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Computation of the inf-sup constant for the divergence

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A numerical method for approximating the inf-sup constant of the divergence (LBB constant) is proposed, and some details of the convergence analysis are reported.

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz polytope and let $V := H^1_0(\Omega;\mathbb{R}^n)$ denote the space of L^2 vector fields over Ω with generalized first derivatives in $L^2(\Omega)$ and vanishing trace on the boundary, and let $Q := L^2_0(\Omega)$ denote the space of L^2 functions with vanishing average over Ω. It is known [1, 4] that the divergence operator $\operatorname{div} : V \to Q$ possesses a continuous right-inverse, i.e., there exists a positive constant β such that for any $q \in Q$ there exists some $v \in V$ with $\operatorname{div} v = q$ and $\beta\|Dv\| \leq \|q\|$ (here $\|\cdot\|$ is the $L^2(\Omega)$ norm). The largest number β with this property is characterized by

$$\beta = \inf_{q\in Q\setminus\{0\}} \sup_{v\in V\setminus\{0\}} \frac{(q, \operatorname{div} v)_{L^2(\Omega)}}{\|q\|\|Dv\|}. \tag{1}$$

The numerical approximation of β with stable standard finite element pairings [3] is problematic because convergence cannot be guaranteed in general [2]. Since, with the space of velocity gradients $\Gamma := DV$, the constant β can be rewritten as

$$\beta = \inf_{q\in Q\setminus\{0\}} \sup_{\gamma\in \Gamma\setminus\{0\}} \frac{(q, \operatorname{tr} \gamma)_{L^2(\Omega)}}{\|q\|\|\gamma\|}, \tag{2}$$

numerical schemes that directly approximate the space Γ are applicable. The classical Helmholtz decomposition [with $\perp$ denoting L^2 orthogonality in $\Sigma := L^2(\Omega;\mathbb{R}^{n\times n})$] reads

$$\Gamma := \mathfrak{Z}^\perp, \qquad \text{with } \mathfrak{Z} := [H(\operatorname{div}^0,\Omega)]^n = \{\sigma \in \Sigma : \text{all rows of } \sigma \text{ are divergence-free}\}.$$

The work [7] proposed a discrete analogue in $\Sigma_h := P_k(\mathcal{T}_h;\mathbb{R}^{n\times n})$, the space of piecewise polynomial (of degree $\leq k$) tensor fields with respect to a simplicial triangulation $\mathcal{T}_h$ of Ω, as follows

$$\Gamma_h := \mathfrak{Z}_h^\perp, \qquad \text{with } \mathfrak{Z}_h := (RT_k(\mathcal{T}_h)^n \cap \mathfrak{Z}) \subseteq (\mathfrak{Z} \cap \Sigma_h),$$

where $\perp$ denotes L^2 orthogonality in Σ_h and $RT_k(\mathcal{T}_h)^n$ denotes the subspace of Σ whose rows belong to the Raviart–Thomas finite element space [3] of degree k. The property $\mathfrak{Z}_h \subseteq \Sigma_h$ is proved in [5]. One should note that in general $\Gamma_h \not\subseteq \Gamma$.

Let Q_h denote the subspace of Q consisting of $\mathcal{T}_h$-piecewise polynomial functions of degree $\leq k$. The approximation β_h is defined as

$$\beta_h = \inf_{q_h\in Q_h\setminus\{0\}} \sup_{\gamma_h\in \Gamma_h\setminus\{0\}} \frac{(q_h, \operatorname{tr} \gamma_h)_{L^2(\Omega)}}{\|q_h\|\|\gamma_h\|}. \tag{3}$$

Lemma A. *Let $\mathcal{T}_h$ be a regular refinement of a (coarser) mesh $\mathcal{T}_H$. Then, $\beta \leq \beta_h \leq \beta_H$.*

P r o o f. From (2) and $Q_h \subseteq Q$ it is obvious that

$$\beta \leq \inf_{q_h\in Q_h\setminus\{0\}} \sup_{\gamma\in \Gamma\setminus\{0\}} \frac{(q_h, \operatorname{tr} \gamma)_{L^2(\Omega)}}{\|q_h\|\|\gamma\|}.$$

With the L^2 projection Π_h onto Σ_h, it furthermore follows for any nonzero $q_h \in Q_h$ that

$$\sup_{\gamma\in \Gamma\setminus\{0\}} \frac{(q_h, \operatorname{tr} \gamma)_{L^2(\Omega)}}{\|q_h\|\|\gamma\|} = \sup_{\gamma\in \Gamma\setminus\{0\}} \frac{(q_h, \operatorname{tr} \Pi_h\gamma)_{L^2(\Omega)}}{\|q_h\|\|\gamma\|} \leq \sup_{\substack{\gamma\in\Gamma\\ \Pi_h\gamma\neq 0}} \frac{(q_h, \operatorname{tr} \Pi_h\gamma)_{L^2(\Omega)}}{\|q_h\|\|\Pi_h\gamma\|} \leq \sup_{\gamma_h\in \Gamma_h\setminus\{0\}} \frac{(q_h, \operatorname{tr} \gamma_h)_{L^2(\Omega)}}{\|q_h\|\|\gamma_h\|},$$

where the last estimate holds because $\Pi_h\Gamma \subseteq \Gamma_h$ (proof: $\forall\gamma \in \Gamma\ \forall\mathfrak{z}_h \in \mathfrak{Z}_h\ (\Pi_h\gamma, \mathfrak{z}_h)_{L^2(\Omega)} = (\gamma, \mathfrak{z}_h)_{L^2(\Omega)} = 0$). Note that in the pathological case $\{\gamma \in \Gamma : \Pi_h\gamma \neq 0\} = \emptyset$, where the third expression in the displayed formula equals $-\infty$, the left-hand side equals zero, and the desired estimate is obviously still valid.

The infimum over all nonzero $q_h \in Q_h$ in combination with the first upper bound of β shows $\beta \leq \beta_h$. The second asserted inequality is obtained in an analogous way. $\square$

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Lemma A establishes a monotonically decreasing approximation under mesh refinement. For a convergence proof, the following equivalent formulation of the problem turns out useful. It is well known [2] that

$$\beta^2 = \inf_{v\neq 0} \frac{\|\operatorname{div} v\|^2}{\|Dv\|^2} \tag{4}$$

where the infimum is taken over the V-orthogonal complement (that is, with respect to the inner product $(D\cdot, D\cdot)_{L^2(\Omega)}$) of the divergence-free functions in V. The discrete analogue of this space is

$$X_h := \{\tau_h \in \Gamma_h : (\tau_h, \eta_h)_{L^2(\Omega)} = 0 \text{ for all } \eta_h \in \Gamma_h \text{ with } \operatorname{tr}\eta_h = 0\}.$$

It can be shown that, in analogy to the infinite-dimensional setting, β_h satisfies

$$\beta_h^2 = \inf_{\xi_h \in X_h\setminus\{0\}} \frac{\|\operatorname{tr}\xi_h\|^2}{\|\xi_h\|^2}. \tag{5}$$

Let $\mathfrak{P}_h : \Sigma \to X_h$ denote the L^2-orthogonal projection onto the space X_h. For the sake of simple exposition assume that $u \in V$ is an eigenfunction corresponding to (4) with $\|Du\| = 1$. Since $\Pi_h\Gamma \subseteq \Gamma_h$, the projection Π_h maps Γ to Γ_h and, thus, $\mathfrak{P}_h \circ \Pi_h = \mathfrak{P}_h$. The function $\Pi_h Du \in \Gamma_h$ can hence be decomposed as

$$\Pi_h Du = \mathfrak{P}_h Du + (1 - \mathfrak{P}_h)\Pi_h Du.$$

By definition, $(1 - \mathfrak{P}_h)$ is the orthogonal projection from Γ_h to the trace-free elements of Γ_h. Thus, taking the trace in the above relation reveals $\operatorname{tr}\mathfrak{P}_h Du = \operatorname{tr}\Pi_h Du$. The Rayleigh–Ritz principle and this conservation property show

$$\beta_h^2\|\mathfrak{P}_h Du\|^2 \leq \|\operatorname{tr}\mathfrak{P}_h Du\|^2 = \|\operatorname{tr}\Pi_h Du\|^2 \leq \|\operatorname{tr} Du\|^2 = \|\operatorname{div} u\|^2 = \beta.$$

The Pythagoras rule with $\|Du\|^2 = 1$ reads $\|\mathfrak{P}_h Du\|^2 = 1 - \|(1 - \mathfrak{P}_h)Du\|^2$, so that rearranging terms in the last displayed formula yields:

Lemma B. *Any eigenfunction $u \in V$ corresponding to (4) with $\|Du\| = 1$ satisfies $(1 - \|(1 - \mathfrak{P}_h)Du\|^2)\beta_h \leq \beta$.* □

The same lower bound (with some further technical steps in the proof [6]) holds in the case that β is not an eigenvalue. Lemmas A–B show that the convergence $\beta_h \searrow \beta$ as $h \to 0$ is solely determined by the approximation properties of the projection $\mathfrak{P}_h$. These can be quantified with arguments from the theory of the approximation of saddle-point problems. The main result, a detailed proof of which can be found in [6], reads as follows.

Theorem. *Let $(\mathcal{T}_h)_h$ be a sequence of nested partitions such that the mesh size function h uniformly converges to zero. Then the sequence $(\beta_h)_h$ converges monotonically from above towards the inf-sup constant β from (1), i.e.,*

$$\beta_h \searrow \beta \quad \textit{under mesh refinement}.$$

Any $v \in V$ that is V-orthogonal to all the divergence-free elements of V is approximated under mesh refinement: $\|(1 - \mathfrak{P}_h)Du\| \to 0$. Provided that the square of the inf-sup constant β^2 is an eigenvalue of (4) with normalized eigenfunction $u \in H^{1+s}(\Omega;\mathbb{R}^n)$ for some $0 < s < \infty$, any $\mathcal{T}_h$ satisfies

$$(1 - \|(1 - \mathfrak{P}_h)Du\|^2)\frac{\beta_h^2 - \beta^2}{\beta^2} \leq \|(1 - \mathfrak{P}_h)Du\|^2 \leq C\|h\|^{2r}_{L^\infty(\Omega)}\|u\|^2_{H^{1+s}(\Omega)}$$

for the rate $r := \min\{k + 1, s\}$ and some mesh-size independent constant $C > 0$.

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