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Squared-down passivity based H_∞ almost synchronization of homogeneous continuous-time multi-agent systems with partial-state coupling via static protocol

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Abstract— **This paper studies H_∞ almost state and output synchronizations of homogeneous multi-agent systems (MAS) with partial-state coupling with general linear agents affected by external disturbances. We will characterize when static linear protocols can be designed for state and output synchronization for a MAS such that the impact of disturbances on the network disagreement dynamics, expressed in terms of the H_∞ norms of the corresponding closed-loop transfer function, is reduced to any arbitrarily small value. Meanwhile, the static protocol only needs rough information on the network graph, that is a lower bound for the real part and an upper bound for the modulus of the non-zero eigenvalues of the Laplacian matrix associated with the network graph. Our study focuses on three classes of agents which are squared-down passive, squared-down passifiable via output feedback and squared-down minimum-phase with relative degree 1.**

I. Introduction

The problem of synchronization among agents in a multi-agent system has received substantial attention in recent years, because of its potential applications in cooperative control of autonomous vehicles, distributed sensor network, swarming and flocking and others. The objective of synchronization is to secure an asymptotic agreement on a common state or output trajectory through decentralized control protocols (see [1], [15], [21], [33] and references therein).

State synchronization basically requires homogeneous MAS (i.e. agents have identical dynamics). State synchronization based on diffusive partial-state coupling has also been considered in many papers (e.g. see [11], [12], [24], [25], [26], [30], [31]). The case where the full state is shared over the network, will be referred to as full-state coupling. If only part of the state is shared over the network, we refer to it as partial-state coupling. For partial-state coupling, the synchronization can be achieved via a dynamic protocol or a static protocol. State synchronization via a dynamic protocol imposes restrictions on the agent dynamics. Agents are assumed to be at most weakly unstable (all poles in the closed-left half plane) in e.g. [26], [32] and references therein. Alternatively, agents are assumed to be at most weakly non-minimum-phase (all invariant zeros in closed left half plane) in e.g. [4], [8], [28], [29], [37] and references therein. For state synchronization via a static protocol, agents are usually required to be passive or passifiable via output feedback. For example, [34] considers linear agents which are either passive or passifiable via output feedback. In that case a certain set of graphs is identified for which state synchronization can be achieved. In [5], agents are strictly G-passifiable via output feedback while [6] deals with linear agents which are either passive or passifiable via state/output feedback agent. In [10], [19], [20], input feedforward passivity is studied in connection with output synchronization. Nonlinear input-affine passive agents are considered in [3], [27], [35], [39], [41] while general nonlinear passive agents are studied in [9], [16], [36]. The main objective is to derive conditions for synchronization for different classes of network graphs: undirected, balanced, directed or time-varying graphs.

On the other hand, most research works have focused on the idealized case where the agents are not affected by external disturbances. In the literature where external disturbances are considered, γ-suboptimal H_∞ design is developed for MAS to achieve an H_∞ norm from an external disturbance to the synchronization error among agents less to a priori given γ. In particular, [12], [40] considered the H_∞ norm from an external disturbance to the output error among agents. [23] considered the H_∞ norm from an external disturbance to the state error among agents, whereas [13] and [14] try to obtain an H_∞ norm from a disturbance to the average of the states in a network of single or double integrators.

By contrast, [17] introduced the notion of H_∞ *almost synchronization* for homogeneous MAS, where the goal is to reduce the H_∞ norm from an external disturbance to the synchronization *error*, to any arbitrary desired level. This work is extended later in [18], [37], and [38].

In this paper, we study H_∞ *almost state and output synchronization* for a MAS with partial-state coupling via static protocol design for passifiable agents affected by external disturbances. We will show that the solvability of these problems depends on two classifications:

- Input-matched disturbances or not
- Minimum-phase agents or not

Due to space limitation, the proof of the main results are

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omitted, However the proofs are available online in the full version of this paper.

Notations and definitions: Given a matrix $A \in \mathbb{R}^{m\times n}$, A^{T} denotes the transpose of A, and $\|A\|$ denotes the induced 2-norm of A. A square matrix A is said to be Hurwitz stable if all its eigenvalues are in the open left half complex plane. $A \otimes B$ depicts the Kronecker product between A and B. I_n denotes the n-dimensional identity matrix and 0_n denotes $n \times n$ zero matrix; we will use I or 0 if the dimension is clear from the context.

A *weighted directed graph* $\mathcal{G}$ is defined by a triple $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, \dots, N\}$ is a node set, $\mathcal{E}$ is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the weighting matrix, where $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. We have $a_{ii} = 0$ while $a_{ij} > 0$ denotes an *edge* from node j to node i. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1, \dots, i_k\}$ such that $(i_j, i_{j+1} \in \mathcal{E}$ for $j = 1, \dots, k-1$. A *directed tree* is a subgraph (subset of nodes and edges) in which every node has exactly one parent node except for one node, called the *root*, which has no parent node. In this case, the root has a directed path to every other node in the tree. A *directed spanning tree* is a directed tree containing all the nodes of the graph. For a weighted graph $\mathcal{G}$, a matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^{N} a_{ik}, & i = j, \\ -a_{ij}, & i \neq j, \end{cases}$$

is called the *Laplacian matrix* associated with the graph $\mathcal{G}$. The matrix L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector **1**. A specific class of graphs needed in this paper is presented below:

Definition 1 *For any given $\alpha \geqslant \beta > 0$, let $\mathbb{G}^N_{\alpha,\beta}$ denote the set of directed graphs with N nodes that contain a directed spanning tree and for which the corresponding Laplacian matrix L satisfies $\|L\| < \alpha$ while its nonzero eigenvalues have a real part larger than or equal to β.*

II. Review of squared-down passivity and passifiability

Consider a system

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. In this paper, we first define *squared-down passive* and *squared-down passifiable via output feedback* for a non-square system (1) based on the idea of *squaring down* in [22].

A system (1) is called *squared-down passive* with a pre-compensator $G_1 \in \mathbb{R}^{m\times q}$ and a post-compensator $G_2 \in \mathbb{R}^{q\times p}$ if the interconnection with input $\hat{u}$ and output $\hat{y}$ is passive where $u = G_1\hat{u}$ and $\hat{y} = G_2 y$. Assuming G_1 and G_2 are such that (A, BG_1) is stabilizable, (A, G_2C) is detectable while BG_1 and G_2C have full column- and row-rank, respectively, then this is equivalent to the existence of a positive definite matrix P, such that

$$PA + A^{\mathrm{T}}P \leqslant 0, \qquad PBG_1 = C^{\mathrm{T}}G_2^{\mathrm{T}}. \tag{2}$$

Remark 1 *Note that when $G_1 = I$, squared-down passivity is reduced to G-passivity as used in [7]. For a square system, we can choose $G_1 = G_2 = I$ and squared-down passivity becomes conventional passivity.*

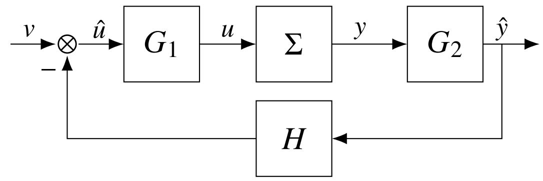

Fig. 1. A squared-down passive system via output feedback

Similarly, a system (1) is called *squared-down passifiable via output feedback* with a pre-compensator $G_1 \in \mathbb{R}^{m\times q}$ and a post-compensator $G_2 \in \mathbb{R}^{q\times p}$ if there exists an output feedback

$$\hat{u} = -H\hat{y} + v \tag{3}$$

which makes the system (1) squared-down passive with respect to the new input v, as shown in Figure 1.

A system (1) is squared-down passifiable via an output feedback (3) if there exist a matrix H and a positive definite matrix P such that

$$\begin{array}{c} P(A - BG_1HG_2C) + (A - BG_1HG_2C)^{\mathrm{T}}P \leqslant 0, \\ PBG_1 = C^{\mathrm{T}}G_2^{\mathrm{T}}. \end{array} \tag{4}$$

This sufficient condition is also necessary for a system to be squared-down passifiable via output feedback if (A, BG_1) is stabilizable, (A, G_2C) is detectable while BG_1 and G_2C have full column- and row-rank, respectively.

Finally, we will define a class of agents, which are *squared-down minimum-phase with relative degree* 1.

A system (1) is called *squared-down minimum-phase with relative degree* 1 with a pre-compensator $G_1 \in \mathbb{R}^{m\times q}$ and a post-compensator $G_2 \in \mathbb{R}^{q\times p}$ if the square system (A, BG_1, G_2C) is minimum-phase with relative degree 1, i.e. $\det(G_2CBG_1) \neq 0$.

Remark 2 *It is easy to show that if the system* (1) *is squared-down minimum-phase with relative degree* 1*, one can choose G_1 such that $G_2CBG_1 = I$.*

In this paper we will use the following lemma which makes the structure of a system more explicit when it is squared-down passifiable via static output feedback.

Lemma 1 *Consider system* (1) *and assume it is squared-down passifiable via static output feedback with compensator G_1 and G_2 and output feedback gain H as in Figure 1, then for the system (A, BG_1, G_2C), with input $\hat{u}$, with $u = G_1\hat{u}$, and output $\hat{y} = G_2y$, there exist non-singular transformation matrices T_x, $T_{\hat{u}}$ and $T_{\hat{y}}$ with*

$$\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = T_x x, \qquad \tilde{u} = T_{\hat{u}}\hat{u}, \qquad \tilde{y} = T_{\hat{y}}\hat{y}$$

where $T_{\hat{y}} = (T_{\hat{u}}^{-1})^T$, such that the dynamics of $\tilde{x}$ is represented by

$$\begin{aligned}\dot{\tilde{x}}_1 &= A_{11}\tilde{x}_1 + A_{12}\tilde{x}_2,\\ \dot{\tilde{x}}_2 &= A_{21}\tilde{x}_1 + A_{22}\tilde{x}_2 + \tilde{u},\\ \tilde{y} &= \tilde{x}_2,\end{aligned} \tag{5}$$

where $\tilde{x}_1 \in \mathbb{R}^{n-q}$ and $\tilde{x}_2 \in \mathbb{R}^q$. To be more specific, we have:

$$A_{11} = \begin{pmatrix} A_{11s} & 0 \\ 0 & A_{110} \end{pmatrix}, A_{12} = \begin{pmatrix} A_{121} \\ A_{122} \end{pmatrix}, A_{21} = \begin{pmatrix} A_{211} & A_{212} \end{pmatrix} \tag{6}$$

with:

$$A_{11s} + A_{11s}^T < 0, \qquad A_{110} + A_{110}^T = 0, \qquad A_{212} = -A_{122}^T$$

Remark 3 *If the system is squared-down passive, i.e. $H = 0$ in Figure 1, then we can additionally guarantee that*

$$A_{22} + A_{22}^T \leqslant 0 \tag{7}$$

Proof: Obviously the system (A, BG_1, G_2C) is at most weakly non-minimum phase with relative degree 1. Note that there exists $P > 0$ such that (4) is satisfied. Choose a unitary matrix U such that

$$UP^{1/2}BG_1 = \bar{B} = \begin{pmatrix} 0 \\ \bar{B}_2 \end{pmatrix}$$

with $\bar{B}_2$ invertible which is possible since BG_1 is injective.

We first apply a state space transformation $\check{x} = T_{x1}x$ with $T_{x1} = UP^{1/2}$ and we get:

$$\Sigma : \begin{cases} \dot{\check{x}} = \bar{A}\check{x} + \bar{B}\hat{u}, \\ \hat{y} = \bar{C}\check{x}, \end{cases}$$

where

$$\begin{aligned}(\bar{A} - \bar{B}H\bar{C}) + (\bar{A} - \bar{B}H\bar{C})^{\mathrm{T}} &\leqslant 0 \\ \bar{B} &= \bar{C}^{\mathrm{T}}\end{aligned} \tag{8}$$

We decompose $\bar{A}$ compatibly with $\bar{B}$:

$$\bar{A} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix}$$

Next, (8) implies that

$$\bar{A}_{11} + \bar{A}_{11}^{\mathrm{T}} \leqslant 0$$

Choose a unitary matrix U_1 such that:

$$U_1\bar{A}_{11}U_1^{\mathrm{T}} = A_{11} = \begin{pmatrix} A_{11s} & 0 \\ 0 & A_{110} \end{pmatrix}$$

with $A_{11s} + A_{11s}^{\mathrm{T}} < 0$ and $A_{110} + A_{110}^{\mathrm{T}} = 0$. Then, it is easily verified that

$$T_x = \begin{pmatrix} U_1 & 0 \\ 0 & I \end{pmatrix} T_{x1}, \qquad T_u = \bar{B}_2$$

yields (5) and (6). Remains to verify that $A_{212} = -A_{122}^{\mathrm{T}}$. If we look at (8) then we get:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} - \hat{H} \end{pmatrix} + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} - \hat{H} \end{pmatrix}^{\mathrm{T}} =$$

$$\begin{pmatrix} A_{11s} + A_{11s}^{\mathrm{T}} & 0 & A_{121} + A_{211}^{\mathrm{T}} \\ 0 & 0 & A_{122} + A_{212}^{\mathrm{T}} \\ A_{121}^{\mathrm{T}} + A_{211} & A_{122}^{\mathrm{T}} + A_{212} & A_{22} + A_{22}^{\mathrm{T}} - \hat{H} - \hat{H}^{\mathrm{T}} \end{pmatrix} \leqslant 0 \tag{9}$$

where $\hat{H} = \bar{B}_2 H \bar{B}_2^{\mathrm{T}}$ from which it is immediately clear that we must have $A_{212} = -A_{122}^{\mathrm{T}}$. ■

III. Problem formulation

Consider a MAS composed of N identical linear time-invariant agents of the form,

$$\begin{aligned}\dot{x}_i &= Ax_i + Bu_i + E\omega_i, \\ y_i &= Cx_i,\end{aligned} \qquad (i = 1, \dots, N) \tag{10}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are respectively the state, input, and output vectors of agent i, and $\omega_i \in \mathbb{R}^r$ is the external disturbance.

The communication network provides each agent with a linear combination of its own outputs relative to that of other neighboring agents. In particular, each agent $i \in \{1, \dots, N\}$ has access to the quantity,

$$\zeta_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j) = \sum_{j=1}^{N} \ell_{ij} y_j. \tag{11}$$

The communication topology of the network can be described by a weighted and directed graph $\mathcal{G}$ with corresponding Laplacian matrix L. We will primarily focus on *partial-state coupling* where C does not have full-column rank.

If the graph $\mathcal{G}$ describing the communication topology of the network contains a directed spanning tree, then it follows from [2] that the Laplacian matrix L has a simple eigenvalue at the origin, with the corresponding right eigenvector $\mathbf{1}$ and all the other eigenvalues are in the open right-half complex plane. Let $\lambda_1, \dots, \lambda_N$ denote the eigenvalues of L such that $\lambda_1 = 0$ and $\mathrm{Re}(\lambda_i) > 0$, $i = 2, \dots, N$.

Let N be any agent and define

$$\bar{x}_i = x_N - x_i, \quad \bar{u}_i = u_N - u_i \text{ and } \bar{y}_i = y_N - y_i$$

and

$$\bar{x} = \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_{N-1} \end{pmatrix}, \; \bar{u} = \begin{pmatrix} \bar{u}_1 \\ \vdots \\ \bar{u}_{N-1} \end{pmatrix}, \; \bar{y} = \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_{N-1} \end{pmatrix} \text{ and } \omega = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_N \end{pmatrix}.$$

Obviously, output synchronization is achieved if $\bar{y} = 0$. In other words one of the prime objectives is to achieve:

$$\lim_{t\to\infty}(x_i(t) - x_N(t)) = 0, \quad \forall i \in \{1, \dots, N-1\}, \tag{12}$$

$$\lim_{t\to\infty}(y_i(t) - y_N(t)) = 0, \quad \forall i \in \{1, \dots, N-1\}. \tag{13}$$

Remark 4 *The agent N is not necessarily a root agent. Obviously,* (12) *is equivalent to the condition that*

$$\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$$

for all $i, j \in \{1, \dots, N\}$ and a similar connection holds for (13).

In this paper we focus on static protocols parameterized in a parameter ε:

$$u_i = F_\varepsilon \zeta_i, \qquad i = 1, \dots N \tag{14}$$

We formulate below two H_∞ almost state/output synchronization problems.

Problem 1 *Consider a MAS described by* (10) *and* (11). *Let* $\mathbf{G}$ *be a given set of graphs such that* $\mathbf{G} \subseteq \mathbb{G}^N$. *The* ***$H_\infty$ almost state synchronization problem via static protocol*** *with a set of network graphs* $\mathbf{G}$ *is to find, if possible, a linear static protocol parameterized in terms of a parameter* ε, *of the form* (14) *such that, for any given real number* $\delta > 0$, *there exists an* ε^* *such that for any* $\varepsilon \in (0, \varepsilon^*]$ *and for any graph* $\mathcal{G} \in \mathbf{G}$, (12) *is satisfied for all initial conditions in the absence of disturbances and the closed loop transfer matrix* $T_{\omega\bar{x}}$ *satisfies*

$$\|T_{\omega\bar{x}}\|_\infty < \delta. \tag{15}$$

Problem 2 *Consider a MAS described by* (10) *and* (11). *Let* $\mathbf{G}$ *be a given set of graphs such that* $\mathbf{G} \subseteq \mathbb{G}^N$. *The* ***$H_\infty$ almost output synchronization problem with bounded state errors via static protocol*** *with a set of network graphs* $\mathbf{G}$ *is to find, if possible, a linear static protocol parameterized in terms of a parameter* ε, *of the form* (14) *and a constant* M *(independent of* ε*) such that, for any given real number* $\delta > 0$, *there exists an* ε^* *such that for any* $\varepsilon \in (0, \varepsilon^*]$ *and for any graph* $\mathcal{G} \in \mathbf{G}$, (13) *is satisfied for all initial conditions in the absence of disturbances and the closed loop transfer matrices* $T_{\omega\bar{x}}$ *and* $T_{\omega\bar{y}}$ *satisfy*

$$\|T_{\omega\bar{x}}\|_\infty < M, \qquad \text{and} \qquad \|T_{\omega\bar{y}}\|_\infty < \delta. \tag{16}$$

IV. H_∞ ALMOST DISTURBANCE DECOUPLING

In this section, we establish a connection between H_∞ almost state/output synchronization among agents in the network via static protocol and a robust H_∞ almost state/output disturbance decoupling problem via static output feedback with internal stability.

Preliminary results: The MAS system described by (10) and (11) after implementing the linear static protocol (14) is described by

$$\begin{aligned}\dot{x}_i &= Ax_i + BF_\varepsilon \zeta_i + E\omega_i,\\ y_i &= Cx_i\end{aligned}$$

for $i = 1, \dots, N$. Let

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}, \quad \omega = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_N \end{pmatrix}.$$

Then, the overall dynamics of the N agents can be written as

$$\dot{x} = (I_N \otimes A + L \otimes BF_\varepsilon C)x + (I_N \otimes E)\omega. \tag{17}$$

Firstly, we define the **robust H_∞ almost output disturbance decoupling problem with bounded input via static output feedback** as follows. Given $\Lambda \subset \mathbb{C}$, there should exist a parameterized controller

$$u = F_\varepsilon y \tag{18}$$

and $M > 0$ such that, for any given $\delta > 0$, there exists $\varepsilon^* > 0$ for which the interconnection of (18) and the system,

$$\begin{aligned}\dot{x} &= Ax + \lambda Bu + E\omega,\\ y &= Cx,\end{aligned} \tag{19}$$

has the property that for any $\lambda \in \Lambda$ and for any $0 < \varepsilon < \varepsilon^*$ we have:

1) The interconnection of the systems (18) and (19) and is internally stable;
2) The resulting closed-loop transfer function
$$T^\lambda_{\omega y} = C(sI - A - \lambda BF_\varepsilon C)^{-1}E \tag{20}$$
from ω to y has an H_∞ norm less than δ.
3) The resulting closed-loop transfer function
$$T^\lambda_{uy} = C(sI - A - \lambda BF_\varepsilon C)^{-1}B \tag{21}$$
has an H_∞ norm less than δ.
4) The resulting closed-loop transfer function
$$T^\lambda_{\omega u} = F_\varepsilon C(sI - A - \lambda BF_\varepsilon C)^{-1}E \tag{22}$$
from ω to u has an H_∞ norm less than M.
5) The resulting closed-loop transfer function
$$T^\lambda_{uu} = F_\varepsilon C(sI - A - \lambda BF_\varepsilon C)^{-1}B \tag{23}$$
has an H_∞ norm less than M.

In the above, Λ denotes all possible locations for the nonzero eigenvalues of the Laplacian matrix L when the graph varies over the set $\mathbf{G}$. It is also important to note that M is independent of the choice for δ and independent of $\lambda \in \Lambda$.

The **robust H_∞ almost state disturbance decoupling problem with bounded input via static output feedback** is equivalent to the above with the only modification being that instead of (20) and (21), the closed-loop transfer functions

$$T^\lambda_{\omega x} = (sI - A - \lambda BF_\varepsilon C)^{-1}E \tag{24}$$
$$T^\lambda_{ux} = (sI - A - \lambda BF_\varepsilon C)^{-1}B \tag{25}$$

both have an H_∞ norm less than δ.

The connection of the above problem with the H_∞ almost output synchronization problem via static protocol as defined in Problem 2 is given below.

Lemma 2 *Let* $\mathbf{G}$ *be a set of graphs such that the associated Laplacian matrices are uniformly bounded and let* Λ *consist of all possible nonzero eigenvalues of Laplacian matrices associated with graphs in* $\mathbf{G}$. *The* H_∞ *almost output synchronization problem via static protocol for the MAS described by* (10) *and* (11) *given* $\mathbf{G}$ *is solved by a parameterized protocol* $u_i = F_\varepsilon \zeta_i$ *if the robust* H_∞ *almost output disturbance decoupling problem with bounded input via static output feedback for the system* (19) *with* $\lambda \in \Lambda$ *is solved by the parameterized controller* $u = F_\varepsilon y$.

The next lemma establishes the similar connection between the H_∞ almost state disturbance decoupling problem with H_∞ almost state synchronization problem.

Lemma 3 *Let $\mathbf{G}$ be a set of graphs such that the associated Laplacian matrices are uniformly bounded and let Λ consists of all possible nonzero eigenvalues of Laplacian matrices associated with graphs in $\mathbf{G}$. The H_∞ almost state synchronization problem via static protocol for the MAS described by* (10) *and* (11) *given $\mathbf{G}$ is solved by a parameterized protocol $u_i = F_\varepsilon \zeta_i$ if the robust H_∞ almost state disturbance decoupling problem with bounded input via static output feedback for the system* (19) *with $\lambda \in \Lambda$ is solved by the parameterized controller $u = F_\varepsilon y$.*

V. H_∞ ALMOST SYNCHRONIZATION

In this section, we will consider a static protocol design to achieve H_∞ almost output synchronization. We consider a MAS described by (10) and (11).

We split the analysis in two cases. The first case considers agents which are squared-down passifiable given G_1, G_2 and H. Clearly this included squared-down passive agents as a special case. A squared-down passifiable agent given G_1, G_2 and H is such that (A, BG_1, G_2C) is weakly minimum-phase with relative degree 1. However, if the system is actually minimum-phase instead of only weakly minimum-phase then we actually obtain stronger results. Therefore, this case is analyzed separately. The problems formulated earlier in this paper were in terms of an arbitrary set of graphs $\mathbf{G}$. The results in this section are obtained for specific classes of graphs where:

$$\mathbf{G} = \mathbb{G}_{\alpha,\beta}^N$$

for some $\alpha, \beta > 0$ which has been defined in Definition 1. For all the problems in this paper we consider the same parameterized protocol

$$u_i = -\tfrac{1}{\varepsilon} G_1 G_2 \zeta_i, \tag{26}$$

The first result regarding H_∞ almost output synchronization problem via static protocol is stated as follows.

Theorem 1 *Consider a MAS described by* (10) *and* (11)*. Assume (A, B, C) is squared-down passifiable with respect to G_1, G_2 and H such that (A, BG_1) is stabilizable, (A, G_2C) is detectable while BG_1 and G_2C have full column- and row-rank, respectively. Let any real numbers $\alpha, \beta > 0$ and a positive integer N be given, and hence a set of network graphs $\mathbb{G}_{\alpha,\beta}^N$ be defined. The H_∞ almost output synchronization problem via static protocol problem as defined in Problem 2 with respect to fictitious output $\hat{y} = G_2 y$ where $\mathbf{G} = \mathbb{G}_{\alpha,\beta}^N$ is solvable if*

$$\operatorname{Im} E \subseteq \operatorname{Im} BG_1. \tag{27}$$

In particular, there exists an M such that for any given real number $\delta > 0$, there exists an ε^*, such that for any $\varepsilon \in (0, \varepsilon^*)$, the protocol (26) *achieves state synchronization (without disturbances), an H_∞ norm from ω to $\hat{y}_i - \hat{y}_j$ less than δ and an H_∞ norm from ω to $x_i - x_j$ less than M for any $i, j \in 1, \dots, N$ and for any graph $\mathcal{G} \in \mathbb{G}_{\alpha,\beta}^N$.*

The above shows that for squared-down passifiable agents we can achieve almost output synchronization with respect to the fictitious output $\hat{y} = G_2 y$ if (27). The following example illustrates that this result is no longer valid if we consider almost state synchronization, if we use the original output y or if the condition (27) is not satisfied.

Example 1 *Consider the system:*

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x + \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u + \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \omega \\ y &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x \end{aligned} \tag{28}$$

It is easily verified that this system is squared-down passive with respect to

$$G_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

The transfer matrix from ω to x when $u = -\rho G_1 G_2 y$ is given by:

$$G_{\omega x}^{\lambda}(s) = \frac{1}{s^2 + \lambda\rho s + 1} \begin{pmatrix} (s + \lambda\rho)e_1 & e_2 \\ -e_1 & se_2 \end{pmatrix}$$

Given that:

$$G_{\omega x}^{\lambda}(0) = \begin{pmatrix} \lambda\rho e_1 & e_2 \\ -e_1 & 0 \end{pmatrix},$$

we note that the second condition of the robust H_∞ almost state disturbance decoupling problem with bounded input via static output feedback is not satisfied unless we are in the trivial case $e_1 = e_2 = 0$.

Next, consider the transfer matrix from ω to $\hat{y} = G_2 y$ when $u = -\rho G_1 G_2 y$. We obtain:

$$G_{\omega\hat{y}}^{\lambda}(s) = \frac{1}{s^2 + \lambda\rho s + 1} \begin{pmatrix} -e_1 & se_2 \end{pmatrix}$$

Given that:

$$G_{\omega\hat{y}}^{\lambda}(0) = \begin{pmatrix} -e_1 & 0 \end{pmatrix},$$

we note that the second condition of the robust H_∞ almost state disturbance decoupling problem with bounded input via static output feedback is not satisfied unless $e_1 = 0$. The latter is equivalent to the condition $\operatorname{Im} E \subseteq \operatorname{Im} BG_1$.

By Lemma 2, robust H_∞ almost output disturbance decoupling is equivalent to H_∞ almost output synchronization of a MAS whose agents have the dynamics given in (28)*. Hence, if H_∞ almost output disturbance decoupling is not solvable, then robust H_∞ output synchronization is clearly not solvable either.*

The next theorem shows that if the system is minimum-phase instead of only weakly minimum-phase then we can achieve almost **state** synchronization if condition (27) is satisfied. Moreover, if condition (27) is not satisfied then we still achieve almost **output** synchronization.

Theorem 2 *Consider a MAS described by* (10) *and* (11)*. Assume (A, B, C) is squared-down minimum-phase with relative degree 1 with G_1 and G_2 such that (A, BG_1) is controllable and (A, G_2C) is observable. Assume that without loss of generality G_1 is chosen such that Remark 2 is satisfied. Let any real numbers $\alpha, \beta > 0$ and a positive integer N be given,* *and hence a set of network graphs $\mathbb{G}^N_{\alpha,\beta}$ be defined. The H_∞ almost state synchronization problem via static protocol, as defined in 1 where $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$, is solvable if* (27) *is satisfied. If* (27) *is not satisfied then the H_∞ almost output synchronization via static protocol, as defined in 2 with respect to the fictitious output $\hat{y} = G_2 y$ where $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$, is solvable. In particular, for any given real number $\delta > 0$, there exists an ε^*, such that for any $\varepsilon \in (0, \varepsilon^*)$, the protocol* (26) *achieves output synchronization and an H_∞ norm from ω to $\hat{y}_i - \hat{y}_j$ less than δ for any $i, j \in 1, \dots, N$ and for any graph $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$. If* (27) *is satisfied then the protocol* (26) *achieves almost state synchronization and an H_∞ norm from ω to $x_i - x_j$ less than δ for any $i, j \in 1, \dots, N$ and for any graph $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$.*

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