stated in Problem 3 with $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ is solvable. In particular, for any given real number $r > 0$, there exists an ε^*, such that for any $\varepsilon \in (0, \varepsilon^*)$, the protocol (21) achieves state synchronization and the resulting system from ω to $x_i - x_j$ has an H_∞ norm less than r for any $i, j \in 1, \dots, N$ and for any graph $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$.

Proof. Using Lemma 2, we know that we only need to verify that $u = \rho Fx$ solves the robust H_2-ADDPSS with bounded input for the system (12) with $\lambda \in \Lambda$. We use the same feedback as in the proof of Theorem 3. In the proof of Theorem 3 it is already shown that the closed loop system is asymptotically stable and the H_∞ norm of the transfer function from ω to u is bounded. The only remaining part of the proof is to show that the H_2 norm from ω to x can be made arbitrarily small. Using the algebraic Riccati equation it is easy to see that we have:

$$(A - \rho\lambda BB'P)'P + P(A - \rho\lambda BB'P) + \rho\beta PBB'P \le 0$$

for large ρ. But then we have:

$$Q_\varepsilon(A - \rho\lambda BB'P)' + (A - \rho\lambda BB'P)Q_\varepsilon + BB' \le 0$$

for $Q_\varepsilon = \varepsilon\beta^{-1}P^{-1}$. Then Lemma 3 immediately yields that we can make the H_2 norm from ω to x arbitrarily small by choosing a sufficiently small ε. □

3.3.2. ATEA-based method

The ATEA-based design is basically a method of time-scale structure assignment in linear multivariable systems by high-gain feedback [19]. In the current case, we do not need the full structure presented in the above method. It is sufficient to note that there exists non-singular transformation matrix $T_x \in \mathbb{R}^{n\times n}$ (See [17, Theorem 1]) such that

$$\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = T_x x, \tag{23}$$

and the dynamics of $\hat{x}$ is represented as

$$\begin{aligned} \dot{\hat{x}}_1 &= \bar{A}_{11}\hat{x}_1 + \bar{A}_{12}\hat{x}_2, \\ \dot{\hat{x}}_2 &= \bar{A}_{21}\hat{x}_1 + \bar{A}_{22}\hat{x}_2 + \lambda \bar{B}u + \bar{B}\omega, \end{aligned} \tag{24}$$

with $\bar{B}$ invertible, and that (A, B) is stabilizable implies that $(\bar{A}_{11}, \bar{A}_{12})$ is stabilizable.

Choose F_1 such that $\bar{A}_{11} + \bar{A}_{12}F_1$ is asymptotically stable. In that case a suitable protocol for (1) is

$$u_i = F_\varepsilon \zeta_i, \tag{25}$$

where F_ε is designed as

$$F_\varepsilon = \frac{1}{\varepsilon}\bar{B}^{-1}\begin{pmatrix} F_1 & -I \end{pmatrix} T_x \tag{26}$$

The main result regarding H_∞-ASSFS is stated as follows. The result is basically the same as Theorem 3 except for a different design protocol.

Theorem 5. *Consider a MAS described by (1) and (4) such that (13) is satisfied. Let any real numbers α, $\beta > 0$ and a positive integer N be given, and hence a set of network graphs $\mathbb{G}^N_{\alpha,\beta}$ be defined.*

If (A, B) is stabilizable then the H_∞-ASSFS stated in Problem 1 with $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ is solvable. In particular, for any given real number $r > 0$, there exists an ε^*, such that for any $\varepsilon \in (0, \varepsilon^*)$, the protocol (25) achieves state synchronization and the resulting system from ω to $x_i - x_j$ has an H_∞ norm less than r for any $i, j \in 1, \dots, N$ and for any graph $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$.

Proof. Similarly to the proof of Theorem 3, we only need to establish that $u = F_\varepsilon x$ solves the robust H_∞-ADDPSS with bounded input for the system (12) with $\lambda \in \Lambda$. Given $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$, we know that $\lambda \in \Lambda$ implies $\operatorname{Re}\lambda \geq \beta$.

After a basis transformation, the interconnection of (12) and $u = F_\varepsilon x$ is equal to the interconnection of (24) and (25). We obtain:

$$\begin{aligned} \dot{\hat{x}}_1 &= \bar{A}_{11}\hat{x}_1 + \bar{A}_{12}\hat{x}_2, \\ \varepsilon\dot{\hat{x}}_2 &= (\varepsilon\bar{A}_{21} + \lambda F_1)\hat{x}_1 + (\varepsilon\bar{A}_{22} - \lambda I)\hat{x}_2 + \varepsilon\bar{B}\omega. \end{aligned} \tag{27}$$

Define

$$\tilde{x}_1 = \hat{x}_1, \quad \tilde{x}_2 = \hat{x}_2 - F_1\hat{x}_1.$$

Then we can write this system (27) in the form:

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{A}_{11}\tilde{x}_1 + \tilde{A}_{12}\tilde{x}_2, \\ \varepsilon\dot{\tilde{x}}_2 &= \varepsilon\tilde{A}_{21}\tilde{x}_1 + (\varepsilon\tilde{A}_{22} - \lambda I)\tilde{x}_2 + \varepsilon\bar{B}\omega, \end{aligned} \tag{28}$$

where

$$\begin{aligned} \tilde{A}_{11} &= \bar{A}_{11} + \bar{A}_{12}F_1, \quad \tilde{A}_{12} = \bar{A}_{12}, \\ \tilde{A}_{21} &= \bar{A}_{21} - F_1\bar{A}_{11} + \bar{A}_{22} - F_1\bar{A}_{12}, \quad \tilde{A}_{22} = \bar{A}_{22} - F_1\bar{A}_{12}. \end{aligned}$$

In the absence of the external disturbances, the above system (28) is asymptotically stable for small enough ε.

Since $\tilde{A}_{11} = \bar{A}_{11} + \bar{A}_{12}F_1$ is Hurwitz stable, there exists $P > 0$ such that the Lyapunov equation $P\tilde{A}_{11} + \tilde{A}_{11}'P = -I$ holds. For the dynamics $\tilde{x}_1$, we define a Lyapunov function $V_1 = \tilde{x}_1'P\tilde{x}_1$. Then the derivative of V_1 can be bounded

$$\begin{aligned} \dot{V}_1 &\leq -\|\tilde{x}_1\|^2 + \tilde{x}_2'\tilde{A}_{12}'P\tilde{x}_1 + \tilde{x}_1'P\tilde{A}_{12}\tilde{x}_2 \\ &\leq -\|\tilde{x}_1\|^2 + 2\operatorname{Re}(\tilde{x}_1'P\tilde{A}_{12}\tilde{x}_2) \\ &\leq -\|\tilde{x}_1\|^2 + r_1\|\tilde{x}_1\|\|\tilde{x}_2\|, \end{aligned}$$

where $2\|P\tilde{A}_{12}\| \leq r_1$. Now define a Lyapunov function $V_2 = \varepsilon\tilde{x}_2'\tilde{x}_2$ for the dynamics $\tilde{x}_2$. The derivative of V_2 can then also be bounded.

$$\begin{aligned} \dot{V}_2 &\leq -2\operatorname{Re}(\lambda)\|\tilde{x}_2\|^2 + 2\varepsilon\operatorname{Re}(\tilde{x}_2'\tilde{A}_{21}\tilde{x}_1) + 2\varepsilon\tilde{x}_2'\tilde{A}_{22}\tilde{x}_2 + 2\varepsilon\operatorname{Re}(\tilde{x}_2'\bar{B}\omega) \\ &\leq -2\operatorname{Re}(\lambda)\|\tilde{x}_2\|^2 + \varepsilon r_2\|\tilde{x}_1\|\|\tilde{x}_2\| + \varepsilon r_3\|\tilde{x}_2\|^2 + \varepsilon r_4\|\omega\|\|\tilde{x}_2\| \\ &\leq -\beta\|\tilde{x}_2\|^2 + \varepsilon r_2\|\tilde{x}_1\|\|\tilde{x}_2\| + \varepsilon r_4\|\omega\|\|\tilde{x}_2\| \end{aligned}$$

for a small enough ε, where we choose r_2, r_3, r_4 such that

$$2\|\tilde{A}_{21}\| \leq r_2, \qquad 2\|\tilde{A}_{22}\| \leq r_3, \qquad \text{and} \qquad 2\|\bar{B}\| \leq r_4.$$

Let $V = V_1 + \gamma V_2$ for some $\gamma > 0$. Then, we have

$$\begin{aligned} \dot{V} \leq &-\|\tilde{x}_1\|^2 + r_1\|\tilde{x}_1\|\|\tilde{x}_2\| - \gamma\beta\|\tilde{x}_2\|^2 + \varepsilon\gamma r_2\|\tilde{x}_1\|\|\tilde{x}_2\| \\ &+ \varepsilon\gamma r_4\|\omega\|\|\tilde{x}_2\|. \end{aligned}$$

We have that

$$\begin{aligned} r_1\|\tilde{x}_1\|\|\tilde{x}_2\| &\leq r_1^2\|\tilde{x}_2\|^2 + \frac{1}{4}\|\tilde{x}_1\|^2, \\ \varepsilon\gamma r_2\|\tilde{x}_1\|\|\tilde{x}_2\| &\leq \varepsilon^2\gamma^2 r_2^2\|\tilde{x}_1\|^2 + \frac{1}{4}\|\tilde{x}_2\|^2, \\ \varepsilon\gamma r_4\|\omega\|\|\tilde{x}_2\| &\leq \varepsilon^2\gamma^2 r_4^2\|\omega\|^2 + \frac{1}{4}\|\tilde{x}_2\|^2. \end{aligned}$$

Now we choose γ such that $\gamma\beta = 1 + r_1^2$ and $r_5 = \gamma r_4$. Then, we obtain

$$\begin{aligned} \dot{V} &\leq -\frac{1}{2}\|\tilde{x}_1\|^2 - \frac{1}{2}\|\tilde{x}_2\|^2 + \varepsilon^2 r_5^2\|\omega\|^2 \\ &\leq -\frac{1}{2}\|\tilde{x}\|^2 + \varepsilon^2 r_5^2\|\omega\|^2, \end{aligned}$$

for a small enough ε. From the above, we have that $\|T_{\omega\tilde{x}}\|_\infty < 2\varepsilon r_5$, which immediately leads to $\|T_{\omega x}\|_\infty < r$ for any real number $r > 0$ as long as we choose ε small enough. On the other hand:

$$T_{\omega u}(s) = -\frac{1}{\varepsilon}\begin{pmatrix} 0 & \bar{B}^{-1} \end{pmatrix} T_{\omega\tilde{x}}(s)$$

and hence:

$$\|T_{\omega u}\|_\infty \leq \|\bar{B}^{-1}\| r_5.$$

Therefore, $u = F_\varepsilon x$ solves the robust H_∞-ADDPSS with bounded input for the system (12) as required. □

The main result regarding H_2-ASSFS is stated as follows.

Theorem 6. *Consider a MAS described by (1) and (4) such that (13) is satisfied. Let any real numbers α, $\beta > 0$ and a positive integer N be given, and hence a set of network graphs $\mathbb{G}^N_{\alpha,\beta}$ be defined.*

If (A, B) is stabilizable then the H_2-ASSFS stated in Problem 1 with $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ is solvable. In particular, for any given real number $r > 0$, there exists an ε^*, such that for any $\varepsilon \in (0, \varepsilon^*)$, the protocol (25) achieves state synchronization and the resulting system from ω to $x_i - x_j$ has an H_2 norm less than r for any $i, j \in 1, \dots, N$ and for any graph $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$.

Proof. Using Lemma 2, we know that we only need to verify that the feedback solves the robust H_2-ADDPSS with bounded input for the system (12) with $\lambda \in \Lambda$. We use the same feedback as in the proof of Theorem 5. In the proof of Theorem 5 it is already shown that the closed loop system is asymptotically stable and the H_∞ norm of the transfer function from ω to u is bounded. The only remaining part of the proof is to show that the H_2 norm from ω to x can be made arbitrarily small. This clearly is equivalent to showing that the system (28) has an arbitrary small H_2 norm from ω to $\tilde{x}_1$ and $\tilde{x}_2$ for sufficiently small ε. Choose Q such that

$$Q\tilde{A}_{11}' + \tilde{A}_{11}Q = -I$$

In that case we have:

$$A_{cl}\begin{pmatrix}\sqrt{\varepsilon}Q & 0\\ 0 & \sqrt{\varepsilon}I\end{pmatrix}+\begin{pmatrix}\sqrt{\varepsilon}Q & 0\\ 0 & \sqrt{\varepsilon}I\end{pmatrix}A_{cl}'+\begin{pmatrix}0 & 0\\ 0 & \bar{B}\bar{B}'\end{pmatrix}$$
$$\leq\begin{pmatrix}\sqrt{\varepsilon} & \sqrt{\varepsilon}(\tilde{A}_{12}+Q\tilde{A}_{21}')\\ \sqrt{\varepsilon}(\tilde{A}_{12}'+\tilde{A}_{21}Q) & -\frac{\beta}{\sqrt{\varepsilon}}I\end{pmatrix}$$

for sufficiently small ε where:

$$A_{cl}=\begin{pmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ \tilde{A}_{21} & \tilde{A}_{22}-\frac{\lambda}{\varepsilon}I\end{pmatrix}$$

and we used that

$$\lambda+\lambda'\geq 2\beta.$$

We then obtain for sufficiently small ε that:

$$A_{cl}\begin{pmatrix}\sqrt{\varepsilon}Q & 0\\ 0 & \sqrt{\varepsilon}I\end{pmatrix}+\begin{pmatrix}\sqrt{\varepsilon}Q & 0\\ 0 & \sqrt{\varepsilon}I\end{pmatrix}A_{cl}'$$
$$+\begin{pmatrix}0 & 0\\ 0 & \bar{B}\bar{B}'\end{pmatrix}\leq 0$$

Then Lemma 3 immediately yields that we can make the H_2 norm from ω to x arbitrarily small by choosing a sufficiently small ε. □

4. MAS with partial-state coupling

In this section, similar to the approach of the previous section, we show first that the almost state synchronization among agents in the network with partial-state coupling can be solved by equivalently solving a robust H_∞ or H_2 almost disturbance decoupling problem via measurement feedback with internal stability (in short H_∞ or H_2-ADDPMS). Then, we design a controller for such a robust H_∞ or H_2-ADDPMS with bounded input.

4.1. Necessary and sufficient condition for H_∞-ASSPS

The MAS system described by (1) and (3) after implementing the linear dynamical protocol (8) is described by

$$\begin{cases}\dot{\hat{x}}_i=\begin{pmatrix}A & BC_c(\varepsilon)\\ 0 & A_c(\varepsilon)\end{pmatrix}\hat{x}_i+\begin{pmatrix}BD_c(\varepsilon)\\ B_c(\varepsilon)\end{pmatrix}\zeta_i+\begin{pmatrix}E\\ 0\end{pmatrix}\omega_i,\\ y_i=\begin{pmatrix}C & 0\end{pmatrix}\hat{x}_i,\\ \zeta_i=\sum_{j=1}^{N}\ell_{ij}y_j,\end{cases}\tag{29}$$

for $i=1,\ldots,N$, where

$$\hat{x}_i=\begin{pmatrix}x_i\\ \chi_i\end{pmatrix}.$$

Define

$$\hat{x}=\begin{pmatrix}\hat{x}_1\\ \vdots\\ \hat{x}_N\end{pmatrix},\qquad \omega=\begin{pmatrix}\omega_1\\ \vdots\\ \omega_N\end{pmatrix},$$

and

$$\bar{A}=\begin{pmatrix}A & BC_c(\varepsilon)\\ 0 & A_c(\varepsilon)\end{pmatrix},\quad \bar{B}=\begin{pmatrix}BD_c(\varepsilon)\\ B_c(\varepsilon)\end{pmatrix},\quad \bar{E}=\begin{pmatrix}E\\ 0\end{pmatrix},$$
$$\bar{C}=\begin{pmatrix}C & 0\end{pmatrix}.$$

Then, the overall dynamics of the N agents can be written as

$$\dot{\hat{x}}=(I_N\otimes\bar{A}+L\otimes\bar{B}\bar{C})\hat{x}+(I_N\otimes\bar{E})\omega.\tag{30}$$

We define a robust H_∞-ADDPMS with bounded input as follows. Given $\Lambda\subset\mathbb{C}$, there should exist $M>0$ such that for any given real number $r>0$, we can find a parameterized controller

$$\begin{aligned}\dot{\chi}&=A_c(\varepsilon)\chi+B_c(\varepsilon)y,\\ u&=C_c(\varepsilon)\chi+D_c(\varepsilon)y,\end{aligned}\tag{31}$$

where $\chi\in\mathbb{R}^{n_c}$, for the following system,

$$\begin{aligned}\dot{x}&=Ax+\lambda Bu+B\omega,\\ y&=Cx\end{aligned}\tag{32}$$

such that the following holds for any $\lambda\in\Lambda$:

1. The closed-loop system of (31) and (32) is internally stable
2. The resulting closed-loop transfer function $T_{\omega x}$ from ω to x has an H_∞ norm less than r.
3. The resulting closed-loop transfer function $T_{\omega u}$ from ω to u has an H_∞ norm less than M.

In the above, Λ denotes all possible locations for the nonzero eigenvalues of the Laplacian matrix L when the graph varies over the set **G**. It is also important to note that M is independent of the choice for r.

In order to obtain our main result, we will need the following lemma:

Lemma 4. *Consider the system:*

$$\begin{aligned}\dot{x}&=Ax+Bu+E\omega,\\ y&=Cx\\ z&=x\end{aligned}$$

with (A, B) stabilizable and (C, A) detectable. The H_∞-ADDPMS for the above system is defined as the problem to find for any $r>0$ a controller of the form (31) such that the closed loop system is internally stable while the H_∞ norm from ω to z is less than r. The H_∞-ADDPMS is solvable if and only if:

1. $\operatorname{im}E\subset\operatorname{im}B$,
2. $(A, E, C, 0)$ *is left-invertible,*
3. $(A, E, C, 0)$ *is minimum-phase.*

Proof. From [18] we immediately obtain that the H_∞-ADDPMS is solvable if and only if:

1. $\operatorname{im}E\subset\operatorname{im}B$
2. $(A, E, C, 0)$ is at most weakly non-minimum-phase and left-invertible.
3. For any $\delta>0$ and every invariant zero s_0 of $(A, E, C, 0)$, there exists a matrix K such that $sI-A-BKC$ is invertible and

$$\|(s_0I-A-BKC)^{-1}E\|_\infty<\delta\tag{33}$$

Choose a suitable basis such that:

$$A=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix},\quad B=\begin{pmatrix}B_1\\ B_2\end{pmatrix},\quad E=\begin{pmatrix}E_1\\ E_2\end{pmatrix},$$
$$C=\begin{pmatrix}I & 0\end{pmatrix}$$

Assume s_0 is an imaginary axis zero of $(A, E, C, 0)$. In that case the rank of the matrix:

$$\begin{pmatrix}sI-A_{11} & -A_{12} & E_1\\ -A_{21} & sI-A_{22} & E_2\\ I & 0 & 0\end{pmatrix}$$

drops for $s=s_0$. This implies the existence of $p\neq 0$ and $q\neq 0$ such that

$$\begin{pmatrix}-A_{12}\\ s_0I-A_{22}\end{pmatrix}p=\begin{pmatrix}E_1\\ E_2\end{pmatrix}q.$$

The final condition for H_∞ almost disturbance decoupling requires for any $\delta > 0$ the existence of a K such that (33) is satisfied. However:

$$
\begin{aligned}
&(s_0 I - A - BKC)^{-1} E q \\
&= \begin{pmatrix} s_0 I - A_{11} - B_1 K & -A_{12} \\ -A_{21} - B_2 K & s_0 I - A_{22} \end{pmatrix}^{-1} \begin{pmatrix} -A_{12} \\ s_0 I - A_{22} \end{pmatrix} p \\
&= \begin{pmatrix} s_0 I - A_{11} - B_1 K & -A_{12} \\ -A_{21} - B_2 K & s_0 I - A_{22} \end{pmatrix}^{-1} \begin{pmatrix} s_0 I - A_{11} - B_1 K & -A_{12} \\ -A_{21} - B_2 K & s_0 I - A_{22} \end{pmatrix} \\
&\begin{pmatrix} 0 \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ p \end{pmatrix},
\end{aligned}
$$

which yields a contradiction if δ is such that

$$\|p\| > \delta \|q\|.$$

Therefore we cannot have any invariant zeros in the imaginary axis. In other words, the system $(A, E, C, 0)$ needs to be minimum-phase instead of weakly minimum-phase. Conversely, if $(A, E, C, 0)$ is minimum-phase it is easy to verify that for any $\delta > 0$ there exists K such that (33) is satisfied. □

Theorem 7. *Consider the MAS described by (1) and (3) with (A, B) stabilizable and (C, A) detectable.*

(Part I) Let $\alpha, \beta > 0$ be given such that a set of graphs $\mathbb{G}^N_{\alpha,\beta}$ be defined. Then, the H_∞-ASSPS for the MAS with any graph $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$ is solvable by a parameterized protocol (8) for any $\alpha > \beta > 0$ if and only if

$$\operatorname{im} E \subset \operatorname{im} B \tag{34}$$

while $(A, E, C, 0)$ is minimum phase and left-invertible.

*(Part II) Let **G** be a set of graphs such that the associated Laplacian matrices are uniformly bounded and let Λ consist of all possible nonzero eigenvalues of Laplacian matrices associated with graphs in **G**. Then, the H_∞-ASSPS for the MAS with any graph $\mathcal{G} \in$ **G** is solved by a parameterized protocol (8) if the robust H_∞-ADDPMS with bounded input for the system (32) with $\lambda \in \Lambda$ is solved by the parameterized controller (31).*

Proof. By using $L = T S_L T^{-1}$, we define

$$\eta := (T^{-1} \otimes I_n)\hat{x} = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_N \end{pmatrix}, \qquad \bar{\omega} = (T^{-1} \otimes I)\omega = \begin{pmatrix} \bar{\omega}_1 \\ \vdots \\ \bar{\omega}_N \end{pmatrix}$$

where $\eta_i \in \mathbb{C}^{n+n_c}$ and $\bar{\omega}_i \in \mathbb{C}^q$. In the new coordinates, the dynamics of η can be written as

$$\dot{\eta}(t) = (I_N \otimes \bar{A} + S_L \otimes \bar{B}\bar{C}\eta + (T^{-1} \otimes E)\omega, \tag{35}$$

which is rewritten as

$$
\begin{aligned}
\dot{\eta}_1 &= \bar{A}\eta_1 + \sum_{j=2}^{N} s_{1j} \bar{B}\bar{C}\eta_j + \bar{E}\bar{\omega}_1, \\
\dot{\eta}_i &= (\bar{A} + \lambda_i \bar{B}\bar{C})\eta_i + \sum_{j=i+1}^{N} s_{ij} \bar{B}\bar{C}\eta_j + \bar{E}\bar{\omega}_i, \\
\dot{\eta}_N &= (\bar{A} + \lambda_N \bar{B}\bar{C})\eta_N + \bar{E}\bar{\omega}_N,
\end{aligned} \tag{36}
$$

with $i \in \{2, \ldots, N-1\}$ where

$$\bar{E} = \begin{pmatrix} 0 \\ E \end{pmatrix}, \qquad S_L = [s_{ij}].$$

As in the case of full-state coupling, we can show that:

$$\bar{x} = (V \otimes I_n) \begin{pmatrix} \eta_2 \\ \vdots \\ \eta_N \end{pmatrix}, \tag{37}$$

for some suitably chosen matrix V which is uniformly bounded. Therefore the H_∞ norm of the transfer matrix from ω to $\bar{x}$ can be made arbitrarily small if and only if the H_∞ norm of the transfer matrix from $\bar{\omega}$ to η can be made arbitrarily small.

In order for the H_∞ norm from $\bar{\omega}$ to η to be arbitrarily small we need the H_∞ norm from $\bar{\omega}_N$ to η_N to be arbitrarily small. In other words, the robust H_∞-ADDPMS with bounded input has to be solvable for the system

$$
\begin{aligned}
\dot{x} &= Ax + \lambda B u + E\omega, \\
y &= Cx
\end{aligned}
$$

From the results of Lemma 4, we find that this is only possible if (34) is satisfied and $(A, E, C, 0)$ is left-invertible and minimum phase.

On the other hand, suppose (31) solves the robust H_∞-ADDPMS with bounded input of (32) and assume (34) is satisfied. We need to show that (8) solves the H_∞-ASSFS for the MAS described by (1) and (3). This follows directly from arguments very similar to the approach used in the proof of Lemma 1. □

4.2. Necessary and sufficient condition for H_2-ASSPS

The MAS system described by (1) and (3) after implementing the linear dynamical protocol (8) is described by (29) for $i = 1, \ldots, N$, and, as before, the overall dynamics of the N agents can be written as

$$\dot{\hat{x}} = (I_N \otimes \bar{A} + L \otimes \bar{B}\bar{C})\hat{x} + (I_N \otimes \bar{E})\omega. \tag{38}$$

We define a robust H_2-ADDPMS with bounded input as follows. Given $\Lambda \subset \mathbb{C}$, there should exist $M > 0$ such that for any given real number $r > 0$, we can find a parameterized controller

$$
\begin{aligned}
\dot{\chi} &= A_c(\varepsilon)\chi + B_c(\varepsilon)y, \\
u &= C_c(\varepsilon)\chi + D_c(\varepsilon)y,
\end{aligned} \tag{39}
$$

where $\chi \in \mathbb{R}^{n_c}$, for the following system,

$$
\begin{aligned}
\dot{x} &= Ax + \lambda B u + B\omega, \\
y &= Cx
\end{aligned} \tag{40}
$$

such that the following holds for any $\lambda \in \Lambda$:

1. The closed-loop system of (39) and (40) is internally stable
2. The resulting closed-loop transfer function $T_{\omega x}$ from ω to x has an H_2 norm less than r.
3. The resulting closed-loop transfer function $T_{\omega u}$ from ω to u has an H_∞ norm less than M.

In the above, Λ denotes all possible locations for the nonzero eigenvalues of the Laplacian matrix L when the graph varies over the set **G**. It is also important to note that M is independent of the choice for r.

The following lemma, provides a necessary condition for the H_2-ADDPMS:

Lemma 5. *Consider the system:*

$$
\begin{aligned}
\dot{x} &= Ax + Bu + E\omega, \\
y &= Cx \\
z &= x
\end{aligned}
$$

with (A, B) stabilizable and (C, A) detectable. The H_2-ADDPMS for the above system is defined as the problem to find for any $r > 0$ a controller of the form (39) such that the closed loop system is internally stable while the H_∞ norm from ω to z is less than r. The H_2-ADDPMS is solvable only if:

1. *$\operatorname{im} E \subset \operatorname{im} B$*
2. *$(A, E, C, 0)$ is at most weakly non-minimum-phase and left-invertible.*

Proof. This follows directly from [18]. □

Theorem 8. *Consider the MAS described by (1) and (3) with (A, B) stabilizable and (C, A) detectable.*

(Part I) Let α, $\beta > 0$ be given such that a set of graphs $\mathbb{G}^N_{\alpha,\beta}$ be defined. Then, the H_2-ASSPS for the MAS with any graph $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$ is solvable by a parameterized protocol (8) for any $\alpha > \beta > 0$ only if

$$\operatorname{im} E \subset \operatorname{im} B \tag{41}$$

while $(A, E, C, 0)$ is at most weakly non-minimum phase and left-invertible .

*(Part II) Let **G** be a set of graphs such that the associated Laplacian matrices are uniformly bounded and let Λ consist of all possible nonzero eigenvalues of Laplacian matrices associated with graphs in **G**. Then, the H_2-ASSPS for the MAS with any graph $\mathcal{G} \in \mathbf{G}$ is solved by a parameterized protocol (8) if the robust H_2-ADDPMS with bounded input for the system (40) with $\lambda \in \Lambda$ is solved by the parameterized controller (39).*

Proof. Similar, to the proof of Theorem 7, the dynamics can be written in the form (36).

Using (37), we note the H_2 norm of the transfer matrix from ω to $\bar{x}$ can be made arbitrarily small if and only if the H_2 norm of the transfer matrix from $\bar{\omega}$ to η can be made arbitrarily small.

In order for the H_2 norm from $\bar{\omega}$ to η to be arbitrarily small we need the H_2 norm from $\bar{\omega}_N$ to η_N to be arbitrarily small. In other words, the robust H_2-ADDPMS with bounded input has to be solvable for the system

$$\dot{x} = Ax + \lambda Bu + E\omega,$$
$$y = Cx$$

From the results of Lemma 5, we find that this is only possible if (41) is satisfied, $(A, E, C, 0)$ is left-invertible and at most weakly non-minimum phase.

On the other hand, suppose (39) solves the robust H_2-ADDPMS with bounded input of (40) and assume (41) is satisfied. We need to show that (8) solves the H_2-ASSFS for the MAS described by (1) and (3). This follows directly from arguments very similar to the approach used in the proof of Lemma 1. □

4.3. Protocol design for H_∞-ASSPS

We present below two protocol design methods based on robust stabilization for the case $E = B$ and therefore the case where $(A, B, C, 0)$ is minimum-phase. One relies on an algebraic Riccati equation (ARE) method, and the other is based on the direct eigenstructure assignment method.

4.3.1. ARE-based method

Using an algebraic Riccati equation, we can design a suitable protocol. As in the full-state coupling case, we choose $F = -B'P$ with $P = P' > 0$ being the unique solution of the continuous-time algebraic Riccati equation

$$A'P + PA - 2\beta PBB'P + I = 0, \tag{42}$$

where β is a lower bound for the real part of the non-zero eigenvalues of all Laplacian matrices associated with a graph in $\mathbb{G}^N_{\alpha,\beta}$.

Since $(A, B, C, 0)$ is minimum-phase then for any ε there exists δ small enough such that

$$AQ + QA' + BB' + \varepsilon^{-4}Q^2 - \delta^{-2}QC'CQ = 0 \tag{43}$$

has a solution $Q > 0$. We then consider the following protocol:

$$\begin{aligned} \dot{\chi}_i &= (A + K_\varepsilon C)\chi_i - K_\varepsilon \zeta_i, \\ u_i &= F_\varepsilon \chi_i, \end{aligned} \tag{44}$$

where

$$F_\varepsilon = -\frac{1}{\varepsilon}B'P, \qquad K_\varepsilon = -\frac{1}{\delta^2}QC'$$

The main result in this section is stated as follows.

Theorem 9. *Consider a MAS described by (1) and (3) with (A, B) stabilizable and (C, A) detectable. Let any real numbers α, $\beta > 0$ and a positive integer N be given, and hence a set of network graphs $\mathbb{G}^N_{\alpha,\beta}$ be defined.*

The H_∞-ASSPS stated in Problem 2 with $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ is solvable. In particular, for any given real number $r > 0$, there exists an ε^*, such that for any $\varepsilon \in (0, \varepsilon^*)$, the protocol (44) achieves state synchronization and the resulting system from ω to $x_i - x_j$ has an H_∞ norm less than r for any $i, j \in 1, \ldots, N$ and for any graph $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$.

Proof. Using Theorem 7, we know that we only need to verify that

$$\begin{aligned} \dot{\chi} &= (A + K_\varepsilon C)\chi - K_\varepsilon y, \\ u &= F_\varepsilon \chi, \end{aligned} \tag{45}$$

solves the robust H_∞-ADDPMS with bounded input for the system (32) with $\lambda \in \Lambda$. Given $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$, we know that $\lambda \in \Lambda$ implies $\operatorname{Re}\lambda \geq \beta$. Obviously $A + BF_\varepsilon$ and $A + K_\varepsilon C$ are both asymptotically stable by construction and hence the intersection of (32) and (45) is asymptotically stable. The closed loop transfer function from ω to x is equal to:

$$T_{\omega x} = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} I & -T_2 \\ \tilde{T}_3 & I \end{pmatrix}^{-1} \begin{pmatrix} T_1 \\ -\tilde{T}_4 \end{pmatrix}$$

where:

$$T_1(s) = (sI - A - \lambda BF_\varepsilon)^{-1}B \tag{46a}$$

$$T_2(s) = \lambda(sI - A - \lambda BF_\varepsilon)^{-1}BF_\varepsilon \tag{46b}$$

$$\tilde{T}_3(s) = \lambda(sI - A - K_\varepsilon C + \lambda BF_\varepsilon)^{-1}BF_\varepsilon \tag{46c}$$

$$\tilde{T}_4(s) = (sI - A - K_\varepsilon C + \lambda BF_\varepsilon)^{-1}B \tag{46d}$$

As argued in the proof of Theorem 3, we have:

$$\|T_1\|_\infty < \frac{\varepsilon}{\beta}, \qquad \|T_2\|_\infty < \frac{2|\lambda|}{\beta^2} \leq \frac{2\alpha}{\beta^2}.$$

On the other hand, (43) implies according to the bounded real lemma:

$$\|T_3\|_\infty < \varepsilon^2 \tag{47}$$

where

$$T_3(s) = (sI - A - K_\varepsilon C)^{-1}B$$

Note that:

$$\tilde{T}_3 = \lambda(I + \lambda T_3 F_\varepsilon)^{-1}T_3 F_\varepsilon$$

which yields, using (47), that

$$\|\tilde{T}_3\|_\infty < (1 - \varepsilon M_1)^{-1}\varepsilon M_1 < 2\varepsilon M_1$$

for small ε where M_1 is such that:

$$|\lambda|\|B'P\| < \alpha\|B'P\| = M_1$$

The above yields:

$$\|T_{\omega x}\|_\infty < \varepsilon M_2$$

for some suitable constant M_2. The closed loop transfer function from ω to u is equal to:

$$T_{\omega u} = \begin{pmatrix} F_\varepsilon & F_\varepsilon \end{pmatrix} \begin{pmatrix} I & -T_2 \\ \tilde{T}_3 & I \end{pmatrix}^{-1} \begin{pmatrix} T_1 \\ -\tilde{T}_4 \end{pmatrix}$$

which yields using similar arguments as above that:

$$\|T_{\omega u}\|_\infty < M_3$$

for some suitable constant M_3 independent of ε. Therefore the H_∞ norm of the transfer matrix $T_{\omega x}$ becomes arbitrarily small for sufficiently small ε while the H_∞ norm of the transfer matrix $T_{\omega u}$ remains bounded. □

4.3.2. Direct method

For ease of presentation, we only consider the case $q = 1$, i.e. the case where we have a scalar measurement. We consider the state feedback gain F_ε given in (26), that is

$$F_\varepsilon = \frac{1}{\varepsilon}\bar{B}^{-1}\begin{pmatrix} F_1 & -I \end{pmatrix} T_x,$$

where T_x is defined in (23).

Next, we consider the observer design. Note that the system $(A, B, C, 0)$ is minimum-phase and left-invertible. In that case there is a nonsingular matrix Γ_x such that, by defining $\bar{x} = \Gamma_x x$, we obtain the system

$$\begin{aligned} \dot{x}_a &= A_a x_a + L_{ad} y, \\ \dot{x}_d &= A_d x_d + B_d(u + \omega + E_{da} x_a + E_{dd} x_d), \\ y &= C_d x_d. \end{aligned} \tag{48}$$

where

$$\bar{x} = \Gamma_x x = \begin{pmatrix} x_a \\ x_d \end{pmatrix},$$

with $x_a \in \mathbb{R}^{n-\rho}$ and $x_d \in \mathbb{R}^\rho$ and where the matrices $A_d \in \mathbb{R}^{\rho\times\rho}$, $B_d \in \mathbb{R}^{\rho\times 1}$, and $C_d \in \mathbb{R}^{1\times\rho}$ have the special form

$$A_d = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \quad B_d = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$
$$C_d = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}. \tag{49}$$

Furthermore, the eigenvalues of A_a are the invariant zeros of (A, B, C) and hence A_a is asymptotically stable. The transformation Γ_x can be calculated using available software, either numerically [7] or symbolically [2].

Next, define a high-gain scaling matrix

$$S_\varepsilon := \mathrm{diag}(1, \varepsilon^2, \ldots, \varepsilon^{2\rho-2}), \tag{50}$$

and define the output injection matrix

$$K_\varepsilon = \Gamma_x \begin{pmatrix} 0 \\ \varepsilon^{-2} S_\varepsilon^{-1} K \end{pmatrix}. \tag{51}$$

where K is such that $A_d + B_d K$ is asymptotically stable. We then consider the following protocol:

$$\begin{aligned} \dot{\chi}_i &= (A + K_\varepsilon C)\chi_i - K_\varepsilon \zeta_i, \\ u_i &= F_\varepsilon \chi_i, \end{aligned} \tag{52}$$

The main result in this section is stated as follows.

Theorem 10. *Consider a MAS described by a SISO system (1) and (3). Let any real numbers $\alpha, \beta > 0$ and a positive integer N be given, and hence a set of network graphs $\mathbb{G}^N_{\alpha,\beta}$ be defined.*

If (A, B) is stabilizable then the H_∞-ASSPS stated in Problem 2 with $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ is solvable. In particular, for any given real number $r > 0$, there exists an ε^*, such that for any $\varepsilon \in (0, \varepsilon^*)$, the protocol (52) achieves state synchronization and the resulting system from ω to $x_i - x_j$ has an H_∞ norm less than r for any $i, j \in 1, \ldots, N$ and for any graph $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$.

Proof. We use a similar argument as in the proof of Theorem 10. We know that we only need to verify that

$$\begin{aligned} \dot{\chi} &= (A + K_\varepsilon C)\chi - K_\varepsilon y, \\ u &= F_\varepsilon \chi, \end{aligned} \tag{53}$$

solves the robust H_∞-ADDPMS with bounded input for the system (32) with $\lambda \in \Lambda$. Given $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$, we know that $\lambda \in \Lambda$ implies $\mathrm{Re}\,\lambda \geq \beta$. Obviously $A + BF_\varepsilon$ and $A + K_\varepsilon C$ are both asymptotically stable by construction and hence the intersection of (32) and (45) is asymptotically stable. As in the proof of Theorem 9, the closed loop transfer function from ω to x is equal to:

$$T_{\omega x} = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} I & -T_2 \\ \tilde{T}_3 & I \end{pmatrix}^{-1} \begin{pmatrix} T_1 \\ -\tilde{T}_4 \end{pmatrix} \tag{54}$$

where, as before, we use the definitions in (46) but with our modified F_ε and K_ε. As argued in the proof of Theorem 5, we have:

$$\|T_1\|_\infty < M_1\varepsilon, \qquad \|T_2\|_\infty < M_2.$$

for suitable constants $M_1, M_2 > 0$. Finally

$$T_3(s) = \Gamma_x \begin{pmatrix} sI - A_a & L_{ad}C_d \\ -B_d E_{da} & Z_1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ B_d \end{pmatrix}$$

where

$$Z_1 = sI - A_d - \varepsilon^{-2} S_\varepsilon^{-1} K C_d - B_d E_{dd}$$

We obtain:

$$T_3(s) = \varepsilon^{2n}\Gamma_x \begin{pmatrix} I & 0 \\ 0 & S_\varepsilon^{-1} \end{pmatrix} \begin{pmatrix} sI - A_a & L_{ad}C_d \\ \varepsilon^{2n} B_d E_{da} & Z_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ B_d \end{pmatrix}$$

with

$$Z_2 = sI - \varepsilon^{-2} A_d - \varepsilon^{-2} K C_d + \varepsilon^{2n} B_d E_{dd} S_\varepsilon^{-1},$$

using that

$$\varepsilon^{-2} A_d = S_\varepsilon A_d S_\varepsilon^{-1}, \quad S_\varepsilon B_d = \varepsilon^{2n} B_d \quad \text{and} \quad C_d S_\varepsilon^{-1} = C_d.$$

Note that $E_{d\varepsilon}$ is bounded for $\varepsilon < 1$. Next, we note that:

$$\begin{aligned} X_\varepsilon(s) &= \begin{pmatrix} sI - A_a & L_{ad}C_d \\ 0 & sI - \varepsilon^{-2}(A_d + KC_d) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ B_d \end{pmatrix} \\ &= \varepsilon^2 \begin{pmatrix} -(sI - A_a)^{-1} L_{ad} \\ I \end{pmatrix} (\varepsilon^2 sI - A_d - KC_d)^{-1} B_d \end{aligned}$$

From the above we can easily conclude that there exists M such that $\|X_\varepsilon\|_\infty < M\varepsilon^2$. We have:

$$T_3 = \varepsilon^{2n}\Gamma_x \begin{pmatrix} I & 0 \\ 0 & S_\varepsilon^{-1} \end{pmatrix} (I + \varepsilon^2 X_\varepsilon E_{d\varepsilon})^{-1} X_\varepsilon$$

where

$$E_{d\varepsilon} = \begin{pmatrix} \varepsilon^{2n-2} E_{da} & \varepsilon^{2n-2} E_{dd} S_\varepsilon^{-1} \end{pmatrix}$$

which is clearly bounded for $\varepsilon < 1$. This clearly implies, using our bounds for X_ε and $E_{d\varepsilon}$, that there exists $M_3 > 0$ such that:

$$\|T_3\| \leq \varepsilon^2 M_3$$

for small ε since

$$\varepsilon^{2n} \begin{pmatrix} I & 0 \\ 0 & S_\varepsilon^{-1} \end{pmatrix} < \varepsilon^2 I.$$

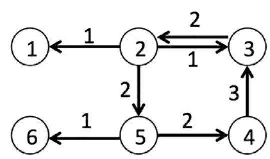

Fig. 1. The communication topology.

Our bound for T_3 guarantees that

$$\|\tilde{T}_3\|_\infty < \varepsilon M_4, \qquad \|\tilde{T}_4\|_\infty < \varepsilon M_5,$$

for suitable M_4 and M_5. Moreover

$$\|F_\varepsilon\| < \varepsilon^{-1} M_0$$

Given our bounds, we immediately obtain from (54) that there exists M_6 such that

$$\|T_{\omega x}\|_\infty < M_6 \varepsilon.$$

The closed loop transfer function from ω to u is equal to:

$$T_{\omega u} = \begin{pmatrix} F_\varepsilon & F_\varepsilon \end{pmatrix} \begin{pmatrix} I & -T_2 \\ \tilde{T}_3 & I \end{pmatrix}^{-1} \begin{pmatrix} T_1 \\ -\tilde{T}_4 \end{pmatrix}$$

which yields, using similar arguments as above, that:

$$\|T_{\omega u}\|_\infty < M_7$$

for some suitable constant M_7 independent of ε. In other words, the transfer function from ω to x is arbitrarily small for sufficiently small ε while the transfer function from ω to u is bounded which completes the proof. □

4.4. Protocol design for H_2-ASSFS

We present below two protocol design methods based on robust stabilization for the case $E = B$. The necessary condition provided earlier shows that $(A, B, C, 0)$ need only be at most weakly non-minimum-phase. The following designs are provided under the stronger assumption that $(A, B, C, 0)$ is minimum-phase.

4.4.1. ARE-based method

We consider the protocol (44) already used in the case of H_∞-ASSFS. It is easy to verify using a similar proof that this protocol also solves the robust H_2-ADDPMS with bounded input and therefore solves H_2-ASSPS for the MAS. Using the same notation as before, this relies on the fact that we have N_1 such that

$$\|T_1\|_2 < \varepsilon N_1$$

which follows directly from the full-state coupling case. On the other hand we have N_2 such that

$$\|T_3\|_2 < \varepsilon^2 N_2$$

since $Q \to 0$ for $\delta \to 0$ and

$$(A - KC)Q + Q(A - KC)' + BB' \leq 0.$$

It is then easily shown that

$$\|\tilde{T}_4\|_2 < \varepsilon^2 N_3$$

for some $N_3 > 0$. The rest of the proof is then as before in the case of H_∞-ASSPS.

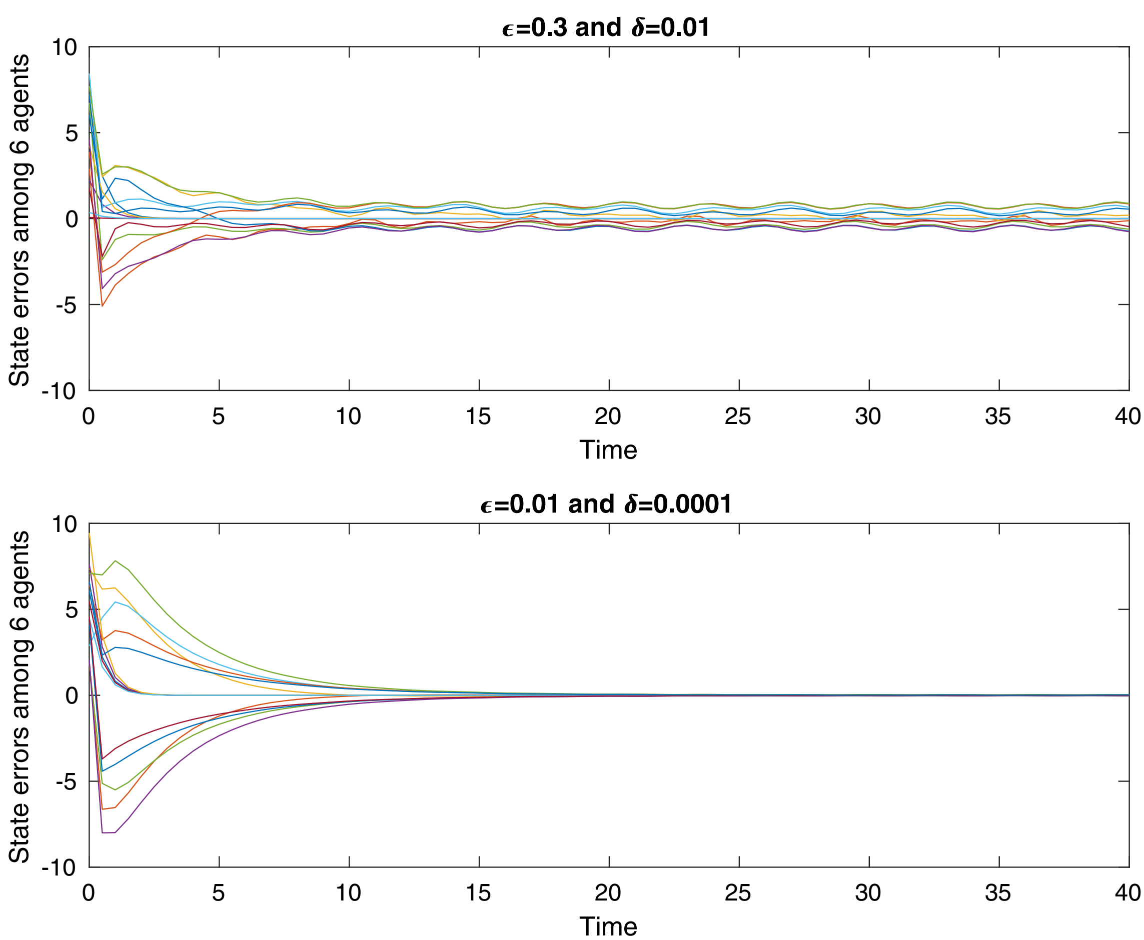

Fig. 2. State errors among $N = 6$ agents.

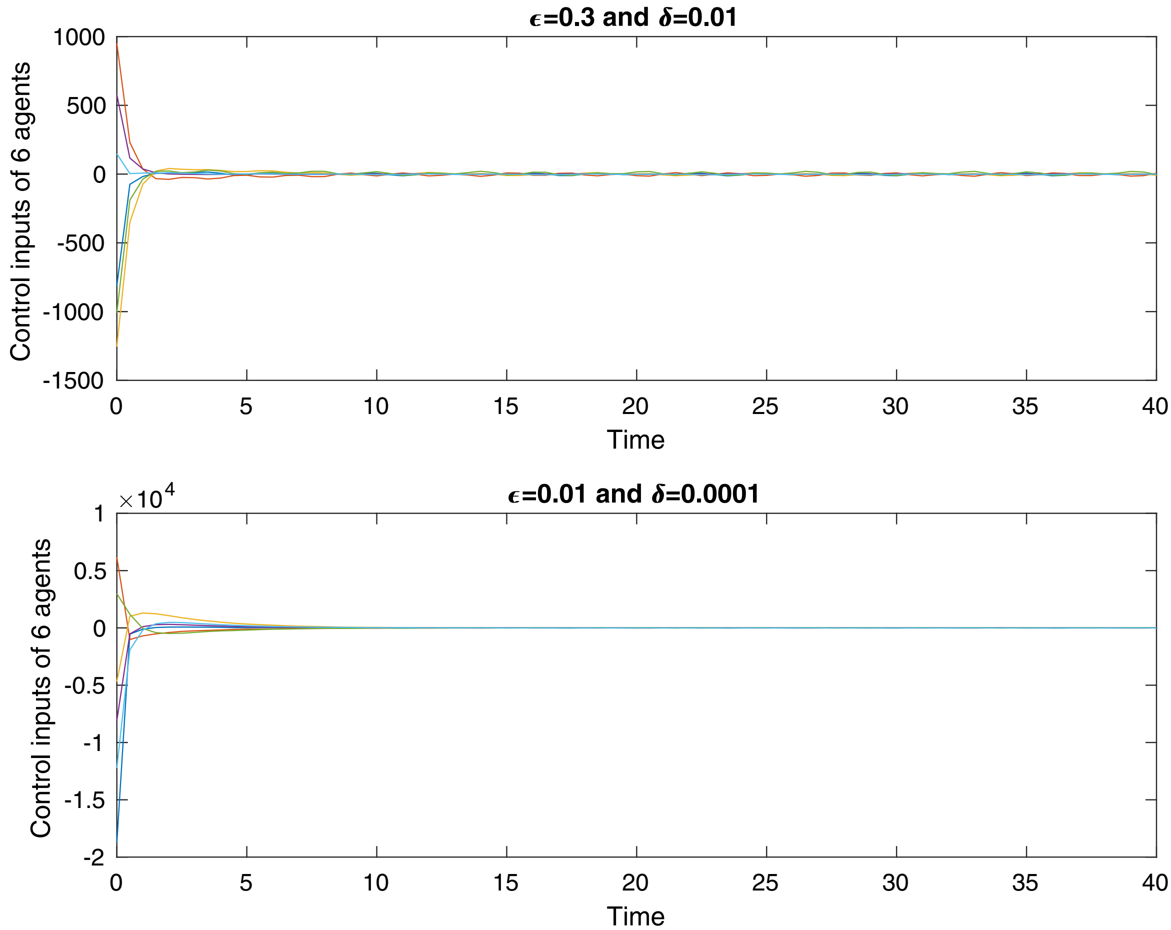

Fig. 3. The controller inputs of $N = 6$ agents.

4.4.2. Direct method

We consider the protocol (52) already used in the case of H_∞-ASSFS. It is easy to verify using a similar proof that this protocol also solves the robust H_2-ADDPMS with bounded input and therefore solves H_2-ASSPS for the MAS. Using the same notation as before, this relies on the fact that we have N_1 such that

$$\|T_1\|_2 < \varepsilon N_1$$

which follows directly from the full-state coupling case. On the other hand we have N_2 such that

$$\|\tilde{T}_4\|_2 < \varepsilon N_2$$

using that X_ε has an H_2 norm of order ε. The rest of the proof is then as before in the case of H_∞-ASSPS.

5. Example

In this section, we illustrate our results on a homogeneous MAS of $N = 6$ agents. We consider the H_∞ almost state synchronization problem via partial-state coupling.

The agent model is given by:

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 2 & 2 & 0 \\ 5 & 4 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \quad C' = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

$$E = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix},$$

with disturbances

$$\omega_1 = \sin(3t), \ \omega_2 = \cos(t), \ \omega_3 = 0.5,$$

$$\omega_4 = \sin(2t) + 1, \ \omega_5 = \sin(t), \ \omega_6 = \cos(2t).$$

The communication topology is shown in Fig. 1 with the Laplacian matrix

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & -1 & 4 & -3 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & -2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

We design a controller of the form (44) based on an ARE-based method. The feedback gain $F_\varepsilon = -\frac{1}{\varepsilon}B'P$ with P given by the algebraic Riccati equation (22) and $K_\varepsilon = -\frac{1}{\delta^2}QC'$ given by the algebraic Riccati equation (43). When choosing $\varepsilon = 0.3$ and $\delta = 0.01$, we get the controller

$$\dot{\chi}_i = \begin{pmatrix} -2 & 0 & 0 \\ 2 & -299.214 & -301.214 \\ 5 & -199.194 & -201.194 \end{pmatrix} \chi_i + \begin{pmatrix} 0 \\ 301.214 \\ 203.194 \end{pmatrix} \zeta_i,$$

$$u_i = \begin{pmatrix} -34.068 & -30.5702 & -27.0943 \end{pmatrix} \chi_i;$$

while when choosing $\varepsilon = 0.01$ and $\delta = 0.0001$, the controller is

$$\dot{\chi}_i = \begin{pmatrix} -2 & 0 & 0 \\ 2 & -29999 & -30001 \\ 5 & -19999 & -20001 \end{pmatrix} \chi_i + \begin{pmatrix} 0 \\ 30001 \\ 20003 \end{pmatrix} \zeta_i,$$

$$u_i = \begin{pmatrix} -1022 & -917.1 & -812.8 \end{pmatrix} \chi_i.$$

The results are shown in Fig. 2. It is clear that when ε goes smaller, the H_∞ norm from the disturbance to the relative error between the states of the different agents gets smaller. The controller inputs for all agents are shown in Fig. 3.

6. Conclusion

In this paper, we have studied H_∞ and H_2 almost state synchronization for MAS with identical linear agents affected by external disturbances. The communication network is directed and coupled through agents' states or outputs. We have first developed the necessary and sufficient conditions on agents' dynamics for the solvability of H_∞ and H_2 almost state synchronization problems. Then, we have designed protocols to achieve H_∞ and H_2 almost state synchronization among agents based on two methods. One is ARE-based method and the other is ATEA-baed method. The future work could be to extend the results of this paper to nonlinear agents, that is, H_∞ and H_2 almost state synchronization for MAS with identical nonlinear agents affected by external disturbances.

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