Passivity based state synchronization of multi-agent systems via static or adaptive nonlinear dynamic protocols

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Abstract: This paper studies state synchronization of homogeneous multi-agent systems (MAS) with partial-state coupling (i.e., agents are coupled through part of states). We identify three classes of agents, for which static linear protocols can be designed. They are agent which are squared-down passive, squared-down passifiable via output feedback, or G-minimum-phase with relative degree 1. We find that, for squared-down passive agents, the static protocol does not need any network information, as long as the network graph contains a directed spanning tree, while for the other two classes of agents, the static protocol needs rough information on the network graph, in particular, a lower bound of the non-zero eigenvalues of the Laplacian matrix associated with the network graph. However, when adaptive nonlinear dynamic protocols are utilized, even this rough information about the network is no longer needed for the other two classes of agents.

Key Words: Multi-agent system; Squared-down passivity; State synchronization

1 Introduction

The problem of synchronization among agents in a multi-agent system has received substantial attention in recent years, because of its potential applications in cooperative control of autonomous vehicles, distributed sensor network, swarming and flocking and others. The objective of synchronization is to secure an asymptotic agreement on a common state or output trajectory through decentralized control protocols (see [1, 9, 16, 25] and references therein). State synchronization inherently requires homogeneous MAS (i.e. agents have identical dynamics). Therefore, in this paper we focus on homogeneous MAS. So far most work has focused on state synchronization based on diffusive full-state coupling, where the agent dynamics progress from single- and double-integrator dynamics (e.g. [11, 12, 13, 14, 15]) to more general dynamics (e.g. [18, 22, 24, 27]). State synchronization based on diffusive partial-state coupling has also been considered (e.g. [7, 8, 18, 19, 20, 21, 23]).

For partial-state coupling, a linear dynamic protocol is generally designed for agents to achieve state synchronization. When a static protocol is required, a restriction is always imposed on the agentn. We require agents to be passive, passifiable via state feedback, or passifiable via output feedback (it is called output feedback passive in [26]). For example, [4] and [26] deal with linear passive and passifiable via state/output feedback agent, [2, 28, 30] deal with input-affine nonlinear passive agents, and [6, 10, 29] deal with general nonlinear passive agents. Their objectives are mainly to derive synchronization conditions for network graphs with different kind of communication such as undirected graph, balanced graph, directed graph, and time-varying graph. On the other hand, in [26], where agents are square, a synchronization region is identified. That is, given output feedback passive agents, a certain set of graphs is identified such that state synchronization can be achieved. The passivity assumption in the agent dynamics is replaced in [3] by assuming that the agent system is weakly minimum-phase and with relative degree one.

In this paper, we will study state synchronization of homogeneous MAS with partial-state coupling, where agents are general, linear, and non-square. We assume that the network graph always has a directed spanning tree. The contribution of this paper is threefold:

1. We identify three classes of agents, for which static linear protocols can be designed. They are agents which are squared-down passive, squared-down passifiable via output feedback, and G-minimum-phase with relative degree 1.
2. We develop static protocols for these three classes of agents. We find that, for square-down passive agents, the static protocol does not need any network information, as long as the network graph contains a di-

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rected spanning tree, while for the other two classes of agents, the static protocol needs rough information of the network graph, i.e., a lower bound of the non-zero eigenvalues of the Laplacian matrix associated with the network graph.

3. For squared-down passifiable via output feedback agents and G-minimum-phase agents with relative degree 1, we develop adaptive nonlinear dynamic protocols such that the protocol is independent of the communication networks. In other words, the adaptive nonlinear dynamic protocol can work for any communication network with any number of agents, as long as the graph is undirected.

Notations and definitions: Given a matrix $A \in \mathbb{R}^{m\times n}$, A^{T} denotes the conjugate transpose of A, while $\|A\|$ denotes the induced 2-norm of A. A square matrix A is said to be Hurwitz stable if all its eigenvalues are in the open left half complex plane. $A \otimes B$ depicts the Kronecker product between A and B. I_n denotes the n-dimensional identity matrix and 0_n denotes $n \times n$ zero matrix; we will use I or 0 if the dimension is clear from the context.

A *weighted directed graph* $\mathcal{G}$ is defined by a triple $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, \dots, N\}$ is a node set, $\mathcal{E}$ is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the weighting matrix, and $a_{ij} > 0$ iff $(i, j) \in \mathcal{E}$. Each pair in $\mathcal{E}$ is called an *edge*. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1, \dots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, \dots, k-1$. A *directed tree* is a subgraph (subset of nodes and edges) in which every node has exactly one parent node except for one node. A *directed spanning tree* is a subgraph which is a directed tree containing all the nodes of the original graph. For a weighted graph $\mathcal{G}$, a matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^{N} a_{ik}, & i = j, \\ -a_{ij}, & i \neq j, \end{cases}$$

is called the *Laplacian matrix* associated with the graph $\mathcal{G}$. In the case where $\mathcal{G}$ has non-negative weights, L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $\mathbf{1}$.

2 Passivity and passifiability via output feedback

Consider a general system $\Sigma(A, B, C)$:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p$. Assume $\Sigma(A, B, C)$ is a system with minimal realization where B and C have full row and column rank respectively. In that case, a square system (1) is called *passive* if there exists a positive definite matrix P such that

$$PA + A^{\mathrm{T}}P \leqslant 0, \qquad PB = C^{\mathrm{T}}, \tag{2}$$

The square system (1) with $m = p$ is called *passifiable via output feedback* if there exists an output feedback

$$u(t) = -Hy(t) + v(t), \tag{3}$$

which makes the square system (1) passive with respect to the new input $v(t)$ (see [26]), i.e., the system with minimal realization $(A - BHC, B, C)$ is passive where B and C have full column and row rank respectively.

For a non-square system (1) with $m \neq p$, Fradkov defines G-passivity and G-passifiability in [5]. Given a prespecified $m \times p$-matrix G, a system (1) is called *G-passive*, if there exist a positive definite matrix P, such that

$$PA + A^{\mathrm{T}}P \leqslant 0, \qquad PB = C^{\mathrm{T}}G^{\mathrm{T}}. \tag{4}$$

Similarly, a non-square system (1) with $m \neq p$ is called *G-passifiable* if there exists an output feedback (3) which makes the non-square system (1) G-passive with respect to the new input v.

In this paper, we define *squared-down passive* and *squared-down passifiable via output feedback* for a non-square system (1) based on the idea of *squared down* in [17]. Suppose there exist a static pre-compensator $G_1 \in \mathbb{R}^{m\times q}$ and a static post-compensator $G_2 \in \mathbb{R}^{q\times p}$, where q is the relative degree of the system (1). A non-square system (1) with $m \neq p$ is called *squared-down passive* with a pre-compensator G_1 and a post-compensator G_2, if there exist a positive definite matrix P, such that

$$PA + A^{\mathrm{T}}P \leqslant 0, \qquad PBG_1 = C^{\mathrm{T}}G_2^{\mathrm{T}}. \tag{5}$$

Similarly, a non-square system (1) with $m \neq p$ is called *squared-down passifiable via output feedback* if there exists an output feedback (3), which makes the non-square system (1) squared-down passive with respect to the new input v.

Remark 1 *Note that when $G_1 = I$, our squared-down passivity is the G-passivity in [5]. Moreover, for a square system, $G_1 = G_2 = I$, squared-down passivity becomes conventional passivity.*

Next, we will define a class of agents, which is called *G-minimum-phase agents with relative degree* 1. For a matrix $G \in \mathbb{R}^{m\times p}$, a non-square system (1) is called G-minimum-phase agent with relative degree 1 if the square system (A, B, GC) is minimum-phase with relative degree 1. Note that for such a system (A, B, GC), there exist non-singular state transformation matrices T_x and T_u such that

$$\tilde{x}(t) = \begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{pmatrix} = T_x x(t), \qquad \tilde{u}(t) = T_u u(t)$$

and the dynamics of $\bar{x}$ is represented as

$$\begin{aligned} \dot{\tilde{x}}_1(t) &= A_{11}\tilde{x}_1(t) + A_{12}\tilde{x}_2(t), \\ \dot{\tilde{x}}_2(t) &= A_{21}\tilde{x}_1(t) + A_{22}\tilde{x}_2(t) + \tilde{u}(t), \\ \hat{y}(t) &= \tilde{x}_2(t), \end{aligned} \tag{6}$$

where $\tilde{x}_1(t) \in \mathbb{R}^{n-m}$ and $\tilde{x}_2(t) \in \mathbb{R}^m$. Moreover, A_{11} is Hurwitz stable.

3 Problem formulation

Consider a MAS consisting of N identical non-square agents:

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + Bu_i(t), \\ y_i(t) = Cx_i(t), \end{cases} \tag{7}$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$, and $y_i(t) \in \mathbb{R}^p$.

The communication network provides agent i ($i = 1, \ldots, N$) with the following information,

$$\zeta_i(t) = \sum_{j=1}^{N} a_{ij}(y_i(t) - y_j(t)) \tag{8}$$

where $a_{ij} \geqslant 0$ and $a_{ii} = 0$. This communication topology of the network can be described by a weighted graph $\mathcal{G}$ with nodes corresponding to the agents in the network and the weight of edges given by the coefficient a_{ij}. Specifically, $a_{ij} > 0$ indicates that there exists an edge from agent j to agent i with weight a_{ij} in the graph. In terms of the coefficients of L, ζ_i can be rewritten as

$$\zeta_i(t) = \sum_{j=1}^{N} \ell_{ij} y_j(t). \tag{9}$$

As noted in [16, Corollary 2.5], if the graph contains a directed spanning tree then the Laplacian matrix L has a simple eigenvalue at the origin with corresponding right eigenvector $\mathbf{1}$. Let $\lambda_1, \ldots, \lambda_N$ denote the eigenvalues of L such that $\lambda_1 = 0$ and $\operatorname{Re}(\lambda_i) > 0$, $i = 2, \ldots, N$. We can then define a set of network graphs as follows.

Definition 1 *For any $\beta > 0$, let $\mathbb{G}_\beta^N$ denote the set of directed graphs with N nodes which contains a directed spanning tree for which the corresponding Laplacian matrix L has the property that $\operatorname{Re}(\lambda_i) > \beta$, $i = 2, \ldots, N$. Moreover, let $\mathbb{G}^N$ denote the set of directed graphs with N nodes which contains a directed spanning tree.*
Let $\mathbb{G}^{N,u}$ denote the set of undirected graphs with N nodes that is connected.

Our goal in this paper is to achieve state synchronization among agents in MAS, that is

$$\lim_{t\to\infty} (x_i(t) - x_j(t)) = 0, \tag{10}$$

for all $i, j \in \{1, \ldots, N\}$.

We formulate two state synchronization problems as follows.

Problem 1 *Consider a MAS described by agents* (7) *and* (8). *Let $\mathcal{G}$ be a given set of graphs such that $\mathcal{G} \subseteq \mathbb{G}^N$. The* state synchronization *problem via static protocol with a set of network graph $\mathcal{G}$ is to find, if possible, a linear static protocol of the form*

$$u_i(t) = F\zeta_i(t), \tag{11}$$

for $i = 1, \ldots, N$ such that, for any graph $\mathcal{G} \subseteq \mathbb{G}^N$ and for all initial conditions for the agents, state synchronization among agents is achieved.

Problem 2 *Consider a MAS described by agents* (7) *and* (8). *The* state synchronization *problem via dynamic protocol with a set of network graph $\mathcal{G}$ is to find, if possible, a nonlinear dynamic protocol of the form*

$$\begin{cases} \dot{x}_{ci}(t) = f_i(x_{ci}(t), \zeta_i(t)), \\ u_i(t) = g_i(x_{ci}(t), \zeta_i(t)), \end{cases} \tag{12}$$

for $x_{ci}(t) \in \mathbb{R}^{n_i}$ and $i = 1, \ldots, N$ such that, for any N and any graph $\mathcal{G} \in \mathbb{G}^{N,u}$ and for all initial conditions for the agents, state synchronization among agents is achieved.

4 Static protocol design

In this section, we will consider a static protocol design for a MAS with squared down-passive agents, squared down-passifiable via output feedback agents, or G-minimum-phase and with relative degree 1 agents.

4.1 Preliminary results

The MAS system described by (7) and (8) after implementing the linear static protocol (11) is written as

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + BF\zeta_i(t), \\ y_i(t) = Cx_i(t), \\ \zeta_i(t) = \sum_{j=1}^{N} \ell_{ij} y_j(t), \end{cases} \quad i = 1, \ldots, N \tag{13}$$

Define $x(t) = \begin{pmatrix} x_1^{\mathrm{T}}(t) & x_2^{\mathrm{T}}(t) & \cdots & x_N^{\mathrm{T}}(t) \end{pmatrix}^{\mathrm{T}}$. Then the overall dynamics of the N agents can be written as

$$\dot{x}(t) = (I_N \otimes A + L \otimes BFC)x(t). \tag{14}$$

We immediately have the following result.

Lemma 1 *The MAS* (14) *achieves state synchronization if and only if the following $N-1$ subsystems,*

$$\dot{\eta}_i(t) = (A + \lambda_i BFC)\eta_i(t), \qquad i = 2, \ldots, N \tag{15}$$

are asymptotically stable, where λ_i ($i = 2, \ldots, N$) are the non-zero eigenvalues of L. Moreover, the synchronization trajectory is given by

$$\dot{\eta}_1(t) = A\eta_1(t), \qquad \eta_1(0) = (w \otimes I_n)x(0), \tag{16}$$

where w is the normalized left eigenvector of L with row sum equal to 1 and associated with the zero eigenvalue.

4.2 Squared-down passive

For a MAS with squared-down passive agents, we design the static protocol as,

$$u_i(t) = -G_1 K G_2 \zeta_i(t), \tag{17}$$

where K is any symmetric and positive definite matrix and G_1, G_2 are given in (5).

Theorem 1 *Consider a MAS described by* (7) *and* (8). *Let a set of network graphs $\mathbb{G}^N$ be defined.*
Assume (A, B, C) is squared-down passive with G_1, G_2 given in (5)*, the state synchronization problem stated in Problem 1 via a static protocol is solvable for any graph $\mathcal{G} \in \mathbb{G}^N$. In particular, the static protocol* (17) *solves the state synchronization problem for any graph $\mathcal{G} \in \mathbb{G}^N$. Moreover, the synchronized trajectory is given by* (16).

Proof: According to Lemma 1, we only need to prove that $A - \lambda BG_1KG_2C$ is asymptotically stable for all λ with $\operatorname{Re}(\lambda) > 0$.

Because we assume the system has a minimal realization, we know that $(A,\ G_2C)$ is detectable. Moreover, we have

$$\begin{aligned} P(A - \lambda BG_1KG_2C) &+ (A - \lambda BG_1KG_2C)^{\mathrm{T}}P \\ &= PA + A^{\mathrm{T}}P - 2\operatorname{Re}(\lambda)PBG_1KG_2C \\ &\leqslant -2\operatorname{Re}(\lambda)C^{\mathrm{T}}G_2^{\mathrm{T}}KG_2C. \end{aligned}$$

This implies that $A - \lambda BG_1KG_2C$ is Hurwitz stable for any $\lambda \in \mathbb{C}^+$. In particular, we have stability for any non-zero eigenvalue of the Laplacian matrix associated with a graph in the set $\mathbb{G}^N$. This proves the result. ■

Remark 2 *When the agents are square, $G_1 = G_2 = I$, the above result is presented in [26].*

4.3 Squared-down passifiable via output feedback

A system (1) is squared-down passifiable via an output feedback (3) if and only if there exist a matrix H and positive-definite matrix P such that

$$\begin{aligned} P(A - BG_1HG_2C)+(A - BG_1HG_2C)^{\mathrm{T}}P \leqslant 0, \\ PBG_1 = C^{\mathrm{T}}G_2^{\mathrm{T}}. \end{aligned} \tag{18}$$

Next we will show that the static protocol in the form of (17) still works in this case, but part of the knowledge of network graphs (that is parameter β) is required. In other words, the problem can be solvable for a set of graphs $\mathbb{G}_\beta^N$ via the static protocol (11).
We design the static protocol of the form (11) as

$$u_i(t) = -\rho G_1KG_2\zeta_i(t), \tag{19}$$

where $\rho > 0$ is a parameter dependent on β and K is any symmetric and positive definite matrix. Note that β is the lower bound of the real part of nonzero eigenvalues of all Laplacian matrices associated with the graphs

Theorem 2 *Consider a MAS described by agents* (7) *and* (8). *Let any $\beta > 0$ be given, and hence a set of network graphs $\mathbb{G}_\beta^N$ be defined.*
Assume (A, B, C) is squared down-passifiable via output feedback with G_1, G_2 given in (5)*, then the state synchronization problem stated in Problem 1 is solvable. In particular, there exists a ρ^* such that for any $\rho > \rho^*$, protocol* (19) *solves the state synchronization problem for any graph $\mathcal{G} \in \mathbb{G}_\beta^N$. Moreover, the synchronized trajectory is given by* (16).

Proof: Similar to Theorem 1, we only need to prove $A - \lambda\rho BG_1KG_2C$ is Hurwitz stable for any λ that satisfies $\mathrm{Re}(\lambda) > \beta$. Since K is symmetric and positive definite, $G_2^{\mathrm{T}}KG_2$ is also symmetric and positive definite for any injective matrix G_2. Since the system is squared-down passifiable there exists a matrix H and positive-definite matrix P such that (18) is satisfied. Moreover, for a fixed K, there exists a real number $b > 0$ such that

$$H + H^{\mathrm{T}} \leqslant 2bK. \tag{20}$$

Then, we find that

$$\begin{aligned} &P(A - \lambda\rho BG_1KG_2C) + (A - \lambda\rho BG_1KG_2C)^{\mathrm{T}}P \\ &\quad\leqslant PBG_1HG_2C + C^{\mathrm{T}}G_2^{\mathrm{T}}H^{\mathrm{T}}G_1^{\mathrm{T}}B^{\mathrm{T}}P \\ &\qquad - \mathrm{Re}(\lambda)\rho(PBG_1KG_2C + C^{\mathrm{T}}G_2^{\mathrm{T}}H^{\mathrm{T}}G_1^{\mathrm{T}}B^{\mathrm{T}}P) \\ &\quad\leqslant 2bC^{\mathrm{T}}G_2^{\mathrm{T}}KG_2C - 2\,\mathrm{Re}(\lambda)\rho C^{\mathrm{T}}G_2^{\mathrm{T}}KG_2C \\ &\quad= -2(\mathrm{Re}(\lambda)\rho - b)C^{\mathrm{T}}G_2^{\mathrm{T}}KG_2C. \end{aligned}$$

Choosing $\rho^* = b/\beta$ immediately ensures that $A - \lambda\rho BG_1KG_2C$ is Hurwitz stable for $\rho > \rho^*$, which proves the result. ■

4.4 *G*-minimum-phase agents with relative degree 1

For a MAS with G-minimum-phase agents with relative degree 1, we design the static protocol as

$$u_i(t) = -\rho T_u^{-1}G\zeta_i(t) \tag{21}$$

where $\rho > 0$ is a real number.
The main result in this subsection can be stated as follows.

Theorem 3 *Consider a MAS described by agents* (7) *and* (8). *Let any $\beta > 0$ be given, and hence a set of network graphs $\mathbb{G}_\beta^N$ be defined.*
Assume (A, B, C) is G-minimum-phase with injective G, then the state synchronization problem stated in Problem 1 is solvable. In particular, there exists a $\rho^* > 0$ such that for any $\rho > \rho^*$, protocol (21) *solves the state synchronization problem for any graph $\mathcal{G} \in \mathbb{G}_\beta^N$. Moreover, the synchronized trajectory is given by* (16).

Proof: According to Lemma 1, we only need to prove that $A - \lambda\rho BT_u^{-1}GC$ is asymptotically stable for all λ with $\mathrm{Re}(\lambda) > \beta$.
Since the agent is G-minimum-phase, the stability of $A - \lambda\rho BT_u^{-1}GC$ is equivalent to the stability of the system

$$\begin{aligned} \dot{\tilde{x}}_1(t) &= A_{11}\tilde{x}_1(t) + A_{12}\tilde{x}_2(t), \\ \dot{\tilde{x}}_2(t) &= A_{21}\tilde{x}_1(t) + A_{22}\tilde{x}_2(t) + \lambda\tilde{u}(t), \\ \hat{y}(t) &= GCT_x^{-1}\tilde{x}(t) = [0 \;\; I]\tilde{x}(t) = \tilde{x}_2(t), \end{aligned} \tag{22}$$

via a controller

$$\tilde{u}(t) = -\rho\hat{y}(t) \tag{23}$$

for all λ with $\mathrm{Re}(\lambda) > \beta$. The closed-loop system of (22) and (23) is written as

$$\begin{aligned} \dot{\tilde{x}}_1(t) &= A_{11}\tilde{x}_1(t) + A_{12}\tilde{x}_2(t), \\ \dot{\tilde{x}}_2(t) &= A_{21}\tilde{x}_1(t) + (A_{22} - \lambda\rho I)\tilde{x}_2(t), \end{aligned} \tag{24}$$

Since A_{11} is Hurwitz stable, there exists a $P_1 > 0$ such that

$$P_1A_{11} + A_{11}^{\mathrm{T}}P_1 = -I.$$

Now choose $\rho^* > 0$ such that for any $\rho \geqslant \rho^*$, $A_{22} - \lambda\rho I$ is Hurwitz stable. Therefore, there exists $P_2 > 0$ such that

$$P_2(A_{22} - \lambda\rho^*I) + (A_{22} - \lambda\rho^*I)^{\mathrm{T}}P_2 = -I$$

Then for any $\rho > \rho^*$, we have

$$P_2(A_{22} - \lambda\rho I) + (A_{22} - \lambda\rho I)^*P_2 \leqslant -\kappa I$$

for all λ with $\mathrm{Re}(\lambda) > \beta$ where κ is such that:

$$\kappa = 1 + 2\beta(\rho - \rho_*)\|P_2^{-1}\|^{-1}$$

Note that κ is an increasing function of ρ. Define a Lyapunov function

$$V(t) = \tilde{x}_1^{\mathrm{T}}(t)P_1\tilde{x}_1(t) + \tilde{x}_2^{\mathrm{T}}(t)P_2\tilde{x}_2(t).$$

Then, the derivative of V is obtained as

$$\begin{aligned} \dot{V}(t) = &- \|\tilde{x}_1(t)\|^2 - \kappa\|\tilde{x}_2(t)\|^2 + 2\,\mathrm{Re}(\tilde{x}_2^{\mathrm{T}}(t)A_{12}^{\mathrm{T}}P_1\tilde{x}_1(t)) \\ &+ 2\,\mathrm{Re}(\tilde{x}_1^{\mathrm{T}}(t)A_{21}^{\mathrm{T}}P_2\tilde{x}_2(t)) \\ \leqslant &- \|\tilde{x}_1(t)\|^2 - \kappa\|\tilde{x}_2(t)\|^2 + 2r_1\|\tilde{x}_1(t)\|\|\tilde{x}_2(t)\| \\ &+ 2r_2\|\tilde{x}_1(t)\|\|\tilde{x}_2(t)\| \\ = &\begin{pmatrix}\|\tilde{x}_1(t)\| & \|\tilde{x}_2(t)\|\end{pmatrix}\begin{pmatrix}-1 & r_1 + r_2 \\ r_1 + r_2 & -\kappa\end{pmatrix}\begin{pmatrix}\|\tilde{x}_1(t)\| \\ \|\tilde{x}_2(t)\|\end{pmatrix} \end{aligned}$$

where $r_1 \geqslant \|A_{12}^{\mathrm{T}} P_1\|$ and $r_2 \geqslant \|A_{21}^{\mathrm{T}} P_2\|$. It is clear that by choosing ρ sufficiently large will guarantee that $\kappa > (r_1 + r_2)^2$, the closed-loop system (24) is asymptotically stable. This proves the result. ■

5 Nonlinear dynamic protocol design

For agents that are squared-down passifiable via output feedback or G-minimum-phase with relative degree 1, when linear static (or dynamic) protocols are used, it has been shown in the above section (or in the literature) that some knowledge about the network is required. That is the lower bound of the real part of the nonzero eigenvalues of any Laplacian matrices associated with the graph in the graph set. In this section, we will investigate nonlinear dynamic protocols that do not use any knowledge about the network graph. Not even the number of agents needs to be known. We only assume the network graph is undirected and strongly connected. In particular, nonlinear dynamic protocols are designed based on the adaptation of parameter $\rho(t)$ in the protocol (19) using network information $\zeta_i(t)$.

5.1 Squared-down passifiable via output feedback

Let the protocol of agent $i \in \{1, \ldots, N\}$ be

$$\begin{cases} \dot{\rho}_i(t) = \zeta_i^{\mathrm{T}}(t) G_2^{\mathrm{T}} K G_2 \zeta_i(t), \\ u_i(t) = -\rho_i(t) G_1 K G_2 \zeta_i(t), \end{cases} \tag{25}$$

where $\rho_i(t)$ is a time-varying function with $\rho_i(0) > 0$, and K is any positive definite symmetric matrix.

The main result for a MAS with squared-down passifiable via output feedback agents via adaptive nonlinear dynamic protocols is stated in the following theorem.

Theorem 4 *Consider a MAS described by agents* (7) *and* (9) *where* (A, B, C) *is squared-down passifiable via output feedback with* G_1 *and* G_2 *while* (A, BG_1) *is controllable and* (A, G_2C) *is observable. Let a set of undirected network graphs* $\mathbb{G}^{N,u}$ *be defined.*

The state synchronization problem stated in Problem 2 is solvable for for the set of graphs $\mathcal{G} \in \mathbb{G}^{N,u}$. *In particular, the adaptive dynamic protocol* (25) *solves the state synchronization problem for any N with measurements given by* (8) *for any underlying undirected graph* $\mathcal{G} \in \mathbb{G}^{N,u}$.

Proof: Since the system is squared-down passifiable via output feedback with G_1, G_2 while (A, BG_1) is controllable and (A, G_2C) is observable, there exists a matrix H and positive-definite matrix P such that (18) is satisfied. There exists a real number $b > 0$ such that

$$L \otimes (H + H^{\mathrm{T}}) \leqslant 2b(L^2 \otimes K). \tag{26}$$

Define $x(t) = \begin{pmatrix} x_1^{\mathrm{T}}(t) & \cdots & x_N^{\mathrm{T}}(t) \end{pmatrix}^{\mathrm{T}}$ and $\rho(t) = \operatorname{diag}(\rho_1(t), \cdots, \rho_N(t))$. Then, the overall dynamics of the network is written as

$$\dot{x}(t) = (I \otimes A)x(t) - (\rho L \otimes BG_1 K G_2 C)x(t), \tag{27}$$

Moreover, $x(t)^{\mathrm{T}}(L\rho L \otimes C^{\mathrm{T}} G_2^{\mathrm{T}} K G_2 C)x(t) = \sum_{i=1}^N \dot{\rho}_i(t)\rho_i(t)$. Consider the following candidate Lyapunov function $V(x(t)) = x^{\mathrm{T}}(t)(L \otimes P)x(t)$ with L symmetric. We obtain using (18) that $\dot{V}(x(t)) \leqslant -2\sum_{i=1}^N (\rho_i(t) - b)\dot{\rho}_i(t)$. This implies that

$$\sum_{i=1}^N (\rho_i(t) - b)^2 \leqslant V(x(0)) + \sum_{i=1}^N (\rho_i(0) - b)^2$$

which guarantees that all $\rho_i(t)$ for $i = 1, \ldots, N$ are bounded. This implies in particular that

$$\hat{\zeta}(t) = (I \otimes G_2)\zeta(t) = (L \otimes G_2 C)x(t) \in \mathcal{L}_2$$

Choose W such that $A - WG_2C$ is asymptotically stable which is possible since (A, G_2C) is detectable. Then $p = (L \otimes I)x(t)$ satisfies:

$$\dot{p} = [I \otimes (A - WG_2C)]\, p + [(I \otimes W) - (\rho L \otimes BG_1K)]\, \hat{\zeta}(t)$$

This yields $p \in \mathcal{L}_2$ since $I \otimes (A - WG_2C)$ is asymptotically stable, $\hat{\zeta}(t) \in \mathcal{L}_2$ and $(I \otimes W) - (\rho L \otimes BG_1K)$ is bounded since ρ is bounded. Clearly $p \in \mathcal{L}_2$ guarantees that state synchronization is achieved since L is connected and hence the kernel of the matrix L is equal to $\mathbf{1}$. ■

5.2 *G*-minimum-phase with relative degree 1

For the MAS with G-minimum-phase with relative degree 1 agents and unknown communication graph, we design a dynamic protocol of agent $i \in \{1, \ldots, N\}$ as

$$\begin{cases} \dot{\rho}_i(t) = \zeta_i^{\mathrm{T}}(t) G^{\mathrm{T}} G \zeta_i(t), \\ u_i(t) = -\rho_i(t) T_u^{-1} G \zeta_i(t), \end{cases} \tag{28}$$

where $\rho_i(t)$ $(i = 1, \ldots, N)$ is a time-varying function with $\rho_i(0) > 0$. The main result for a MAS with G-minimum-phase with relative degree 1 agents is stated as follows.

Theorem 5 *Consider a MAS described by agents* (7) *and* (9) *where the agents are G-minimum-phase with relative degree* 1 *while* (A, GC) *is observable. Let a set of undirected network graphs* $\mathbb{G}^{N,u}$ *be defined.*

The state synchronization problem stated in Problem 2 is solvable for for the set of graphs $\mathcal{G} \in \mathbb{G}^{N,u}$. *In particular, the adaptive dynamic protocol* (28) *solves the state synchronization problem for any N and for any undirected graph* $\mathcal{G} \in \mathbb{G}^{N,u}$.

Proof: Based on (6), the system can be written as:

$$\dot{\tilde{x}}(t) = (I \otimes \tilde{A})\tilde{x}(t) - (\rho L \otimes E)\tilde{x}(t)$$

where $\tilde{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and $E = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$. Moreover, $x^{\mathrm{T}}(t)(L\rho L \otimes E)x(t) = \sum_{i=1}^N \dot{\rho}_i(t)\rho_i(t)$. Since A_{11} is Hurwitz stable, there exists $b > 0$ such that $\tilde{A} - bE$ is Hurwitz stable. Choose $P > 0$ such that

$$(\tilde{A} - bE)^{\mathrm{T}} P + P(\tilde{A} - bE) \leqslant 0.$$

It is easily seen that for sufficiently large b we can choose $P = \operatorname{diag}(P_1, aI)$. Note that in that case we have $EP = PE = EPE = aE$. Finally choose $c > 0$ such that $L \leqslant cL^2$. Consider the following candidate Lyapunov function

$V(x(t)) = \tilde{x}^{\mathrm{T}}(t)(L \otimes P)\tilde{x}(t)$ with L symmetric. We obtain $\dot{V}(x(t)) \leqslant -2a \sum_{i=1}^{N} (\rho_i(t) - bc)\dot{\rho}_i(t)$. This implies that

$$\sum_{i=1}^{N} (\rho_i(t) - bc)^2 \leqslant V(x(0)) + \sum_{i=1}^{N} (\rho_i(0) - bc)^2$$

which guarantees that all ρ_i for $i = 1, \ldots, N$ are bounded. Similarly to the proof of Theorem 4, we can then establish that we achieve state synchronization. ■

Remark 3 *Note that the adaptive nonlinear dynamic protocols* (25) *and* (28) *are universal. They work for any communication network with any number of agents as long as the associated network graph is undirected and strongly connected. Moreover, for square agents which are passifiable via output feedback (i.e., when $G_1 = G_2 = I$), no additional knowledge about the model for the agents is needed. The same remark applies to square passive agents.*

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