

Passivity based state synchronization of homogeneous discrete-time multi-agent systems via static protocol in presence of input delay

Zhenwei Liu, Meirong Zhang, Ali Saberi, and Anton A. Stoorvogel

Abstract—This paper studies state synchronization of homogeneous discrete-time multi-agent systems (MAS) with partial-state coupling (i.e., agents are coupled through part of their states) via static protocol in presence of input delay. Both uniform input delay and nonuniform input delay are considered. We identify one class of agents for which static linear protocol can be designed, which is named squared-down passifiable via input feedforward. A parameterized static protocol is proposed for each agent such that state synchronization is achieved among agents with uniform or nonuniform input delay. Moreover, we derive upper bounds for uniform and nonuniform input delay that can be tolerated.

I. INTRODUCTION

The problem of synchronization among agents in a multi-agent system has received substantial attention in recent years, because of its potential applications in cooperative control of autonomous vehicles, distributed sensor network, swarming and flocking and others. The objective of synchronization is to secure an asymptotic agreement on a common state or output trajectory through decentralized control protocols (see [1], [14], [19], [26], [35] and references therein).

So far a lot of work in state synchronization for MAS has focused on state synchronization based on diffusive full-state coupling, i.e., agents are coupled through their states. In this case, universally static protocols are considered.

For partial-state coupling (i.e., agents are coupled through their output which consists of part of their states), state synchronization can be achieved via a dynamic protocol or a static protocol. The standard approach leads to dynamic, observer-based protocols. In [10] a purely decentralized solution was presented which did not rely on this extra communication.

Moreover, in several papers the protocol is introspective where the protocol does not make use of only relative information between the output of the agent and the output of other agents but also uses absolute information about the state of their agent.

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State synchronization via a static protocol with partial-state coupling imposes restrictions on the agent dynamics. For state synchronization via a static protocol, agents are usually required to be passive or passifiable via output feedback. For example, [36] considers linear agents which are either passive or passifiable via output feedback. In [8], agents are strictly G -passifiable via output feedback while [9] deals with linear agents which are either passive or passifiable via output feedback. Nonlinear input-affine passive agents are considered in [5], [28], [41], [43], [44] while general nonlinear passive agents are studied in [11], [20], [42].

Most references assume an idealized network model. But, in practical applications, the network model is always imperfect. In particular, time-delay effects are ubiquitous in any communication scheme. As clarified in [4], we can identify two kinds of time delay: input delay and communication delay. Input delay results from processing time to generate an input for each agent while communication delay refers to the time consumed during the transfer of information between agents. Most effort has been put into input delay problems (see [2], [13], [15], [16], [17], [23], [30], [31], [39] for example). These references, although including results on linear and non-linear agents, are mostly restricted to simple agent models such as first/second-order dynamics. Recently, in [18], [33] and [34], the synchronization problem under unknown uniform constant or time-varying input delay is solved for both discrete- and continuous-time high-order linear agents that are critically unstable. For discrete-time homogeneous MAS, we refer to [7], [12], [24], [29], [40] for example. When a static protocol is required, a restriction is always imposed on the agents. For communication delay, see [5], [6], [16], [21], [22], [30], [38].

In this paper, we will study state synchronization of discrete-time homogeneous MAS in the presence of input delay via static protocols. We focus on the class of squared down passifiable via input feedforward agents. Both uniform input delay and nonuniform input delay are considered. The communication network is assumed to contain a directed spanning tree in the uniform case, and assumed to be undirected in the nonuniform case. The parameterized static protocols are designed for this class of agents based on discrete-time passivity. These designed static protocols will be able to tolerate both uniform and nonuniform input delay.

Notations and definitions: Given a matrix $A \in \mathbb{R}^{m \times n}$, A^T and A^* denote the transpose and conjugate transpose of A , respectively while $\|A\|$ denotes the induced 2-norm of A . A square matrix A is said to be Schur stable if all its eigenvalues are in the open unit disc.

A *weighted directed graph* \mathcal{G} is defined by a triple $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, \dots, N\}$ is a node set, \mathcal{E} is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is the weighting matrix, and $a_{ij} > 0$ iff $(i, j) \in \mathcal{E}$. Each pair in \mathcal{E} is called an *edge*. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1, \dots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, \dots, k-1$. A *directed tree* is a subgraph (subset of nodes and edges) in which every node has exactly one parent node except for one node, called the *root*, which has no parent node. In this case, the root has a directed path to every other node in the tree. A *directed spanning tree* is a subgraph which is a directed tree containing all the nodes of the original graph. An agent is called a *root agent* if it is the root of some directed spanning tree of the associated graph. Let $\Pi_{\mathcal{G}}$ denote the set of all root agents for a graph. For a weighted graph \mathcal{G} , a matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^N a_{ik}, & i = j, \\ -a_{ij}, & i \neq j, \end{cases}$$

is called the *Laplacian matrix* associated with the graph \mathcal{G} . In the case where \mathcal{G} has non-negative weights, L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $\mathbf{1}$, the vector with all elements equal to 1.

II. PASSIVITY AND SQUARED DOWN PASSIFIABILITY VIA INPUT FEEDFORWARD FOR DISCRETE-TIME MAS

Consider a general discrete-time system Σ :

$$\Sigma : \begin{cases} x(k+1) = Ax(k) + Bu(k), \\ y(k) = Cx(k) + Du(k), \end{cases} \quad (1)$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$.

Definition 1 ([3]): The discrete-time system (1) is called *passive* if the system is square and for initial condition $x(0) = 0$, for any input u and for any $k \geq 0$ we have:

$$\sum_{i=0}^k y^T(i)u(i) \geq 0.$$

The positive real lemma (see e.g. [3] and [37]) gives an easy characterization whether systems are passive or not.

Lemma 1: Assume that (A, B) is controllable and (A, C) is observable. The system (1) is passive if and only if there exists a matrix $P > 0$ such that:

$$\begin{pmatrix} A^T P A - P & A^T P B - C^T \\ B^T P A - C & -D - D^T + B^T P B \end{pmatrix} \leq 0. \quad (2)$$

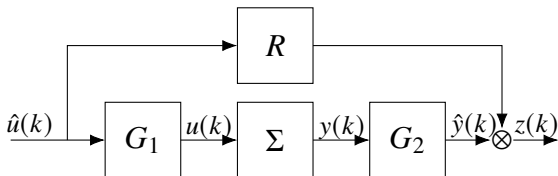


Fig. 1. A squared-down passive system via input feedforward

For non-square systems, we use the idea of *squaring down* in [27]. A system (1) is called *squared-down passive* with

respect to a pre-compensator G_1 and a post-compensator G_2 if the interconnection with input \hat{u} and output \hat{y} is passive.

Since strictly proper systems are never passive, we need a different concept. A system (1) with $D = 0$ is called *squared-down passifiable via input feedforward* with respect to a pre-compensator $G_1 \in \mathbb{R}^{m \times q}$ and a post-compensator $G_2 \in \mathbb{R}^{q \times p}$ if there exists an input feedforward

$$z(k) = R\hat{u}(k) + \hat{y}(k) \quad (3)$$

which makes the system (1) with $D = 0$ squared-down passive with respect to input \hat{u} and the new output $z(k)$, as shown in Figure 1. For this class of systems, we find:

Lemma 2: Consider a system (1) with $D = 0$ such that (A, BG_1) is controllable and (G_2C, A) observable. The system is squared-down passifiable via input feedforward with pre-compensator $G_1 \in \mathbb{R}^{m \times q}$ and post-compensator $G_2 \in \mathbb{R}^{q \times p}$ if and only if there exist matrices R and $P > 0$ such that

$$G(P) = \begin{pmatrix} A^T P A - P & A^T P B G_1 - C^T G_2^T \\ G_1^T B^T P A - G_2 C & -R - R^T + G_1^T B^T P B G_1 \end{pmatrix} \leq 0. \quad (4)$$

III. PROBLEM DESCRIPTION

We will study a MAS consisting of N identical agents:

$$\begin{aligned} x_i(k+1) &= Ax_i(k) + Bu_i(k), \\ y_i(k) &= Cx_i(k), \end{aligned} \quad (5)$$

where $x_i(k) \in \mathbb{R}^n$, $u_i(k) \in \mathbb{R}^m$ and $y_i(k) \in \mathbb{R}^p$ are state, input and output of agent i ($i = 1, \dots, N$), respectively.

The communication network provides each agent with a linear combination of its own output relative to that of other neighboring agents. In particular, each agent $i \in \{1, \dots, N\}$ has access to the quantity,

$$\zeta_i(k) = \frac{1}{1 + \sum_{j=1}^N a_{ij}} \sum_{j=1}^N a_{ij}(y_i(k) - y_j(k)), \quad (6)$$

where $a_{ij} \geq 0$, $a_{ii} = 0$ for $i, j \in \{1, \dots, N\}$. The topology of the network can be described by a graph \mathcal{G} with nodes corresponding to the agents in the network and edges given by the nonzero coefficients a_{ij} . In particular, $a_{ij} > 0$ implies that an edge exists from agent j to i . The weight of the edge equals the magnitude of a_{ij} . Next we write ζ_i as

$$\zeta_i(k) = \sum_{j=1}^N d_{ij}(y_i(k) - y_j(k)), \quad (7)$$

where $d_{ij} \geq 0$, and we choose $d_{ii} = 1 - \sum_{j=1, j \neq i}^N d_{ij}$ such that $\sum_{j=1}^N d_{ij} = 1$ with $i, j \in \{1, \dots, N\}$. Note that $d_{ii} > 0$ for any i . The weight matrix $D = [d_{ij}]$ is then a, so-called, row stochastic matrix. Let $D_{in} = \text{diag}\{d_{in}(i)\}$ with $d_{in}(i) = \sum_{j=1}^N a_{ij}$. Then the relationship between the row stochastic matrix D and the Laplacian matrix L is

$$(I + D_{in})^{-1} L = I - D. \quad (8)$$

As noted in [25, Corollary 3.5], the existence of a directed spanning tree guarantees that the row stochastic matrix D has a simple eigenvalue at 1 with corresponding right eigenvector $\mathbf{1}$ and all other eigenvalues are strictly within the unit disc.

Let $\lambda_1, \dots, \lambda_N$ denote the eigenvalues of D such that $\lambda_1 = 1$ and $|\lambda_i| < 1$, $i = 2, \dots, N$. We can then define a set of network graphs as follows.

Definition 2: For $\delta \in (0, 1)$, let \mathbb{G}_δ^N denote the set of directed graphs with N nodes which contain a directed spanning tree and for which the corresponding row stochastic matrix D has the property that $|\lambda_i| < \delta$ for $i = 2, \dots, N$.

Definition 3: For $\delta \in (0, 1)$, let $\mathbb{G}_\delta^{N,u}$ denote the set of undirected graphs with N nodes which are strongly connected and for which the corresponding row stochastic matrix D has the property that $|\lambda_i| < \delta$, $i = 2, \dots, N$.

Our goal in this paper is to achieve state synchronization among agents in a MAS, that is $\lim_{k \rightarrow \infty} (x_i(k) - x_j(k)) = 0$ for all $i, j \in \{1, \dots, N\}$. When input delays exist in systems (5), the discrete time MAS can be expressed as

$$\begin{aligned} x_i(k+1) &= Ax_i(k) + Bu_i(k - \kappa_i), \\ y_i(k) &= Cx_i(k), \end{aligned} \quad (i = 1, \dots, N) \quad (9)$$

where $\kappa_1, \dots, \kappa_N$ are unknown integers satisfying $\kappa_i \in [0, \bar{\kappa}]$ ($i = 1, \dots, N$) with $\bar{\kappa}$ the upper bound of the input delay. In case of uniform delay, there exists an integer κ satisfying $\kappa = \kappa_1 = \kappa_2 = \dots = \kappa_N$ and the MAS (9) will be of the form

$$\begin{aligned} x_i(k+1) &= Ax_i(k) + Bu_i(k - \kappa), \\ y_i(k) &= Cx_i(k), \end{aligned} \quad (i = 1, \dots, N) \quad (10)$$

where κ is an unknown integer satisfying $\kappa \in [0, \bar{\kappa}]$ with $\bar{\kappa}$ the upper bound of the input delay.

For the above MAS, we formulate three problems:

Problem 1 (No delays): Consider a MAS described by (5) and (7). The *state synchronization* problem via static protocol with a set of network graphs \mathbb{G}_δ^N is to find, if possible, a linear static protocol of the form

$$u_i(k) = F\zeta_i(k), \quad (i = 1, \dots, N) \quad (11)$$

such that, for any graph $\mathcal{G} \in \mathbb{G}_\delta^N$ and for all initial conditions for the agents, state synchronization is achieved.

Problem 2 (Uniform input delay): Consider a MAS described by (7) and (10). The *state synchronization* problem with a set of network graphs \mathbb{G}_δ^N is to find, if possible, a linear protocol in the form of (11), for $i = 1, \dots, N$ such that, for any graph $\mathcal{G} \in \mathbb{G}_\delta^N$, for any $\kappa \leq \bar{\kappa}$, and for all the initial conditions of agents, state synchronization is achieved.

Problem 3 (Non-uniform input delay): Consider a MAS described by (7) and (9). The *state synchronization* problem with a set of network graphs $\mathbb{G}_\delta^{N,u}$ is to find, if possible, a linear protocol in the form of (11), for $i = 1, \dots, N$ such that, for any graph $\mathcal{G} \in \mathbb{G}_\delta^{N,u}$, for any $\kappa_1, \dots, \kappa_N \leq \bar{\kappa}$, and for all the initial conditions of agents, state synchronization is achieved.

IV. STATE SYNCHRONIZATION FOR AGENTS WITHOUT INPUT DELAY THAT ARE SQUARED-DOWN PASSIFIABLE VIA INPUT FEEDFORWARD

The MAS system described by (5) and (7) after implementing the linear static protocol (11) is written as

$$\begin{cases} x_i(k+1) = Ax_i(k) + BF\zeta_i(k), \\ y_i(k) = Cx_i(k), \\ \zeta_i(k) = \sum_{j=1}^N d_{ij}(y_i(k) - y_j(k)), \end{cases} \quad (12)$$

for $i = 1, \dots, N$. Define $x(k) = (x_1^T(k) \ x_2^T(k) \ \dots \ x_N^T(k))^T$. Then the overall dynamics of the N agents can be written as

$$x(k+1) = (I \otimes A + (I - D) \otimes BFC)x(k). \quad (13)$$

We have the following result.

Lemma 3: The MAS (13) achieves state synchronization if and only if the following $N - 1$ subsystems,

$$\eta_i(k+1) = (A + (1 - \lambda_i)BFC)\eta_i(k), \quad i = 2, \dots, N \quad (14)$$

are asymptotically stable, where λ_i with $|\lambda_i| < 1$ for $i = 2, \dots, N$ are the eigenvalues of D unequal to 1.

For a MAS with squared-down passifiable agents via input feedforward, we design a static protocol of the form:

$$u_i(k) = -\varepsilon G_1 G_2 \zeta_i(k), \quad (15)$$

where $\varepsilon > 0$ is a parameter to be designed. We have

Theorem 1: Consider a MAS described by agents (5) and (7) where the agents are squared-down passifiable via input feedforward with respect to G_1 and G_2 such that (A, BG_1) is controllable and (A, G_2C) is observable. Let any $\delta \in (0, 1)$ be given.

The state synchronization problem stated in Problem 1 is solvable for the set of graphs \mathbb{G}_δ^N . In particular, there exists an ε_* such that for any $\varepsilon < \varepsilon_*$, protocol (15) solves the state synchronization problem for any graph $\mathcal{G} \in \mathbb{G}_\delta^N$.

Remark 1: It can be shown that the *synchronized trajectory* is given by $x_s(k) = \eta_1(k)$, which is governed by

$$\eta_1(k+1) = A\eta_1(k), \quad \eta_1(0) = (w \otimes I_n)x(0), \quad (16)$$

where w is the normalized (in the sense that $\sum w_i = 1$) left eigenvector of the row stochastic matrix D associated with the eigenvalue 1. This shows that the modes of the synchronized trajectory are determined by the eigenvalues of A and the complete dynamics depends on both A and a weighted average of the initial conditions of agents.

One can further show that $\eta_1(0)$ is only a linear combination of initial conditions of root agents. As such, the synchronized trajectory given by (16) can be written explicitly as

$$\eta_1(k) = A^k \sum_{i \in \Pi_{\mathcal{G}}} w_i x_i(0), \quad (17)$$

which is a weighted average of the trajectories of root agents. *Proof of Theorem 1:* From Lemma 3, it is clear that we only need to prove that

$$\eta(k+1) = (A - \varepsilon(1 - \lambda)BG_1G_2C)\eta(k) \quad (18)$$

is asymptotically stable for all λ with $|\lambda| < \delta$. By choosing a Lyapunov function $V(k) = \eta^*(k)P\eta(k)$, we obtain

$$\begin{aligned} & V(k+1) - V(k) \\ &= \begin{pmatrix} \eta(k) \\ -\varepsilon(1 - \lambda)G_2C\eta(k) \end{pmatrix}^* G(P) \begin{pmatrix} \eta(k) \\ -\varepsilon(1 - \lambda)G_2C\eta(k) \end{pmatrix} \\ &\quad + \varepsilon^2 |1 - \lambda|^2 \eta^*(k) C^T G_2^T (R + R^T) G_2 C \eta(k) \\ &\quad - 2\varepsilon \text{Re}(1 - \lambda) \eta^*(k) C^T G_2^T G_2 C \eta(k) \\ &\leq \eta^*(k) C^T G_2^T [\varepsilon^2 |1 - \lambda|^2 (R + R^T) - 2\varepsilon \text{Re}(1 - \lambda)I] G_2 C \eta(k), \end{aligned}$$

where $G(P)$ is given by (4). Choosing a large enough $r > 0$ such that $R + R^T < rI$ and using $|\lambda| < \delta$, we find that

$$\begin{aligned} \varepsilon|1 - \lambda|^2(R + R^T) - 2\operatorname{Re}(1 - \lambda)I &\leq \varepsilon(1 + \delta)^2 rI - 2(1 - \delta)I \\ &= (\varepsilon(1 + \delta)^2 r - 2(1 - \delta))I. \end{aligned}$$

For all $\varepsilon < \varepsilon_*$ with $\varepsilon_* = \frac{2(1-\delta)}{r(1+\delta)^2}$, we have for some $\nu > 0$

$$V(k+1) - V(k) \leq \nu \eta^*(k) C^T G_2^T G_2 C \eta(k),$$

which proves the required stability since $(A, G_2 C)$ is detectable. ■

V. STATE SYNCHRONIZATION FOR AGENTS WITH INPUT DELAY THAT ARE SQUARED-DOWN PASSIFIABLE VIA INPUT FEEDFORWARD

A. Uniform input delay

In this subsection, we will consider the case with uniform input delay. The MAS system described by (7) and (10). We still use a static protocol of the form (15). The overall dynamics of the N agents can be written as

$$x(k+1) = (I \otimes A)x(k) + ((I - D) \otimes BFC)x(k - \kappa). \quad (19)$$

As observed in Section IV, the synchronization for the system (19) is equivalent to the asymptotic stability of the following $N - 1$ subsystems,

$$\tilde{\eta}_i(k+1) = A\tilde{\eta}_i(k) + (1 - \lambda_i)BFC\tilde{\eta}_i(k - \kappa), \quad (20)$$

for any integer $\kappa \in [0, \bar{\kappa}]$, where λ_i , $i = 2, \dots, N$ are those eigenvalues of D inside the unit disc.

Lemma 4: The MAS (19) achieves state synchronization if and only if the system (20) is asymptotically stable for $i = 2, \dots, N$ and for any integer $\kappa \in [0, \bar{\kappa}]$.

In light of the definition of Problem 2 that synchronization is formulated for a set of graphs, we obtain a *robust* stabilization problem, i.e. the stabilization of the system

$$x(k+1) = Ax(k) + (1 - \lambda)Bu(k - \kappa), \quad (21)$$

via a protocol (11) for any λ which is an eigenvalue inside the unit disc of a stochastic row matrix D associated with a graph in the set \mathbb{G}_δ . Define

$$\omega_{\max} = \begin{cases} 0, & A \text{ is Schur stable.} \\ \max\{\omega \in [0, \pi] \mid \det(e^{j\omega}I - A) = 0\}, & \text{otherwise} \end{cases}$$

The main result for agents with uniform input delay can be stated as follows.

Theorem 2: Consider a MAS described by (7) and (9) where the agents are squared-down passifiable via input feedforward given G_1 and G_2 such that (A, BG_1) is controllable and $(A, G_2 C)$ is observable. Let any $\delta \in (0, 1)$ be given.

The state synchronization problem stated in Problem 2 is solvable if

$$\bar{\kappa}\omega_{\max} < \arccos(\delta). \quad (22)$$

In particular, for any given $\bar{\kappa}$ satisfying (22), there exist a $\varepsilon_* > 0$ such that for any $\varepsilon \in (0, \varepsilon_*)$, static protocol (15) solves the state synchronization problem for any graph $\mathcal{G} \in \mathbb{G}_\delta^N$ and any $\kappa \in [0, \bar{\kappa}]$.

Remark 2: In the case of uniform delay, it is still true that the *synchronized trajectory* is given by $x_s(k) = \eta_1(k)$ which is governed by (17) where w is the first row of T^{-1} and independent of the delays.

In order to prove our main result the following lemma will be employed.

Lemma 5: Consider a linear time-delay system

$$x(k+1) = Ax(k) + A_1 x(k - \kappa), \quad (23)$$

where $x(k) \in \mathbb{R}^n$ and $\kappa \in \mathbb{N}$. Suppose $A + A_1$ is Schur stable. Then, (23) is asymptotically stable if

$$\det[e^{j\omega}I - A - e^{-j\omega\kappa_r}A_1] \neq 0,$$

for all $\omega \in [-\pi, \pi]$, and for all $\kappa_r \in \mathbb{R}$ with $0 < \kappa_r \leq \kappa$.

Proof: By extending Lemma 1 of [32] to discrete-time and making a minor modification, we can obtain this lemma. ■

Proof of Theorem 2: By using the static protocol (15), we can obtain the closed-loop system as

$$x(k+1) = Ax(k) - \varepsilon(1 - \lambda)BG_1G_2Cx(k - \kappa), \quad (24)$$

where $|\lambda| < \delta$ and $\kappa \in [0, \bar{\kappa}]$.

Since $\bar{\kappa}$ satisfies condition (22), there exists a $\rho > 0$ and $\phi > 0$ such that

$$\cos(\bar{\kappa}\omega) > \delta + \frac{1}{\rho} \text{ for all } \omega \text{ such that } |\omega| < \omega_{\max} + \phi.$$

We will use Lemma 5 to prove the stability of (24). We first note that in the proof of Theorem 1 it has been shown that $A - \varepsilon(1 - \lambda)BG_1G_2C$ is Schur stable. Remains to establish

$$\det[e^{j\omega}I - A + e^{-j\omega\kappa_r}\varepsilon(1 - \lambda)BG_1G_2C] \neq 0, \quad (25)$$

for all $\omega \in [-\pi, \pi]$, for all $\kappa_r \in \mathbb{R}$ with $0 < \kappa_r \leq \bar{\kappa}$ and for all λ with $|\lambda| < \delta$.

We will split the proof into two cases where $|\omega| < \omega_{\max} + \phi$ and $|\omega| \geq \omega_{\max} + \phi$ respectively.

If $|\omega| \geq \omega_{\max} + \phi$, there exists a $\mu > 0$ such that

$$\sigma_{\min}(e^{j\omega}I - A) > \mu, \quad \text{for all } \omega \text{ such that } |\omega| \geq \omega_{\max} + \phi,$$

where $\sigma_{\min}(\cdot)$ denote the smallest singular value. We have:

$$|e^{-j\omega\kappa_r}(1 - \lambda)| < 2$$

and hence $\|e^{-j\omega\kappa_r}\varepsilon(1 - \lambda)BG_1G_2C\| < 2\varepsilon\|BG_1G_2C\|$ and therefore there exists ε_1 such that

$$\|e^{-j\omega\kappa_r}\varepsilon(1 - \lambda)BG_1G_2C\| < \frac{\mu}{2}$$

for all $\varepsilon \in (0, \varepsilon_1)$. In that case,

$$\sigma_{\min}(e^{j\omega}I - A + e^{-j\omega\kappa_r}\varepsilon(1 - \lambda)BG_1G_2C) > \frac{\mu}{2}$$

for all $\omega_{\max} + \phi \leq |\omega| \leq \pi$ and hence, condition (25) holds for $|\omega| \geq \omega_{\max} + \phi$. If $|\omega| < \omega_{\max} + \phi$, we find that

$$\operatorname{Re}[(1 - \lambda)e^{-j\omega\kappa_r}] > \operatorname{Re}[(1 - \lambda)e^{-j\omega_{\max}\bar{\kappa}}] > \frac{1}{\rho},$$

and $|(1 - \lambda)e^{-j\omega\kappa_r}| < 1 + \delta$. With the same arguments as in Theorem 1, we can show that for $\varepsilon \in [0, \varepsilon_2]$ with $\varepsilon_2 = \frac{2}{r\rho(1+\delta)^2}$ such that

$$A - \varepsilon(1 - \lambda)e^{-j\omega\kappa_r}BG_1G_2C$$

is Schur stable for all $|\omega| < \omega_{\max} + \phi$. Choosing $\varepsilon_* = \min\{\varepsilon_1, \varepsilon_2\}$, the result follows. ■

B. Non-uniform input delays

In this subsection we will show that the protocol design for uniform delay also works for non-uniform delay. However, unlike the directed graph in the previous subsection, the graphs considered in this subsection are undirected.

The main result can be stated as follows.

Theorem 3: Consider a MAS described by agents (7) and (9) where the agents are squared-down passifiable via input feedforward with respect to G_1 and G_2 such that (A, BG_1) is controllable and (A, G_2C) is observable. Let any $\delta \in (0, 1)$ be given. The state synchronization problem stated in Problem 3 is solvable if

$$\bar{\kappa}\omega_{\max} < \frac{\pi}{2}. \quad (26)$$

In particular, for any given $\bar{\kappa}$ satisfying (26), there exist a $\varepsilon_* > 0$ such that for any $\varepsilon \in (0, \varepsilon_*)$, static protocol (15) solves the state synchronization problem for any graph $\mathcal{G} \in \mathbb{G}_{\delta}^{N,u}$ and any $\kappa_1, \dots, \kappa_N \in [0, \bar{\kappa}]$.

Remark 3: In the case of uniform delay, it is still true that the *synchronized trajectory* is given by $x_s(k) = \eta_1(k)$ where

$$\eta_1(k+1) = A\eta_1(k), \quad \eta_1(0) = (\tilde{w} \otimes I_n)x(0), \quad (27)$$

however the weights \tilde{w} are no longer equal to the first row of T^{-1} . The specific weights are dependent on the actual values of the delays $\kappa_1, \dots, \kappa_N$.

The following lemma is used in the proof.

Lemma 6: Consider a linear time-delay system

$$x(k+1) = Ax(k) + \sum_{i=1}^m A_i x(k - \kappa_i), \quad (28)$$

where $x(k) \in \mathbb{R}^n$ and $\kappa_i \in \mathbb{N}$. Suppose $A + \sum_{i=1}^m A_i$ is Schur stable. Then, (28) is asymptotically stable if

$$\det[e^{j\omega}I - A - \sum_{i=1}^m e^{-j\omega\kappa_i} A_i] \neq 0,$$

for all $\omega \in [-\pi, \pi]$ and for all $\kappa_i^r \in \mathbb{R}$ with $0 < \kappa_i^r \leq \kappa_i$ ($i = 1, \dots, N$).

Proof: By extending Lemma 5 to nonuniform delays, we can obtain this lemma. ■

Proof of Theorem 3: To clarify the protocol design, we introduce here a delay operator S_i for the agent i such that $(S_i x_i)(k) = x_i(k - \kappa_i)$. In the frequency domain,

$$\tilde{S}_i(\omega) = z^{-\kappa_i} = e^{-j\omega\kappa_i}.$$

We define $T_1 \in \mathbb{R}^{(N-1) \times N}$ and $T_2 \in \mathbb{R}^{N \times (N-1)}$

$$T_1 = (I \quad -\mathbf{1}), \quad T_2 = (I \quad 0)^T. \quad (29)$$

Since $D\mathbf{1} = \mathbf{1}$ we get that

$$(I - D) = (I - D)T_2T_1. \quad (30)$$

We define $\bar{x}_i := x_i - x_N$ as the state synchronization error for agent $i = 1, \dots, N-1$, and $\bar{x} = (\bar{x}_1^T \quad \bar{x}_2^T \quad \dots \quad \bar{x}_{N-1}^T)^T$. Using

$$(I - D)T_2\bar{x} = (I - D)x, \quad (31)$$

we can write the full closed-loop system as

$$\bar{x}(k+1) = (I \otimes A)\bar{x}(k) - \varepsilon[T_1 S_k(I - D)T_2] \otimes (BG_1 G_2 C)\bar{x}(k), \quad (32)$$

where S_k represents the delays,

$$S_k = \text{diag}\{S_i\} \quad \text{where} \quad (S_i z_i)(k) = z_i(k - \kappa_i). \quad (33)$$

We first need to show that the system is stable without the delays, i.e. when $S_k = I$, which yields the system:

$$\bar{x}(k+1) = (I \otimes A)\bar{x} - \varepsilon(T_1(I - D)T_2 \otimes BG_1 G_2 C)\bar{x}(k),$$

Note that from (31) it is easy to see that the eigenvalues of $T_1(I - D)T_2$ are exactly the nonzero eigenvalue of $I - D$. This system is Schur stable if the $N-1$ systems:

$$\eta_i(k+1) = (A - \varepsilon(1 - \lambda_i)BG_1 G_2 C)\eta_i(k),$$

are Schur stable where $\lambda_2, \dots, \lambda_N$ are the eigenvalues unequal to 1 of D which satisfy $|\lambda_i| < \delta$. This follows from an argument similar to the proof of Theorem 1.

Next, we consider the system with delay. From Lemma 5 it follows that system (32) is Schur stable if

$$\det[e^{j\omega}I - I \otimes A + \varepsilon T_1 \tilde{S}_k(\omega)(I - D)T_2 \otimes BG_1 G_2 C] \neq 0, \quad (34)$$

for all $\omega \in [-\pi, \pi]$, for all $\kappa_1^r, \dots, \kappa_N^r \in \mathbb{R}$ with $0 < \kappa_i^r \leq \bar{\kappa}$ ($i = 1, \dots, N$) and all possible D associated to a network graph in $\mathbb{G}_{\delta}^{N,u}$. Choose $\gamma > 0$ and $\phi > 0$ such that

$$(1 - \delta) \cos(\bar{\kappa}(\omega_{\max} + \phi)) > \gamma.$$

Next, we split the proof of (34) into two cases where $|\omega| < \omega_{\max} + \phi$ and $|\omega| \geq \omega_{\max} + \phi$ respectively

If $|\omega| \geq \omega_{\max} + \phi$, there exists a $\mu > 0$ such that

$$\sigma_{\min}(e^{j\omega}I - A) > \mu, \quad \forall \omega \text{ such that } |\omega| \geq \omega_{\max} + \phi.$$

The boundedness of D implies that there exists a $\varepsilon_* > 0$ such that

$$\|\varepsilon T_1 \tilde{S}_k(\omega)(I - D)T_2 \otimes BG_1 G_2 C\| \leq \frac{\mu}{2},$$

for any $\varepsilon < \varepsilon_*$. Therefore, (34) holds for $|\omega| \geq \omega_{\max} + \phi$.

Next, for $|\omega| < \omega_{\max} + \phi$, we use a Lyapunov argument. Define

$$P_D = T_2^T(I - D)T_2 \otimes P, \quad (35)$$

where P satisfies (4). Since $T_1(I - D)T_2$ is invertible, it is easily verified that $P_D > 0$. We need to prove A_{ω} is stable where

$$A_{\omega} = (I \otimes A) - \varepsilon[T_1 \tilde{S}_k(\omega)(I - D)T_2] \otimes (BG_1 G_2 C).$$

We get,

$$\begin{aligned} P_D - A_{\omega}^* P_D A_{\omega} &= - \begin{pmatrix} I \\ -w \end{pmatrix}^* [(T_2^T(I - D)T_2) \otimes G(P)] \begin{pmatrix} I \\ -w \end{pmatrix} \\ &\quad + \varepsilon[T_2^T(I - D)(\tilde{S}_k(\omega) + \tilde{S}_k(\omega)^*)(I - D)T_2] \otimes C^T G_2^T G_2 C \\ &\quad - \varepsilon^2[T_2^T(I - D)\tilde{S}_k(\omega)^*(I - D)\tilde{S}_k(\omega)(I - D)T_2] \\ &\quad \quad \otimes C^T G_2^T (R + R^T) G_2 C \\ &\geq \varepsilon[T_2^T(I - D)(\tilde{S}_k(\omega) + \tilde{S}_k(\omega)^*)(I - D)T_2] \otimes C^T G_2^T G_2 C \\ &\quad - \varepsilon^2[T_2^T(I - D)\tilde{S}_k(\omega)^*(I - D)\tilde{S}_k(\omega)(I - D)T_2] \\ &\quad \quad \otimes C^T G_2^T (R + R^T) G_2 C. \end{aligned}$$

where $w = \varepsilon(T_1 \tilde{S}_k(\omega)(I - D)T_2 \otimes G_2 C)$. We have:

$$(I - D)[\tilde{S}_k(\omega) + \tilde{S}_k(\omega)^*](I - D) \geq 2\gamma(I - D)$$

and we choose ε small enough such that:

$$\varepsilon[(I - D)\tilde{S}_k(\omega)^*(I - D)\tilde{S}_k(\omega)(I - D)] \otimes (R + R^T) \leq \gamma(I - D) \otimes I$$

which yields

$$P_D - A_\omega^* P_D A_\omega \geq \varepsilon \gamma T_2^T (I - D) T_2 \otimes C^T G_2^T G_2 C$$

which results in the required asymptotic stability of A_ω . ■

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