Degree Conditions for Hamiltonian Properties of Claw-free Graphs

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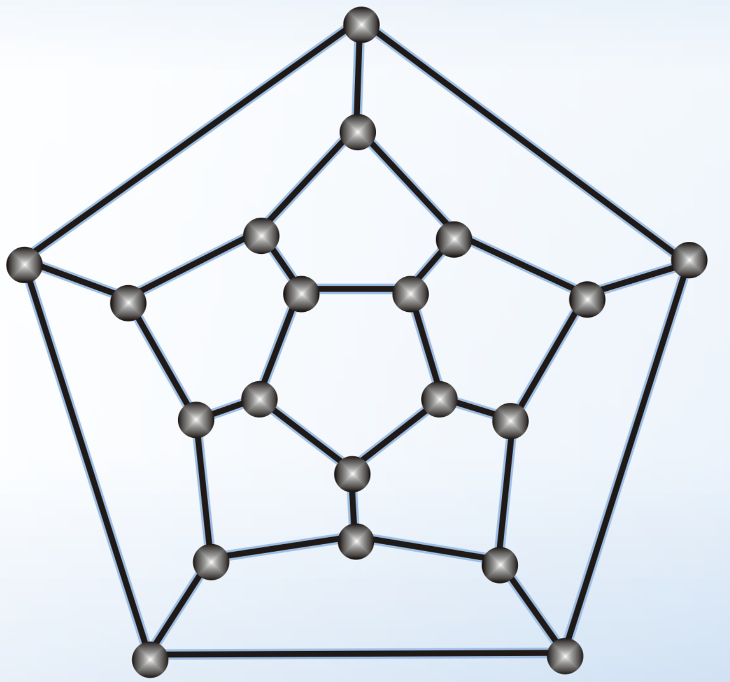

University of Twente

Degree Conditions for Hamiltonian Properties of Claw-free Graphs

Tao Tian

DEGREE CONDITIONS FOR HAMILTONIAN PROPERTIES OF CLAW-FREE GRAPHS

DISSERTATION

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Preface

This thesis contains a selection of the research results the author obtained within the field of hamiltonian graph theory since September 2015. After an introductory chapter, the reader will find five chapters that contain more or less independent, but highly interrelated topics within this research field.

The first chapter contains a brief introduction and discussion, with some background and motivation for the research in this field, as well as an account of some of the main research methods in this research area. In this chapter, we also list some general and specific terminology and notation that will be used in the succeeding chapters. Several more specific terms and particular notations that are not defined in the introductory chapter can be found in the chapters where they are first needed and introduced.

The second chapter deals with conditions on degree sums of adjacent vertices that guarantee the traceability of claw-free graphs. This chapter is mainly based on the research that the author has completed while he was working as a PhD student in the Beijing Institute of Technology, China.

The other chapters are mainly based on research results that the author obtained during his stay as a visiting scholar at the University of Twente, sponsored by the China Scholarship Council.

The third chapter deals with the hamiltonicity of the line graph of a given graph under sufficient degree sum conditions of adjacent vertices. This research was motivated by recent similar results about traceability which were already obtained by the author in Beijing.

The fourth to sixth chapter are all concerned with the hamiltonicity and traceability of claw-free graphs, involving both degree conditions as well as neighborhood conditions. The results presented there are motivated by and are based on other recent results for hamiltonicity.

Chapters 2 to 6 all have the structure of a journal paper. However, in order to avoid too much repetition, some frequently used theorems and lemmas are stated in Chapter 1, and all references are presented at the end of this thesis. The whole work is based on the following joint papers, which have been submitted to journals.

Papers underlying this thesis

[1] Degree sums of adjacent vertices for traceability of claw-free graphs, submitted (with L. Xiong, Z.-H Chen and S. Wang). (Chapter 2)

[2] Hamiltonicity of line graphs, submitted (with H.J. Broersma and L. Xiong). (Chapter 3)

[3] 2-connected hamiltonian claw-free graphs involving degree and neighborhood conditions, submitted (with H.J. Broersma and L. Xiong). (Chapter 4)

[4] Sufficient degree and neighborhood conditions for traceability of claw-free graphs, submitted (with H.J. Broersma and L. Xiong). (Chapter 5)

[5] A note on sufficient degree conditions for traceability of claw-free graphs, submitted (with H.J. Broersma and L. Xiong). (Chapter 5)

[6] Generalized Dirac conditions for traceability of claw-free graphs, submitted (with H.J. Broersma and L. Xiong). (Chapter 6)

Some other recent joint papers by the author

[1] Some physical and chemical indices of the Union Jack lattice, *Journal of Statistical Mechanics: Theory and Experiment,* **2** (2015), P02014 (with S. Li and W. Yan).

[2] On the minimal energy of trees with a given number of vertices of odd degree, *MATCH Communications in Mathematical and in Computer Chemistry,* **73** (2015), 3–10 (with W. Yan and S. Li).

[3] The spectrum and Laplacian spectrum of the dice lattice, *Journal of Statistical Physics,* **164** (2016), 449–462 (with S. Li and W. Yan).

[4] Traceability on 2-connected line graphs, *Applied Mathematics and Computation,* **321** (2018), 463–471 (with L. Xiong).

[5] 2-connected hamiltonian claw-free graphs involving degree sum of adjacent vertices, *Discuss. Math. Graph Theory*. doi:10.7151/dmgt.2125 (with L. Xiong).

[6] On the maximal energy of trees with at most two vertices of even degree, *Acta Mathematica Sinica, English Series*, preprint (with W. Yan and S. Li).

[7] Number of vertices of degree three in spanning 3-trees in square graphs, submitted (with W. Aye and L. Xiong).

Notation

Let G be a (simple) graph with vertex set $V(G)$, edge set $E(G)$, and let $v \in V(G)$ and $U, V \subseteq V(G)$.

$\|E(G)\|$	the number of edges of G
$\|V(G)\|$	the number of vertices of G
$N_G(v)$	the set of neighbors of v
$N_G(U)$	$\cup_{x \in U} N_G(x)$
$N_G[U]$	$N_G(U) \cup U$
$d_G(v)$ (or $d(v)$)	the degree of v (the number of neighbors of v)
$\delta(G)$	the minimum degree of G
$\alpha(G)$	the independence number of G
$\alpha'(G)$	the matching number of G
$\kappa(G)$	the (vertex) connectivity of G
$\kappa'(G)$	the edge connectivity of G
$g(G)$	the length of a shortest cycle in G
$c(G)$	the length of a longest cycle in G
$E[U, V]$	$\{uv \in E(G) \mid u \in U, v \in V\}$
$e(U, V)$	$\|E[U, V]\|$
$G[U]$	the subgraph induced by vertex set U in G

Contents

Chapter 1

Introduction

In this introductory chapter, we will describe our main contributions to the field of hamiltonian graph theory, and we will also present some common results that are repeatedly used in the succeeding chapters. But we start this introduction with some general background and terminology. We assume that the reader is familiar with the basics of mathematics, in particular with the basic definitions of graph theory. Most of the terminology we use in this thesis is standard and can be found in any textbook on graph theory. We use the most recent version of the textbook of Bondy and Murty [8] as our main source for terminology and notation.

1.1 General introduction

The graphs we consider in this thesis are finite and undirected, i.e., they consist of a finite set of vertices and a finite set of (undirected) edges, where each edge joins an unordered pair of distinct vertices (so we do not allow loops). Sometimes we allow multiple edges, i.e., edges that join the same pair of vertices. We will specify these concepts later, but for the moment we can do without any formal definitions or notation.

1.1.1 Hamiltonian and traceable graphs

Two of the central concepts in this thesis are the *hamiltonicity* of graphs and the *traceability* of graphs. Intuitively, these concepts deal with the way one can traverse the vertices and edges of a graph in such a way that one passes through all of its vertices exactly once.

To make this more precise, let G be a graph without multiple edges consisting of a vertex set $V(G)$ and an edge set $E(G)$. Then this graph G is called *hamiltonian* if G contains a *Hamilton cycle*, sometimes also referred to as a *spanning cycle*. This means there exists a sequence $v_1e_1v_2e_2\dots v_{n-1}e_{n-1}v_ne_nv_1$ such that $V(G) = \{v_1, v_2, \dots, v_n\}$, $|V(G)| = n$, each e_i is an edge of G joining the pair of vertices $\{v_i, v_{i+1}\}$, for $i = 1, 2, \dots, n-1$, and e_n is an edge of G joining the pair $\{v_n, v_1\}$ (so, in particular all $e_i \in E(G)$ for $i = 1, 2, \dots, n$). Similarly, this graph G is called *traceable* if G contains a *Hamilton path*, i.e., a sequence $v_1e_1v_2e_2\dots v_{n-1}e_{n-1}v_n$ in the above sense.

The hamiltonian problem, i.e., the problem of deciding whether a given graph is hamiltonian or not, is a long-standing and well-studied problem within graph theory and computational complexity. Named after Sir William Rowan Hamilton, this problem finds its origins in the 1850s as a two-person game, in which a player has to produce a Hamilton cycle in a graph (representing a dodecahedron) after another player has prescribed five consecutive vertices of it. The existence of Hamilton cycles is also related to early attempts of Peter Guthrie Tait to prove the well-known Four Colour Conjecture (now Four Colour Theorem), and it is also a special case of the well-known Travelling Salesman Problem. We omit the details here, because the research reported in this thesis bears no close relationship to the above problem areas. Nevertheless, these problem areas have spurred the interest in hamiltonian graph theory in general, leading to a wealth of publications.

Today, hamiltonian graph theory is a very active research field within graph theory, resulting in a lot of papers, dealing with many variations on this subject, and with many related problems. These developments have supplied the graph theory community with many new results, as well as with many new open problems and questions involving cycles and paths in graphs. This is also the motivation for our research. We will come back to this later.

Within computational complexity, the hamiltonian problem of deciding whether a given graph is hamiltonian (or traceable) is generally NP-complete, implying that to date there does not exist an easily verifiable necessary and sufficient condition for the existence of a Hamilton cycle (or Hamilton path). This is one of the main reasons why people have focussed on either sufficient conditions or necessary conditions for hamiltonicity or traceability of graphs. This enables the identification of YES-instances and NO-instances of the hamiltonian problem. Without going into detail, we note here that the far majority of published results is on sufficient conditions for hamiltonicity.

1.1.2 Degree conditions for hamiltonian properties

Intuitively, it is obvious that a graph is more likely to contain a Hamilton cycle or path if each of its vertices has many *neighbors*, i.e., is joined to many other vertices by edges; this number of neighbors is usually called the *degree* of a vertex.

Degree conditions are by now known as the classic approach to hamiltonian problems. In [41], Dirac proved that if the degree of each vertex of a graph is at least half of the *order*, i.e., the number of vertices, of the graph (Dirac-type condition), then it contains a Hamilton cycle. As a generalization of Dirac's Theorem, Ore in [68] proved that if the degree sum of any two *independent* vertices (not joined by an edge) is at least the order of the graph (Ore-type condition), then it contains a Hamilton cycle. Both results are best possible, in the sense that the conclusion is no longer valid if we lower the bound on the minimum degree or minimum degree sum in the above statements. Obviously, Ore's Theorem implies Dirac's Theorem, and can in fact be shown to be more generally applicable. It inspired others to introduce other sufficient conditions for hamiltonian properties based on the degrees and neighborhoods of the vertices of a graph.

Motivated by Dirac's Theorem and Ore's Theorem, the related concept of the minimum degree sum over all independent sets of t vertices of a graph was introduced (See, e.g., [10, 35, 49, 50, 52, 58, 65, 85]), as well as the minimum cardinality of the neighborhood union over all independent sets of t vertices of a graph (See, e.g., [1, 45, 47, 64]), and the minimum cardinality of the neighborhood union over all sets of t vertices of a graph (See, e.g., [36, 46, 48]). Apart from the above concepts, other variations involve the maximum degree of pairs of vertices with *distance two* (independent pairs that have a common neighbor; Fan's condition; see, e.g., [43]), the minimum degree sum of any pairs of *adjacent* vertices (joined by an edge; see, e.g., [14, 34, 79]), and the maximum degree of pairs of adjacent vertices (Lai's condition; see, e.g., [59]). There exist several survey papers on hamiltonian graph theory in which the interested reader can find more details on the above concepts and conditions (See, e.g., the surveys in [3, 5, 7, 11, 44, 53–55, 57, 62]).

As we mentioned earlier, the results of Dirac and Ore are best possible, in the sense that the degree conditions cannot be relaxed without violating the conclusion that the graphs are hamiltonian. One way to extend such results is to try to characterize the *exceptional graphs*, i.e., to find a nice description that identifies the structure of the nonhamiltonian graphs that meet the relaxed degree condition. We will encounter many examples of such results in this thesis. Another way to extend known results on the hamiltonicity of general graphs is to focus on restricted graph classes, i.e., to impose some limitation on the structure of the graphs. As we will see, degree conditions for hamiltonicity of general graphs can be relaxed considerably if we consider a certain subclass of graphs.

In this thesis, we mainly concentrate on sufficient degree conditions for the existence of Hamilton cycles and Hamilton paths in claw-free graphs, to be defined in the next section. Intuitively, a graph is *claw-free* if among any three neighbors of each vertex of the graph, there is at least one pair that is joined by an edge.

1.1.3 Basic terminology and notation

In the remainder of this introduction, we will describe our results and present some common approaches, techniques and results that we will repeatedly use in the succeeding chapters. We recall that most of the terminology we use in this thesis is standard and can be found in any textbook on graph theory, and that we use [8] as our main source for terminology and notation. We continue with some basic definitions and conventions that we use throughout the thesis.

As we noted before, we consider finite, undirected and loopless graphs only, but we sometimes allow multiple edges. To distinguish the situations, a graph without multiple edges will be called a *simple graph* or simply a graph. A graph is called a *multigraph* if it may contain multiple edges. Some of the concepts that we define next pertain to simple graphs as well as to multigraphs, whereas others have clear counterparts for multigraphs, but we only define them for (simple) graphs here.

In the next paragraphs, we let G denote a graph with vertex set $V(G)$ and edge set $E(G)$. Let X and Y be nonempty sets of vertices (not necessarily disjoint) of G. Then $E[X,Y]$ denotes the set of edges of G with one end in X and the other end in Y, and $e(X,Y) = |E[X,Y]|$. For a vertex x of G, we denote by $N_G(x)$ the *neighborhood* of x in G, i.e., the set of vertices adjacent to x in G, and by $d_G(x) = |N_G(x)|$ (or simply $d(x)$ if no confusion can arise) the *degree* of x in G. For a vertex set $S \subseteq V(G)$, we define $N_G(S) = \cup_{x\in S} N_G(x)$ and $N_G[S] = N_G(S) \cup S$. To distinguish vertex sets with different degrees, we use $D_i(G) = \{v \in V(G) \mid d(v) = i\}$, and we let $D(G) = D_1(G) \cup D_2(G)$. An edge $e = uv \in E(G)$ is called a *pendant edge* of G if $\min\{d(u), d(v)\} = 1$. The *circumference* of G, denoted by $c(G)$, is the length of a longest cycle of G. The *girth* of G, denoted by $g(G)$, is the length of a shortest cycle of G.

Given a nonempty subset $S \subseteq V(G)$, the *induced subgraph* $G[S]$ of G is the subgraph with vertex set S and edge set $\{uv \in E(G) \mid \{u, v\} \subseteq S\}$. We say that H is an induced subgraph of G if H is isomorphic to $G[S]$ for some nonempty subset $S \subseteq V(G)$. A graph is *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. A graph is *triangle-free* if it contains no cycle with exactly three vertices.

If G is a connected graph, then the *distance* between two vertices u and v of G is the length (i.e., the number of edges) of a shortest path between u and v, and is denoted by $dist(u, v)$. As in [8], the *independence number*, the *matching number*, the *connectivity* and the *edge-connectivity* of G are denoted by $\alpha(G)$, $\alpha'(G)$, $\kappa(G)$ and $\kappa'(G)$, respectively.

A subset $X \subseteq E(G)$ is called an *edge-cut* of G if $G - X$ has at least two components. An edge-cut X of G is called *essential* if $G - X$ has at least two *non-trivial* components, i.e., components that contain at least one edge. For an integer $k \geq 1$, the graph G is said to be *essentially k-edge-connected* if G is connected and does not admit an essential edge-cut X with $|X| < k$.

The *line graph* of G, denoted by $L(G)$, has $E(G)$ as its vertex set, while two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have a vertex in common. It is well-known and easy to check that line graphs are claw-free graphs. We also note without proof that a graph G is essentially k-edge-connected if and only if $L(G)$ is k-connected (or complete).

1.1.4 Key concepts and auxiliary results

Next, we are going to shortly review some key concepts that we use throughout the thesis. The first concept yields a way to shift attention and considerations from a claw-free graph H to a closely related line graph $L(G)$ of a triangle-free graph G. This will enable us to show the validity of statements about the hamiltonicity and traceability of H by proving equivalent statements about G. Since we will mainly deal with the latter, we find it convenient to use H for the original claw-free graph for which we will establish hamiltonicity and traceability results, and G for the graph we will deal with in our proofs. We apologize for any confusion this may cause.

Let H be a graph and let t be a positive integer. Below, we use t-set as shorthand for a subset with t vertices. Formally, the degree concepts we informally introduced earlier are defined as follows.

- $\delta(H) = \min\{d(v) \mid v \in V(H)\}$ (Dirac-type);
- $\sigma_2(H) = \min\{d(u) + d(v) \mid uv \notin E(H)\}$ (Ore-type);
- $\sigma_t(H) = \min\{\sum_{i=1}^{t} d_H(v_i) \mid \{v_1, v_2, \ldots, v_t\}$ is an independent t-set of $H\}$ (if $t > \alpha(H)$, we set $\sigma_t(H) = \infty$);
- $U_t(H) = \min\{|\bigcup_{i=1}^{t} N_H(v_i)| \mid \{v_1, v_2, \ldots, v_t\}$ is an independent t-set of $H\}$ (if $t > \alpha(H)$, we set $U_t(H) = \infty$);
- $\delta_t(H) = \min\{|\bigcup_{i=1}^{t} N_H(v_i)| \mid \{v_1, v_2, \ldots, v_t\}$ is a t-set in $H\}$;

- $\delta_F(H) = \min\{\max\{d(u), d(v)\} \mid u, v \in V(H) \text{ with } \mathrm{dist}(u, v) = 2\}$ (Fan-type);
- $\overline{\sigma}_2(H) = \min\{d(u) + d(v) \mid uv \in E(H)\}$ (Brualdi and Shanny-type);
- $\delta_L(H) = \min\{\max\{d(u), d(v)\} \mid uv \in E(H)\}$ (Lai-type).

Obviously, $\delta(H) = \sigma_1(H) = U_1(H) = \delta_1(H)$, and $\sigma_t(H) \geq U_t(H) \geq \delta_t(H)$. We let $\Omega(H, t) = \{\delta(H), \sigma_2(H), \sigma_t(H), U_t(H), \sigma_t(H), \delta_F(H), \overline{\sigma}_2(H), \delta_L(H)\}$.

A connected subgraph Ψ of a graph G is called a *closed trail* of G if the degree of each vertex of Ψ is even (in Ψ); it is called an *open trail* (or just trail) if $\Psi + e$ is a closed trail for an edge e not belonging to Ψ but joining two vertices of Ψ (In case we consider multigraphs, e may join two vertices that are already adjacent in Ψ). A (closed) trail Ψ of G is called a *spanning (closed) trail* (ST and SCT for short) of G if $V(G) = V(\Psi)$, and it is called a *dominating (closed) trail* (DT and DCT for short) of G if $E(G - V(\Psi)) = \emptyset$. So, every edge of G has at least one end vertex on a DT or DCT of G, and every ST (SCT) is also a DT (DCT), but not the other way around. A graph is *eulerian* if it is connected and each vertex has even degree. A graph is *supereulerian* if it contains an SCT. The family of supereulerian graphs is denoted by $\mathscr{SL}$.

The supereulerian graph problem, raised by Boesch, Suffel, and Tindell [4], is similar to the hamiltonian problem we mentioned before. It reflects the quest to find an easily verifiable characterization of supereulerian graphs. It is also partly motivated by the hamiltonian problem. Pulleyblank [69] showed that determining whether a graph is supereulerian is NP-complete, even when restricted to *planar* graphs (We refrain from giving the definition because we will not encounter planar graphs in the sequel). Degree conditions have also been considered in the context of studying supereulerian graphs. Numerous sufficient conditions for $G \in \mathscr{SL}$ in terms of lower bounds on degrees in G have been established (See, e.g., [2, 15, 16, 18–21, 27, 29–32, 37, 39, 80]). For more literature on supereulerian graphs, we refer the interested reader to the surveys [22, 38, 60]. Sufficient conditions for guaranteeing that a graph has a spanning trail also attracted several authors' attention (See, e.g., [18, 24, 25, 40, 42, 67, 77, 80, 82]). Supereulerian graphs, spanning (closed) trails, eulerian subgraphs, and dominating (closed) trails with certain properties have many applications to other areas, but in the sequel we will focus our attention to applications related to hamiltonian properties of line graphs and claw-free graphs.

Most of the results on hamiltonicity of line graphs are based on the following well-known result of Harary and Nash-Williams [56]. It shows a nice equivalence between the existence of a DCT in a graph G and a Hamilton cycle in its line graph $L(G)$.

Theorem 1.1. (Harary and Nash-Willians [56]). *The line graph $L(G)$ of a graph G with at least three edges is hamiltonian if and only if G has a DCT.*

We also need the following counterpart, showing the equivalence between the existence of a DT in a graph G and a Hamilton path in its line graph $L(G)$.

Theorem 1.2. (Li, Lai and Zhan [61]). *Let G be a graph with $|E(G)| \geq 1$. Then the line graph $L(G)$ of G is traceable if and only if G has a DT.*

As we mentioned before, the class of line graphs forms a subclass of the class of claw-free graphs. In the next section, we will see that studying hamiltonian properties of claw-free graphs and line graphs is in fact equivalent, in a particular sense determined by a closure operation due to Ryjáček [71].

1.2 Ryjáček's closure for claw-free graphs

In the context of investigating the hamiltonicity or traceability of claw-free graphs, Ryjáček [71] introduced the following very useful *closure* operation. A vertex v of a graph H is called *locally connected* if $N_H(v)$ induces a connected subgraph in H. The closure of a claw-free graph H is the graph obtained from H by joining all pairs of nonadjacent vertices in the neighborhood of a locally connected vertex by edges, and repeating this procedure in the newly obtained (claw-free) graph as long as this is possible. The (unique) *closure of the claw-free graph H* that is obtained this way is denoted by $cl(H)$.

Obviously, $\kappa(cl(H)) \geq \kappa(H)$, a fact that we will use implicitly without specifying it. A claw-free graph H is said to be *closed* if $H = cl(H)$. The following theorem summarizes the basic properties of $cl(H)$.

Theorem 1.3. (Ryjáček [71]). *Let H be a claw-free graph. Then*

(i) $cl(H)$ is well-defined;

(ii) there is a triangle-free graph G such that $cl(H) = L(G)$;

(iii) H and $cl(H)$ have the same circumference.

It is known that a connected line graph $H \neq K_3$ can be determined by a unique graph G with $H = L(G)$. In this case, we call G the *preimage* graph of the graph H. For a claw-free graph H, the closure $cl(H)$ of H can be obtained in polynomial time [71], and the preimage graph of a line graph can be obtained in linear time [70]. So, we can compute G efficiently for $cl(H) = L(G)$.

Later, the above theorem was extended to an analogous statement for traceability of claw-free graphs.

Theorem 1.4. (Brandt, Favaron and Ryjáček [9]). *Let H be a claw-free graph. Then H is traceable if and only if $cl(H)$ is traceable.*

By combining Theorem 1.1 with Theorem 1.3, investigating the hamiltonicity of a claw-free graph H is equivalent to investigating the existence of a DCT in a graph G for which $L(G) = cl(H)$. Similarly, by combining Theorem 1.2 with Theorem 1.4, investigating the traceability of a claw-free graph H is equivalent to investigating the existence of a DT in a graph G for which $L(G) = cl(H)$. These equivalences enable the application of powerful reduction methods based on the seminal work due to Catlin [20] and later refinements. Originally, these methods and tools developed by Catlin were invented to study the existence of SCTs and DCTs. For more information about closure concepts in claw-free graphs, the interested reader is referred to [6, 9, 12, 13, 72–74, 83].

1.3 Catlin's reduction method

Let G be a connected multigraph. For $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by successively identifying the two end vertices of each edge $e \in X$ and deleting the resulting loops. Note that, in general G/X is a multigraph, also in case G is a simple graph. If Γ is a connected sub(multi)graph of G, then we write G/Γ for $G/E(\Gamma)$; in this case, we use v_Γ to denote the only remaining vertex of Γ in G/Γ, i.e., the vertex in G/Γ to which Γ is contracted, and we call this vertex v_Γ a *contracted vertex* if $\Gamma \neq K_1$ in order to distinguish it from the remaining vertices of G.

Let $O(G)$ be the set of vertices of odd degree in G. A graph in which each vertex has even degree is called an *even graph*. Adopting the terminology of [20], a multigraph G is called *collapsible* if for every even subset $R \subseteq V(G)$, there is a spanning connected sub(multi)graph Γ_R of G with $O(\Gamma_R) = R$. The graph K_1 is regarded as a collapsible and supereulerian graph.

In [20], Catlin showed that every multigraph G can be partitioned into a unique collection of vertex-disjoint maximal collapsible sub(multi)graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_c$. Based on this, he defined the *reduction* of G as $G' = G/(\cup_{i=1}^{c} \Gamma_i)$, i.e., the graph obtained from G by successively contracting each Γ_i into a single vertex v_i $(1 \leq i \leq c)$. So for each vertex $v \in V(G')$, there is a unique maximal collapsible sub(multi)graph (possibly consisting of only v itself), denoted by $\Gamma_0(v)$, such that v is the contraction image of $\Gamma_0(v)$; we call this $\Gamma_0(v)$ the *preimage* of v. Recall that we call v a contracted vertex if $\Gamma_0(v) \neq K_1$. A multigraph G is called *reduced* if $G' = G$. In fact, in that case G is simple (as stated in Theorem 1.5(c) below). We have gathered some of the main results of Catlin et al. in the following theorem and lemma.

Theorem 1.5. (Catlin et al. [20, 23]). *Let G be a connected multigraph and let G' be the reduction of G.*

(a) G is collapsible if and only if $G' = K_1$, and G has an SCT if and only if G' has an SCT.

(b) G has a DCT if and only if G' has a DCT containing all the contracted vertices of G'.

(c) *If G is a reduced graph, then G is simple and triangle-free, and $\delta(G) \leq 3$. Moreover, then any subgraph Ψ of G is reduced, and either $\Psi \in \{K_1, K_2, K_{2,t}\ (t \geq 2)\}$ or $|E(\Psi)| \leq 2|V(\Psi)| - 5$.*

Lemma 1.6. (Catlin [17,20]). *The graphs K_3 and $K_{3,3} - e$ are collapsible.*

Here, $K_{3,3} - e$ denotes $K_{3,3}$ minus an arbitrary edge. Later, the above Theorem 1.5(a) was extended to an analogous statement for spanning trail of a graph.

Theorem 1.7. (Xiong and Zong [84]). *Let G be a connected graph of order n, and let G' be the reduction of G. Then G has an ST if and only if G' has an ST.*

1.4 The reduction of the core of a graph

Let H be a k-connected claw-free graph with $\delta(H) \geq 3$ ($k \in \{2,3\}$). By Theorem 1.3, there is a triangle-free graph G such that $cl(H) = L(G)$. By the definition of $cl(H)$, $V(cl(H)) = V(H)$ and $d_{cl(H)}(v) \geq d_H(v)$ for any $v \in V(cl(H))$, and so $d_{cl(H)}(v) \geq d_H(v) \geq 3$. For an edge $e = xy$ in G, let v_e be the vertex in $cl(H)$ corresponding to e in G. Then $d_{cl(H)}(v_e) = d_G(x) + d_G(y) - 2$. Thus, if $cl(H) = L(G)$ is a k-connected graph with $\delta(cl(H)) \geq 3$, then G is an essentially k-edge-connected graph with $\overline{\sigma}_2(G) \geq 5$.

Now let G be an essentially 2-edge-connected graph with $\overline{\sigma}_2(G) \geq 5$. Then, obviously $X = D_1(G) \cup D_2(G)$ is an independent set in G. Let E_1 denote the set of pendant edges in G. For each $x \in D_2(G)$, there are two edges e_x^1 and e_x^2 incident with x. Let $X_2(G) = \{e_x^1 \mid x \in D_2(G)\}$. Then, adopting the terminology of [75], the *core* of G, denoted by G_0, is defined by

$$G_0 = G/(E_1 \cup X_2(G)).$$

In fact, this concept was already defined in an earlier paper [81], where the notation $I_X(G)$ was used instead of G_0. In our situation, G_0 is simply the multigraph obtained from G by deleting the vertices of $D_1(G)$ and replacing each path of length 2 whose internal vertex has degree 2 in G by an edge. Hence, we can regard the vertex set $V(G_0)$ as a subset of $V(G)$. A vertex in G_0

is called *nontrivial* if it is obtained by contracting some edge(s) in $E_1 \cup X_2(G)$ or if it is adjacent to a vertex in $D_2(G)$ in G. For instance, if $v \in D_2(G)$ and $N_G(v) = \{x, y\}$, and if x_v is the vertex in G_0 obtained by contracting the edge xv, then both x_v and y are nontrivial in G_0 (although x_v is a contracted vertex and y is not a contracted vertex of G_0). Since $\overline{\sigma}_2(G) \geq 5$, all vertices in $D_2(G_0)$ are nontrivial.

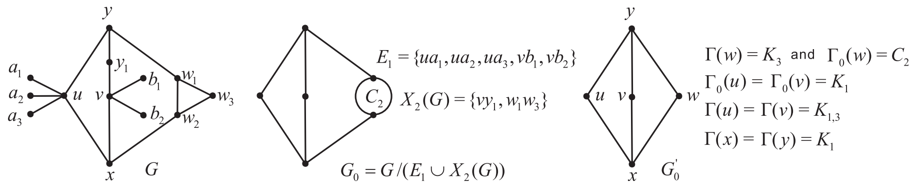

FIGURE 1.1: The reduction G_0' of the core G_0 of a graph G.

Let G_0' be the reduction of G_0. For a vertex $v \in V(G_0')$, let $\Gamma_0(v)$ be the maximal collapsible preimage of v in G_0, and let $\Gamma(v)$ be the preimage of v in G. Note that $\Gamma(v)$ is the graph induced by edge(s) composed of $E(\Gamma_0(v))$ and possibly some edge(s) of $E_1 \cup X_2(G)$ (For an example, see Figure 1.1). A vertex v in G_0' is a *nontrivial vertex* if v is a contracted vertex (i.e., if $|E(\Gamma(v))| \geq 1$ or $|V(\Gamma(v))| > 1$) or if v is adjacent to a vertex in $D_2(G)$.

Using Theorem 1.5, Veldman [81] and Shao [75] proved the following.

Theorem 1.8. *Let G be a connected and essentially k-edge-connected graph such that $\overline{\sigma}_2(G) \geq 5$, $k \in \{2, 3\}$, and $L(G)$ is not complete. Let G_0' be the reduction of the core G_0 of G. Then each of the following holds:*

(a) G_0 is well-defined, nontrivial, $\delta(G_0) \geq \kappa'(G_0) \geq k$, and $\kappa'(G_0') \geq \kappa'(G_0)$.

(b) (Lemma 5 in [81]) *G has a DCT if and only if G_0' has a DCT containing all the nontrivial vertices.*

We have the following similar result.

Theorem 1.9. *Let G be a connected and essentially k-edge-connected graph such that $\overline{\sigma}_2(G) \geq 5$, $k \in \{2, 3\}$, and $L(G)$ is not complete. Let G_0' be the reduction of the core G_0 of G. Then the following holds:*

(c) G has a DT if and only if G_0' has a DT containing all the nontrivial vertices.

Proof of Theorem 1.9. Clearly, if G has a dominating trail, then G_0' has a dominating trail containing all the nontrivial vertices of G_0'. Conversely, we assume that G_0' has a dominating trail T' containing all the nontrivial vertices of G_0'. Set $G_s' = G_0'[V(T')]$ and $U = V(G_0') - V(T')$. Then U is an independent subset of both $V(G_0')$ and $V(G)$, $U \cap N_G[D_1(G) \cup D_2(G)] = \emptyset$ and T' is a spanning trail of G_s'. Set $G_s = G_0 - U$ and $G_t = G - (U \cup D_1(G))$. By our definitions, G_t is a subdivision of G_s and G_s' is the reduction of G_s. Since G_s' has a spanning trail, by Theorem 1.7, G_s has a spanning trail. Since G_t is a subdivision of G_s with each edge of G_s subdivided at most once, it follows that G_t has a dominating trail T such that $V(G_t) - V(T) \subseteq D_2(G)$. Then $V(G) - V(T) \subseteq U \cup D_1(G) \cup D_2(G)$. Since $U \cup D_1(G) \cup D_2(G)$ is an independent subset of $V(G)$, T is a dominating trail of G. This completes the proof. □

1.5 Main results of this thesis

In Chapter 2, we consider the traceability of a 2-connected claw-free graph H of order n with a given degree sum condition on adjacent vertices. We obtain that if $\overline{\sigma}_2(H) \geqslant \frac{2n-5}{7}$ and $\delta(H) \geq 3$, and n is sufficiently large, then either H is traceable or H belongs to one class of well-characterized exceptional graphs. We also show that if $\overline{\sigma}_2(H) > \frac{n-6}{3}$ and $\delta(H) \geq 3$, and n is sufficiently large, then H is traceable, and that the lower bound $\frac{n-6}{3}$ is sharp.

In Chapter 3, it is conjectured (by Chen et al. [39]) that a 3-edge-connected simple graph G with sufficiently large order n and with $\overline{\sigma}_2(G) > \frac{n}{9} - 2$ is either supereulerian or contractible to the Petersen graph. We show that the conjecture is true for $\overline{\sigma}_2(G) \geq \frac{2n}{15} - 2$. Furthermore, we show that, for an essentially k-edge-connected simple graph G with sufficiently large order n ($k \in \{2, 3\}$), each of the following holds: (i) if $k = 2$ and $\overline{\sigma}_2(G) \geq 2(\lfloor n/8 \rfloor - 1)$, then either $L(G)$ is hamiltonian or G can be contracted to one of a set of six graphs that are not supereulerian; (ii) if $k = 3$ and $\overline{\sigma}_2(G) \geq 2(\lfloor n/15 \rfloor - 1)$, then either $L(G)$ is hamiltonian or G can be contracted to the Petersen graph.

In Chapter 4, we consider sufficient minimum degree and degree sum conditions that imply that graphs admit a Hamilton cycle, unless they have a small order or they belong to well-defined classes of exceptional graphs. Our main result implies that a 2-connected claw-free graph H of sufficiently large order n with minimum degree $\delta(H) \geq 3$ ($\delta(H) \geq 18$, respectively) is hamiltonian if the degree sum of any set of t independent vertices of G is at least $\frac{t(n+5)}{5}$ ($\frac{tn}{6}$, respectively), where $t \in \{1, 2, \ldots, 5\}$ ($t \in \{1, 2, \ldots, 6\}$, respectively), unless G belongs to one of a number of well-defined classes of exceptional graphs depending on t. Our results unify and extend several known earlier results.

In Chapter 5, we consider sufficient minimum degree and degree sum conditions that imply that graphs admit a Hamilton path, unless they have a small order or they belong to well-defined classes of exceptional graphs. Firstly, one of our results implies that a 2-connected claw-free graph H of sufficiently large order n with minimum degree $\delta(H) \geq 3$ is traceable if the degree sum of any set of t independent vertices of H is at least $\frac{t(n+6)}{6}$, where $t \in \{1, 2, \ldots, 6\}$. Secondly, one of our results implies that a 2-connected claw-free graph H of sufficiently large order n with minimum degree $\delta(H) \geq 22$ is traceable if the degree sum of any set of t independent vertices of H is at least $\frac{t(2n-5)}{14}$, where $t \in \{1, 2, \ldots, 7\}$, unless H belongs to one of a number of well-defined classes of exceptional graphs depending on t. Our third result implies that a 2-connected claw-free graph H of sufficiently large order n with $\delta(H) \geq 18$ is traceable if the degree sum of any set of 6 independent vertices is larger than $n-6$, and we show that this lower bound on the degree sums is sharp. Our results unify and extend several known earlier results.

In Chapter 6, we consider sufficient generalized Dirac-type conditions that imply that graphs admit a Hamilton path. Our result implies that a 2-connected claw-free graph H of sufficiently large order n with minimum degree $\delta(H) \geq 3$ is traceable if $\delta_2(H) \geq \frac{2(n+8)}{12}$ (or $\delta_3(H) \geq \frac{3(n+6)}{15}$, or $\delta_4(H) \geq \frac{4(n+4)}{16}$, or $\delta_5(H) \geq \frac{5(n-1)}{15}$).

Chapter 2

Degree sums of adjacent vertices for traceability

In this chapter, we first recall some known results on hamiltonicity and traceability for general graphs and claw-free graphs. This culminates in results of Brualdi and Shanny [14] and Chen [34] that form the main motivation for our results that we present and prove in this chapter. In fact, we establish traceability analogues of the hamiltonicity results obtained in [34], based on degree conditions that originate from [14].

2.1 Introduction

We start this introductory section with a short overview of known results that constitute the main motivation for the research that is reported in the remainder of this chapter.

2.1.1 Motivation

In the study of hamiltonicity of graphs, the following theorem due to Dirac [41] is well-known and the starting point of a development that has resulted in a vast amount of publications.

Theorem 2.1. (Dirac, [41]) *Every graph of order $n \geq 3$ with minimum degree $\delta(G) \geq \frac{n}{2}$ is hamiltonian.*

Theorem 2.1 has the following easy corollary for traceability.

Theorem 2.2. *Every graph of order n with minimum degree $\delta(G) \geq \frac{n-1}{2}$ is traceable.*

The above results are best possible in the sense that the lower bounds on the minimum degree cannot be relaxed without violating the conclusion. This can be seen, e.g., from the complete bipartite graph $K_{\frac{n-1}{2},\frac{n+1}{2}}$ in case of Theorem 2.1 and $K_{\frac{n-2}{2},\frac{n+2}{2}}$ in case of Theorem 2.2. However, if we impose additional restrictions on the structure of the graphs, these lower bounds can be improved considerably, as demonstrated by the following result in [66].

Theorem 2.3. (Matthews and Sumner [66]). *Let G be a connected claw-free graph of order n with $\delta(G) \geq \frac{n-2}{3}$. Then G is hamiltonian.*

As we have seen in Chapter 1, in addition to Dirac's minimum degree condition, various degree and neighborhood conditions have been used in subsequent studies on hamiltonicity and traceability of graphs. Here, we look at one particular type of conditions, inspired by the early work of Brualdi and Shanny from the 1980s. In [14], they considered a degree sum condition on adjacent pairs of vertices of graphs guaranteeing that their line graphs are hamiltonian. Here we look at such degree sum conditions imposed on claw-free graphs. But we first note that for general graphs, a sufficient degree sum condition on adjacent pairs for hamiltonicity and traceability can easily be deduced from Theorems 2.1 and 2.2.

Corollary 2.1. Every connected graph G of order $n \geq 3$ with $\overline{\sigma}_2(G) \geq \frac{3n-2}{2}$ is hamiltonian.

Proof. Let G be a connected graph of order $n \geq 3$ with $\overline{\sigma}_2(G) \geq \frac{3n-2}{2}$. Then, for any vertex x of G, we can choose a neighbor y of x, since G is assumed to be connected. Hence, $d(x)+d(y) \geq \overline{\sigma}_2(G) \geq \frac{3n-2}{2}$. This implies that $d(x) \geq \frac{n}{2}$, since $d(y) \leq n-1$. Therefore, Corollary 2.1 is implied by Theorem 2.1. $\square$

Similarly, it easy to check that the following traceability result is implied by Theorem 2.2.

Corollary 2.2. Every connected graph G of order n with $\overline{\sigma}_2(G) \geq \frac{3n-3}{2}$ is traceable.

We already mentioned that Theorems 2.1 and 2.2 are sharp, in the sense that we cannot lower the degree bound without violating the conclusion. Unfortunately, the same holds for Corollaries 2.1 and 2.2. For Corollary 2.1, this can be seen from the graphs $G_m = (m+1)K_1 \vee K_m$, the join of $m+1$ disjoint copies of a K_1 (so a set of $m+1$ independent vertices) with a disjoint complete graph K_m on m vertices ($m \geq 1$). One easily checks that with $n = |V(G_m)| = 2m+1$, $\delta(G_m) = \frac{n-1}{2}$, and $\overline{\sigma}_2(G_m) = \frac{3n-3}{2}$, while G_m is not hamiltonian since the number of the components of $G_m - V(K_m)$ is $m+1$. Similarly, the nontraceable graphs $G_m^1 = (m+2)K_1 \vee K_m$ with $n = |V(G_m^1)| = 2m+2$, $\delta(G_m^1) = \frac{n-2}{2}$, and $\overline{\sigma}_2(G_m^1) = \frac{3n-4}{2}$ show that Corollary 2.2 is sharp.

The above discussion reveals that considering degree sum conditions on adjacent pairs of vertices for general graphs does not provide anything relevant, in the sense of essentially new and more general results. However, if we consider claw-free graphs, this picture changes. This was first observed by Chen [34] who considered the Brualdi-Shanny condition for guaranteeing hamiltonicity of claw-free graphs (as reflected in Theorems 2.4 and 2.5 of this section). To formulate Chen's results, we need some additional notation.

We let $\mathcal{Q}_0(r,k)$ denote the class of k-edge-connected graphs of order at most r that do not admit an SCT. It is known that $\mathcal{Q}_0(5,2) = \{K_{2,3}\}$, and that for $k \geq 4$ these classes are empty, but for other appropriate values of k and r these classes are usually not easy to describe explicitly. In [34], Chen proved the following general result.

Theorem 2.4. (Chen [34]). *Let $p > 0$ be a given integer, let ϵ be a given real number, and let $k \in \{2,3\}$. Suppose H is a k-connected claw-free graph of order n with $\delta(H) \geq 3$. If $\overline{\sigma}_2(H) \geq \frac{2n+\epsilon}{p}$ and n is sufficiently large, then either H is hamiltonian or $cl(H) = L(G)$, where G is an essentially k-edge-connected triangle-free graph that can be contracted to a graph in $\mathcal{Q}_0(5p-10,k)$ for some $p \geq 3$.*

In order to present a more concrete application of the above general result, we need some additional notation.

For a $K_{2,3}$, suppose $D_2(K_{2,3}) = \{v_1, v_2, v_3\}$ and $D_3(K_{2,3}) = \{u_1, u_2\}$. Let $\mathcal{K}_{2,3}(s_1, s_2, s_3, r)$ denote the family of essentially 2-edge-connected graphs of size n (so, with n edges) obtained from a $K_{2,3}$ by replacing each $v_i \in D_2(K_{2,3})$ by a connected triangle-free subgraph of size $s_i \geq 1$ and replacing one vertex in $D_3(K_{2,3})$ by a connected triangle-free subgraph of size $r \geq 0$ such that $\sum_{i=1}^{3} s_i + r + 6 = n$. Note that each graph in $\mathcal{K}_{2,3}(s_1, s_2, s_3, r)$ can be contracted to a $K_{2,3}$, and that the line graph of each of these graphs has order n. These line graphs will be used in the formulation of the next result.

Let $\mathcal{Q}_{2,3}(s_1, s_2, s_3, r)$ be the set of 2-connected claw-free graphs H whose Ryjáček closure $cl(H)$ is the line graph $L(G)$ of a graph G in $\mathcal{K}_{2,3}(s_1, s_2, s_3, r)$.

As a special case of Theorem 2.4 with fixed given values for p and ϵ, the following was obtained in [34], and independently in [79].

Theorem 2.5. (Chen [34]). *Let H be a 2-connected claw-free graph of order n with $\delta(H) \geq 3$. If $\overline{\sigma}_2(H) \geq \frac{2n-4}{4}$ and n is sufficiently large, then one of the following holds:*

(a) H is hamiltonian;

(b) $H \in \mathcal{Q}_{2,3}(s_1, s_2, s_3, r)$ and $\frac{2n-4}{4} \leq \overline{\sigma}_2(H) \leq \frac{2n-2}{4}$, where $min\{s_1, s_2, s_3\} \geq \frac{n-6}{4}$, $r \geq \frac{n-10}{4}$; or

(c) $H \in \mathcal{Q}_{2,3}(s_1, s_2, s_3, 0)$ and $\frac{2n-4}{4} \leq \overline{\sigma}_2(H) \leq \frac{2n-6}{3}$, where $min\{s_1, s_2, s_3\} \geq \frac{n-6}{4}$.

Motivated by the above results, in this chapter we give best possible degree sum conditions on adjacent pairs of vertices for claw-free graphs G with $\delta(G) \geq 3$ to be traceable.

2.2 Our results

Let F_1 and F_2 be the graphs depicted in Figure 2.1, and let $G_1, G_2, \ldots, G_6$ be the graphs that are depicted in Figure 2.2. Denote by $\mathcal{R}_0(r, k)$ the family of k-edge-connected graphs of order at most r that do not admit a spanning

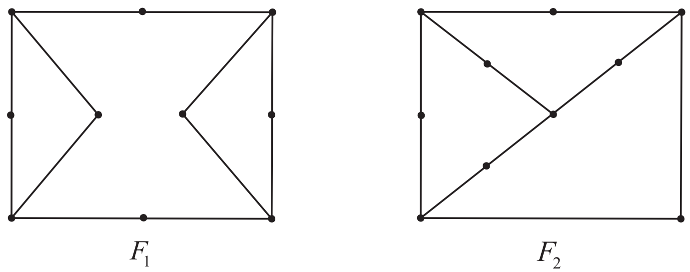

FIGURE 2.1: Two graphs of order 10 without a spanning trail.

trail. Since some graphs in $\mathcal{Q}_0(r,k)$ contain a spanning trail, like $K_{2,3}$ for $k=2$ and the Petersen graph for $k=3$, $\mathcal{R}_0(r,k) \subseteq \mathcal{Q}_0(r,k)$. By Theorem 2.9 below, we know that $\mathcal{R}_0(11,2) = \{F_1, F_2, G_1, G_2, \ldots, G_6\}$. These graphs will play a key role in the results that we are going to present and prove in the remainder of this chapter.

Our first main result is the following analogue of Theorem 2.4 for traceability.

Theorem 2.6. *Let $p > 0$ be a given integer, let ϵ be a given real number, and let $k \in \{2,3\}$. Suppose H is a k-connected claw-free graph of order n with $\delta(H) \geq 3$. If $\overline{\sigma}_2(H) \geq \frac{2n+\epsilon}{p}$ and n is sufficiently large, then either H is traceable or $cl(H) = L(G)$, where G is an essentially k-edge-connected triangle-free graph that can be contracted to a graph in $\mathcal{R}_0(5p-10,k)$ for some $p \geq 4$.*

We postpone all the proofs to later sections of this chapter in order to increase the readability. As an application of Theorem 2.6, we obtain the following result.

Theorem 2.7. *Let H be a 2-connected claw-free graph of order n with $\delta(H) \geq 3$. If $\overline{\sigma}_2(H) \geq \frac{2n-5}{7}$ and n is sufficiently large, then either H is traceable or $cl(H) = L(G)$, where G is an essentially 2-edge-connected triangle-free graph that can be contracted to either F_1 or F_2 such that all vertices of degree two are nontrivial.*

For a graph $F \in \{F_1, F_2\}$, let $D_2(F) = \{v_1, v_2, \ldots, v_6\}$. Let $\mathcal{F}(n,s)$ be the family of essentially 2-edge-connected graphs in which each graph is obtained from such a graph F by replacing each $v_i \in D_2(F)$ by a triangle-free

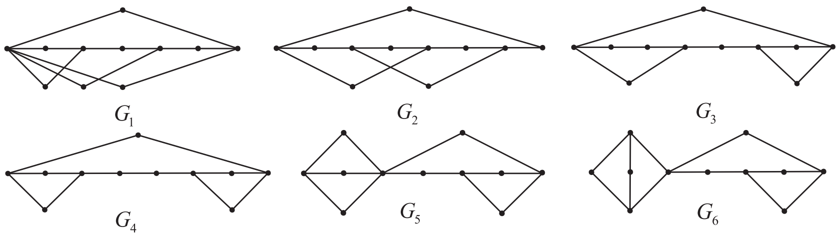

FIGURE 2.2: Six graphs of order 11 without a spanning trail.

subgraph Φ_i of size $s_i \geq s$ such that $n = 12+\sum_{i=1}^{6} s_i$. In particular, if $s = \frac{n-12}{6}$, then we let $\mathscr{F}(n, \frac{n-12}{6})$ be the family of essentially 2-edge-connected graphs in which each graph is obtained from F by adding $\frac{n-12}{6}$ pendant edges to each vertex of degree two of F.

Let $\mathscr{R}_{\mathscr{F}}(n,s)$ be the set of 2-connected claw-free graphs H whose Ryjáček closure $cl(H)$ is the line graph $L(G)$ of a graph G in $\mathscr{F}(n,s)$.

Theorem 2.7 in fact can be deduced from the following result.

Theorem 2.8. *Let H be a 2-connected claw-free graph of order n with $\delta(H) \geq 3$. If $\overline{\sigma}_2(H) \geq \frac{2n-5}{7}$ and n is sufficiently large, then either H is traceable or $\overline{\sigma}_2(H) \leq \frac{n-6}{3}$ and $H \in \mathscr{R}_{\mathscr{F}}(n, \frac{2n-19}{14})$.*

Remark 2.1. Let G^* be a graph obtained from the graph G_1 of Figure 2.2 by adding $\frac{n-14}{7} \geq 2$ pendant edges at each vertex of degree two of G_1. Then $\overline{\sigma}_2(L(G^*)) = \frac{2n-14}{7} < \frac{2n-5}{7}$. Clearly, $L(G^*) \notin \mathscr{R}_{\mathscr{F}}(n, \frac{2n-19}{14})$. Note that G^* cannot be contracted to a graph in $\{F_1, F_2\}$. This example shows that the bound $\frac{2n-5}{7}$ in Theorems 2.7 and 2.8 is asymptotically sharp.

To prove our main results, we need the following key ingredient which is a useful result by itself.

Theorem 2.9. *If G is a 2-edge-connected graph of order at most 11, then either G has a spanning trail or $G \in \{F_1, F_2, G_1, G_2, \ldots, G_6\}$.*

Since all the graphs depicted in Figures 2.1 and 2.2 are not 3-edge-connected, Theorem 2.9 implies the following result.

Corollary 2.3. If G is a 3-edge-connected graph of order at most 11, then G has a spanning trail.

Theorem 2.8 implies the following result immediately.

Corollary 2.4. Let H be a 2-connected claw-free graph of order n. If $\delta(H) \geq \frac{2n-5}{14}$ and n is sufficiently large, then either H is traceable or $\delta(H) \leq \frac{n-6}{6}$ and $H \in \mathscr{R}_{\mathscr{F}}(n, \frac{2n-19}{14})$.

From our proof of Theorem 2.8 (which will be given in Section 2.6), we also obtain the following results.

Theorem 2.10. *Let H be a 2-connected claw-free graph of order n with $\delta(H) \geq 3$. If $\overline{\sigma}_2(H) \geq \frac{n-6}{3}$ and n is sufficiently large, then either H is traceable or $\overline{\sigma}_2(H) = \frac{n-6}{3}$ and $H \in \mathscr{R}_{\mathscr{F}}(n, \frac{n-12}{6})$.*

From Theorem 2.10, we immediately get the following corollary.

Corollary 2.5. Let H be a 2-connected claw-free graph of order n. If $\delta(H) \geq \frac{n-6}{6}$ and n is sufficiently large, then either H is traceable or $\delta(H) = \frac{n-6}{6}$ and $H \in \mathscr{R}_{\mathscr{F}}(n, \frac{n-12}{6})$.

Define $\mathscr{F} = \{H \mid H = L(G)$, where G is obtained from F_1 or F_2 by adding at least one pendant edge to each vertex of degree two of F_1 or $F_2\}$.

In [82], Wang and Xiong proved the following result.

Theorem 2.11. (Wang and Xiong [82]). *Let H be a 2-connected claw-free graph of order $n \geq 137$ such that $\delta(H) > \frac{n}{7} + 4$. Then H is traceable or $H \in \mathscr{F}$.*

Corollary 2.4 is an improvement of Theorem 2.11, and a substantial improvement of the following result for 2-connected claw-free graphs of order n, when n is sufficiently large.

Theorem 2.12. (Matthews and Sumner [66]). *Let H be a connected claw-free graph of order n with $\delta(H) \geq \frac{n-2}{3}$. Then H is traceable.*

The remainder of this chapter is organized as follows. In Section 2.3, we present some useful auxiliary results. In Section 2.4, the proof of Theorem 2.9 is given. In Section 2.6, our proofs of Theorems 2.6 and 2.8 are given.

2.3 Preliminaries and auxiliary results

Niu, Xiong and Zhang in [67] defined the *smallest graph* in a collection of graphs as a graph that has the least order and subject to that has the least size amongst all graphs of that order in the collection. In particular, they considered the smallest order and size of 2-edge-connected graphs without spanning trails. They proved the following result, in which both F_1 and F_2 (of Figure 2.1) are graphs with order 10 and size 12 that do not admit a spanning trail.

Theorem 2.13. (Niu, Xiong and Zhang [67]). *If G is a 2-edge-connected graph of order at most 10, then either G has a spanning trail or $G \in \{F_1, F_2\}$.*

In [82], Wang and Xiong proved the following two useful results.

Theorem 2.14. (Wang and Xiong [82]). *Let G be a 2-connected graph with circumference $c(G)$.*

(a) If $c(G) \leq 5$, then G has a spanning trail that starts from any given vertex.

(b) If $c(G) \leq 7$, then G has a spanning trail.

The following result will be needed in our proof of Theorem 2.8.

Theorem 2.15. (Wang and Xiong [82]). *Let G be a 2-edge-connected graph. Then for any subset $S \subseteq V(G)$ with $|S| \leq 6$ and $E(G-S) = \emptyset$, either G has a trail passing through all vertices of S or $G \in \{F_1, F_2\}$.*

In the next section, we continue with our proof of Theorem 2.9.

2.4 Proof of Theorem 2.9

Before we present the proof, we need some conventions. In a connected graph G, let $C = v_0 v_1 v_2 \cdots v_{c(G)-1} v_0$ denote a longest cycle containing the vertices $v_0, v_1, \ldots, v_{c(G)-1}$ of G. For convenience, in the following, the subscripts are taken modulo $c(G)$. For any $v_i, v_j \in V(C)$ (with $v_i \neq v_j$), without loss of generality, we assume that $i < j$. We use $v_i \overrightarrow{C} v_j$ to denote the segment

$v_i v_{i+1} \cdots v_{j-1} v_j$ of C, i.e., $v_i \overrightarrow{C} v_j$ is a trail (path) along the edges of C starting from the vertex v_i and terminating at the vertex v_j. Note that $v_i \overrightarrow{C} v_j$ contains the vertices v_i and v_j exactly once.

Proof of Theorem 2.9. Let G be a 2-edge-connected simple graph of order at most 11. If G has a spanning trail, then we are done. In the following, we assume that G has no spanning trail.

Assume first that G has a triangle. Then we let G' be the reduction of G. By Theorem 1.5(c), G' is triangle-free. Then, since $|V(G)| \leq 11$, we obtain that $|V(G')| \leq 9$. Now, since G is 2-edge-connected, G' is also 2-edge-connected. By Theorem 2.13, G' has a spanning trail. Then by Theorem 1.7, G has a spanning trail, a contradiction.

Therefore, we next assume that G is triangle-free. If $|V(G)| \leq 10$, then by Theorem 2.13, G is isomorphic to one of the graphs F_1 and F_2 depicted in Figure 2.1. Hence, in the remainder of the proof, we only need to consider the case that $|V(G)| = 11$. We distinguish two cases based on the connectivity $\kappa(G)$ of G.

Case 1. $\kappa(G) \geq 2$.
Since G has no spanning trail then by Theorem 2.14, $c(G) \geq 8$. Therefore, $8 \leq c(G) \leq 9$; otherwise, $G - V(C)$ has at most one vertex, and we can find a spanning trail of G, a contradiction. Here, $C = v_0 v_1 v_2 \cdots v_{c(G)-1} v_0$ denotes a longest cycle of G (and $c(G) = 8$ or 9). By deleting all the chords of C, the resulting 2-connected graph G_0 is a spanning subgraph of G. Thus, G_0 has no spanning trail; otherwise, G has a spanning trial, a contradiction. We first prove the following claim.

Claim 1. $G_0 - V(C)$ is an independent set.

Proof. It suffices to prove that $|V(D)| = 1$ for each component D of $G_0 - V(C)$.

First assume that $c(G_0) = 8$. Then $|V(D)| < 3$; otherwise, since G_0 is a 2-connected triangle-free graph, there exists a path xyz of D with $v_i \in N_{G_0}(x) \cap V(C)$. Since $|V(G_0)| = 11$, $V(G_0) = V(D) \cup V(C)$ and so $zyxv_i \overrightarrow{C} v_{i-1}$ is a spanning trail of G_0, a contradiction. If $|V(D)| = 2$, then $D = K_2$. Since G_0 is 2-connected, we assume that xy is an edge of D with $v_i \in N_{G_0}(x) \cap V(C)$,

$v_j \in N_{G_0}(y) \cap V(C)$ (and $v_i \neq v_j$). Let G^* be the spanning subgraph of G_0 with edge set $E(G_0 - \{x, y\}) \cup \{xv_i, xy, yv_j\}$. Then G^* is 2-connected, and $v_i x y v_j$ is an induced path of length 3 in G^*. Let $\widetilde{G} = G^*/\{xy\}$. Then, by Theorem 2.13, either $\widetilde{G}$ has a spanning trail or $\widetilde{G} \in \{F_1, F_2\}$. In the first case, G^* has a spanning trail thus G_0 has a spanning trail as well, a contradiction. In the second case, so if $\widetilde{G} \in \{F_1, F_2\}$, then by the construction of $\widetilde{G}$, G^* has a cycle of length 9, a contradiction. Hence, $|V(D)| = 1$, as required.

Next assume that $c(G_0) = 9$. Then $|V(D)| = 1$; otherwise, there exists an edge xy in D with $v_i \in N_{G_0}(x) \cap V(C)$. Then $yxv_i \overrightarrow{C} v_{i-1}$ is a spanning trail of G_0, a contradiction. □

Using Claim 1, let $V(G_0) \setminus V(C) = \{u_1, u_2, \ldots, u_t\}$. Then, since $|V(G_0)| = 11$ and by $8 \leq c(G_0) \leq 9$, $2 \leq t \leq 3$. We prove another claim.

Claim 2. For any two vertices $x, y \in V(G_0) \setminus V(C)$, $|N_{G_0}(x) \cap N_{G_0}(y)| \leq 1$.

Proof. By contradiction, we assume that $v_i, v_j \in N_G(x) \cap N_G(y)$ (with $v_i \neq v_j$). Then the spanning subgraph G^τ of $G_0[V(C) \cup \{x, y\}]$ with edge set $E(C) \cup \{xv_i, xv_j, yv_i, yv_j\}$ is an even subgraph. Since $8 \leq |V(C)| \leq 9$, $G_0 - (V(C) \cup \{x, y\})$ has at most one vertex. Then G_0 has a spanning trail containing all edges of G^τ, a contradiction. □

Since $\kappa(G) \geq 2$, for any $x \in V(G_0) \setminus V(C)$, $|N_{G_0}(x) \cap V(C)| \geq 2$, and we consider exactly two edges e_x, e'_x that are incident with x. Let $E_1 = \{e_x, e'_x \mid x \in V(G_0) \setminus V(C)\}$, and let $G^\star$ be the spanning subgraph of G_0 with edge set $E(G_0 - \cup_{i=1}^{t} u_i) \cup E_1$. Then $G^\star$ is 2-connected. Let V_1 be the set of all vertices of odd degree in $G^\star$. Then $V_1 \subseteq V(C)$. Since $|V_1| \leq 6$, $|V_1| \in \{0, 2, 4, 6\}$, and it suffices to consider the cases when $|V_1| = 4$ or 6 (since, if $|V_1| = 0$ or 2, it is immediate that $G^\star$ has a spanning trail, a contradiction).

We distinguish the two remaining subcases for Case 1.

Subcase 1.1. $|V_1| = 6$.

Then $c(G^\star) = 8$ and $|V(G^\star) \setminus V(C)| = 3$. Then $N_{G^\star}(x) \cap N_{G^\star}(y) = \emptyset$, for any $x, y \in V(G^\star) \setminus V(C)$ with $x \neq y$. Since $|V(C)| = 8$ and $|V_1| = 6$, there exist at least three consecutive vertices of V_1 on C. Without loss of generality, we assume that $v_i, v_{i+1}, \ldots, v_{i+l} \in V_1 \cap V(C)$, with $2 \leq l \leq 5$.

First suppose that V_1 has exactly three consecutive vertices on C. Then $l = 2$ and $V_1 = \{v_i, v_{i+1}, v_{i+2}, v_{i+4}, v_{i+5}, v_{i+6}\}$. Then, since $G^\star - \{v_i v_{i+1}, v_{i+4} v_{i+5}\}$ is connected and has exactly two vertices of odd degree, $G^\star$ has a spanning trail, a contradiction.

Next suppose that V_1 has at least four consecutive vertices on C. Then $3 \leq l \leq 5$. Since $G^\star$ is triangle-free, $G^\star - \{v_i v_{i+1}, v_{i+2} v_{i+3}\}$ is connected and has exactly two vertices of odd degree. Then $G^\star$ has a spanning trail, a contradiction.

Subcase 1.2. $|V_1| = 4$.
We prove another claim.

Claim 3. For any pair of vertices $v_i, v_j \in V_1$, v_i, v_j are nonadjacent on C.

Proof. By contradiction, we assume that $v_i, v_{i+1} \in V_1$. Then $G^\star - \{v_i v_{i+1}\}$ has exactly two vertices of odd degree. Then $G^\star$ has a spanning trail, a contradiction. □

Using Claim 3, and by $8 \leq c(G^\star) \leq 9$, without loss of generality, we assume that $V_1 = \{v_i, v_{i+2}, v_{i+4}, v_{i+6}\}$. Note that $|V(G^\star) \setminus V(C)| \leq 3$ and $|N_{G^\star}(x) \cap V(C)| = 2$ for any $x \in V(G^\star) \setminus V(C)$. Then by Claim 2, and using that $V_1 = \{v_i, v_{i+2}, v_{i+4}, v_{i+6}\}$, it is easy to check that $G^\star$ is isomorphic to one of the graphs in $\{G_1, G_2, G_3, G_4\}$ as depicted in Figure 2.2. Since joining any two nonadjacent vertices of a graph in $\{G_1, G_2, G_3, G_4\}$ by an edge will result in a triangle or a spanning trail in the new graph, $G = G_0 = G^\star$. Hence, in this situation, $G \in \{G_1, G_2, G_3, G_4\}$. This completes the proof for Case 1.

Case 2. $\kappa(G) = 1$.
Let $B_1, B_2, \ldots, B_t$ $(t \geq 2)$ be the blocks of G. Since G is triangle-free, $|V(B_i)| \geq 4$ for $1 \leq i \leq t$. We first prove two claims.

Claim 4. Each end-block of G has at least 5 vertices.

Proof. If there exists an end-block B_i of G with 4 vertices, then $G[V(B_i)]$ is a cycle of length 4. Obviously, G/B_i is a 2-edge-connected triangle-free (simple) graph of order 8. By Theorem 2.13, G/B_i has a spanning trail. Since B_i and G/B_i have a vertex in common, the spanning trail of G/B_i can be extended to a spanning trail of G, a contradiction. □

Claim 5. $t = 2$.

Proof. By contradiction, without loss of generality, we assume that B_1 and B_t are two end-blocks of G and B_k is a third distinct block of G. By Claim 4, $|V(B_1)| \geq 5$ and $|V(B_t)| \geq 5$. Since B_1 and B_t have at most one vertex in common with B_k, respectively, $11 = |V(G)| \geq |V(B_1)|+|V(B_t)|+|V(B_k)|-2 \geq 5+5+4-2 = 12$, a contradiction. □

Since $|V(G)| = 11$ and $t = 2$, either $|V(B_1)| = |V(B_2)| = 6$ or $|V(B_1)| = 5$ and $|V(B_2)| = 7$. Then $B_i \notin \mathscr{SL}$; otherwise, the spanning trail of G/B_i can be extended to be a spanning trail of G, a contradiction.

First suppose that $|V(B_1)| = |V(B_2)| = 6$. Since $B_i \notin \mathscr{SL}$, $c(B_i) \leq 5$. By Theorem 2.14(a), both B_1 and B_2 have a spanning trail that starts from any given vertex. Since B_1 and B_2 have a vertex in common, there exists a spanning trail of G, a contradiction.

Next suppose that $|V(B_1)| = 5$ and $|V(B_2)| = 7$. Since B_1 is 2-connected, triangle-free and $B_1 \notin \mathscr{SL}$, $B_1 = K_{2,3}$. Since $B_2 \notin \mathscr{SL}$, $c(B_2) = 6$; otherwise, by Theorem 2.14(a), both B_1 and B_2 have a spanning trail that starts from any given vertex, and there exists a spanning trail of G, a contradiction. We assume that $C = v_0v_1v_2v_3v_4v_5v_0$ is a longest cycle of B_2.

Then $V(B_1) \cap V(C) \neq \emptyset$; otherwise, there exists a vertex $u \in V(B_2) \setminus V(C)$ such that $V(B_1) \cap V(B_2) = \{u\}$. Since B_2 is 2-connected, there exists a vertex $v_i \in N_G(u) \cap V(C)$. Then $T_2 = uv_i \overrightarrow{C} v_{i+5}$ is a spanning trail of B_2. Since B_1 has a spanning trail T_1 starting from vertex u, by combining T_1 and T_2, we can get a spanning trail of G, a contradiction.

Without loss of generality, we assume that $V(B_1) \cap V(C) = \{v_0\}$ and $V(B_2) \setminus V(C) = \{u\}$. Then $v_0, v_1, v_5 \notin N_G(u)$; otherwise, let T_1 be a spanning trail of B_1 starting from vertex v_0, and $T_2 = v_0 \overrightarrow{C} v_5v_0u$ or $v_0v_5v_4v_3v_2v_1u$ or $v_0 \overrightarrow{C} v_5u$ be a spanning trail of B_2 starting from vertex v_0. By combining T_1 with T_2, we can get a spanning trail of G, a contradiction. Since G is 2-edge-connected and triangle-free, $N_G(u) = \{v_2, v_4\}$.

Then G has a spanning subgraph isomorphic to the graph G_5 or G_6, as depicted in Figure 2.2. Furthermore, by joining any two nonadjacent vertices of G_5 or G_6 by an edge, the new graph will contain a triangle or a spanning trail. Hence, $G \in \{G_5, G_6\}$. This completes the proof. □

2.5 More notation and a lemma due to Chen [34]

Before we can state the technical lemma of Chen [34] that is essential for the other proofs in this chapter, we need some additional terminology and notation.

In the following, let $H = L(G)$ and assume that H is not complete. Then $|V(H)| = |E(G)|$ and $\overline{\sigma}_2(G) = \delta(H) + 2$. If $H = L(G)$ is k-connected with $\delta(H) \geq 3$, then G is essentially k-edge-connected with $\overline{\sigma}_2(G) \geq 5$. For each $v \in V(H)$, there is an edge xy in G corresponding to v with $d_H(v) = d_G(x) + d_G(y) - 2$. We call a path of length k a k-path. For each edge uv in H, there is a 2-path, $P_2 = xyz$ in G such that the edge xy is corresponding to the vertex u, and the edge yz is corresponding to the vertex v in H. Then $d_H(u) + d_H(v) = d_G(x) + 2d_G(y) + d_G(z) - 4$.

For any 2-path $P_2 = xyz$ in G, we define $d_G(P_2) = d_G(x) + 2d_G(y) + d_G(z)$. We also define

$$\delta_2(G) = \min\{d_G(P_2)\} \text{ taken over all 2-paths } P_2 \text{ of } G.$$

Thus, for a graph $H = L(G)$,

$$\delta_2(G) = \overline{\sigma}_2(H) + 4. \tag{2.1}$$

For given integer $p > 0$ and real number ϵ, if $\overline{\sigma}_2(H) \geq \frac{2n+\epsilon}{p}$, then the preimage G of $H = L(G)$ has

$$\delta_2(G) \geq \frac{2n+\epsilon}{p} + 4. \tag{2.2}$$

Let G, G_0 and G_0' be the graphs defined in Section 1.4. For $v \in V(G_0')$, let $\Gamma_0(v)$ be the collapsible preimage of v in G_0, and let $\Gamma(v)$ be the preimage of v in G. For convenience, we use the following notation:

- ◇ $V^* = \{v \in V(G_0') \mid |V(\Gamma(v))| \geq 3\}$;
- ◇ $V_1 = \{v \in V(G_0') \mid |V(\Gamma(v))| = 1$ and v is not adjacent to any vertices in $D_1(G) \cup D_2(G)\}$;

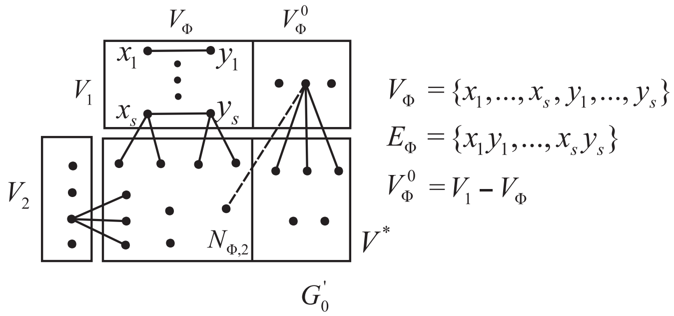

FIGURE 2.3: Decompositions of $V(G_0')$.

- ◇ $V_2 = \{v \in V(G_0') \mid |V(\Gamma(v))| = 2$ or $|V(\Gamma(v))| = 1$ and v is adjacent to a vertex in $D_2(G)\}$;

  (Note that $V^* \cup V_2$ is the set of all nontrivial vertices in G_0').

- ◇ $\Phi = G_0'[V_1]$, the subgraph induced by V_1 in G_0' if $V_1 \neq \emptyset$;

- ◇ $E_\Phi = E(\Phi)$, which is a matching under the conditions of Lemma 2.16 (See below);

- ◇ $V_\Phi = \{v \in V_1 \mid v$ is incident with an edge in $E_\Phi\}$;

- ◇ $V_\Phi^0 = V_1 \setminus V_\Phi$;

- ◇ $N_{\Phi,2} = \bigcup_{v \in V_\Phi \cup V_2} (N_{G_0'}(v) \cap V^*)$ if $V_\Phi \cup V_2 \neq \emptyset$ (otherwise, $N_{\Phi,2} = \emptyset$).

In the following, for given integer $p > 0$ and real number ϵ, we use "$n \gg p$" to reflect that "n is sufficiently large relative to p and ϵ". We are going to rely heavily on the next result due to Chen [34].

Lemma 2.16. (Chen [34]). *Let G be an essentially 2-edge-connected triangle-free graph, and suppose $G \neq K_{1,t}$, G has size $n \gg p$, $\overline{\sigma}_2(G) \geq 5$, and G satisfies (2.2). Assume that $G_0' \notin \mathcal{SL}$. For $V^*, N_{\Phi,2}, V_1, V_2, \Phi, E_\Phi, V_\Phi$, and V_Φ^0 as defined above, we have the following:*

(a) For each $v \in V^*$, $|V(\Gamma(v))| \geq \frac{\delta_2(G)}{2} - d_{G_0'}(v)$ and $|E(\Gamma(v))| \geq \frac{\delta_2(G)}{2} - d_{G_0'}(v) - 1$.

(b) *$D_2(G_0') \subseteq V^*$ and so $d_{G_0'}(v) \geq 3$ for $v \in V_1 \cup V_2$.*

(c) *If $E_\Phi \neq \emptyset$, for each $xy \in E_\Phi$, $(N_{G_0'}(x) \setminus \{y\}) \cup (N_{G_0'}(y) \setminus \{x\}) \subseteq N_{\Phi,2}$, and so E_Φ is a matching.*

(d) *For each vertex v in $V_\Phi^0 \cup V_2$, $N_{G_0'}(v) \subseteq V^*$, and so $V_\Phi^0 \cup V_2$ is an independent set.*

(e) *If $|V_1 \cup V_2| \geq 3$, then $|V_\Phi^0 \cup V_2| + \frac{|V_\Phi|}{2} \leq 2|V^*| - 5$. If $|V_2| \geq 3$ or $V_\Phi \neq \emptyset$, then $|V_2| + \frac{|V_\Phi|}{2} \leq 2|N_{\Phi,2}| - 5$.*

(f) *$|V^*| \leq p$. Furthermore, if $|V^*| = p$ and $G_0' \neq K_{2,t}$ for $t \geq 2$, then $|V(G_0')| \leq 2p - 5 - \frac{\epsilon}{2}$.*

(g) *For $v \in N_{\Phi,2}$, $|E(\Gamma(v))| \geq \delta_2(G) - 5p - 3$ and $|V^*| + |N_{\Phi,2}| \leq p$.*

(h) *If $V_2 \neq \emptyset$, then $|N_{\Phi,2}| \geq 3$. If $V_\Phi \neq \emptyset$, then $|N_{\Phi,2}| \geq 4$. Thus, $|N_{\Phi,2}| \geq 3$ if $|V_2 \cup V_\Phi| \neq 0$.*

2.6 Proofs of Theorems 2.6 and 2.8

In this section, we will present the proofs of Theorems 2.6 and 2.8.

Proof of Theorem 2.6. If H is traceable, then we are done. Thus, in the following, we assume that H is not traceable, and so H is not hamiltonian and H is not complete. By Theorem 1.3, $cl(H)$ is not complete and there exists an essentially k-edge-connected triangle-free graph G such that $cl(H) = L(G)$ and $|E(G)| = |V(H)|$. Let G_0' be the reduction of the core G_0 of G. By Theorem 1.8, $\kappa'(G_0') \geq \kappa'(G_0) \geq k$. Since H is not traceable, by Theorem 1.2, G has no dominating trail. By Theorem 1.9, G_0' has no dominating trail containing all the nontrivial vertices. Then G_0' has no dominating closed trail containing all the nontrivial vertices. Then by Theorem 2.4, $G_0' \in \mathcal{Q}_0(5p-10, k)$. Note that $\mathcal{R}_0(r,k) \subseteq \mathcal{Q}_0(r,k)$. Since G_0' has no spanning trail, and by Theorem 2.9, $G_0' \in \mathcal{R}_0(5p-10, k)$ and $|V(G_0')| \geq 10$. Then $5p - 10 \geq 10$. We conclude that $p \geq 4$. This completes the proof. □

Proof of Theorem 2.8. This is a special case of Theorem 2.6 with $p = 7$, $\epsilon = -5$ and $k = 2$. By Theorem 1.3, there is an essentially 2-edge-connected triangle-free graph G such that the closure $cl(H) = L(G)$ and $|E(G)| = |V(H)| = n$. Since $\delta(H) \geq 3$ and $\overline{\sigma}_2(H) \geq \frac{2n-5}{7}$, $\overline{\sigma}_2(G) \geq 5$ and $\delta_2(G) \geq \frac{2n-5}{7} + 4 = \frac{2n+23}{7}$ by (2.2).

Suppose that H is not traceable. Then $G \neq K_{1,t}$; otherwise, by Theorems 1.2 and 1.4, H is traceable, a contradiction. By Theorems 2.6 and 2.9, G_0' has no spanning trail and $|V(G_0')| \geq 10$. Therefore, $G_0' \notin \mathcal{SL}$.

Let $V^*, V_2, V_\Phi, V_\Phi^0$ and $N_{\Phi,2}$ be the sets relating to G_0' as defined in Section 2.5. If $V_\Phi \cup V_2 \neq \emptyset$, then by the definition, $N_{\Phi,2} \neq \emptyset$. By Lemma 2.16(h), $|N_{\Phi,2}| \geq 3$. Since $N_{\Phi,2} \subseteq V^*$, by Lemma 2.16(g), $|N_{\Phi,2}| \leq 3$. So, $|N_{\Phi,2}| = 3$. Then $V_\Phi = \emptyset$; otherwise, by Lemma 2.16(h), $|N_{\Phi,2}| \geq 4$, a contradiction. Since $N_{\Phi,2} \subseteq V^*$, by Lemma 2.16(g), $3 \leq |V^*| \leq 4$. Then $|V_\Phi^0 \cup V_2| \leq 3$; otherwise, by Lemma 2.16(e), $|V^*| \geq 5$, a contradiction. Therefore, $|V(G_0')| = |V_\Phi^0 \cup V_2| + |V^*| \leq 3 + 4 = 7$, a contradiction. Hence, $V_\Phi = V_2 = \emptyset$ and $V(G_0') = V_\Phi^0 \cup V^*$. Then V^* is the set of all nontrivial vertices of G_0'. By Lemma 2.16(f), $|V^*| \leq 7$.

Claim 1. $|V^*| \leq 6$.

Proof. By contradiction, suppose that $|V^*| = 7$. Then G_0' can not be isomorphic to a $K_{2,t}$ for any $t \geq 2$; otherwise, G_0' has a spanning trail, a contradiction. By Lemma 2.16(f), $|V(G_0')| \leq 2p - 5 - \frac{\epsilon}{2} = 2 \times 7 - 5 - \frac{(-5)}{2} = 11.5$. Then $|V(G_0')| \leq 11$. Then by Theorem 2.9, $G_0' \in \{F_1, F_2, G_1, G_2, \ldots, G_6\}$. By Lemma 2.16(b), $D_2(G_0') \subseteq V^*$.

Suppose first that $G_0' \in \{F_1, F_2\}$. Let $V^* = D_2(G_0') \cup \{v\} = \{v_1, v_2, \ldots, v_6, v\}$, where $v \in D_3(G_0')$. Then $d_{G_0'}(v_i) = 2$ and $d_{G_0'}(v) = 3$. By Lemma 2.16(a), and since $\delta_2(G) \geq \frac{2n+23}{7}$, $s_i = |E(\Gamma(v_i))| \geq \frac{\delta_2(G)}{2} - 3 \geq \frac{2n-19}{14}$, $s = |E(\Gamma(v))| \geq \frac{\delta_2(G)}{2} - 4 \geq \frac{2n-33}{14}$. Furthermore, since $n = 12 + s + \sum_{i=1}^{6} s_i \geq 12 + (\frac{\delta_2(G)}{2} - 4) + 6(\frac{\delta_2(G)}{2} - 3) = \frac{7}{2}\delta_2(G) - 10$, $\delta_2(G) \leq \frac{2n+20}{7}$, contradicting $\delta_2(G) \geq \frac{2n+23}{7}$.

Next suppose that $G_0' \in \{G_1, G_2, \ldots, G_5\}$. Then $V^* = D_2(G_0')$. Let $V^* = D_2(G_0') = \{v_1, v_2, \ldots, v_7\}$. Then $d_{G_0'}(v_i) = 2$. By Lemma 2.16(a), and since $\delta_2(G) \geq \frac{2n+23}{7}$, $s_i = |E(\Gamma(v_i))| \geq \frac{\delta_2(G)}{2} - 3 \geq \frac{2n-19}{14}$. Furthermore, since $n \geq 13 + \sum_{i=1}^{7} s_i \geq 13 + 7(\frac{\delta_2(G)}{2} - 3) = \frac{7}{2}\delta_2(G) - 8$, $\delta_2(G) \leq \frac{2n+16}{7}$, contradicting $\delta_2(G) \geq \frac{2n+23}{7}$.

Finally suppose that $G_0' = G_6$. Since $|D_2(G_0')| = 6$ and $|V^*| = 7$, there exists one vertex $v \in V(G_0') \setminus D_2(G_0')$ such that $v \in V^*$. By Lemma 2.16(d), V_Φ^0 is an independent set. Then $v \in D_4(G_0')$. Let $V^* = D_2(G_0') \cup \{v\} = \{v_1, v_2, \ldots, v_6, v\}$. Then $d_{G_0'}(v_i) = 2$ and $d_{G_0'}(v) = 4$. By Lemma 2.16(a), and since $\delta_2(G) \geq \frac{2n+23}{7}$, $s_i = |E(\Gamma(v_i))| \geq \frac{\delta_2(G)}{2} - 3 \geq \frac{2n-19}{14}$, $s = |E(\Gamma(v))| \geq \frac{\delta_2(G)}{2} - 5 \geq \frac{2n-47}{14}$. Furthermore, since $n = 14 + s + \sum_{i=1}^{6} s_i \geq 14 + (\frac{\delta_2(G)}{2} - 5) + 6(\frac{\delta_2(G)}{2} - 3) = \frac{7}{2}\delta_2(G) - 9$, $\delta_2(G) \leq \frac{2n+18}{7}$, contradicting $\delta_2(G) \geq \frac{2n+23}{7}$. □

Using Claim 1, and by Lemma 2.16(d), $E(G_0' - V^*) = \emptyset$. Note that G_0' is 2-edge-connected. Then by Theorem 2.15, either G_0' has a trail passing through all vertices of V^* or $G_0' \in \{F_1, F_2\}$. For the first case, G_0' has a dominating trail containing all vertices of V^*. Then by Theorems 1.2, 1.4 and 1.9, H is traceable, a contradiction.

Hence, $G_0' \in \{F_1, F_2\}$. By Lemma 2.16(b), $D_2(G_0') \subseteq V^*$. Then, since $|D_2(G_0')| = 6$, $|V^*| = 6$. Let $V^* = D_2(G_0') = \{v_1, v_2, \ldots, v_6\}$. Then $d_{G_0'}(v_i) = 2$. By Lemma 2.16(a), and since $\delta_2(G) \geq \frac{2n+23}{7}$, $s_i = |E(\Gamma(v_i))| \geq \frac{\delta_2(G)}{2} - d_{G_0'}(v_i) - 1 = \frac{\delta_2(G)}{2} - 3 \geq \frac{2n-19}{14}$. Since $n = |E(G)| = |E(G_0')| + \sum_{i=1}^{6} s_i \geq 12 + 6(\frac{\delta_2(G)}{2} - 3) = 3\delta_2(G) - 6$, $\delta_2(G) \leq \frac{n+6}{3}$. Then by (2.1), $\overline{\sigma}_2(H) = \delta_2(G) - 4 \leq \frac{n-6}{3}$. Thus, $G \in \mathscr{F}(n, \frac{2n-19}{14})$, and so $\overline{\sigma}_2(H) \leq \frac{n-6}{3}$ and $H \in \mathscr{R}_{\mathscr{F}}(n, \frac{2n-19}{14})$.

In particular, if $\overline{\sigma}_2(H) = \frac{n-6}{3}$, then by (2.1), $\delta_2(G) = \overline{\sigma}_2(H) + 4 = \frac{n+6}{3}$. By Lemma 2.16(a), and since $\delta_2(G) \geq \frac{n+6}{3}$, $s_i = |E(\Gamma(v_i))| \geq \frac{\delta_2(G)}{2} - d_{G_0'}(v_i) - 1 = \frac{\delta_2(G)}{2} - 3 \geq \frac{n-12}{6}$. Since $n = |E(G)| = |E(G_0')| + \sum_{i=1}^{6} s_i$, $s_i = \frac{n-12}{6}$, for $v_i \in V^*$. By Lemma 2.16(a), $|V(\Gamma(v_i))| \geq \frac{\delta_2(G)}{2} - d_{G_0'}(v_i) = |E(\Gamma(v_i))| + 1$. Thus, $|V(\Gamma(v_i))| = |E(\Gamma(v_i))| + 1$ and so $\Gamma(v_i)$ is a tree. Since G is essentially 2-edge-connected, $\Gamma(v_i) = K_{1,s}$, where $s = \frac{n-12}{6}$. Because $G \in \mathscr{F}(n, \frac{n-12}{6})$, $H \in \mathscr{R}_{\mathscr{F}}(n, \frac{n-12}{6})$. This completes the proof. □

2.7 Concluding remarks

In this chapter, we have been mainly concerned with the traceability of 2-connected claw-free graphs. In order to prove our main results, one of the essential elements we needed was a characterization of all the 2-edge-connected graphs of order at most 11 that have no spanning trail. This has

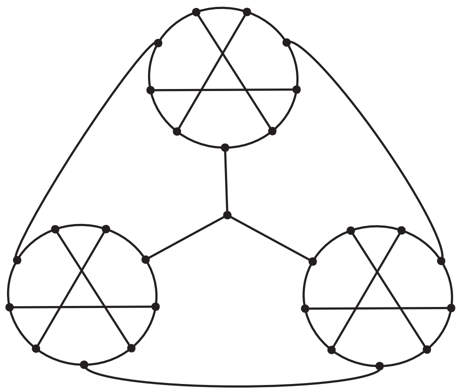

FIGURE 2.4: A 3-edge-connected cubic graph without an ST.

resulted in a more or less explicit description of the obstructions that prevent graphs satisfying the degree conditions from being traceable for given values of $p \leq 7$. For given values of $p \geq 8$, it is much harder to obtain (and write down) such an explicit description, but our main result still implies that there are only a finite number of these obstructions. In principle, for given p, this finite set of obstructions can be found with the help of a computer, but the numbers grow fast with increasing values of p.

As far as we know, the smallest 3-edge-connected graph without a spanning trail is still unknown, but a likely candidate is the cubic (i.e., 3-regular) graph on 28 vertices that is shown in Figure 2.4. In [76], the author proved that this graph has no spanning path. Since the graph is 3-regular, it is easy to prove that it has no spanning trail either. If one would be able to characterize the smallest 3-edge-connected graphs without spanning trails, then, using a similar approach, one can deduce a best possible adjacent degree sum condition for the traceability of 3-connected claw-free graphs.

Chapter 3

Hamiltonicity of line graphs

In this chapter, we continue with degree sum conditions on pairs of adjacent vertices, but here we are mainly concerned with hamiltonicity instead of traceability. Moreover, instead of considering claw-free graphs, we consider line graphs, a proper subclass of the class of claw-free graphs. The work reported here is motivated by results that we recently published in [78] and in [79], but that are not part of this thesis. It is based on earlier results for supereulerian graphs in [28], and inspired by an old conjecture by Benhocine et al. in [2], that was proved by Veldman in the 1990s [81].

3.1 Introduction

We refer to the previous chapters for relevant definitions and notation, and for general background on degree conditions for hamiltonian properties of general graphs and claw-free graphs. In this introductory section, we start by listing the earlier work that inspired many more recent developments in the area of degree conditions for hamiltonian properties of line graphs.

We start with the following result that was conjectured by Benhocine et al. in [2], and proved by Veldman in [81].

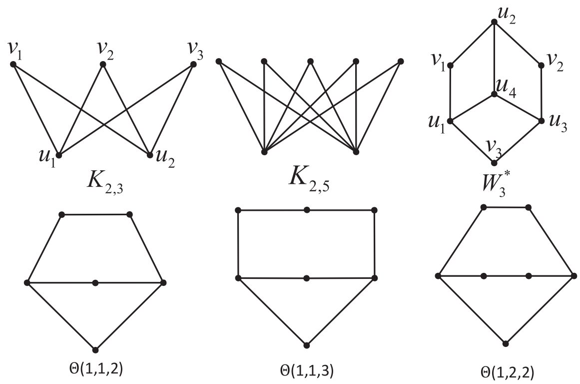

FIGURE 3.1: The graphs $K_{2,3}$, $K_{2,5}$, W_3^*, $\theta(1,1,2)$, $\theta(1,1,3)$, and $\theta(1,2,2)$.

Theorem 3.1. (Veldman [81]). *Let G be an essentially 2-edge-connected graph of order n such that $\overline{\sigma}_2(G) > \frac{2n}{5} - 2$. If n is sufficiently large, then $L(G)$ is hamiltonian.*

In the same paper, Veldman also obtained the following related result, showing that the lower bound in the above result can be improved, but only by allowing a class of exceptional graphs.

Theorem 3.2. (Veldman [81]). *Let G be an essentially 2-edge-connected graph of order n such that $\overline{\sigma}_2(G) > 2(\lfloor \frac{n}{7} \rfloor - 1)$. If n is sufficiently large, then either $L(G)$ is hamiltonian or G is contractible to a $K_{2,3}$ such that all vertices of degree two in $K_{2,3}$ are nontrivial.*

We use $\theta(i,j,k)$ to denote the graph that is obtained from the multigraph consisting of two vertices and three multiple (parallel) edges by subdividing the three edges i, j, and k times, respectively. For example, $\theta(1,1,1) \cong K_{2,3}$. We also define the following two classes of graphs, referring to Figures 3.1 and 3.2:

- $\mathscr{G}_1 = \{K_{2,3}, K_{2,5}, W_3^*, \theta(1,1,2), \theta(1,1,3), \theta(1,2,2)\}$ and
- $\mathscr{G}_2 = \{J(2,2), J(2,3), K_{2,5}^*, K_{2,5}^{**}, C(6,2), C(6,2)', C(6,4), C(6,4)',$
  $\theta(1,1,4), \theta(1,1,4)', \theta(1,1,4)'', \theta(1,1,4)''',$
  $\theta(2,2,2), \theta(1,2,3), \theta(1,2,3)', W_3^{**}\}$.

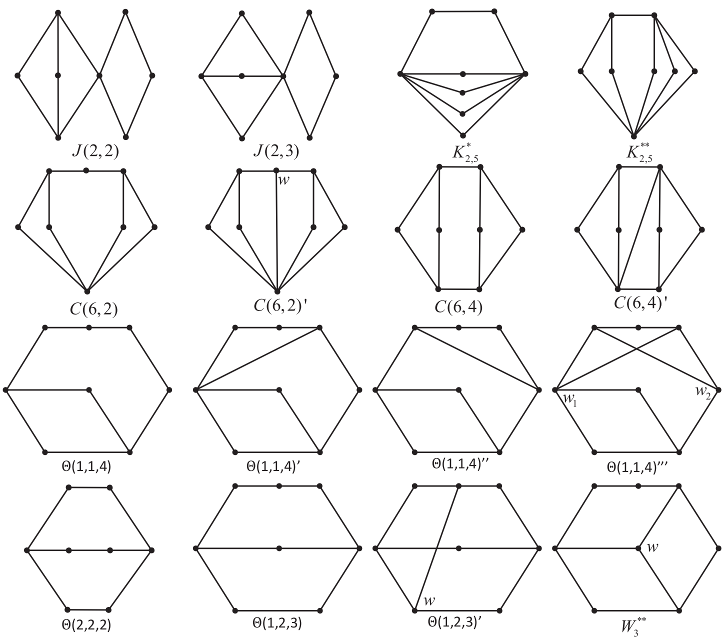

FIGURE 3.2: The sixteen graphs in $\mathscr{G}_2$ that have no SCT.

The graphs in $\mathscr{G}_1$ and $\mathscr{G}_2$ are depicted in Figures 3.1 and 3.2, respectively.

As already mentioned in [81], Theorem 3.2 is best possible in the sense that there exist infinitely many essentially 2-edge-connected graphs G with $\overline{\sigma}_2(G) = 2(\lfloor \frac{n}{7} \rfloor - 1)$ such that $L(G)$ is nonhamiltonian and G is not contractible to $K_{2,3}$. Examples of such graphs can be found among the graphs contractible to $K_{2,5}$ or the 3-cube minus a vertex (W_3^*). The following result confirms this, and shows that we can lower the bound slightly by excluding these two classes of exceptional graphs.

Theorem 3.3. (Tian and Xiong [79]) *Let G be an essentially 2-edge-connected graph of order n such that $\overline{\sigma}_2(G) \geq 2(\lfloor \frac{n}{7} \rfloor - 1)$. If n is sufficiently large, then either $L(G)$ is hamiltonian or G is contractible to a $K_{2,3}$, a $K_{2,5}$, or a W_3^* such that all vertices of degree two in $K_{2,3}$, $K_{2,5}$, and W_3^* are nontrivial.*

The next result shows that for 3-edge-connected graphs, the lower bound in Theorem 3.3 can be improved considerably, even with a stronger conclusion. Here, the Petersen graph is the graph $P(10)$ depicted in Figure 3.3.

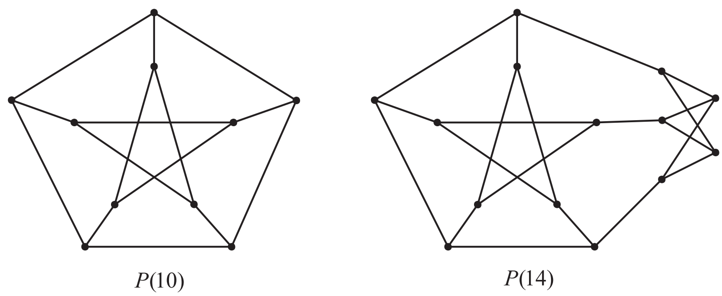

FIGURE 3.3: The Petersen graph $P(10)$, and $P(14)$.

Theorem 3.4. (Chen and Lai [37]) *Let G be a 3-edge-connected graph of order n such that*

$$\overline{\sigma}_2(G) \geq \frac{n}{5} - 2. \tag{3.1}$$

If n is sufficiently large, then either G is supereulerian or G can be contracted to the Petersen graph.

In [39], Chen and Lai improved the lower bound in Theorem 3.4 even further, and obtained the following result.

Theorem 3.5. (Chen and Lai [39]) *Let G be a 3-edge-connected graph of order n such that*

$$\overline{\sigma}_2(G) \geq \frac{n}{6} - 2. \tag{3.2}$$

If n is sufficiently large, then either G is supereulerian or G can be contracted to the Petersen graph.

A natural question in this context is: what is the best possible lower bound for the degree sum condition on pairs of adjacent vertices for which all the exceptional graphs can be contracted to the Petersen graph? Due to a construction based on the so-called Blanuša snarks in [39], there are infinite families of graphs showing that in the following conjecture, if true, the lower bound of (3.3) will be best possible.

Conjecture 3.1. (Chen and Lai [39]) *Let G be a 3-edge-connected graph of order n such that*

$$\overline{\sigma}_2(G) > \frac{n}{9} - 2. \tag{3.3}$$

If n is sufficiently large, then either G is supereulerian or G can be contracted to the Petersen graph.

We define $\mathscr{G}_3 = \{P(10), P(14)\}$, where $P(10)$ and $P(14)$ are the graphs depicted in Figure 3.3. In [27] and [28], Chen et al. improved the result of Theorem 3.5, respectively, and obtained the following two results.

Theorem 3.6. (Chen et al. [27]) *Let G be a 3-edge-connected graph of order n with any given $\epsilon < \frac{16}{13}$ such that*

$$\delta_L(G) \geq \frac{n}{13} - \epsilon. \tag{3.4}$$

If n is sufficiently large, then $L(G)$ is hamiltonian if and only if G does not have the Petersen graph as a nontrivial contraction.

Theorem 3.7. (Chen et al. [28]) *Let G be a 3-edge-connected graph of order n such that*

$$\overline{\sigma}_2(G) > 2(\frac{n}{15} - 1). \tag{3.5}$$

If n is sufficiently large, then either $G \in \mathscr{SL}$ or $G' \in \mathscr{G}_3$. Furthermore, if $\overline{\sigma}_2(G) \geq 2(\frac{n}{14} - 1)$ and $G' = P(14)$, then $n = 14s$ and each vertex in $P(14)$ is obtained by contracting a K_s or $K_s - e$ for some $e \in E(K_s)$.

3.2 Our results

In the following, for given integer $p > 0$, we use "$n \gg p$" to reflect that "n is sufficiently large relative to p". Our main result reads as follows.

Theorem 3.8. *Let G be an essentially k-edge-connected graph of order n (with $k \in \{2, 3\}$), and let $p \geq 2$ be an integer such that*

$$\overline{\sigma}_2(G) \geq 2(\lfloor n/p \rfloor - 1). \tag{3.6}$$

If $n \gg p$, then either $L(G)$ is hamiltonian or G has no DCT and is contractible to a graph in $\mathscr{Q}_0(max\{p, \frac{3}{2}p - 4\}, k)$.

Note that "G is contractible to a graph in $\mathcal{Q}_0(r,k)$" in Theorem 3.8 means that "the reduction of the core G_0 is in $\mathcal{Q}_0(r,k)$". We continue with stating some consequences of our main result, and postpone all proofs to later sections. As applications of Theorem 3.8, we obtain the following results.

Theorem 3.9. *Let G be an essentially 2-edge-connected graph of order n such that*

$$\overline{\sigma}_2(G) \geq 2(\lfloor n/8 \rfloor - 1). \tag{3.7}$$

If n is sufficiently large, then either $L(G)$ is hamiltonian or G is contractible to a graph in $\{K_{2,3}, K_{2,5}, W_3^*, C(6,2)', C(6,4)', \theta(1,1,4)'''\}$.

Theorem 3.10. *Let G be an essentially 3-edge-connected graph of order n such that*

$$\overline{\sigma}_2(G) \geq 2(\lfloor n/15 \rfloor - 1). \tag{3.8}$$

If n is sufficiently large, then either $L(G)$ is hamiltonian or G can be contracted to the Petersen graph.

Theorem 3.11. *Let G be a 3-edge-connected graph of order n such that*

$$\overline{\sigma}_2(G) \geq 2(\lfloor n/15 \rfloor - 1). \tag{3.9}$$

If n is sufficiently large, then either G is supereulerian or G can be contracted to the Petersen graph.

For our proof of Theorem 3.9, we also need the following result.

Theorem 3.12. *Let G be a 2-edge-connected triangle-free graph of order at most 8. Then either G is supereulerian or $G \in \mathcal{G}_1 \cup \mathcal{G}_2$.*

Theorems 3.1, 3.2, 3.3, 3.4, 3.5, and 3.7 are all special cases of Theorem 3.8, with $(p,k) \in \{(5,2),(7,2),(10,3),(12,3),(15,3)\}$. Moreover, with Theorem 3.8, we implicitly provide improvements of Theorem 3.3 and Theorem 3.7, since they are special cases of Theorem 3.8 with $p = 8$ and $k = 2$, and $p = 15$ and $k = 3$, respectively. Even though Theorem 3.6 implies Theorem 3.5, it does not imply our Theorems 3.10 and 3.11. Furthermore, we note that Theorem 3.11 improves Theorem 3.7, and Theorem 3.11 is a special case of Theorem 3.10.

The remainder of this chapter is organized as follows. In Section 3.3, we will present some useful auxiliary results, together with a description of Veldman's reduction method, as well as our proof of Theorem 3.12. In Section 3.4, we present our proofs of Theorems 3.8, 3.9, 3.10, and 3.11.

3.3 Preliminaries

We start this section with some facts we need on reduced graphs, as summarized in the following theorem, where the first fact is folklore and easy to prove.

Theorem 3.13. *Let G be a connected reduced graph of order n. Then each of the following holds:*

(a) *If $G \notin \mathcal{S}\mathcal{L}$ and $\kappa'(G) \geq 2$, then $n \geq 5$, and $n = 5$ only if $G = K_{2,3}$.*

(b) (Corollary 4.11 in [40]) *If $n \leq 15$ and $\delta(G) \geq 3$, then G is supereulerian if and only if $G \notin \mathcal{G}_3$.*

(c) (Lemma 4.8 in [40]) *If $n \geq 15$, $\kappa'(G) \geq 3$ and $\alpha'(G) \leq 7$, then G is supereulerian.*

In [79], Tian and Xiong characterized some small graphs which have no SCT, as follows.

Theorem 3.14. (Tian and Xiong [79]) *Let G be a 2-edge-connected triangle-free graph of order at most 7. Then either G is supereulerian or $G \in \mathcal{G}_1$.*

3.3.1 Proof of Theorem 3.12

Let G be a 2-connected graph, and let $C = v_0v_1v_2 \dots v_{c(G)-1}v_0$ be a longest cycle of G, where the subscripts are taken modulo $c(G)$ throughout. Then any component of $G - V(C)$ has at least two different neighbors on C. Denote by $d_C(v_i, v_j)$ the distance between $v_i, v_j \in V(C)$ (with $v_i \neq v_j$) on C. Obviously, $1 \leq d_C(v_i, v_j) \leq \left\lfloor \frac{|C|}{2} \right\rfloor$.

Proof of Theorem 3.12. Let G be a 2-edge-connected triangle-free graph of order at most 8. If G has an SCT, then we are done. So, in the following, we assume that G has no SCT. If $|V(G)| \le 7$, then by Theorem 3.14, $G \in \mathcal{G}_1$. So, in the following, we only need to consider the case $|V(G)| = 8$.

Suppose first that $\kappa(G) = 1$. Let $B_1, B_2, \ldots, B_t$ $(t \ge 2)$ be the blocks of G. Since G is triangle-free, $|V(B_i)| \ge 4$. Note that B_i and B_j $(i \ne j)$ have at most one vertex in common. Then $t = 2$; otherwise, $8 = |V(G)| \ge 3 \times 4 - 2 = 10$, a contradiction. Without loss of generality, we may assume that $|V(B_1)| = 4$ and $|V(B_2)| = 5$. Since G is triangle-free, $B_1 = C_4$. Then $B_2 \notin \mathcal{SL}$; otherwise G has an SCT, a contradiction. Now, by Theorem 3.13(a), $B_2 = K_{2,3}$. Note that $|V(B_1) \cap V(B_2)| = 1$. This implies that G is isomorphic to the graph $J(2,2)$ or $J(2,3)$.

In the following, we suppose that G is 2-connected. Since G is triangle-free and since $G \notin \mathcal{SL}$, $4 \le c(G) \le 7$.

Observation 1. By deleting all chords of C from G, the resulting 2-connected graph G_0 is a spanning subgraph of G. Obviously, C is also a longest cycle of G_0. Then G_0 has no SCT; otherwise, G has an SCT, a contradiction.

Note that by adding the deleted chords of C to G_0 one by one, by our assumptions, at each step we obtain a spanning subgraph of G which has no SCT, or we derive at a contradiction. Obviously, if $4 \le c(G) \le 5$, then C has no chord. We distinguish the cases that $c(G) = 4$, 5, 6, and 7.

Case 1. $c(G) = 4$.
Then $G - V(C) = 4K_1$, or $K_2 \cup 2K_1$, or $P_3 \cup K_1$, or $2K_2$, or P_4, or $K_{1,3}$, or C_4. Suppose that $G - V(C) = K_2 \cup 2K_1$, or $P_3 \cup K_1$, or $2K_2$, or P_4, or $K_{1,3}$, or C_4. Since G is 2-connected and triangle-free, there exists a path $x_1 \ldots x_k$ $(k = 2$ or 3 or 4) in $G - V(C)$ with $v_i \in N_G(x_1) \cap V(C)$ and $v_j \in N_G(x_k) \cap V(C)$ $(v_i \ne v_j)$. But now we can find a cycle containing the vertices $x_1, \ldots, x_k$ with length more than 4, a contradiction. Thus, $G - V(C) = 4K_1$. Since G is 2-connected and triangle-free, and since $c(G) = 4$, $G = K_{2,6}$. Obviously, $K_{2,6}$ has an SCT, a contradiction.

Case 2. $c(G) = 5$.

Then $G-V(C)=3K_1$, or $K_2\cup K_1$, or P_3. Suppose that $G-V(C)=K_2\cup K_1$ or P_3. Since G is 2-connected and triangle-free, there exists a path $x_1\dots x_k$ ($k=2$ or 3) in $G-V(C)$ with $v_i\in N_G(x_1)\cap V(C)$ and $v_j\in N_G(x_k)\cap V(C)$ ($v_i\neq v_j$). But now we can find a cycle containing the vertices $x_1,\dots,x_k$ with length more than 5, a contradiction. Thus, $G-V(C)=3K_1$. Let $V(G)\setminus V(C)=\{x_1,x_2,x_3\}$. Since G is 2-connected and triangle-free, $d_G(x_i)=2$ ($i=1,2,3$). Without loss of generality, we may assume that $N_G(x_1)=\{v_i,v_{i+2}\}$. Then, by symmetry, and since $c(G)=5$, we can assume that either $N_G(x_j)=\{v_i,v_{i+2}\}$ or $N_G(x_j)=\{v_i,v_{i+3}\}$ ($j=2,3$). Then G is isomorphic to the graph $K^*_{2,5}$ or $K^{**}_{2,5}$.

Case 3. $c(G)=6$.

By Observation 1, without loss of generality, we first assume that C is an induced cycle of G, namely $G=G_0$. Then $G-V(C)=2K_1$ or K_2. Let $V(G)\setminus V(C)=\{x_1,x_2\}$.

Subcase 3.1. $G-V(C)=2K_1$.

Since G is 2-connected and triangle-free, $2\le d_G(x_i)\le 3$ ($i=1,2$). Suppose that $d_G(x_1)=3$ (it is similar for $d_G(x_2)=3$). Without loss of generality, we may assume that $N_G(x_1)=\{v_i,v_{i+2},v_{i+4}\}$. By $2\le d_G(x_2)\le 3$, $2\le |N_G(x_2)\cap V(C)|\le 3$. Then, it is easy to check that G has an SCT, a contradiction. Therefore, $d_G(x_1)=d_G(x_2)=2$. Without loss of generality, we may assume that either $N_G(x_1)=\{v_i,v_{i+2}\}$ or $N_G(x_1)=\{v_i,v_{i+3}\}$.

Suppose first that $N_G(x_1)=\{v_i,v_{i+2}\}$. If $v_i\in N_G(x_2)$ (by symmetry, it is similar for $v_{i+2}\in N_G(x_2)$), then $v_{i+1},v_{i+2},v_{i+3},v_{i+5}\notin N_G(x_2)$; otherwise, either G has a triangle or G has an SCT, a contradiction. Then by $d_G(x_2)=2$, $N_G(x_2)=\{v_i,v_{i+4}\}$. Then G has a spanning subgraph isomorphic to the graph $C(6,2)$.

If $v_{i+1}\in N_G(x_2)$, then $v_i,v_{i+2},v_{i+3},v_{i+4},v_{i+5}\notin N_G(x_2)$; otherwise, either G has a triangle or G has a cycle of length more than 6, a contradiction. Then $d_G(x_2)=1$, contrary to $d_G(x_2)=2$. Therefore, in the following, we may assume that $v_i,v_{i+1},v_{i+2}\notin N_G(x_2)$. Since G is 2-connected and triangle-free, and by $d_G(x_2)=2$, $N_G(x_2)=\{v_{i+3},v_{i+5}\}$. Then G has a spanning subgraph isomorphic to the graph $C(6,4)$.

Now suppose that $N_G(x_1) = \{v_i, v_{i+3}\}$. By $d_G(x_2) = 2$, $|N_G(x_2) \cap V(C)| = 2$. Then, it is easy to check that either G has an SCT, or G has a triangle, or G has a cycle of length more than 6, a contradiction.

Subcase 3.2. $G - V(C) = K_2$.
Without loss of generality, we may assume that $N_G(x_1) \cap V(C) = \{v_i\}$. Since G is 2-connected and triangle-free, and since $c(G) = 6$, $N_G(x_2) \cap V(C) = \{v_{i+3}\}$. Then G has a spanning subgraph isomorphic to $\theta(2,2,2)$.

In both subcases, by Observation 1, joining any two nonadjacent vertices of $C(6,2)$ or $C(6,4)$ or $\theta(2,2,2)$ by an edge (step by step) will result in a triangle, or a $C(6,2)'$, or a $C(6,4)'$, or an SCT in the new graph. Hence, $G \in \{C(6,2), C(6,2)', C(6,4), C(6,4)', \theta(2,2,2)\}$.

Case 4. $c(G) = 7$.
By Observation 1, without loss of generality, we first assume that C is an induced cycle of G, namely $G = G_0$. Then $G - V(C) = K_1$. Let $V(G) \setminus V(C) = \{x\}$. Since G is 2-connected and triangle-free, $2 \le d_G(x) \le 3$.

Suppose that $d_G(x) = 2$. We may assume that $N_G(x) = \{v_i, v_j\}$ ($v_i \neq v_j$). Obviously, $2 \le d_C(v_i, v_j) \le 3$. If $d_C(v_i, v_j) = 2$, then, without loss of generality, we may assume that $N_G(x) = \{v_i, v_{i+2}\}$. Then G has a spanning subgraph isomorphic to $\theta(1,1,4)$. If $d_C(v_i, v_j) = 3$, then, without loss of generality, we may assume that $N_G(x) = \{v_i, v_{i+3}\}$. Then G has a spanning subgraph isomorphic to $\theta(1,2,3)$.

Suppose that $d_G(x) = 3$. Without loss of generality, we may assume that $N_G(x) = \{v_i, v_{i+2}, v_{i+4}\}$. Then G has a spanning subgraph isomorphic to W_3^{**}.

By Observation 1, joining any two nonadjacent vertices of $\theta(1,1,4)$ or $\theta(1,2,3)$ or W_3^{**} by an edge (step by step) will result in a triangle, or a graph in $\{\theta(1,1,4)', \theta(1,1,4)'', \theta(1,1,4)''', \theta(1,2,3)', W_3^{**}\}$, or an SCT of the new graph. Hence, G is one of the graphs in $\{\theta(1,1,4), \theta(1,1,4)', \theta(1,1,4)'', \theta(1,1,4)''', W_3^{**}, \theta(1,2,3), \theta(1,2,3)'\}$. This completes the proof. $\square$

3.3.2 Veldman's reduction method

Recall that $D(G) = D_1(G) \cup D_2(G)$ is the set of vertices with degree 1 or 2 in G. For an independent subset X of $D(G)$, define $I_X(G)$ as the graph obtained from G by deleting the vertices in X of degree 1 and replacing each path of length 2 whose internal vertex is a vertex in X of degree 2 by an edge. Note that $I_X(G)$ may not be simple. We call G *X-collapsible* if $I_X(G)$ is collapsible. A subgraph H of G is an *X-subgraph* of G if $d_H(x) = d_G(x)$ for all $x \in X \cap V(H)$. An X-subgraph H of G is called *X-collapsible* if H is $(X \cap V(H))$-collapsible. Let $R(X)$ be the set of vertices in X that are not contained in an X-collapsible X-subgraph of G. Since $I_X(G)$ has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $L_1, \ldots, L_k$ such that $\bigcup_{i=1}^{k} V(L_i) = V(I_X(G))$, the graph G has a unique collection of pairwise vertex-disjoint maximal X-collapsible X-subgraph $H_1, \ldots, H_k$ such that $(\bigcup_{i=1}^{k} V(H_i)) \cup R(X) = V(G)$. The *$X$-reduction* of G is the graph obtained from G by contracting $H_1, \ldots, H_k$. Letting G'' be the X-reduction of G and $v \in V(G'')$, the *preimage* of v is denoted by $\theta^{-1}(v)$. A vertex v of G'' is called *nontrivial* if $\theta^{-1}(v) \neq K_1$ and *trivial* otherwise. The graph G is *X-reduced* if there exists a graph G^* and an independent subset X^* of $D(G^*)$ such that $X = R(X^*)$ and G is the X^*-reduction of G^*. An X-subgraph H of G is called *X-reduced* if H is $(X \cap V(H))$-reduced.

Remark 3.1. If $X = \emptyset$, then the refinement method ($\emptyset$-reduction) is just the reduction method of Catlin. Let G be an essentially 2-edge-connected graph with $\overline{\sigma}_2(G) \geq 5$. Then $D(G)$ is an independent set. For the $D(G)$-reduction of G, if $R(D(G)) = \emptyset$, then the refinement method of the reduction of the core of the graph G is just the $D(G)$-reduction method of Veldman.

In [81], Veldman obtained the following result.

Theorem 3.15. (Veldman [81]) *Let G be a connected graph of order n, and let $p \geq 2$ be an integer such that*

$$\overline{\sigma}_2(G) \geq 2(\lfloor n/p \rfloor - 1). \tag{3.10}$$

If $n \gg p$, then

$$|V(G'')| \leq max\{p, \frac{3}{2}p - 4\}, \tag{3.11}$$

where G'' is the $D(G)$-reduction of G. Moreover, for $p \leq 7$, (3.11) holds with equality only if (3.10) holds with equality.

Using Theorem 3.15, we can easily deduce the following result.

Theorem 3.16. *Let G be an essentially k-edge-connected graph of order n (with $k \in \{2,3\}$), and let $p \geq 2$ be an integer such that*

$$\overline{\sigma}_2(G) \geq 2(\lfloor n/p \rfloor - 1). \tag{3.12}$$

If $n \gg p$, then exactly one of the following holds.

(a) $G_0 \in \mathcal{SL}$;

(b) $G_0' \notin \mathcal{SL}$ *with* $|V(G_0')| \leq max\{p, \frac{3}{2}p - 4\}$ *and* $\kappa'(G_0') \geq k$.

Proof of Theorem 3.16. By Theorem 1.5(a), (a) and (b) of Theorem 3.16 are mutually exclusive. Suppose that $G_0 \notin \mathcal{SL}$. Then $L(G)$ is not complete; otherwise, $G = K_{1,n-1}$ or K_n, and so $G_0 \in \mathcal{SL}$, a contradiction. By Theorem 1.5(a), and since $G_0 \notin \mathcal{SL}$, $G_0' \notin \mathcal{SL}$. Since $\overline{\sigma}_2(G) \geq 2(\lfloor n/p \rfloor - 1)$, if $n \geq 4p$, then $\overline{\sigma}_2(G) \geq 6$, and consequently $D(G)$ is an independent set. Let G'' be the $D(G)$-reduction of G. By Theorem 3.15, $|V(G'')| \leq \max\{p, \frac{3}{2}p - 4\}$. Since G_0' is a refinement of the $D(G)$-reduction of G, $|V(G_0')| \leq |V(G'')| \leq \max\{p, \frac{3}{2}p - 4\}$. By Theorem 1.8(a), $\kappa'(G_0') \geq k$. This completes the proof. □

3.4 Supereulerian graphs and hamiltonian line graphs

Before we continue with the remaining proofs of our results, we mention one other result of Chen [30] and an application of our Theorem 3.12 in order to obtain an Ore-type analogue of the results in this chapter.

Let G be a graph, and let $k \geq 0$ be an integer. If there is a graph G^* such that G can be obtained from G^* by removing at most k edges, then G is said to be *at most k edges short* of being G^*.

Theorem 3.17. (Chen [30]) *Let G be a 2-edge-connected graph with girth $g \in \{3,4\}$, and let $p \geq 2$ be an integer. If*

$$\sigma_2(G) \geq \frac{2}{g-2}(\frac{n}{p} + g - 4),$$

and if

$$n \geq 4(g-2)p^2,$$

then exactly one of the following holds:

(a) $G \in \mathscr{SL}$;

(b) $G' \notin \mathscr{SL}$ *and* $|V(G')| \leq p$, *where G' is the reduction of G. Further, if $|V(G')| = p$, then $n = (g-2)ps$, for some integer s, and $\delta(G) = \frac{1}{g-2}(\frac{n}{p} + g - 4)$, and either*

(i) $g = 3$, *and the preimage H_i of each vertex v_i of G' is at most $\frac{1}{2}d_{G'}(v_i)$ edges short of being K_s, or*

(ii) $g = 4$, *and the preimage H_i of each vertex v_i of G' is at most $\frac{1}{2}d_{G'}(v_i)$ edges short of being $K_{s,s}$.*

As an application of Theorems 3.12 and 3.17, we obtain the following result.

Theorem 3.18. *Let G be a 2-edge-connected graph with girth $g \in \{3,4\}$. If*

$$\sigma_2(G) \geq \frac{2}{g-2}(\frac{n}{8} + g - 4),$$

and if

$$n \geq 256(g-2),$$

then exactly one of the following holds:

(a) $G \in \mathscr{SL}$;

(b) $G' \in \mathscr{G}_1 \cup \mathscr{G}_2$, *where G' is the reduction of G. In particular, if $|V(G')| = 8$, then $n = 8(g-2)s$, for some integer s, and either*

(i) $g=3$, *and the preimage* H_i *of each vertex* v_i *of* G' *is at most 2 edges short of being* K_s*, or*

(ii) $g=4$, *and the preimage* H_i *of each vertex* v_i *of* G' *is at most 2 edges short of being* $K_{s,s}$.

Proof of Theorem 3.18. If G has an SCT, then we are done. In the following, we assume that G has no SCT. By Theorem 3.17(b), $G' \notin \mathcal{SL}$ and $|V(G')| \le 8$. By the definition of contraction, $\kappa'(G') \ge \kappa'(G) \ge 2$. By Theorem 1.5(c), G' is simple and triangle-free. Then by Theorem 3.12, $G' \in \mathcal{G}_1 \cup \mathcal{G}_2$. So, $d_{G'}(v) \le 5$ for any $v \in V(G')$. By Theorem 3.17(b), Theorem 3.18(b) holds. This completes the proof. $\square$

Next, we continue with the remaining proofs of our results.

3.4.1 Proof of Theorem 3.8 and a useful proposition

Proof of Theorem 3.8. If $L(G)$ is hamiltonian, then we are done. In the following, we assume that $L(G)$ is not hamiltonian, and so $L(G)$ is not complete. Then by Theorem 1.1, G has no DCT. Since $\overline{\sigma}_2(G) \ge 2(\lfloor n/p \rfloor - 1)$, if $n \ge 4p$, then $\overline{\sigma}_2(G) \ge 6$, and consequently $D(G)$ is an independent set. Let G'' be the $D(G)$-reduction of G. By Theorem 3.15, $|V(G'')| \le \max\{p, \frac{3}{2}p-4\}$. Let G_0' be the reduction of the core G_0 of G. By Theorem 1.5(c), G_0' is simple and triangle-free. By Theorem 1.8, $G_0' \notin \mathcal{SL}$ and $\kappa'(G_0') \ge k$. Since G_0' is a refinement of the $D(G)$-reduction of G, $|V(G_0')| \le |V(G'')| \le \max\{p, \frac{3}{2}p-4\}$. This completes the proof. $\square$

Let G_0' be the reduction of the core of G. For $v \in V(G_0')$, let $\Gamma(v)$ be the preimage of v in G. For convenience, we define the following sets, and we prove a useful proposition that we use in some of the later proofs.

- $S_0 = \{v \in V(G_0') \mid v \text{ is a nontrivial vertex in } G_0'\}$;
- $S_1 = \{v \in S_0 \mid |V(\Gamma(v))| > 1\}$;
- $S_2 = S_0 \setminus S_1$, the set of vertices v with $\Gamma(v) = K_1$ and adjacent to some vertices in $D_2(G)$;

- $V_0 = V(G_0') \setminus S_0$.

Proposition 3.1. Let G be an essentially k-edge-connected graph of order n ($k \in \{2,3\}$) with $\overline{\sigma}_2(G) \geq 2(\lfloor n/p \rfloor - 1)$, where $p \geq 2$ is an integer. Let G_0' be the reduction of the core of G, and suppose $G_0' \notin \mathcal{SL}$. Let S_0, S_1 and V_0 be the sets defined above. If $n \gg p$, then each of the following holds:

(a) If $v \in S_1$, then $|V(\Gamma(v))| \geq \lfloor \frac{n}{p} \rfloor - l + 1$, where $l = \max\{p, \frac{3p}{2} - 4\}$.

(b) $S_1 = S_0$.

(c) V_0 is an independent set, and $N_{G_0'}(v) \subseteq S_1$ for any $v \in V_0$.

(d) $|S_0| \leq p$. Furthermore, if $|S_0| = p$, then $V(G_0') = S_0$.

Proof. As the assumptions of Proposition 3.1 imply the assumptions of Theorem 3.16, it follows from Theorem 3.16 that $|V(G_0')| \leq \max\{p, \frac{3p}{2} - 4\}$ and $\kappa'(G_0') \geq k \geq 2$. For convenience, in the following, let $l = \max\{p, \frac{3p}{2} - 4\}$. For $v \in V(G_0')$, let $\Gamma(v)$ be the preimage of v in G. By Theorem 1.5(c), G_0' is simple and triangle-free. Then

$$d_{G_0'}(v) \leq |V(G_0')| - 2 \leq l - 2, \text{for any } v \in V(G_0'). \tag{3.13}$$

(a) For each $v \in S_1$, by (3.12) and since $n \gg p$, there exists a vertex $u \in V(\Gamma(v))$ with $d_G(u) \geq \lfloor \frac{n}{p} \rfloor - 1$. Then by (3.13),

$$|V(\Gamma(v))| \geq |N_G(u) \cap V(\Gamma(v))| \geq d_G(u) - d_{G_0'}(v) \geq \left\lfloor \frac{n}{p} \right\rfloor - l + 1.$$

(b) By contradiction, suppose that $S_1 \neq S_0$. Let $v \in S_2 = S_0 \setminus S_1$. Thus, $d_G(v) = d_{G_0'}(v)$, and v is adjacent to a vertex $u \in D_2(G)$. By (3.12) and (3.13),

$$2(\lfloor n/p \rfloor - 1) \leq \overline{\sigma}_2(G) \leq d_G(v) + d_G(u) = d_{G_0'}(v) + 2 \leq l,$$

contrary to the fact that $n \gg p$, and so (b) is proved.

(c) By contradiction, suppose that there are two vertices $v_1, v_2 \in V_0$ such that $v_1v_2 \in E(G'_0)$. Since $v_i \in V_0$ $(i = 1, 2)$, $d_G(v_i) = d_{G'_0}(v_i)$. By (3.12) and (3.13),

$$2(\lfloor n/p \rfloor - 1) \le \overline{\sigma}_2(G) \le d_G(v_1) + d_G(v_2) \le 2l - 4,$$

contrary to the fact that $n \gg p$, and so (c) is proved.

(d) By contradiction, suppose that $s = |S_0| > p$. By (b) above, $S_1 = S_0$. Let $S_1 = \{v_1, v_2, \dots, v_s\}$. Then by (a),

$$s(\left\lfloor \frac{n}{p} \right\rfloor - l + 1) \le \left| \bigcup_{i=1}^{s} V(\Gamma(v_i)) \right| \le n,$$

a contradiction if $n \gg p$.

Now suppose that $|S_1| = p$ and $V(G'_0) \backslash S_1 \ne \emptyset$. Let $v \in V_0 = V(G'_0) \backslash S_1$. By (c), we can assume that $N_{G'_0}(v) = \{v_1, v_2, \dots, v_t\}$ and $N_G(v) = \{w_1, w_2, \dots, w_t\}$ such that $w_i \in \Gamma(v_i)$ $(1 \le i \le t)$. Note that $D_2(G'_0) \subseteq S_1$. Then $d_{G'_0}(v) \ge 3$ and so $t \ge 3$. By (3.13), $d_G(v) = d_{G'_0}(v) \le l - 2$. Then

$$d_G(w_i) \ge 2(\left\lfloor \frac{n}{p} \right\rfloor - 1) - d_G(v) \ge 2\left\lfloor \frac{n}{p} \right\rfloor - l. \tag{3.14}$$

Since G'_0 is 2-edge-connected and triangle-free, and since $t \ge 3$, $d_{G'_0}(v_i) \le l - 3$. Then by (3.14),

$$|V(\Gamma(v_i))| \ge |N_G(w_i) \cap V(\Gamma(v_i))| \ge d_G(w_i) - d_{G'_0}(v_i) \ge 2\left\lfloor \frac{n}{p} \right\rfloor - 2l + 3. \tag{3.15}$$

By (a) and (3.15),

$$|V_0| + t(2\left\lfloor \frac{n}{p} \right\rfloor - 2l + 3) + (p - t)(\left\lfloor \frac{n}{p} \right\rfloor - l + 1) \le \left| \bigcup_{u \in V(G'_0)} V(\Gamma(u)) \right| = n.$$

Then

$$|V_0| + t + (p + t)(\left\lfloor \frac{n}{p} \right\rfloor - l + 1) \le n,$$

a contradiction if $n \gg p$, and so (d) is proved. $\square$

3.4.2 Proof of Theorem 3.9

Proof of Theorem 3.9. This is the special case of Theorem 3.8 with $p = 8$ and $k = 2$. Suppose that $L(G)$ is not hamiltonian. Because $\overline{\sigma}_2(G) \geq 2(\lfloor n/8 \rfloor - 1)$, if $n \geq 32$, then $\overline{\sigma}_2(G) \geq 6$, and consequently $D(G)$ is an independent set. Let G_0' be the reduction of the core G_0 of G. By Theorem 1.5(c), G_0' is simple and triangle-free. Then by Theorems 3.8 and 3.12, G has no DCT and $G_0' \in \mathscr{G}_1 \cup \mathscr{G}_2$. Note that each of the graphs in the set $\{J(2,2), J(2,3), C(6,2), C(6,4), \theta(1,1,2), \theta(1,1,3), \theta(1,2,2), \theta(1,1,4), \theta(1,1,4)', \theta(1,1,4)'', \theta(2,2,2), \theta(1,2,3), \theta(1,2,3)'\}$ can be contracted to a $K_{2,3}$, each graph in $\{K_{2,5}^*, K_{2,5}^{**}\}$ can be contracted to a $K_{2,5}$, and W_3^{**} can be contracted to a W_3^*. We conclude that G_0' can be contracted to a graph in $\{K_{2,3}, K_{2,5}, W_3^*, C(6,2)', C(6,4)', \theta(1,1,4)'''\}$. This completes the proof. □

3.4.3 Proof of Theorem 3.10

Proof of Theorem 3.10. This is the special case of Theorem 3.8 with $p = 15$ and $k = 3$. Suppose that $L(G)$ is not hamiltonian. Because $\overline{\sigma}_2(G) \geq 2(\lfloor n/15 \rfloor - 1)$, if $n \geq 60$, then $\overline{\sigma}_2(G) \geq 6$, and consequently $D(G)$ is an independent set. Let G_0' be the reduction of the core G_0 of G. By Theorem 3.8, $G_0' \notin \mathscr{S}\mathscr{L}$ with $|V(G_0')| \leq 18$ and $\kappa'(G_0') \geq 3$.

By Proposition 3.1(b), $S_1 = S_0$ and so $V(G_0') = V_0 \cup S_1$. If $|V(G_0')| \leq 15$, then, by Theorem 3.13(b), $G_0' \in \{P(10), P(14)\}$. Obviously, in this case, G can be contracted to the Petersen graph. In the following, we only need to consider the case that $16 \leq |V(G_0')| \leq 18$.

Let $V_0 = \{v_1, v_2, \dots, v_t\}$ and $S_1 = \{v_{t+1}, v_{t+2}, \dots, v_{|V(G_0')|}\}$. Without loss of generality, we can assume that

$$|V(\Gamma(v_{t+1}))| \leq |V(\Gamma(v_{t+2}))| \leq \cdots \leq |V(\Gamma(v_{|V(G_0')|}))|.$$

Note that $\sum_{i=t+1}^{|V(G_0')|} |V(\Gamma(v_i))| \leq n$. Hence $|V(\Gamma(v_{t+1}))| \leq \frac{n}{|V(G_0')|-t}$.

By Proposition 3.1(d), $|S_1| \leq 15$. Then, since $|V(G_0')| \geq 16$, $t \geq 1$. By Proposition 3.1(c), V_0 is an independent set and $\bigcup_{i=1}^{t} N_{G_0'}(v_i) \subseteq S_1$. Since G_0' is

3-edge-connected,

$$d_{G'_0}(v_i) \geq 3, \text{for } v_i \in V(G'_0). \tag{3.16}$$

Let $\{w_1, w_2, \ldots, w_s\}$ be a maximal subset of $\bigcup_{i=1}^{t} N_G(v_i)$ which satisfies the following two conditions:

(i) for any pair of vertices $\{w_i, w_j\} \subset \{w_1, w_2, \ldots, w_s\}$ $(w_i \neq w_j)$, there exists a pair of vertices $\{z_i, z_j\} \subset \bigcup_{i=1}^{t} N_{G'_0}(v_i)$ $(z_i \neq z_j)$ such that $w_i \in \Gamma(z_i), w_j \in \Gamma(z_j)$, and $\Gamma(z_i) \cap \Gamma(z_j) = \emptyset$;

(ii) for each $w_i \in \{w_1, w_2, \ldots, w_s\}$, there is a vertex v_j $(j \leq t)$ that is adjacent to w_i in G.

Note that in this case $\left|\bigcup_{i=1}^{t} N_{G'_0}(v_i)\right| = s$. Then, since $t \geq 1$ and (3.16),

$$s \geq 3.$$

Claim 1. $d_{G'_0}(v_i) \leq 15$, for $v_i \in V(G'_0)$.

Proof. By Theorem 1.5(c), G'_0 is simple and triangle-free. Then, since $\kappa'(G'_0) \geq 3$ and $|V(G'_0)| \leq 18$, the claim holds immediately. □

By Claim 1,

$$d_G(w_i) \geq 2(\left\lfloor \frac{n}{15} \right\rfloor - 1) - d_G(v_j) \geq 2(\frac{n-14}{15} - 1) - 15 = \frac{2n-283}{15}, \tag{3.17}$$

where v_j $(j \leq t)$ is adjacent to w_i in G.

By (i), for each w_i, there is a vertex $z_i \in \bigcup_{i=1}^{t} N_{G'_0}(v_i)$ such that $w_i \in V(\Gamma(z_i))$. Hence by (3.17) and Claim 1,

$$\begin{aligned} |V(\Gamma(z_i))| \geq |N_G(w_i) \cap V(\Gamma(z_i))| &\geq d_G(w_i) - d_{G'_0}(z_i) \\ &\geq \frac{2n-283}{15} - 15 \\ &= \frac{2n-508}{15}. \end{aligned} \tag{3.18}$$

Hence,

$$\left|\bigcup_{i=1}^{s} V(\Gamma(z_i))\right| = \sum_{i=1}^{s} |V(\Gamma(z_i))| \geq \frac{s(2n-508)}{15}.$$

Since $\left|\bigcup_{i=1}^{s} V(\Gamma(z_i))\right| \leq n$ and n is sufficiently large,

$$s \leq 7.$$

Therefore, $3 \leq s \leq 7$.

For $x \in S_1$, by Proposition 3.1(a),

$$|V(\Gamma(x))| \geq \left\lfloor \frac{n}{15} \right\rfloor - 18 + 1 \geq \frac{n-14}{15} - 17 \geq \frac{n-269}{15}. \tag{3.19}$$

In particular, if $x \in \bigcup_{i=1}^{t} N_{G_0'}(v_i)$, then $|V(\Gamma(x))| = |V(\Gamma(z_i))|$ for some $z_i \in \bigcup_{i=1}^{t} N_{G_0'}(v_i)$. By (3.18), $|V(\Gamma(x))| \geq \frac{2n-508}{15}$.

Without loss of generality, we let $V(G_0') = \{v_1, \ldots, v_t, v_{t+1}, \ldots, v_{t+s}, v_{t+s+1}, \ldots, v_{t+s+r}\}$, where

$$\bigcup_{i=1}^{t} N_{G_0'}(v_i) = \{v_{t+1}, \ldots, v_{t+s}\},$$

$S_1 = \{v_{t+1}, \ldots, v_{t+s}, v_{t+s+1}, \ldots, v_{t+s+r}\}$ and $t+s+r = |V(G_0')|$.

By (3.18) and (3.19),

$$t + \frac{s(2n-508)}{15} + \frac{r(n-269)}{15} \leq \left|\bigcup_{v_i \in V(G_0')} V(\Gamma(v_i))\right| = n.$$

Since n is sufficiently large,

$$2s + r \leq 15. \tag{3.20}$$

Since $s \geq 3$, and by (3.20), $s + r \leq 12$. Then, since $|V(G_0')| \geq 16$, $t \geq 4$. Let $G^* = G_0'[\{v_1, \ldots, v_t, v_{t+1}, \ldots, v_{t+s}\}]$. By (3.16), $d_{G^*}(v_i) \geq 3$, for $i \leq t$. By Theorem 1.5(c), and since $t \geq 4$, $G^* \notin \{K_1, K_2, K_{2,l}\ (l \geq 2)\}$. Then $3t \leq$

$|E(G^*)| \leq 2(t+s)-5$. So,

$$t \leq 2s-5. \tag{3.21}$$

Using $t \geq 4$, (3.20) and (3.21), we obtain $5 \leq s \leq 7$.

If $s=5$ ($s=6$ or $s=7$), then, by (3.20), $r \leq 5$ ($r \leq 3$ or $r \leq 1$, respectively). Note that $\bigcup_{i=1}^{t} N_{G_0'}(v_i) = \{v_{t+1}, \ldots, v_{t+s}\}$ and V_0 is an independent set. Then $\alpha'(G_0') \leq 7$. Using Theorem 3.13(c), we conclude that G_0' has an SCT, a contradiction. This completes the proof. □

3.4.4 Proof of Theorem 3.11

Proof of Theorem 3.11. Since G is a 3-edge-connected graph, G is also an essentially 3-edge-connected graph. If G is supereulerian, then we are done. In the following, we assume that $G \notin \mathscr{SL}$. Since G is 3-edge-connected, $D(G)=\emptyset$. Let G_0' be the reduction of the core G_0 of G. Then $G=G_0$ and so $G_0 \notin \mathscr{SL}$. By Theorem 3.16, $G_0' \notin \mathscr{SL}$, and $|V(G_0')| \leq 18$ and $\kappa'(G_0') \geq 3$. Then, similarly as in the proof of Theorem 3.10, we conclude that G can be contracted to the Petersen graph. This completes the proof. □

Chapter 4

Neighborhood and degree conditions for hamiltonicity

In this chapter, we are mainly interested in degree and neighborhood conditions for hamiltonicity of 2-connected claw-free graphs, motivated by recent results in [35], and in an attempt to unify and extend several existing results.

4.1 Introduction

We start by listing a number of existing hamiltonicity results for claw-free graphs, the first one involving a Dirac-type condition.

In [66], Matthews and Sumner proved the following.

Theorem 4.1. (Matthews and Sumner [66]). *If H is a 2-connected claw-free graph of order n with $\delta(H) \geq \frac{n-2}{3}$, then H is hamiltonian.*

The Ore-type counterpart of Theorem 4.1 is the following result due to Flandrin et al. [50].

Theorem 4.2. (Flandrin et al. [50]). *If H is a 2-connected claw-free graph of order n with $\sigma_2(H) \geq \frac{2n-5}{3}$, then H is hamiltonian.*

With $U_2(H)$ instead of $\sigma_2(H)$, Bauer et al. proved the following more general result in [1].

Theorem 4.3. (Bauer, Fan and Veldman [1]). *If H is a 2-connected claw-free graph of order n with $U_2(H) \geq \frac{2n-5}{3}$, then H is hamiltonian.*

Another generalization of Theorem 4.1 involving $\sigma_3(H)$, was obtained independently by Liu et al. [65], Zhang [85], and Broersma [10].

Theorem 4.4. (Liu et al. [65], Zhang [85] and Broersma [10]). *If H is a 2-connected claw-free graph of order n with $\sigma_3(H) \geq n-2$, then H is hamiltonian.*

In an attempt to further generalize the above results to a condition involving $\sigma_4(H)$, Frydrych proved the following result in [52].

Theorem 4.5. (Frydrych [52]). *If H is a 2-connected claw-free graph of order n with $\sigma_4(H) \geq n+3$, then either H is hamiltonian or $cl(H) \in \mathcal{Q}_{2,3}(s_1, s_2, s_3, 0)$.*

The final result in the above list indicates that one has to exclude certain graph classes if one is trying to extend and generalize these results, a phenomenon that we have encountered before in the earlier chapters.

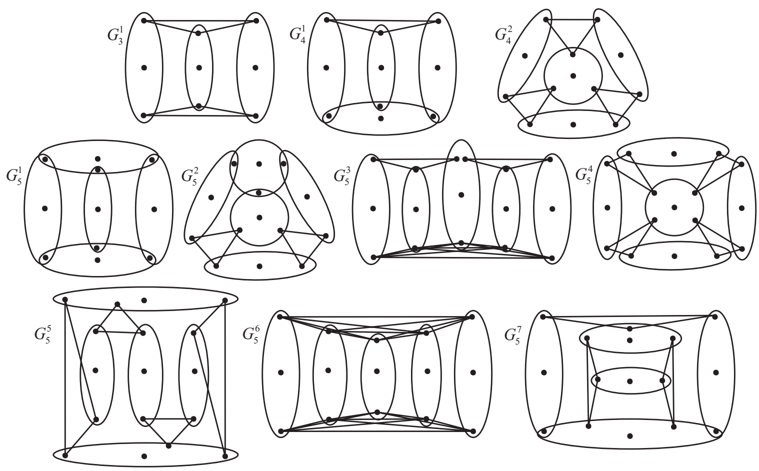

FIGURE 4.1: Ten classes of nonhamiltonian graphs.

The graphs G_3^1, G_4^1, G_4^2, and $G_5^1, G_5^2, \ldots, G_5^7$ are shown in Figure 4.1 (where the circular and elliptical parts represent cliques of arbitrary positive order, but at least the number of black dots indicated in these parts). Let $\widetilde{\mathcal{G}}_3$, $\widetilde{\mathcal{G}}_4$, and

$\widetilde{\mathscr{G}}_5$ be the sets of all spanning subgraphs of G_3^1, G_4^1 and G_4^2, and $G_5^1, G_5^2, \ldots, G_5^7$, respectively.

In [63], Li et al. improved Theorem 4.1 by obtaining the following result.

Theorem 4.6. (Li et al. [63]). *If H is a 2-connected claw-free graph of order n with $\delta(H) \geq \frac{n+5}{5}$, then either H is hamiltonian or $H \in \widetilde{\mathscr{G}}_3 \cup \widetilde{\mathscr{G}}_4$.*

In [49], Favaron et al. got the following two closely related results.

Theorem 4.7. (Favaron et al. [49]). *If H is a 2-connected claw-free graph of order $n \geq 77$ such that $\delta(H) \geq 14$ and $\sigma_6(H) > n + 19$, then either H is hamiltonian or $H \in \widetilde{\mathscr{G}}_3 \cup \widetilde{\mathscr{G}}_4 \cup \widetilde{\mathscr{G}}_5$.*

Theorem 4.8. (Favaron et al. [49]). *If H is a 2-connected claw-free graph of connectivity $\kappa(H) = 2$ and order $n \geq 78$ with $\delta(H) > \frac{n+16}{6}$, then either H is hamiltonian or $H \in \widetilde{\mathscr{G}}_3 \cup \widetilde{\mathscr{G}}_4 \cup \widetilde{\mathscr{G}}_5$.*

Degree conditions for hamiltonicity in claw-free graphs were studied further in [58], where the authors gave a general algorithm that allows one to generate all classes of exceptions, roughly speaking, for a degree condition of the form $\sigma_p(H) \geq n + c(p)$ (or, as a corollary, $\delta(H) \geq \frac{n+c(p)}{p}$), for arbitrary positive integer p and a constant $c(p)$ only depending on p. In [58], with the help of a computer, the computation was performed for $p = 8$, and Kovářík et al. obtained a result for $\sigma_8(H) > n + 39$ with an exceptional family of graphs consisting of 318 infinite classes.

For the formulation of the next results, we also need a lower bound on the order of the graphs we consider, depending on the specific degree or neighborhood union condition we apply, as follows.

For $d_t(H) \in \{\sigma_t(H), U_t(H)\}$, we consider claw-free graphs H that satisfy the following condition:

$$d_t(H) \geq \frac{t(n+\epsilon)}{p}. \tag{4.1}$$

Here $t \geq 1$ and $p \geq t$ are positive integers, and ϵ is a given real number. Depending on the values of p and ϵ, we define $N(p,\epsilon) = \max\{36p^2 - 34p - \epsilon(p+1), 20p^2 - 10p - \epsilon(p+1), (3p+1)(-\epsilon - 4p)\}$. In the following result, we let H be a k-connected claw-free graph of order $n > N(p,\epsilon)$ with $k \in \{2,3\}$.

In [35], Chen proved the following hamiltonicity result.

Theorem 4.9. (Chen [35]). *If $\delta(H) \geq 3$ and $d_t(H) \geq \frac{t(n+\epsilon)}{p}$, then either H is hamiltonian or $cl(H) = L(G)$, where G is an essentially k-edge-connected triangle-free graph without a DCT, and G satisfies one of the following:*

(a) $k = 2$ *and G is contractible to a graph in* $\mathcal{Q}_0(c, 2)$*, where* $c \leq max\{4p - 5, 2p + 1\}$*;*

(b) $k = 3$ *and G is contractible to a graph in* $\mathcal{Q}_0(c, 3)$*, where* $c \leq max\{3p - 5, 2p + 1\}$*.*

Here "G is contractible to a graph in $\mathcal{Q}_0(c, k)$" means that "the reduction of the core of G is in $\mathcal{Q}_0(c, k)$".

As a special case of Theorem 4.9 with given values of p and ϵ, Chen obtained the following result in [35].

Theorem 4.10. (Chen [35]). *Let H be a 2-connected claw-free graph of sufficiently large order n with $\delta(H) \geq 3$. If $d_t(H) \geq \frac{tn}{4}$ for $t \in \{1, 2, 3, 4\}$, then either H is hamiltonian or $cl(H) \in \mathcal{Q}_{2,3}(s_1, s_2, s_3, 0)$.*

The next section contains analogous results we obtained in our attempts to further generalize some of the above results.

4.2 Our results

For a W_3^*, let $D_2(W_3^*) = \{v_1, v_2, v_3\}$ and $D_3(W_3^*) = \{u_1, u_2, u_3, u_4\}$, where the vertices of W_3^* are labeled as in Figure 3.1. Let $\mathcal{W}_3^*(s_1, s_2, s_3, r)$ be the family of essentially 2-edge-connected graphs of size n (Recall that size is used for the number of edges) obtained from a W_3^* by replacing each $v_i \in D_2(W_3^*)$ by a connected triangle-free subgraph of size $s_i \geq 1$ and replacing u_4 by a connected triangle-free subgraph of size $r \geq 0$ such that $\sum_{i=1}^{3} s_i + r + 9 = n$. Note that each graph in $\mathcal{W}_3^*(s_1, s_2, s_3, r)$ is contractible to a W_3^*.

Let $\mathcal{Q}_3(s_1, s_2, s_3, r) = \{H = L(G) \mid G \in \mathcal{W}_3^*(s_1, s_2, s_3, r)\}$.
As another application of Theorem 4.9, we obtain the following result.

Theorem 4.11. *Let H be a 2-connected claw-free graph of sufficiently large order n with $\delta(H) \geq 3$. If $d_t(H) \geq \frac{t(n+5)}{5}$ with $t \in \{1,2,3,4,5\}$, then either H is hamiltonian or $cl(H) \in \mathcal{Q}_{2,3}(s_1,s_2,s_3,r) \cup \mathcal{Q}_3(s_1,s_2,s_3,r)$.*

We postpone the proof of the above result, and continue with our next application of Theorem 4.9. In the following description, we need the graphs T_1, T_2, T_3, T_4, and T_5 that are depicted in Figure 4.2. In order to present our next result, we define six families of nonhamiltonian claw-free graphs as follows.

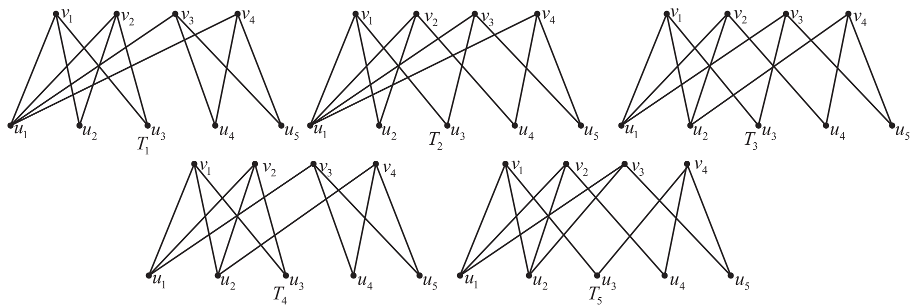

FIGURE 4.2: The graphs T_1, T_2, T_3, T_4 and T_5.

For a $K_{2,3}$, let $D_2(K_{2,3}) = \{v_1,v_2,v_3\}$ and $D_3(K_{2,3}) = \{u_1,u_2\}$. Let $\mathscr{C}_1$ be the family of essentially 2-edge-connected graphs of size n obtained from a $K_{2,3}$ by replacing each $v_i \in D_2(K_{2,3})$ by a connected triangle-free subgraph of size $s_i \geq 1$ and replacing each $u_i \in D_3(K_{2,3})$ by a connected triangle-free subgraph of size $r_i \geq 0$, respectively, such that $\sum_{i=1}^{3} s_i + \sum_{i=1}^{2} r_i + 6 = n$. Note that each graph in $\mathscr{C}_1$ is contractible to a $K_{2,3}$.

For a $K_{2,5}$, let $D_2(K_{2,5}) = \{v_1,v_2,v_3,v_4,v_5\}$. Let $\mathscr{C}_2$ be the family of essentially 2-edge-connected graphs of size n obtained from a $K_{2,5}$ by replacing each $v_i \in D_2(K_{2,5})$ by a connected triangle-free subgraph of size $s_i \geq 1$ such that $\sum_{i=1}^{5} s_i + 10 = n$. Note that each graph in $\mathscr{C}_2$ is contractible to a $K_{2,5}$.

For a W_3^*, let $D_2(W_3^*) = \{v_1,v_2,v_3\}$ and $D_3(W_3^*) = \{u_1,u_2,u_3,u_4\}$. Let $\mathscr{C}_3$ be the family of essentially 2-edge-connected graphs of size n obtained from a W_3^* by replacing each $v_i \in D_2(W_3^*)$ by a connected triangle-free subgraph of size $s_i \geq 1$ and replacing each $u_i \in D_3(W_3^*)$ by a connected triangle-free subgraph of size $r_i \geq 0$, respectively, such that $\sum_{i=1}^{3} s_i + \sum_{i=1}^{4} r_i + 9 = n$. Note that each graph in $\mathscr{C}_3$ is contractible to a W_3^*.

For a $C(6,2)'$, let $D_2(C(6,2)') = \{v_1, v_2, v_3, v_4\}$. Let $\mathscr{C}_4$ be the family of essentially 2-edge-connected graphs of size n obtained from a $C(6,2)'$ by replacing each $v_i \in D_2(C(6,2)')$ by a connected triangle-free subgraph of size $s_i \geq 1$ and replacing w by a connected triangle-free subgraph of size $r \geq 1$, respectively, such that $\sum_{i=1}^{4} s_i + r + 11 = n$. Note that each graph in $\mathscr{C}_4$ is contractible to a $C(6,2)'$.

For a $\theta(1,1,4)'''$, let $D_2(\theta(1,1,4)''') = \{v_1, v_2, v_3\}$. Let $\mathscr{C}_5$ be the family of essentially 2-edge-connected graphs of size n obtained from a $\theta(1,1,4)'''$ by replacing each $v_i \in D_2(\theta(1,1,4)''')$ by a connected triangle-free subgraph of size $s_i \geq 1$ and replacing each w_i $(i = 1,2)$ by a connected triangle-free subgraph of size $r_i \geq 1$, respectively, such that $\sum_{i=1}^{3} s_i + \sum_{i=1}^{2} r_i + 11 = n$. Note that each graph in $\mathscr{C}_5$ is contractible to a $\theta(1,1,4)'''$.

For a T_2 or T_3, let $\mathscr{C}_6$ be the family of essentially 2-edge-connected graphs of size n obtained from a T_2 or T_3 by replacing each u_i by a connected triangle-free subgraph of size $s_i \geq 1$ such that $\sum_{i=1}^{5} s_i + 12 = n$. Note that each graph in $\mathscr{C}_6$ is contractible to a T_2 or T_3.

Let $\mathscr{L}_i = \{H = L(G) \mid G \in \mathscr{C}_i\}$ $(i = 1, 2, \ldots, 6)$. Since $\mathscr{K}_{2,3}(s_1, s_2, s_3, r) \subset \mathscr{C}_1$ and $\mathscr{W}_3^*(s_1, s_2, s_3, r) \subset \mathscr{C}_3$, $\mathscr{Q}_{2,3}(s_1, s_2, s_3, r) \subset \mathscr{L}_1$ and $\mathscr{Q}_3(s_1, s_2, s_3, r) \subset \mathscr{L}_3$.

As another application of Theorem 4.9, we obtain the following result. It shows that, by increasing the (constant) lower bound on $\delta(H)$, in the following sense the degree condition on pairs can be relaxed to a degree condition on larger sets of (independent) vertices.

Theorem 4.12. *Let H be a 2-connected claw-free graph of sufficiently large order n with $\delta(H) \geq 18$. If $d_t(H) \geq \frac{tn}{6}$ with $t \in \{1, 2, \ldots, 6\}$, then either H is hamiltonian or $cl(H) \in \cup_{i=1}^{6} \mathscr{L}_i$.*

The following two results can be deduced from Theorems 4.11 and 4.12 immediately, respectively.

Theorem 4.13. *Let H be a 2-connected claw-free graph of sufficiently large order n with $\delta(H) \geq 3$. If $d_t(H) \geq \frac{t(n+5)}{5}$ with $t \in \{1, 2, 3, 4, 5\}$, then either*

H is hamiltonian or $cl(H) = L(G)$, where G is an essentially 2-edge-connected triangle-free graph that can be contracted to a $K_{2,3}$ or W_3^*.

Theorem 4.14. *Let H be a 2-connected claw-free graph of sufficiently large order n with $\delta(H) \geq 18$. If $d_t(H) \geq \frac{tn}{6}$ with $t \in \{1, 2, \ldots, 6\}$, then either H is hamiltonian or $cl(H) = L(G)$, where G is an essentially 2-edge-connected triangle-free graph that can be contracted to a graph in $\{K_{2,3}, K_{2,5}, W_3^*, C(6,2)', \theta(1,1,4)''', T_2, T_3\}$.*

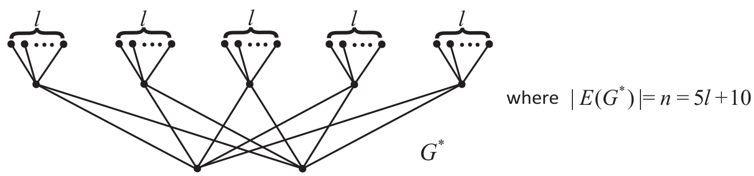

FIGURE 4.3: The graph G^* with $d_t(L(G^*)) = \frac{t(n-5)}{5}$.

Remark 4.1. (a) Let G^* be the graph obtained from a $K_{2,5}$ by adding $l \geq 2$ pendant edges at each vertex of degree two in $K_{2,5}$, that is depicted in Figure 4.3. Similarly, let G^{**} be the graph that is depicted in Figure 4.4. Since G^* and G^{**} have no DCT, by Theorem 1.1, $L(G^*)$ and $L(G^{**})$ are non-hamiltonian. The line graph $L(G^*)$ of order $n = 5l + 10$ ($n \geq 20$) is 2-connected with $d_t(L(G^*)) = t(l+1) = \frac{t(n-5)}{5} < \frac{t(n+5)}{5}$, $\delta(L(G^*)) \geq 3$ and $L(G^*) \notin \mathcal{Q}_{2,3}(s_1, s_2, s_3, r) \cup \mathcal{Q}_3(s_1, s_2, s_3, r)$. The line graph $L(G^{**})$ of order $n = 6l + 13$ ($n \geq 25$) is 2-connected with $\frac{t(n-7)}{6} \leq d_t(L(G^{**})) < \frac{t(n-1)}{6}$, $\delta(L(G^{**})) \geq 3$ and $L(G^{**}) \notin \cup_{i=1}^{6} \mathcal{L}_i$. These examples show that the bounds in Theorems 4.11 and 4.12 are asymptotically sharp, respectively.

(b) The case with $d_t(H) = \sigma_6(H) \geq n$ of Theorem 4.12 is an improvement of the aforementioned "$\sigma_6(H) > n + 19$" theorem due to Favaron et al. in [49]; the case with $d_t(H) = \sigma_4(H) \geq \frac{4n+20}{5}$ ($d_t(H) = \sigma_4(H) \geq \frac{2n}{3}$) of Theorem 4.11 (Theorem 4.12) is an improvement of the aforementioned "$\sigma_4(H) \geq n+3$" theorem obtained by Frydrych in [52]; the case with $d_t(H) = \sigma_3(H) \geq \frac{3n+15}{5}$ ($d_t(H) = \sigma_3(H) \geq \frac{n}{2}$) of Theorem 4.11 (Theorem 4.12) is an improvement of the aforementioned "$\sigma_3(H) \geq n - 2$" theorem obtained by Liu et al. in [65], Zhang in [85], and Broersma in [10]; the case with

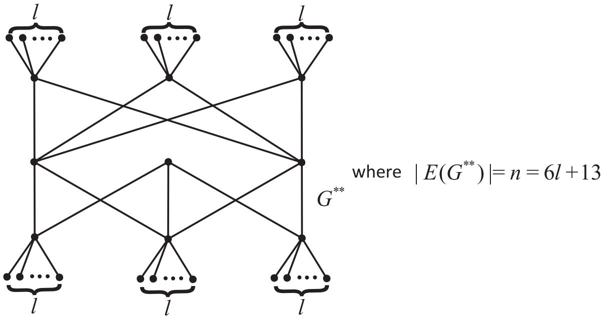

FIGURE 4.4: The graph G^{**} with $\frac{t(n-7)}{6} \leq d_t(L(G^{**})) < \frac{t(n-1)}{6}$.

$d_t(H) = \sigma_2(H) \geq \frac{2n+10}{5}$ ($d_t(H) = \sigma_2(H) \geq \frac{n}{3}$) of Theorem 4.11 (Theorem 4.12) is an improvement of the aforementioned "$\sigma_2(H) \geq \frac{2n-5}{3}$" theorem due to Flandrin et al. in [50]; the case with $d_t(H) = \sigma_1(H) = \delta(H) \geq \frac{n+5}{5}$ ($d_t(H) = \sigma_1(H) = \delta(H) \geq \frac{n}{6}$) of Theorem 4.11 (Theorem 4.12) is an improvement of Theorem 4.6 (Theorems 4.6 and 4.8); the case with $d_t(H)$ for $1 \leq t \leq 5$ ($1 \leq t \leq 6$) of Theorem 4.11 (Theorem 4.12) is an improvement of Theorems 4.3 and 4.10; furthermore, Theorem 4.12 is an improvement of Theorem 4.11, and Theorem 4.14 is an improvement of Theorem 4.13.

(c) Our results also extend earlier results that are based on the notion of the generalized t-degree, $\delta_t(H)$ (the definition was given in Chapter 1), as introduced by Faudree et al. in [48]. Since obviously $\sigma_t(H) \geq U_t(H) \geq \delta_t(H)$, the statements in Theorems 4.11, 4.12, 4.13 and 4.14 are also valid if we replace $d_t(H)$ by $\delta_t(H)$.

The remainder of this chapter is organized as follows. In Section 4.3, we present some useful results. In Section 4.4, we present two technical lemmas. In Section 4.5, our proofs of Theorems 4.11 and 4.12 are given.

4.3 Preliminaries and auxiliary results

Some known facts that we need on reduced graphs are summarized in the following theorem.

Theorem 4.15. *Let G be a connected reduced graph of order n. Then each of the following holds:*

(a) [26] *For $1 < n \leq 9$, if $\kappa'(G) \geq 2$, then $|D_2(G)| \geq 3$.*

(b) [33] *Let M be a maximum matching of G, and let $D_2(G) = l$. If $\delta(G) \geq 2$ and $G \neq K_{2,a}$ $(a \geq 2)$, then $|M| \geq \ min\{\frac{n-1}{2}, \frac{n-l+5}{3}\}$.*

The following lemma will be needed for our proof of the auxiliary result in this section.

Lemma 4.16. (Chen and Chen [26]) *Let G be a 2-edge-connected graph of order n. If $n \leq 10$ and $|D_2(G)| \leq 1$, then either G is collapsible or $G \cong P(10)$.*

The following three lemmas will be needed for our proof of Theorem 4.12 in Section 4.5.

Lemma 4.17. *Let G be a 2-edge-connected reduced graph of order 8. Let $V(G) = V_1 \cup V_2$, where $V_1 = \{v_1, v_2, v_3\}$, $V_2 = \{u_1, u_2, u_3, u_4, u_5\}$, V_1 is an independent set, $\bigcup_{i=1}^{3} N_G(v_i) \subseteq V_2$, and $d_G(v_i) \geq 3$ $(1 \leq i \leq 3)$. Then, either G has a DCT containing V_2 or $G \in \{C(6,2), C(6,2)', \theta(1,1,4)''', \theta(1,2,3)', W_3^{**}\}$.*

Proof of Lemma 4.17. If G has a DCT containing V_2, then we are done. In the following, we assume that G has no DCT containing V_2. Then G has no SCT. Since G is reduced, by Theorem 1.5(c), G is simple and triangle-free. Then by Theorem 3.12, and since $|V(G)| = 8$, $G \in \mathcal{G}_2$. By the assumption of this lemma, $D_2(G) \subseteq V_2$. Then, since $|V_2| = 5$, $|V_1| = 3$ and V_1 is an independent set, we conclude that $G \in \{C(6,2), C(6,2)', \theta(1,1,4)''', \theta(1,2,3)', W_3^{**}\}$. This completes the proof. $\square$

Lemma 4.18. *Let G be a 2-edge-connected reduced graph of order 9. Let $V(G) = V_1 \cup V_2$, where $V_1 = \{v_1, v_2, v_3, v_4\}$, $V_2 = \{u_1, u_2, u_3, u_4, u_5\}$, V_1 is an independent set, $\bigcup_{i=1}^{4} N_G(v_i) \subseteq V_2$, and $d_G(v_i) \geq 3$ $(1 \leq i \leq 4)$. Then, either G has a DCT containing V_2 or $G \in \{T_1, T_2, T_3, T_4, T_5\}$.*

Proof of Lemma 4.18. If G has a DCT containing V_2, then we are done. In the following, we assume that G has no DCT containing V_2. Since G is reduced, by Theorem 1.5(c) and Lemma 1.6, G is simple, triangle-free, $K_{3,3}$-free and $(K_{3,3}-e)$-free. Since V_1 is an independent set, $\bigcup_{i=1}^{4} N_G(v_i) \subseteq V_2$ and $d_G(v_i) \geq 3$ $(1 \leq i \leq 4)$, $|E(G)| \geq 4 \times 3 = 12$. Since $D_2(G) \subseteq V_2$, and since $|V_2| = 5$, $G \neq K_{2,7}$. Since G is reduced and $G \neq K_{2,7}$, by Theorem 1.5(c), $|E(G)| \leq 2|V(G)| - 5 = 13$. So, $12 \leq |E(G)| \leq 13$. Obviously, for any graph,

$$\sum_{v \in V(G)} d_G(v) = 2|E(G)|. \tag{4.2}$$

We first prove the following claim and then distinguish the cases that $|E(G)| = 12$ and $|E(G)| = 13$.

Claim 1. $|D_2(G)| \geq 3$.

Proof. Since $|V(G)| = 9$ and $\kappa'(G) \geq 2$, by Theorem 4.15(a), $|D_2(G)| \geq 3$. □

Case 1. $|E(G)| = 12$.

Then G is a bipartite graph and $d_G(v_i) = 3$ $(1 \leq i \leq 4)$. By Claim 1, without loss of generality, we may assume that $d_G(u_1) \geq d_G(u_2) \geq 2$ and $d_G(u_i) = 2$ $(i = 3,4,5)$. Then, using (4.2), either $d_G(u_1) = 4$ and $d_G(u_2) = 2$ or $d_G(u_1) = d_G(u_2) = 3$.

Suppose first that $d_G(u_1) = 4$ and $d_G(u_2) = 2$. Since $d_G(u_1) = 4$, $N_G(u_1) = V_1$. Since $d_G(u_2) = 2$, without loss of generality, we may assume that $N_G(u_2) = \{v_1, v_2\}$. Then, using $d_G(v_1) = d_G(v_2) = 3$, we get $|N_G(v_1) \cap \{u_3, u_4, u_5\}| = |N_G(v_2) \cap \{u_3, u_4, u_5\}| = 1$. Without loss of generality, we may assume that either $N_G(u_3) = \{v_1, v_2\}$ or $N_G(u_3) = \{v_1, v_3\}$. If $N_G(u_3) = \{v_1, v_2\}$, then $N_G(u_4) = N_G(u_5) = \{v_3, v_4\}$, and $G \cong T_1$. If $N_G(u_3) = \{v_1, v_3\}$, then, since G is a reduced graph, without loss of generality, we may assume that $N_G(u_4) = \{v_2, v_4\}$. Then $N_G(u_5) = \{v_3, v_4\}$, and $G \cong T_2$.

Now suppose that $d_G(u_1) = d_G(u_2) = 3$. Then $2 \leq |N_G(u_1) \cap N_G(u_2) \cap V_1| \leq 3$. Without loss of generality, we may assume that either $N_G(u_1) = \{v_1, v_2, v_3\}$ and $N_G(u_2) = \{v_1, v_2, v_4\}$, or $N_G(u_1) = N_G(u_2) = \{v_1, v_2, v_3\}$. We distinguish these two subcases.

Subcase 1.1. $N_G(u_1) = \{v_1, v_2, v_3\}$ and $N_G(u_2) = \{v_1, v_2, v_4\}$.

Then, since $d_G(v_1) = d_G(v_2) = 3$, there exists a vertex $u_i \in \{u_3, u_4, u_5\}$ such that $1 \le |N_G(u_i) \cap \{v_1, v_2\}| \le 2$.

If $|N_G(u_i) \cap \{v_1, v_2\}| = 1$, then, without loss of generality, we may assume that $N_G(u_3) = \{v_1, v_3\}$. Note that G is a 2-edge-connected reduced bipartite graph. Then, without loss of generality, we may assume that $N_G(u_4) = \{v_2, v_4\}$ and $N_G(u_5) = \{v_3, v_4\}$, so that $G \cong T_3$.

If $|N_G(u_i) \cap \{v_1, v_2\}| = 2$, then, without loss of generality, we may assume that $N_G(u_3) = \{v_1, v_2\}$. Note that again G is a 2-edge-connected reduced bipartite graph. Then $N_G(u_4) = N_G(u_5) = \{v_3, v_4\}$, and $G \cong T_4$.

Subcase 1.2. $N_G(u_1) = N_G(u_2) = \{v_1, v_2, v_3\}$.
Since G is a 2-edge-connected reduced bipartite graph, $|N_G(u_i) \cap \{v_1, v_2, v_3\}| = 1$ $(i = 3, 4, 5)$; otherwise, G has a subgraph isomorphic to $K_{3,3} - e$, a contradiction. Without loss of generality, we may assume that $N_G(u_3) = \{v_1, v_4\}$, $N_G(u_4) = \{v_2, v_4\}$ and $N_G(u_5) = \{v_3, v_4\}$. Then $G \cong T_5$. It is easy to check that T_1, T_2, T_3, T_4 and T_5 have no DCT containing V_2.

Case 2. $|E(G)| = 13$.
Since V_1 is an independent set, and using $|E(G)| = 13$, either $d_G(v_i) = 3$ $(1 \le i \le 4)$, or there exists exactly one vertex $v \in V_1$ with $d_G(v) = 4$. We distinguish these two subcases.

Subcase 2.1. $d_G(v_i) = 3$ $(1 \le i \le 4)$.
Then $|E(G[V_2])| = 1$. By Claim 1, without loss of generality, we assume that $d_G(u_1) \ge d_G(u_2) \ge 2$ and $d_G(u_j) = 2$ $(j = 3, 4, 5)$. Then, using $|E(G[V_2])| = 1$ and (4.2), $d_G(u_1) = d_G(u_2) = 4$. Then $u_1u_2 \notin E(G)$; otherwise, using $d_G(u_1) = d_G(u_2) = 4$, $|E(G[V_2])| = 1$, and $|V_1| = 4$, we obtain that G has a triangle, a contradiction.

Suppose there exists a vertex $u \in \{u_1, u_2\}$ such that $N_G(u) \cap \{u_3, u_4, u_5\} \neq \emptyset$. Using $d_G(u_1) = d_G(u_2) = 4$, without loss of generality, we may assume that $N_G(u_1) = \{v_1, v_2, v_3, u_3\}$ and $N_G(u_2) = V_1$. Since G is triangle-free, and since $d_G(u_3) = 2$ and $|E(G[V_2])| = 1$, $N_G(u_3) = \{u_1, v_4\}$. Since $d_G(v_4) = 3$, there exists some vertex $w \in \{u_4, u_5\}$ such that $|N_G(w) \cap \{v_1, v_2, v_3\}| = 2$. Then $G[\{v_1, v_2, v_3, u_1, u_2, w\}] \cong K_{3,3} - e$, a contradiction. Hence, $N_G(u) \cap V_2 = \emptyset$ $(u \in \{u_1, u_2\})$ and so $N_G(u_1) = N_G(u_2) = V_1$. Using $|E(G[V_2])| = 1$ and since $d_G(u_j) = 2$ ($j = 3,4,5$), there exists exactly one vertex $w \in \{u_3, u_4, u_5\}$ such that $|N_G(w) \cap V_1| = 2$. Without loss of generality, we may assume that $N_G(u_3) = \{v_1, v_2\}$. Then $G[\{v_1, v_2, v_3, u_1, u_2, u_3\}] \cong K_{3,3} - e$, a contradiction.

Subcase 2.2. There exists exactly one vertex $v \in V_1$ with $d_G(v) = 4$.

Then G is a 2-edge-connected reduced bipartite graph. Without loss of generality, we may assume that $d_G(v_1) = 4$. By Claim 1, without loss of generality, we assume that $d_G(u_1) \geq d_G(u_2) \geq 2$ and $d_G(u_i) = 2$ ($i = 3,4,5$). Using $|E(G)| = 13$ and (4.2), we deduce that $d_G(u_1) = 4$ and $d_G(u_2) = 3$. Then $N_G(u_1) = V_1$. Since $d_G(v_1) = 4$ and $d_G(u_2) = 3$, without loss of generality, we may assume that either $N_G(u_2) = \{v_1, v_2, v_3\}$ or $N_G(u_2) = \{v_2, v_3, v_4\}$.

Suppose first that $N_G(u_2) = \{v_1, v_2, v_3\}$. Since $d_G(v_1) = 4$, without loss of generality, we may assume that $N_G(v_1) = V_2 \setminus \{u_5\}$. Then $v_2, v_3 \notin N_G(u_i)$ ($i = 3,4$); otherwise, G has a subgraph isomorphic to $K_{3,3} - e$, a contradiction. Now $N_G(u_3) = N_G(u_4) = \{v_1, v_4\}$, and so $N_G(u_5) = \{v_2, v_3\}$. Hence $G[\{v_1, v_2, v_3, u_1, u_2, u_5\}] \cong K_{3,3} - e$, a contradiction.

Next suppose that $N_G(u_2) = \{v_2, v_3, v_4\}$. Then, since $d_G(v_1) = 4$, $N_G(v_1) = V_2 \setminus \{u_2\}$. Since $d_G(u_i) = 2$ ($i = 3,4,5$), by symmetry and without loss of generality, we may assume that $v_2u_3, v_3u_4, v_4u_5 \in E(G)$. Then $v_1u_1v_2u_3v_1u_4v_3u_2$ $v_4u_5v_1$ is an SCT of G, a contradiction. This completes the proof. □

Lemma 4.19. *Let G be a 2-edge-connected reduced graph of order 10. Let $V(G) = V_1 \cup V_2$, where $V_1 = \{v_1, v_2, v_3, v_4, v_5\}$, $V_2 = \{u_1, u_2, u_3, u_4, u_5\}$, V_1 is an independent set, $\bigcup_{i=1}^{5} N_G(v_i) \subseteq V_2$ and $d_G(v_i) \geq 3$ ($1 \leq i \leq 5$). Then G has an SCT.*

Proof of Lemma 4.19. If G has an SCT, then we are done. In the following, we assume that G has no SCT. Since V_1 is an independent set, and since $\bigcup_{i=1}^{5} N_G(v_i) \subseteq V_2$ and $d_G(v_i) \geq 3$ ($1 \leq i \leq 5$), $|E(G)| \geq 5 \times 3 = 15$. Since $D_2(G) \subseteq V_2$ and since $|V_2| = 5$, $G \neq K_{2,8}$. Then by Theorem 1.5(c), $|E(G)| \leq 2|V(G)| - 5 = 15$. Hence $|E(G)| = 15$, and the equalities hold only if $d_G(v_i) = 3$ ($1 \leq i \leq 5$). So G is a 2-edge-connected reduced bipartite graph. By Lemma 1.6, G is $K_{3,3}$-free and $(K_{3,3} - e)$-free. Since G is a reduced bipartite graph, and using $|E(G)| = 15$ and (4.2), we deduce that $|D_2(G)| \leq 3$. If $|D_2(G)| \leq 1$, then, by Lemma 4.16, $G \cong P(10)$ since G is reduced. Noting that $P(10)$ has a cycle of length 5, we obtain a contradiction to the fact that G is bipartite. So, $2 \leq |D_2(G)| \leq 3$. We distinguish these two cases.

Case 1. $|D_2(G)| = 2$.
Without loss of generality, we may assume that $d_G(u_1) \geq d_G(u_2) \geq d_G(u_3) \geq 3$ and $d_G(u_i) = 2$ $(i = 4,5)$. Then, using $|E(G)| = 15$ and (4.2), either $d_G(u_1) = 5$ and $d_G(u_2) = d_G(u_3) = 3$, or $d_G(u_1) = d_G(u_2) = 4$ and $d_G(u_3) = 3$. We distinguish these two subcases.

Subcase 1.1. $d_G(u_1) = 5$ and $d_G(u_2) = d_G(u_3) = 3$.
Then $N_G(u_1) = V_1$. Without loss of generality, we may assume that $N_G(u_2) = \{v_1, v_2, v_3\}$. Then, since $d_G(u_3) = 3$, $|N_G(u_3) \cap \{v_1, v_2, v_3\}| = 1$; otherwise, $G[\{v_1, v_2, v_3, u_1, u_2, u_3\}] \cong K_{3,3} - e$ or $K_{3,3}$, a contradiction. Without loss of generality, we may assume that $N_G(u_3) = \{v_1, v_4, v_5\}$ and $v_2 \in N_G(u_4)$. Then $v_3 \notin N_G(u_4)$; otherwise, $G[\{v_1, v_2, v_3, u_1, u_2, u_4\}] \cong K_{3,3} - e$, a contradiction. Without loss of generality, we may assume that $N_G(u_4) = \{v_2, v_4\}$. Then $N_G(u_5) = \{v_3, v_5\}$. We deduce that $v_1u_1v_3u_5v_5u_3v_4u_4v_2u_2v_1$ is an SCT of G, a contradiction.

Subcase 1.2. $d_G(u_1) = d_G(u_2) = 4$ and $d_G(u_3) = 3$.
Then $3 \leq |N_G(u_1) \cap N_G(u_2) \leq 4$. Suppose that $|N_G(u_1) \cap N_G(u_2)| = 4$. Without loss of generality, we may assume that $N_G(u_1) = N_G(u_2) = V_1 \setminus \{v_5\}$. Since $d_G(u_3) = 3$, $|N_G(u_3) \cap (V_1 \setminus \{v_5\})| \geq 2$. Then G has a subgraph isomorphic to $K_{3,3} - e$ or $K_{3,3}$, a contraction.

Thus, $|N_G(u_1) \cap N_G(u_2)| = 3$. Without loss of generality, we may assume that $N_G(u_1) = V_1 \setminus \{v_5\}$ and $N_G(u_2) = V_1 \setminus \{v_4\}$. Since $d_G(u_3) = 3$, $|N_G(u_3) \cap \{v_1, v_2, v_3\}| = 1$; otherwise, $G[\{v_1, v_2, v_3, u_1, u_2, u_3\}] \cong K_{3,3} - e$ or $K_{3,3}$, a contraction. Now, without loss of generality, we may assume that $N_G(u_3) = \{v_1, v_4, v_5\}$ and $v_2 \in N_G(u_4)$. Then $v_3 \notin N_G(u_4)$; otherwise, $G[\{v_1, v_2, v_3, u_1, u_2, u_4\}] \cong K_{3,3} - e$, a contraction. Now, either $N_G(u_4) = \{v_2, v_4\}$ and so $N_G(u_5) = \{v_3, v_5\}$, or $N_G(u_4) = \{v_2, v_5\}$ and so $N_G(u_5) = \{v_3, v_4\}$. For the first case, $v_1u_2v_3u_5v_5u_3v_4u_4v_2u_1v_1$ is an SCT of G, a contradiction. For the second case, $v_1u_2v_3u_5v_4u_3v_5u_4v_2u_1v_1$ is an SCT of G, a contradiction.

Case 2. $|D_2(G)| = 3$.
Without loss of generality, we may assume that $d_G(u_1) \geq d_G(u_2) \geq 3$ and $d_G(u_i) = 2$ ($i = 3, 4, 5$). Note that G is a 2-edge-connected reduced bipartite graph. Then, using $|E(G)| = 15$ and (4.2), we get that $d_G(u_1) = 5$ and $d_G(u_2) = 4$. Then $N_G(u_1) = V_1$. Without loss of generality, we may assume that $N_G(u_2) = V_1 \setminus \{v_5\}$. Since $d_G(v_5) = 3$, $|N_G(v_5) \cap \{u_3, u_4, u_5\}| = 2$. Without loss of generality, we assume that $N_G(v_5) \cap V_2 = \{u_1, u_3, u_4\}$. Since $d_G(u_5) = 2$, without loss of generality, we may assume that $N_G(u_5) = \{v_1, v_2\}$. Now $G[\{v_1, v_2, v_3, u_1, u_2, u_5\}] \cong K_{3,3} - e$, a contradiction. This completes the proof. □

Before we present our proofs of Theorems 4.11 and 4.12 in the final section, we first introduce some additional notation and two technical lemmas due to Chen [35].

4.4 Notation and two technical lemmas

In this section, let H be a k-connected claw-free graph of order $n > N(p, \epsilon)$ with $k \in \{2, 3\}$, where $t \geq 1$ and $p \geq t$ are positive integers, and ϵ is a given real number, and $N(p, \epsilon) = \max\{36p^2 - 34p - \epsilon(p+1), 20p^2 - 10p - \epsilon(p+1), (3p+1)(-\epsilon - 4p)\}$. Moreover, we assume that $\delta(H) \geq 3$ and $d_t(H) \geq \frac{t(n+\epsilon)}{p}$, and that $cl(H) = L(G)$, and we let G, G_0 and G_0' be the graphs defined in Section 1.4. For $v \in V(G_0')$, we let $\Gamma_0(v)$ be the collapsible preimage of v in G_0, and we let $\Gamma(v)$ be the preimage of v in G. We also use the following notation.

- ⋄ $S_0 = \{v \in V(G_0') \mid v \text{ is a nontrivial vertex in } G_0'\}$;
- ⋄ $S_1 = \{v \in V(S_0) \mid |E(\Gamma(v))| \geq 1\}$;
- ⋄ $S_2 = S_0 \setminus S_1$, the set of vertices v with $\Gamma(v) = K_1$ and adjacent to some vertices in $D_2(G)$;
- ⋄ $V_0 = V(G_0') \setminus S_1$, the set of vertices v with $\Gamma(v) = K_1$ in G, which includes S_2;

- $\Phi_0 = G'_0[V_0]$;
- M_0 is a maximum matching in Φ_0, and V_{M_0} is the vertex set of M_0;
- $U_0 = V_0 \setminus V_{M_0}$, and so $V(G'_0) = S_1 \cup V_{M_0} \cup U_0$.

We also note the following before presenting the two technical lemmas. Since $\overline{\sigma}_2(G) \geq 5$, and by the definition of G'_0, we deduce that $D_2(G'_0) \subseteq S_1$. Since $\sigma_t(H) \geq U_t(H)$, we note that $U_t(H) \geq \frac{t(n+\epsilon)}{p}$ implies $\sigma_t(H) \geq \frac{t(n+\epsilon)}{p}$. This shows that it suffices to prove Theorems 4.11 and 4.12 for $\sigma_t(H)$.

In [35], Chen proved the following two technical lemmas which will be needed in our main proofs in Section 4.5.

Lemma 4.20. (Chen [35]). *With the above assumptions, the following three statements hold.*

(a) If M is a matching in G with $|M| \geq t$, then $|M|\frac{\sigma_t(H)+2t}{t} \leq \sum\limits_{xy \in M} (d_G(x) + d_G(y))$.

(b) Let $V_r \subseteq S_1$ with $|V_r| = r$, and let M'_b be a matching in G'_0 with $|M'_b| = b$. If $V_r \cap V(M'_b) = \emptyset$ and $r + b \geq t$, then $\sum\limits_{v \in V_r} (|V(\Gamma(v))| + d_{G'_0}(v)) + \sum\limits_{xy \in M'_b} (|V(\Gamma(x))| + |V(\Gamma(y))| + d_{G'_0}(x) + d_{G'_0}(y)) \geq \frac{(r+b)(\sigma_t(H)+2t)}{t} + 2b$.

(c) If $n > -\epsilon(p+1)$, then $|D_2(G'_0)| \leq p$.

Lemma 4.21. (Chen [35]). *With the above assumptions, and the additional assumptions that $p \geq 3(k-1)$, H is nonhamiltonian, and $G'_0 \neq K_{2,a}$, the following four statements hold.*

(a) $|S_1| + |M_0| \leq p$.

(b) If $|S_1| + |M_0| = p$, then $|E(G'_0)| \geq 2p + \epsilon - |S_1| + \sum_{v \in U_0} d_G(v)$. Furthermore, if $|M_0| = 0$, then $V(G'_0) = S_1 \cup U_0$, $|E(G'_0)| \geq \epsilon + p + \sum_{v \in U_0} d_G(v)$ and $|V(G'_0)| \leq 2p - \epsilon - 5$.

(c) $|U_0| \leq 2|S_1| + 3|M_0| - 5$ and $|V(G'_0)| \leq 3|S_1| + 5|M_0| - 5$.

(d) If $\delta(H) \geq 3p - 6$ when $k = 3$, or if $\delta(H) \geq 4p - 6$ when $k = 2$, then $M_0 = \emptyset$ and $S_2 = \emptyset$.

4.5 Proofs of Theorems 4.11 and 4.12

In this section, we shall present our proofs of Theorems 4.11 and 4.12.

Proof of Theorem 4.11. This is the special case of Theorem 4.9 with $k=2$, $p=5$, $1 \le t \le 5$ and $\epsilon = 5$. Suppose that H is not hamiltonian. By Theorem 1.3, $cl(H) = L(G)$, where G is an essentially 2-edge-connected triangle-free graph with $|E(G)| = n$. By Theorem 1.1, G does not have a DCT. Let G_0' be the reduction of the core G_0 of G. Then by Theorem 1.8, $G_0' \notin \mathscr{SL}$ and $\kappa'(G_0') \ge 2$. Then by Theorem 3.13(a), $|V(G_0')| \ge 5$. By Theorems 4.9(a) and 1.5(c), $|E(G_0')| \le 2|V(G_0')| - 4 \le 2(4p-5) - 4 = 26$, and G_0' is a simple triangle-free graph.

Let S_0, S_1, M_0, and U_0 be the sets defined in Section 4.4. By Theorem 1.8, G_0' has no DCT containing S_0. If $n > 26$, then $|E(G_0')| < |E(G)|$ and so $|S_1| \ge 1$. We distinguish the cases that $G_0' \ne K_{2,a}$ and $G_0' = K_{2,a}$.

Case 1. $G_0' \ne K_{2,a}$.
Then by Lemma 4.21(a), $|S_1| + |M_0| \le 5$. We distinguish the subcases that $|S_1| + |M_0| \le 3$, $|S_1| + |M_0| = 4$, and $|S_1| + |M_0| = 5$.

Subcase 1.1. $|S_1| + |M_0| \le 3$.
If $|S_1| = 3$, then, by Lemma 4.21(c), $|V(G_0')| \le 4$, a contradiction. Hence, $1 \le |S_1| \le 2$, and so $|M_0| \le 2$. Then by Lemma 4.21(c), $|V(G_0')| \le 8$. By Theorem 4.15(a), $|D_2(G_0')| \ge 3$, and so $|S_1| \ge 3$, a contradiction.

Subcase 1.2. $|S_1| + |M_0| = 4$.
Suppose first that $|S_1| = 1$ and $|M_0| = 3$ (or $|S_1| = |M_0| = 2$). Then $|D_2(G_0')| \le 2$. Then, by Theorem 4.15(a) and Lemma 4.21(c), $10 \le |V(G_0')| \le 13$. By Theorem 4.15(b), $|\alpha'(G_0')| \ge 5$. If $|S_1| = 1$, then $|\alpha'(G_0' - S_1)| \ge 4$, and so $|M_0| \ge 4$, contrary to $|M_0| = 3$. If $|S_1| = 2$, then $|\alpha'(G_0' - S_1)| \ge 3$, and so $|M_0| \ge 3$, contrary to $|M_0| = 2$.

Next suppose that $|S_1| = 3$ and $|M_0| = 1$. Then by Lemma 4.21(c), $|V(G_0')| \le 9$. If $|V(G_0')| \le 8$, then, by Theorem 3.12 and since $G_0' \notin \mathscr{SL}$, $G_0' \in \mathscr{G}_1 \cup \mathscr{G}_2$. Note that $D_2(G_0') \subseteq S_1$. Using $|S_1| = 3$ and $|M_0| = 1$, we deduce that $G_0' = W_3^*$. Then $S_1 = \{v_1, v_2, v_3\}$ and so $cl(H) \in \mathscr{Q}_3(s_1, s_2, s_3, 0)$. If $|V(G_0')| = 9$, then by Theorem 4.15(a), $|D_2(G_0')| \ge 3$. Since $D_2(G_0') \subseteq S_1$,

$|S_1| = |D_2(G_0')| = 3$. Let M be a maximum matching of G_0'. By Theorem 4.15(b), $|M| = 4$. Then, using $|M_0| = 1$, each vertex in $D_2(G_0')$ is incident with an edge of M, and any two vertices in $D_2(G_0')$ are not adjacent in M; otherwise, $|M_0| = |\alpha'(G_0' - S_1)| \geq 2$, a contradiction. Without loss of generality, we may assume that $M = \{u_1v_1, u_2v_2, u_3v_3, u_4u_5\}$, where $S_1 = \{v_1, v_2, v_3\}$ and $V_0 = \{u_1, u_2, \ldots, u_6\}$. Then $\{u_1, u_2, u_3, u_6\}$ is an independent set of G_0'; otherwise, $|M_0| \geq 2$, a contradiction. Now $|E(G_0')| \geq 3 \times 4 + 1 = 13$. Since $G_0' \neq K_{2,a}$, by Theorem 1.5(c), $|E(G_0')| \leq 2|V(G_0')| - 5 = 13$. We conclude that $|E(G_0')| = 13$, and so S_1 is an independent set of G_0'. Then $|E(G_0'[V_0])| = 7$. Since $d_{G_0'}(u_i) \geq 3$ $(1 \leq i \leq 6)$, using that $\{u_1, u_2, u_3, u_6\}$ is an independent set of G_0', it is easy to prove that $G_0'[V_0]$ contains a triangle, a contradiction.

Finally suppose that $|S_1| = 4$ and $|M_0| = 0$. Then by Lemma 4.21(c), $|V(G_0')| \leq 7$. Then, by Theorem 3.12 and since $G_0' \notin \mathscr{S}\mathscr{L}$, $G_0' \in \mathscr{G}_1$. Since $D_2(G_0') \subseteq S_1$, using $G_0' \neq K_{2,a}$ and $|S_1| = 4$, $G_0' \in \{W_3^*, \theta(1,1,2)\}$. If $G_0' = W_3^*$, then $S_1 = \{v_1, v_2, v_3, u_4\}$, and so $cl(H) \in \mathscr{Q}_3(s_1, s_2, s_3, r)$. Obviously, $\theta(1,1,2)$ can be contracted to a $K_{2,3}$ such that each vertex of degree two of the resulting graph $K_{2,3}$ is nontrivial. So, if $G_0' = \theta(1,1,2)$, then $cl(H) \in \mathscr{Q}_{2,3}(s_1, s_2, s_3, r)$.

Subcase 1.3. $|S_1| + |M_0| = 5$.
Since $|S_1| \geq 1$, $|M_0| \leq 4$. Since $G_0' \neq K_{2,a}$, by Theorem 1.5(c), $|E(G_0')| \leq 2(|S_1| + 2|M_0| + |U_0|) - 5$. By Lemma 4.21(b), $|E(G_0')| \geq 15 - |S_1| + 3|U_0|$. We deduce that $|M_0| \geq 5$, a contradiction. This settles Case 1.

Case 2. $G_0' = K_{2,a}$.
By Lemma 4.20(c), $|D_2(G_0')| \leq 5$. Then since $G_0' \notin \mathscr{S}\mathscr{L}$, $G_0' \in \{K_{2,3}, K_{2,5}\}$. For $v \in S_1$, let $\Gamma(v)$ be the preimage of v in G. Then $|E(G)| = |E(K_{2,a})| + \sum_{v \in S_1} |E(\Gamma(v))|$ $(a \in \{3, 5\})$.

Suppose first that $G_0' = K_{2,3}$. Since $D_2(G_0') \subseteq S_1$, $3 \leq |S_1| \leq 5$. If $|S_1| = 5$, then $V(G_0') = S_1$. By Lemma 4.20, $\sigma_t(H) \geq \frac{t(n+5)}{5}$ $(1 \leq t \leq 5)$, $|E(\Gamma(v))| \geq |V(\Gamma(v))| - 1$ and $n = |E(G)|$,

$$|S_1| \frac{\sigma_t(H) + 2t}{t} \leq \sum_{v \in S_1} (d_{G_0'}(v) + |V(\Gamma(v))|) \leq \sum_{v \in S_1} d_{G_0'}(v) + \sum_{v \in S_1} (|E(\Gamma(v))| + 1).$$

Then $n+15 \leq 12+(n-6)+5=n+11$, a contradiction. Hence $3 \leq |S_1| \leq 4$. Since $D_2(G_0') \subseteq S_1$ and by the definition of $\mathscr{Q}_{2,3}(s_1,s_2,s_3,r)$, $cl(H) \in \mathscr{Q}_{2,3}(s_1,s_2,s_3,r)$.

Next suppose that $G_0' = K_{2,5}$. Since $D_2(G_0') \subseteq S_1$, $5 \leq |S_1| \leq 7$. Let $V_r = D_2(G_0')$. Then $V_r \subseteq S_1$ and $|V_r| = 5$. By Lemma 4.20, $\sigma_t(H) \geq \frac{t(n+5)}{5}$ $(1 \leq t \leq 5)$, $|E(\Gamma(v))| \geq |V(\Gamma(v))| - 1$ and $n = |E(G)|$,

$$|V_r|\frac{\sigma_t(H)+2t}{t} \leq \sum_{v\in V_r}(d_{G_0'}(v)+|V(\Gamma(v))|) \leq \sum_{v\in V_r} d_{G_0'}(v)+\sum_{v\in V_r}(|E(\Gamma(v))|+1).$$

Then $n+15 \leq 10+(n-10)+5 = n+5$, a contradiction. This completes the proof. □

Proof of Theorem 4.12. This is the special case of Theorem 4.9 with $k=2$, $p=6$, $1 \leq t \leq 6$, and $\epsilon = 0$. Suppose that H is not hamiltonian. By Theorem 1.3, $cl(H) = L(G)$, where G is an essentially 2-edge-connected triangle-free graph with $|E(G)| = n$. By Theorem 1.1, G does not have a DCT. Let G_0' be the reduction of the core G_0 of G. Then by Theorem 1.8, $G_0' \notin \mathscr{SL}$ and $\kappa'(G_0') \geq 2$. Then by Theorem 3.13(a), $|V(G_0')| \geq 5$. By Theorems 4.9(a) and 1.5(c), $|E(G_0')| \leq 2|V(G_0')| - 4 \leq 2(4p-5)-4 = 34$, and G_0' is a simple triangle-free graph.

Let S_0, S_1, S_2, M_0, and U_0 be the sets defined in Section 4.4. By Theorem 1.8, G_0' has no DCT containing S_0. If $n > 34$, then $|E(G_0')| < |E(G)|$ and so $|S_1| \geq 1$. We distinguish the cases that $G_0' \neq K_{2,a}$ and $G_0' = K_{2,a}$.

Case 1. $G_0' \neq K_{2,a}$.
Since $\delta(H) \geq 4p-6 = 18$, by Lemma 4.21(d), $M_0 = S_2 = \emptyset$. Then $V(G_0') = S_1 \cup U_0$. By Lemma 4.21(a), $|S_1| \leq 6$. Note that $D_2(G_0') \subseteq S_1$. We distinguish the subcases that $|S_1| \leq 4$, $|S_1| = 5$, and $|S_1| = 6$.

Subcase 1.1. $|S_1| \leq 4$.
By Lemma 4.21(c), $|V(G_0')| \leq 3|S_1| - 5 \leq 7$. Recall that $G_0' \notin \mathscr{SL}$, $\kappa'(G_0') \geq 2$, and that G_0' is a simple triangle-free graph. Then by Theorem 3.12, and since $G_0' \neq K_{2,a}$, we deduce that $G_0' \in \{W_3^*, \theta(1,1,2), \theta(1,1,3), \theta(1,2,2)\}$. If $G_0' \in \{\theta(1,1,3), \theta(1,2,2)\}$, then $|S_1| \geq 5$, a contradiction. Hence, $G_0' \in \{W_3^*, \theta(1,1,2)\}$.

First assume $G_0' = W_3^*$. Then $3 \leq |S_1| \leq 4$. Suppose that $|S_1| = 3$. Then $S_1 = D_2(G_0')$. Now $E(G_0' - S_1) \neq \emptyset$, contradicting the fact that $M_0 = \emptyset$. Hence, $|S_1| = 4$. Since U_0 is an independent set, $S_1 = \{v_1, v_2, v_3, u_4\}$. Then we conclude that $cl(H) \in \mathscr{L}_3$.

Next assume $G_0' = \theta(1,1,2)$. Then $S_1 = D_2(G_0')$. Note that $\theta(1,1,2)$ can be contracted to a $K_{2,3}$ such that the resulting $K_{2,3}$ has three nontrivial vertices. Then $cl(H) \in \mathscr{L}_1$.

Subcase 1.2. $|S_1| = 5$.
By Lemma 4.21(c), $|V(G_0')| \leq 3|S_1| - 5 \leq 10$. We distinguish two subcases.

Suppose first that $|V(G_0')| \leq 7$. Since $G_0' \notin \mathscr{SL}$, by Theorem 3.12 and using $G_0' \notin K_{2,a}$, we deduce that $G_0' \in \{W_3^*, \theta(1,1,2), \theta(1,1,3), \theta(1,2,2)\}$. Since $D_2(G_0') \subseteq S_1$ and using $|S_1| = 5$, $\theta(1,1,2), \theta(1,1,3)$ and $\theta(1,2,2)$ can be contracted to a $K_{2,3}$ such that all vertices of the resulting $K_{2,3}$ are nontrivial. So, we conclude that in case $G_0' \in \{\theta(1,1,2), \theta(1,1,3), \theta(1,2,2)\}$, we have $cl(H) \in \mathscr{L}_1$.

If $G_0' = W_3^*$, then, using that $|S_1| = 5$ and U_0 is an independent set, without loss of generality, we may assume that $S_1 = \{v_1, v_2, v_3, u_1, u_4\}$. Then $cl(H) \in \mathscr{L}_3$.

Next suppose that $8 \leq |V(G_0')| \leq 10$. Let $U_0 = V_1 = \{v_1, v_2, \ldots, v_{|U_0|}\}$ and $S_1 = V_2 = \{u_1, u_2, u_3, u_4, u_5\}$. Note that G_0' is a 2-edge-connected reduced graph, U_0 is an independent set, and G_0' has no DCT containing S_1. Then $\bigcup_{i=1}^{|U_0|} N_{G_0'}(v_i) \subseteq V_2$ and $d_{G_0'}(v_i) \geq 3$ $(1 \leq i \leq |U_0|)$. Then using Lemmas 4.17, 4.18 and 4.19, we deduce that $G_0' \in \{C(6,2), C(6,2)', \theta(1,1,4)''', \theta(1,2,3)'$, $W_3^{**}, T_1, T_2, T_3, T_4, T_5\}$. It is easy to check that $C(6,2)$ and $\theta(1,2,3)'$ can be contracted to a $K_{2,3}$ by contracting a cycle of length four such that the resulting $K_{2,3}$ has four nontrivial vertices. Furthermore, $T_1/G_0'[\{v_3, v_4, u_1, u_4, u_5\}]$, $T_4/G_0'[\{v_1, v_2, u_1, u_2, u_3\}]$ and $T_5/G_0'[\{v_1, v_2, v_3, u_1, u_2\}]$ are isomorphic to a $K_{2,3}$, respectively. So, if $G_0' \in \{C(6,2), \theta(1,2,3)', T_1, T_4, T_5\}$, then $cl(H) \in \mathscr{L}_1$.

Note that $M_0 = \emptyset$, U_0 is an independent set, and $D_2(G_0') \subseteq S_1$. If $G_0' = C(6,2)'$, then $S_1 = D_2(C(6,2)') \cup \{w\}$. So, $cl(H) \in \mathscr{L}_4$. If $G_0' = \theta(1,1,4)'''$, then $S_1 = D_2(\theta(1,1,4)''') \cup \{w_1, w_2\}$. So, $cl(H) \in \mathscr{L}_5$. If $G_0' = W_3^{**}$, then $S_1 = D_2(W_3^{**}) \cup \{w\}$. It is easy to check that W_3^{**} can be contracted to a W_3^*

such that the resulting W_3^* has four nontrivial vertices. So, $cl(H) \in \mathscr{L}_3$. If $G_0' \in \{T_2, T_3\}$, then $S_1 = \{u_1, u_2, u_3, u_4, u_5\}$. So, $cl(H) \in \mathscr{L}_6$.

Subcase 1.3. $|S_1| = 6$.
By Lemma 4.21(b), $6 \le |V(G_0')| \le 2p - \epsilon - 5 = 7$. Since $G_0' \notin \mathscr{SL}$ and $G_0' \neq K_{2,a}$, by Theorem 3.12, $G_0' \in \{W_3^*, \theta(1,1,2), \theta(1,1,3), \theta(1,2,2)\}$. Suppose that $G_0' = W_3^*$. Since $D_2(G_0') \subseteq S_1$, and using that $|S_1| = 6$ and that U_0 is an independent set, either $S_1 = \{v_1, v_2, v_3, u_1, u_2, u_3\}$, or, without loss of generality, we may assume that $S_1 = \{v_1, v_2, v_3, u_1, u_2, u_4\}$. Then $cl(H) \in \mathscr{L}_3$. Suppose that $G_0' \in \{\theta(1,1,2), \theta(1,1,3), \theta(1,2,2)\}$. Since $D_2(G_0') \subseteq S_1$ and using that $|S_1| = 6$, $\theta(1,1,2)$, $\theta(1,1,3)$ and $\theta(1,2,2)$ can be contracted to a $K_{2,3}$ such that all vertices of the resulting $K_{2,3}$ are nontrivial. Then $cl(H) \in \mathscr{L}_1$. This settles Case 1.

Case 2. $G_0' = K_{2,a}$.
By Lemma 4.20(c), $|D_2(G_0')| \le 6$. Then since $G_0' \notin \mathscr{SL}$, $G_0' \in \{K_{2,3}, K_{2,5}\}$. If $G_0' = K_{2,3}$, then since $D_2(G_0') \subseteq S_1$, $3 \le |S_1| \le 5$. Then $cl(H) \in \mathscr{L}_1$. If $G_0' = K_{2,5}$, then since $D_2(G_0') \subseteq S_1$, $5 \le |S_1| \le 7$. For $v \in S_1$, let $\Gamma(v)$ be the preimage of v in G. Then $|E(G)| = |E(K_{2,5})| + \sum_{v \in S_1} |E(\Gamma(v))|$. Suppose that $6 \le |S_1| \le 7$. Let $V_r = D_2(G_0') \cup \{u\}$ such that $V_r \subseteq S_1$, where $d_{G_0'}(u) = 5$. Then $|V_r| = 6$. By Lemma 4.20, $\sigma_t(H) \ge \frac{tn}{6}$ $(1 \le t \le 6)$, $|E(\Gamma(v))| \ge |V(\Gamma(v))| - 1$, and $n = |E(G)|$,

$$|V_r| \frac{\sigma_t(H) + 2t}{t} \le \sum_{v \in V_r} (d_{G_0'}(v) + |V(\Gamma(v))|) \le \sum_{v \in V_r} d_{G_0'}(v) + \sum_{v \in V_r} (|E(\Gamma(v))| + 1).$$

Then $n + 12 \le 15 + (n - 10) + 6 = n + 11$, a contradiction. Hence $|S_1| = 5$. Since $D_2(G_0') \subseteq S_1$ and using $|D_2(G_0')| = 5$, we deduce that $S_1 = D_2(G_0')$. Then $cl(H) \in \mathscr{L}_2$. This completes the proof. $\square$

Chapter 5

Neighborhood and degree conditions for traceability

The results in this chapter are closely related to the results of the previous chapter. Instead of hamiltonicity, in this chapter, we are mainly interested in degree and neighborhood conditions for traceability of 2-connected claw-free graphs. The proofs are similar to the proofs in the previous chapter, but shorter, since we can reuse several results that were obtained in Chapter 4.

5.1 Introduction and main results

We start by recalling some of the notation and conventions of Chapter 4, but we refrain from a long introduction. Instead, we present our main results shortly, and then proceed by some remarks, putting our results in some context of related results.

We let $t \geq 1$ and $p \geq t$ be positive integers, and ϵ be a given real number. Depending on the values of p and ϵ, we define $N(p,\epsilon) = \max\{36p^2 - 34p - \epsilon(p+1), 20p^2 - 10p - \epsilon(p+1), (3p+1)(-\epsilon - 4p)\}$. In the following result, we let H be a k-connected claw-free graph of order $n > N(p,\epsilon)$ with $k \in \{2,3\}$. In this chapter, we first obtain the following analogue of Theorem 4.9 for traceability of claw-free graphs.

Theorem 5.1. *If $\delta(H) \geq 3$ and $d_t(H) \geq \frac{t(n+\epsilon)}{p}$, then either H is traceable or $cl(H) = L(G)$, where G is an essentially k-edge-connected triangle-free graph without a DT, and G_0' satisfies one of the following:*

(a) $k = 2$ and $G_0' \in \mathscr{R}_0(c, 2)$, where $c \leq max\{4p-5, 2p+1\}$ and $p \geq 4$;

(b) $k = 3$ and $G_0' \in \mathscr{R}_0(c, 3)$, where $c \leq max\{3p-5, 2p+1\}$ and $p \geq 7$.

We postpone the proof of Theorem 5.1 to the final section. Let $\mathscr{X}_i$ denote the class of all spanning subgraphs of the nontraceable graphs H_i, $i = 1, \dots, 8$, as depicted in Figure 5.1 (where the circular and elliptical parts represent cliques of arbitrary positive order, but at least the number of black dots indicated in these parts). In [51], Fronček et al. proved the following result.

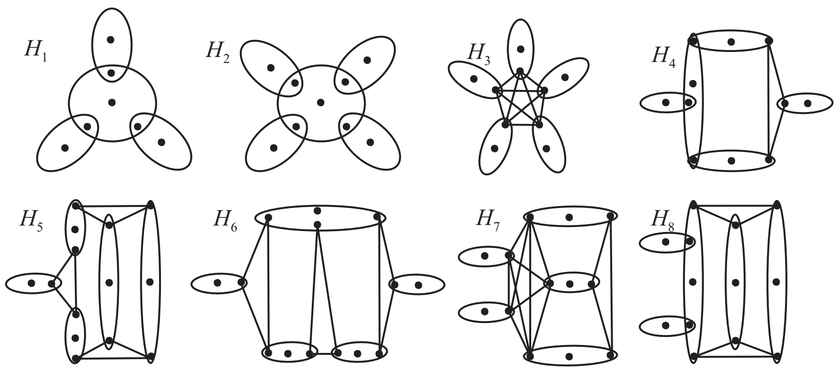

FIGURE 5.1: Eight classes of nontraceable claw-free graphs.

Theorem 5.2. (Fronček, Ryjáček and Skupień [51]). *Let H be a connected claw-free graph of order $n \geq 112 - 7\kappa(cl(H))$ such that $\delta(H) \geq 14$ and $\sigma_6(H) > n + 14 + \kappa(cl(H))$. Then either H is traceable or $cl(H) \in \bigcup_{i=1}^{8} \mathscr{X}_i$.*

As an application of Theorem 5.1, we first obtain the following result.

Theorem 5.3. *Let H be a 2-connected claw-free graph of sufficiently large order n with $\delta(H) \geq 3$. If $d_t(H) \geq \frac{t(n+6)}{6}$ ($t \in \{1, 2, \dots, 6\}$), then H is traceable.*

As another application of Theorem 5.1, we obtain the following related result. It shows that, by increasing the (constant) lower bound on $\delta(H)$, in the following sense the degree condition on larger sets of (independent) vertices can be relaxed.

Theorem 5.4. *Let H be a 2-connected claw-free graph of sufficiently large order n with $\delta(H) \geq 22$. If $d_t(H) \geq \frac{t(n-2.5)}{7}$ with $t \in \{1, 2, \ldots, 7\}$, then either H is traceable or $cl(H) = L(G)$, where G is an essentially 2-edge-connected triangle-free graph that can be contracted to either F_1 or F_2, in such a way that all the vertices of degree two are nontrivial.*

The above theorem can in fact easily be deduced from the following more general result that we are going to prove.

Theorem 5.5. *Let H be a 2-connected claw-free graph of sufficiently large order n with $\delta(H) \geq 22$. If $d_t(H) \geq \frac{t(n-2.5)}{7}$ with $t \in \{1, 2, \ldots, 7\}$, then either H is traceable or $cl(H) \in \mathcal{R}_{\mathcal{F}}(n, 1)$.*

Let $\mathcal{F}_1(n, s)$ denote the subfamily of $\mathcal{F}(n, s)$ in which each Φ_i is isomorphic to K_{1,s_i}. Furthermore, let $\mathcal{R}^1_{\mathcal{F}}(n, s) = \{H = L(G) \mid G \in \mathcal{F}_1(n, s)\}$, a subfamily of $\mathcal{R}_{\mathcal{F}}(n, s)$.

The following result is another application of Theorem 5.1.

Theorem 5.6. *Let H be a 2-connected claw-free graph of sufficiently large order n with $\delta(H) \geq 18$. If $\sigma_6(H) \geq n-6$, then either H is traceable or $\sigma_6(H) = n-6$ and $cl(H) \in \mathcal{R}^1_{\mathcal{F}}(n, 1)$.*

From the proof of Theorem 5.6 (which will be given in Section 5.2), we can easily obtain a result that has already been stated as Corollary 2.5.

Remark 5.1. (a) Let G^* be a graph obtained from F_1 or F_2 by adding $\frac{n-12}{6} \geq 2$ pendant edges (for a suitable choice of n) at each vertex of degree two of F_1 or F_2. Then $d_t(L(G^*)) = \frac{t(n-6)}{6} < \frac{t(n+6)}{6}$. This example shows that for given $t \in \{1, 2, \ldots, 6\}$, the lower bound $\frac{n+6}{6}$ in Theorem 5.3 is asymptotically sharp.

Let G^{**} be a graph obtained from G_1 of Figure 2.2 by adding $\frac{n-14}{7} \geq 2$ pendant edges (for a suitable choice of n) at each vertex of degree two of G_1. Then $d_t(L(G^{**})) = \frac{t(n-7)}{7} < \frac{t(n-2.5)}{7}$. Clearly, $L(G^{**}) \notin \mathcal{R}_{\mathcal{F}}(n, 1)$. Note that G^{**} cannot be contracted to a graph in $\{F_1, F_2\}$. This example shows that, for given $t \in \{1, 2, \ldots, 7\}$, the bound $\frac{n-2.5}{7}$ in Theorems 5.4 and 5.5 is asymptotically sharp.

(b) Our results also extend earlier results that are based on the notion of the generalized t-degree, $\delta_t(H)$. Since obviously $\sigma_t(H) \geq U_t(H) \geq \delta_t(H)$, the statements in Theorems 5.3, 5.4, 5.5 and 5.6 are also valid if we replace $d_t(H)$ by $\delta_t(H)$.

5.2 Proofs of Theorems 5.1, 5.3, 5.5, and 5.6

In this section, we will present the proofs of Theorems 5.1, 5.3, 5.5, and 5.6.

Proof of Theorem 5.1. Suppose that H is not traceable. Then, in particular, H is not hamiltonian and H is not complete. By Theorem 1.3, there exists an essentially k-edge-connected triangle-free graph G such that $cl(H) = L(G)$ and $|E(G)| = |V(H)|$. Let G_0' be the reduction of the core G_0 of G, and let $c = |V(G_0')|$. By Theorem 1.8, $\kappa'(G_0') \geq \kappa'(G_0) \geq k$. Since H is not traceable, by Theorems 1.2 and 1.4, G has no DT. By Theorem 1.9, G_0' has no DT containing all the nontrivial vertices, so, in particular, G_0' has no DCT containing all the nontrivial vertices. By Theorem 4.9, $G_0' \in \mathcal{Q}_0(c, k)$.

If $k = 2$, then, since G_0' has no ST, by Theorem 2.9, $G_0' \in \mathcal{R}_0(c, 2)$ for $c \geq 10$. Using Theorem 4.9(a), we deduce that $p \geq 4$.

If $k = 3$, then, since G_0' has no ST, by Theorem 3.13(b), $G_0' \in \mathcal{R}_0(c, 3)$ for $c \geq 16$. Using Theorem 4.9(b), we deduce that $p \geq 7$. This completes the proof of Theorem 5.1. □

Proof of Theorem 5.3. This is the special case of Theorem 5.1 with $k = 2$, $p = 6$, $1 \leq t \leq 6$ and $\epsilon = 6$. Suppose that H is not traceable. Then H is not hamiltonian. By Theorem 1.3, $cl(H) = L(G)$, where G is an essentially 2-edge-connected triangle-free graph with $|E(G)| = |V(H)|$. Since $\delta(cl(H)) \geq \delta(H) \geq 3$, $\overline{\sigma}_2(G) \geq 5$.

By Theorem 5.1, G does not have a DT. Let G_0' be the reduction of the core G_0 of G. Then by Theorem 1.9, G_0' has no ST, and so $G_0' \neq K_{2,a}$, and $\kappa'(G_0') \geq 2$. Then by Theorem 2.9, $|V(G_0')| \geq 10$. By Theorems 1.5(c) and

5.1(a), $|E(G_0')| \leq 2|V(G_0')| - 5 \leq 2(4p-5) - 5 = 33$, and G_0' is a simple triangle-free graph. By Lemma 1.6, G_0' is $(K_{3,3} - e)$-free and $K_{3,3}$-free.

Let S_0, S_1, V_0, M_0, and U_0 be the sets relating to G_0' as defined in Section 4.4. By Theorem 1.9, G_0' has no DT containing S_0. If $n > 33$, then $|E(G_0')| < |E(G)|$ and so $|S_1| \geq 1$. Let M be a maximum matching in G_0' and $D_2(G_0') = l$. By Lemma 4.21(a), $|S_1| + |M_0| \leq 6$. Note that $D_2(G_0') \subseteq S_1$. We first prove the following claim.

Claim 1. If $|S_1| + |M_0| \leq 5$, then $|V(G_0')| \geq 12$.

Proof. If $|S_1| + |M_0| \leq 5$, then $0 \leq l \leq 5$. Then, by Theorem 2.9, and since G_0' has no ST, $|V(G_0')| \geq 12$. □

We distinguish three cases.

Case 1. $|S_1| + |M_0| \leq 4$.
Then $0 \leq l \leq 4$. Then $|M| \leq 4$; otherwise, $|M_0| = \alpha'(G_0' - S_1) \geq 5 - |S_1|$, and so $|S_1| + |M_0| \geq 5$, a contradiction. However, by Theorem 4.15, by $0 \leq l \leq 4$, and by Claim 1, $|M| \geq 5$, a contradiction.

Case 2. $|S_1| + |M_0| = 5$.
By Claim 1, $|V(G_0')| \geq 12$. We prove another claim.

Claim 2. $|M| \leq 5$.

Proof. By contradiction. Suppose that $|M| \geq 6$. Then $|M_0| = \alpha'(G_0' - S_1) \geq 6 - |S_1|$, and so $|S_1| + |M_0| \geq 6$, contradicting our assumption that $|S_1| + |M_0| = 5$. □

We distinguish five subcases.

Subcase 2.1. $|S_1| = 1$ and $|M_0| = 4$.
Then $0 \leq l \leq 1$. Then, using Theorem 4.15 and $|V(G_0')| \geq 12$, $|M| \geq 6$, contradicting Claim 2.

Subcase 2.2. $|S_1| = 2$ and $|M_0| = 3$.
Then $0 \leq l \leq 2$. Then by Theorem 4.15, by $|V(G_0')| \geq 12$, and by Claim 2, $|M| = 5$ and $\frac{|V(G_0')| - l + 5}{3} \leq 5$. Then $|V(G_0')| \leq 10 + l$. Then by $|V(G_0')| \geq$

12, $|V(G_0')| = 12$ and $l = 2$. Using $|M_0| = 3$, each vertex in S_1 is incident with an edge of M, and the two vertices in S_1 are not adjacent in M; otherwise, $|M_0| = |\alpha'(G_0' - S_1)| \geq 4$, a contradiction. Let $S_1 = \{v_1, v_2\}$ and $V_0 = \{u_1, u_2, \ldots, u_{10}\}$. Without loss of generality, we may assume that $M = \{u_1v_1, u_2v_2, u_3u_4, u_5u_6, u_7u_8\}$. Then $\{u_1, u_2, u_9, u_{10}\}$ is an independent set of G_0'; otherwise, $|M_0| \geq 4$, contrary to $|M_0| = 3$.

We denote $S = \{\{v_1\}, \{v_2\}, \{u_3, u_4\}, \{u_5, u_6\}, \{u_7, u_8\}\}$, $V(S) = \{v_1, v_2, u_3, \ldots, u_8\}$, $E(S) = \bigcup_{S_i \in S} E(S_i, V(S) \setminus S_i)$, and $Q = \{u_1, u_2, u_9, u_{10}\}$. Then $E(S) = E(G_0'[V(S)]) \setminus \{u_3u_4, u_5u_6, u_7u_8\}$.

Since $G_0' \neq K_{2,a}$, by Theorem 1.5(c), $|E(G_0')| \leq 2|V(G_0')| - 5 = 19$. Then $|E(S)| \leq 4$; otherwise, since Q is an independent set of G_0' and $|M_0| = 3$, $|E(G_0')| \geq 3|Q| + |M_0| + |E(S)| \geq 20$, a contradiction.

We first prove three claims before continuing the proof for Subcase 2.2.

Claim 3. $e(S_i, V(S) \setminus S_i) \geq 2$, for any $S_i \in \{\{u_3, u_4\}, \{u_5, u_6\}, \{u_7, u_8\}\}$.

Proof. By contradiction. Without loss of generality, we may assume that $e(\{u_3, u_4\}, V(S) \setminus \{u_3, u_4\}) \leq 1$. Then since $d_{G_0'}(u_i) \geq 3$, $e(\{u_3\}, Q) \geq 1$ and $e(\{u_4\}, Q\}) \geq 1$. Then, using that G_0' is triangle-free, $|M_0| \geq 4$, contrary to $|M_0| = 3$. □

Claim 4. $|E(S)| = 4$.

Proof. Suppose that $|E(S)| \leq 3$. Then by Claim 3, $2 \leq e(S_i, V(S) \setminus S_i) \leq 3$, for any $S_i \in \{\{u_3, u_4\}, \{u_5, u_6\}, \{u_7, u_8\}\}$. Since G_0' is simple and triangle-free, and by Claim 3, it is easy to find a pair of vertices $\{u_i, u_{i+1}\}$ ($i \in \{3, 5, 7\}$) such that $e(\{u_i\}, Q\}) \geq 1$ and $e(\{u_{i+1}\}, Q\}) \geq 1$. Then, since G_0' is triangle-free, $|M_0| \geq 4$, contrary to $|M_0| = 3$. □

Since Q is an independent set of G_0' and $|M_0| = 3$, and using Claim 4, we deduce that $|E(G_0')| \geq 3|Q| + |M_0| + |E(S)| \geq 19$. Then $|E(G_0')| = 19$, and so $d_{G_0'}(u) = 3$ ($u \in Q$). By Claims 3 and 4, $2 \leq e(S_i, V(S) \setminus S_i) \leq 4$, for any $S_i \in \{\{u_3, u_4\}, \{u_5, u_6\}, \{u_7, u_8\}\}$. In the following, for any pair of vertices $\{u_i, u_{i+1}\}$ ($i \in \{3, 5, 7\}$), we always assume that $e(\{u_i\}, V(S) \setminus \{u_i, u_{i+1}\}) \geq e(\{u_{i+1}\}, V(S) \setminus \{u_i, u_{i+1}\})$.

Claim 5. $2 \leq e(\{u_i\}, V(S) \setminus \{u_i, u_{i+1}\}) \leq 4$ ($i \in \{3, 5, 7\}$).

Proof. By Claim 4, $e(\{u_i\}, V(S) \setminus \{u_i, u_{i+1}\}) \leq 4$. Suppose that $e(\{u_i\}, V(S) \setminus \{u_i, u_{i+1}\}) \leq 1$. Then by $d_{G'_0}(u_i) \geq 3$ and $d_{G'_0}(u_{i+1}) \geq 3$, $e(\{u_i\}, Q) \geq 1$ and $e(\{u_{i+1}\}, Q) \geq 1$. Then, since G'_0 is triangle-free, $|M_0| \geq 4$, contrary to $|M_0| = 3$. $\square$

By Claims 4 and 5, we treat the following four subcases separately according to the connection relations of the vertices in $V(S)$.

Suppose first that $e(\{u_i\}, V(S) \setminus \{u_i, u_{i+1}\}) = 4$ and $e(\{u_{i+1}\}, V(S) \setminus \{u_i, u_{i+1}\}) = 0$, or $e(\{u_i\}, V(S) \setminus \{u_i, u_{i+1}\}) = 3$ and $e(\{u_{i+1}\}, V(S) \setminus \{u_i, u_{i+1}\}) = 1$, or $e(\{u_i\}, V(S) \setminus \{u_i, u_{i+1}\}) = e(\{u_{i+1}\}, V(S) \setminus \{u_i, u_{i+1}\}) = 2$. Since G'_0 is simple and triangle-free, it is easy to find a pair of vertices $\{u_j, u_{j+1}\}$ ($j \in \{3, 5, 7\}$) such that $e(\{u_j\}, Q) \geq 1$ and $e(\{u_{j+1}\}, Q) \geq 1$. Then since G'_0 is triangle-free, $|M_0| \geq 4$, contrary to $|M_0| = 3$.

Next suppose that $e(\{u_i\}, V(S) \setminus \{u_i,\ u_{i+1}\}) = 3$ and $e(\{u_{i+1}\}, V(S) \setminus \{u_i, u_{i+1}\}) = 0$. Without loss of generality, we may assume that $e(\{u_3\}, V(S) \setminus \{u_3, u_4\}) = 3$ and $e(\{u_4\}, V(S) \setminus \{u_3, u_4\}) = 0$. Then by Claim 5, $u_3u_5, u_3u_7 \in E(G'_0)$; otherwise, $|E(S)| \geq 5$, contradicting Claim 4. Then by Claims 4 and 5, $u_5u_7 \in E(G'_0)$. Then $G'_0[\{u_3, u_5, u_7\}]$ is a triangle, a contradiction.

Now suppose that $e(\{u_i\}, V(S) \setminus \{u_i,\ u_{i+1}\}) = 2$ and $e(\{u_{i+1}\}, V(S) \setminus \{u_i, u_{i+1}\}) = 1$. Without loss of generality, we may assume that $e(\{u_3\}, V(S) \setminus \{u_3, u_4\}) = 2$ and $e(\{u_4\}, V(S) \setminus \{u_3, u_4\}) = 1$. Then by Claim 5, either $u_3u_5 \in E(G'_0)$ or $u_3u_7 \in E(G'_0)$; otherwise, $|E(S)| \geq 5$, contradicting Claim 4. Without loss of generality, we may assume that $u_3u_5 \in E(G'_0)$. Then $u_4u_5 \notin E(G'_0)$; otherwise, $G'_0[\{u_3, u_4, u_5\}]$ is a triangle, a contradiction. Note that $E(S) = E(\{u_3, u_4\}, V(S) \setminus \{u_3, u_4\}) \cup E(\{u_5\}, V(S) \setminus \{u_5, u_6\})$ under our assumptions. By Claim 5, $e(\{u_7\}, V(S) \setminus \{u_7, u_8\}) \geq 2$. Then, since G'_0 is simple and triangle-free, $u_4u_7, u_5u_7 \in E(G'_0)$. Then $u_3u_6 \notin E(G'_0)$; otherwise, $G'_0[\{u_3, u_5, u_6\}]$ is a triangle, a contradiction. Then $u_6u_7, u_6u_8 \notin E(G'_0)$; otherwise, $|E(S)| \geq 5$, contradicting Claim 4. Then by $d_{G'_0}(u_4) \geq 3$ and $d_{G'_0}(u_6) \geq 3$, $|N_{G'_0}(u_4) \cap Q| \geq 1$ and $|N_{G'_0}(u_6) \cap Q| \geq 2$. Without loss of generality, we assume that $u_4u_a, u_6u_b \in E(G'_0)$ ($u_a, u_b \in Q$ and $u_a \neq u_b$). Then $\{u_3u_5, u_4u_a, u_6u_b, u_7u_8\}$ is a matching of $G'_0[V_0]$, contrary to $|M_0| = 3$.

Finally suppose that $e(\{u_i\}, V(S) \setminus \{u_i, u_{i+1}\}) = 2$ and $e(\{u_{i+1}\}, V(S) \setminus \{u_i, u_{i+1}\}) = 0$ ($i \in \{3, 5, 7\}$). Then $d_{G'_0}(u_i) = 3$ ($i \in \{3, 5, 7\}$); otherwise,

$N_{G_0'}(u_i)\cap Q \neq \emptyset$ for some $i \in \{3,5,7\}$. Then, using $d_{G_0'}(u_{i+1}) \geq 3$, we can find a matching of $G_0'[V_0]$ with cardinality 4, contrary to $|M_0| = 3$. Without loss of generality, we may assume that $d_{G_0'}(u_4) \geq d_{G_0'}(u_6) \geq d_{G_0'}(u_8) \geq 3$. Note that $d_{G_0'}(u) = 3$ $(u \in Q)$, $d_{G_0'}(v_1) = d_{G_0'}(v_2) = 2$, and $|E(G_0')| = 19$. Then $\sum_{i\in\{4,6,8\}} d_{G_0'}(u_i) = 13$. Then by our assumptions, $d_{G_0'}(u_4) = 5$. Then, either $d_{G_0'}(u_6) = 5$ and $d_{G_0'}(u_8) = 3$, or $d_{G_0'}(u_6) = 4$ and $d_{G_0'}(u_8) = 4$. Both cases will result in a subgraph isomorphic to $K_{3,3}$ or $K_{3,3} - e$ of G_0', a contradiction. This settles Subcase 2.2.

Subcase 2.3. $|S_1| = 3$ and $|M_0| = 2$.
Then $0 \leq l \leq 3$. Then, by Theorem 4.15, by $|V(G_0')| \geq 12$, and by Claim 2, $|M| = 5$ and $\frac{|V(G_0')|-l+5}{3} \leq 5$. Hence, $|V(G_0')| \leq 10 + l$. Then, using $|V(G_0')| \geq 12$, we get $2 \leq l \leq 3$. We distinguish two subcases.

Suppose first that $l = 2$. Then $|V(G_0')| = 12$. Then, using $|M_0| = 2$, we get that each vertex in S_1 is incident with an edge of M, and any two vertices in S_1 are not adjacent in M; otherwise, $|M_0| = |\alpha'(G_0' - S_1)| \geq 3$, a contradiction. Let $S_1 = \{v_1, v_2, v_3\}$ and $V_0 = \{u_1, u_2, \dots, u_9\}$. Without loss of generality, we may assume that $M = \{u_1v_1, u_2v_2, u_3v_3, u_4u_5, u_6u_7\}$. Then $\{u_1, u_2, u_3, u_8, u_9\}$ is an independent set of G_0'; otherwise, $|M_0| \geq 3$, contrary to $|M_0| = 2$. Then $|E(G_0')| \geq 3 \times 5 + 2 = 17$. Since $G_0' \neq K_{2,a}$, by Theorem 1.5(c), $|E(G_0')| \leq 2|V(G_0')| - 5 = 19$. Hence, $17 \leq |E(G_0')| \leq 19$. Then $|E[\{u_4, u_5\}, \{v_1, v_2, v_3, u_6, u_7\}] \cup E[\{u_6, u_7\}, \{v_1, v_2, v_3, u_4, u_5\}]| \leq 2$; otherwise, $|E(G_0')| \geq 3 \times 5 + 2 + 3 = 20$, a contradiction. Using that G_0' is simple and triangle-free, and that $d_{G_0'}(u) \geq 3$ $(u \in V_0)$, there exists a pair of vertices $\{u_i, u_{i+1}\} \in \{\{u_4, u_5\}, \{u_6, u_7\}\}$ such that $e(\{u_i\}, \{u_1, u_2, u_3, u_8, u_9\}) \geq 1$ and $e(\{u_{i+1}\}, \{u_1, u_2, u_3, u_8, u_9\}) \geq 1$. Then, since G_0' is triangle-free, $|M_0| = \alpha'(G_0' - S_1) \geq 3$, contrary to $|M_0| = 2$.

Next suppose that $l = 3$. Then $12 \leq |V(G_0')| \leq 13$. Similarly as in the proof for the case $l = 2$, we can prove that this is impossible by considering the cases that $|V(G_0')| = 12$ and 13 separately. This settles Subcase 2.3.

Subcase 2.4. $|S_1| = 4$ and $|M_0| = 1$.
Then $0 \leq l \leq 4$. By Lemma 4.21(c), $|V(G_0')| \leq 12$. Then, by Claim 1, $|V(G_0')| = 12$, and so $|U_0| = 6$. By Theorem 4.15 and Claim 2, $|M| = 5$ and

$\frac{|V(G_0')|-l+5}{3} \leq 5$. This results in $2 \leq l \leq 4$. Since $|M_0| = 1$, each vertex in S_1 is incident with an edge of M, and any two vertices in S_1 are not adjacent in M; otherwise, $|M_0| = |\alpha'(G_0' - S_1)| \geq 2$, a contradiction. Let $S_1 = \{v_1, v_2, v_3, v_4\}$ and $V_0 = \{u_1, u_2, \ldots, u_8\}$. Without loss of generality, we may assume that $M = \{u_1v_1, u_2v_2, u_3v_3, u_4v_4, u_5u_6\}$. Now $\{u_1, u_2, u_3, u_4, u_7, u_8\}$ is an independent set of G_0'; otherwise, $|M_0| \geq 2$, contrary to $|M_0| = 1$. We get that $|E(G_0')| \geq 3 \times 6 + 1 = 19$. Since $G_0' \neq K_{2,a}$, by Theorem 1.5(c), $|E(G_0')| \leq 2|V(G_0')| - 5 = 19$. Hence, $|E(G_0')| = 19$, and so S_1 is an independent set of G_0'.

Now $N_{G_0'}(S_1) \subseteq V_0 \backslash \{u_5, u_6\}$ and $N_{G_0'}(\{u_5, u_6\}) \subseteq V_0$; otherwise, $|E(G_0')| \geq 3 \times 6 + 1 + 1 = 20$, contrary to $|E(G_0')| = 19$. By Theorem 1.5(c), $G_0'[V_0]$ is simple and triangle-free. Then, using $d_{G_0'}(u_i) \geq 3$ $(i \in \{5, 6\})$, $N_{G_0'}(\{u_5, u_6\}) \subseteq V_0$, and that $\{u_1, u_2, u_3, u_4, u_7, u_8\}$ is an independent set of G_0', $\alpha'(G_0'[V_0]) = |M_0| = 2$, contrary to $|M_0| = 1$. This settles Subcase 2.4.

Subcase 2.5. $|S_1| = 5$ and $|M_0| = 0$.
Then by Lemma 4.21(c), $|V(G_0')| \leq 10$, contrary to $|V(G_0')| \geq 12$. This settles Subcase 2.5 and thereby Case 2.

Case 3. $|S_1| + |M_0| = 6$.
Since $|S_1| \geq 1$, $|M_0| \leq 5$. By Lemma 4.21(b), $|E(G_0')| \geq 2p + \epsilon - |S_1| + 3|U_0|$. By Theorem 1.5(c), $|E(G_0')| \leq 2(|S_1| + 2|M_0| + |U_0|) - 5$. We deduce that $|M_0| \geq 5$. Hence, $|M_0| = 5$, and so $|S_1| = 1$ and $|U_0| = 0$. Then $|V(G_0')| = |S_1| + 2|M_0| + |U_0| = 11$. Note that G_0' is simple and triangle-free, and $\kappa'(G_0') \geq 2$. Then using Theorem 2.9 and that G_0' has no ST, $G_0' \in \{G_1, G_2, \ldots, G_6\}$. Then $|D_2(G_0')| \geq 6$, and so $|S_1| \geq 6$, contrary to $|S_1| = 1$.
This completes the proof. □

Proof of Theorem 5.5. This is a special case of Theorem 5.1 with $p = 7$, $t \in \{1, 2, \ldots, 7\}$, $\epsilon = -2.5$, and $k = 2$. By Theorem 1.3, there is an essentially 2-edge-connected triangle-free graph G with $cl(H) = L(G)$ and $|E(G)| = |V(H)|$. Since $\delta(cl(H)) \geq \delta(H) \geq 22$, $\overline{\sigma}_2(G) \geq 24$.

Suppose that H is not traceable. Then H is not hamiltonian, and G has no DT. Let G_0' be the reduction of the core G_0 of G. By Theorems 1.9 and 2.9,

G_0' has no ST and $|V(G_0')| \geq 10$. Then G_0' cannot be isomorphic to a $K_{2,a}$ for any $a \geq 2$; otherwise, G_0' has an ST, a contradiction.

Let S_0, S_1, S_2, M_0, and U_0 be sets relating to G_0' as defined in Section 4.4. Since $\delta(H) \geq 22 = 4p - 6$, by Lemma 4.21(d), $M_0 = S_2 = \emptyset$. Then $V(G_0') = S_1 \cup U_0$, and S_1 is the set of all the nontrivial vertices of G_0'. By Theorem 1.9, G_0' does not have a DT containing S_1. By Lemma 4.21(a), $|S_1| \leq 7$. We distinguish two cases.

Case 1. $|S_1| \leq 6$.
Since $M_0 = \emptyset$, $E(G_0' - S_1) = \emptyset$. Recall that G_0' is 2-edge-connected. By Theorem 2.15, either G_0' has a trail passing through all vertices of S_1, or $G_0' \in \{F_1, F_2\}$. In the first case, G_0' has a DT containing S_1, a contradiction. Hence, $G_0' \in \{F_1, F_2\}$. Since $D_2(G_0') \subseteq S_1$ and $|D_2(G_0')| = 6$, $D_2(G_0') = S_1$. Then, by the definitions of $cl(H)$, G_0', and $\mathscr{R}_{\mathscr{F}}(n,1)$, $cl(H) \in \mathscr{R}_{\mathscr{F}}(n,1)$.

Case 2. $|S_1| = 7$.
Since $|S_1| = 7$ and $|M_0| = 0$, by Lemma 4.21(b), $|V(G_0')| \leq 2p - \epsilon - 5 = 11.5$. By Theorem 2.9, $G_0' \in \{F_1, F_2, G_1, G_2, \ldots, G_6\}$. Note that $|M_0| = 0$, $V(G_0') = S_1 \cup U_0$, and $d_G(v) = d_{G_0'}(v) \geq 3$ for $v \in U_0$. Suppose first that $G_0' \in \{F_1, F_2\}$. Then $|E(G_0')| = 12$. By Lemma 4.21(b), $|E(G_0')| \geq (-2.5) + 7 + 3 \times 3 = 13.5$, a contradiction. Suppose next that $G_0' \in \{G_1, G_2, \ldots, G_6\}$. Then $13 \leq |E(G_0')| \leq 14$. By Lemma 4.21(b), $|E(G_0')| \geq (-2.5) + 7 + 3 \times 4 = 16.5$, a contradiction. This completes the proof of Theorem 5.5. $\square$

Proof of Theorem 5.6. This is a special case of Theorem 5.1 with $p = 6$, $t = 6$, $\epsilon = -6$, and $k = 2$. By Theorem 1.3, there is an essentially 2-edge-connected triangle-free graph G such that $cl(H) = L(G)$ and $|E(G)| = |V(H)|$. Since $\delta(cl(H)) \geq \delta(H) \geq 18$, $\overline{\sigma}_2(G) \geq 20$.

Suppose that H is not traceable. Then H is not hamiltonian, and G has no DT. Let G_0' be the reduction of the core G_0 of G. By Theorems 1.9 and 2.9, G_0' has no ST and $|V(G_0')| \geq 10$. Then G_0' cannot be isomorphic to a $K_{2,a}$ for any $a \geq 2$; otherwise, G_0' has an ST, a contradiction.

Let S_0, S_1, S_2, M_0, and U_0 be sets relating to G_0' as defined in Section 4.4. Since $\delta(H) \geq 18 = 4p - 6$, by Lemma 4.21(d), $M_0 = S_2 = \emptyset$. Then $V(G_0') = S_1 \cup U_0$, and S_1 is the set of all the nontrivial vertices of G_0'. By

Theorem 1.9, G_0' does not have a DT containing S_1. By Lemma 4.21(a), $|S_1| \leq 6$.

Since $M_0 = \emptyset$, $E(G_0' - S_1) = \emptyset$. Recall that G_0' is 2-edge-connected. By Theorem 2.15, either G_0' has a trail passing through all vertices of S_1, or $G_0' \in \{F_1, F_2\}$. In the first case, G_0' has a DT containing S_1, a contradiction. So, $G_0' \in \{F_1, F_2\}$. Since $D_2(G_0') \subseteq S_1$ and $|D_2(G_0')| = 6$, $D_2(G_0') = S_1$. Let $S_1 = \{v_1, v_2, \ldots, v_6\}$. Then $d_{G_0'}(v_i) = 2$.

Let $V_r = S_1$ and $M_b' = \emptyset$. Then $V_r \cap V(M_b') = \emptyset$ and $|V_r| + |M_b'| = r + b = t = 6$, with $r = 6$ and $b = 0$. By Lemma 4.20(b), $\sum\limits_{v \in V_r} (|V(\Gamma(v))| + d_{G_0'}(v)) \geq \frac{(r+b)(\sigma_t(H)+2t)}{t} = n + 6$. Hence,

$$\sum_{v \in V_r} |V(\Gamma(v))| \geq n - 6. \tag{5.1}$$

Since $|E(\Gamma(v_i))| \geq |V(\Gamma(v_i))| - 1$, using (5.1) and $|E(G_0')| = 12$, $n = |E(G)| = |E(G_0')| + \sum\limits_{v \in V_r} |E(\Gamma(v))| \geq 12 + \sum\limits_{v \in V_r} (|V(\Gamma(v))| - 1) \geq 12 + (n - 12) = n$. Thus, all inequalities are in fact equalities, so that $|E(\Gamma(v_i))| = |V(\Gamma(v_i))| - 1$ and $\sigma_6(H) = n - 6$. This implies that each $\Gamma(v_i)$ induces a tree with $|E(\Gamma(v_i))| = |V(\Gamma(v_i))| - 1 = s_i$. Since G is essentially 2-edge-connected, $\Gamma(v_i) = K_{1,s_i}$. By the definitions of $cl(H)$, G_0', and $\mathcal{R}^1_{\mathcal{F}}(n, 1)$, $cl(H) \in \mathcal{R}^1_{\mathcal{F}}(n, 1)$. This completes the proof of Theorem 5.6.

To obtain Corollary 2.5, note that if n is sufficiently large and $\delta(H) \geq \frac{n-6}{6}$, then using $cl(H) \in \mathcal{R}^1_{\mathcal{F}}(n, 1)$, we deduce that $\delta(H) = \frac{n-6}{6}$ and $\Gamma(v_i) = K_{1,s}$, where $s = \frac{n-12}{6}$. In this case, G is exactly the graph obtained from a graph $F \in \{F_1, F_2\}$ by adding $\frac{n-12}{6}$ pendant edges at each vertex of degree two of F. Thus, $G \in \mathcal{F}(n, \frac{n-12}{6})$, and so $cl(H) \in \mathcal{R}_{\mathcal{F}}(n, \frac{n-12}{6})$. $\square$

Chapter 6

Generalized Dirac-type conditions for traceability

Dirac's classic minimum degree result, Theorem 2.1, inspired much, if not all of the research on degree conditions for hamiltonian properties in graphs. In [44, 62], the authors obtained many generalizations of Dirac's Theorem. In this chapter, we are interested in the minimum cardinality of the neighborhood union over all sets of t vertices that guarantees the traceability of a 2-connected claw-free graph.

6.1 Introduction

As we have seen throughout this thesis, starting from Theorem 2.1, many researchers went in different directions in an attempt to obtain more general results, as well as counterparts for restricted graph classes. In this chapter, our starting point is a result due to Faudree et al. from [46]. There the authors used $\delta_2(H)$ to present the following sufficient degree condition for the hamiltonicity of claw-free graphs.

Theorem 6.1. (Faudree et al. [46]). *Let H be a 2-connected claw-free graph of order n with $\delta_2(H) \geq \frac{n+1}{3}$. Then for n sufficiently large, H is hamiltonian.*

As in Chapter 4, we define $N(p,\epsilon) = \max\{36p^2 - 34p - \epsilon(p+1), 20p^2 - 10p - \epsilon(p+1), (3p+1)(-\epsilon - 4p)\}$, where p is an integer, and ϵ is a given real number. In [36], Chen et al. further generalized Dirac's Theorem and obtained the following result.

Theorem 6.2. (Chen et al. [36]). *Let H be a k-connected claw-free graph of order n ($k \in \{2,3\}$) with $\delta(H) \geq 3$. Let $cl(H) = L(G)$. For given integers $p \geq t > 0$ and a given real number ϵ, if $\delta_t(H) \geq \frac{t(n+\epsilon)}{p}$ and $n > N(p,\epsilon)$, then either H is hamiltonian or $G_0' \in \mathcal{Q}_0(c,k)$, where $c \leq \max\{p/t + 2t, 3p/t + 2t - 7\}$ and G_0' does not have a DCT containing all the nontrivial vertices.*

Since the condition $\delta_t(H) \geq \frac{t(n+\epsilon)}{p}$ and conditions involving $\{\sigma_t(H), U_t(H)\}$ have quite different implications on the structure of the graphs under consideration, Chen et al. were able to give a much better upper bound for $|V(G_0')|$ in Theorem 6.2.

For 2-connected claw-free graphs, they obtained the following result.

Theorem 6.3. (Chen et al. [36]). *Let H be a 2-connected claw-free graph of order n, for n sufficiently large. For given integers p and t with $2 \leq t \leq 4$ and $p/t \leq 3$, if $\delta_t(H) \geq \frac{t(n+1)}{p}$ (i.e., $\delta_t(H) \geq \frac{n+1}{3}$), then H is hamiltonian.*

6.2 Our results

In the following, we first obtain the following analogue of Theorem 6.2 for traceability of claw-free graphs.

Theorem 6.4. *Let H be a k-connected claw-free graph of order n ($k \in \{2,3\}$) with $\delta(H) \geq 3$. Let $cl(H) = L(G)$. For given integers $p \geq t > 0$ and a given real number ϵ, if $\delta_t(H) \geq \frac{t(n+\epsilon)}{p}$ and $n > N(p,\epsilon)$, then either H is traceable or $G_0' \in \mathcal{R}_0(c,k)$, where $c \leq \max\{p/t + 2t, 3p/t + 2t - 7\}$ and G_0' does not have a DT containing all the nontrivial vertices.*

We postpone the proof of the above theorem, and continue with listing some applications of it for different values of t. As applications of Theorem 6.4 with given values for p, t, ϵ, k, we prove the following four results.

Theorem 6.5. *Let H be a 2-connected claw-free graph of order n with $\delta(H) \geq 3$, and let n be sufficiently large. If $\delta_2(H) \geq \frac{2(n+8)}{12}$, then H is traceable.*

Theorem 6.6. *Let H be a 2-connected claw-free graph of order n with $\delta(H) \geq 3$, and let n be sufficiently large. If $\delta_3(H) \geq \frac{3(n+6)}{15}$, then H is traceable.*

Theorem 6.7. *Let H be a 2-connected claw-free graph of order n with $\delta(H) \geq 3$, and let n be sufficiently large. If $\delta_4(H) \geq \frac{4(n+4)}{16}$, then H is traceable.*

Theorem 6.8. *Let H be a 2-connected claw-free graph of order n with $\delta(H) \geq 3$, and let n be sufficiently large. If $\delta_5(H) \geq \frac{5(n-1)}{15}$, then H is traceable.*

Remark 6.1. (a) Let G^* be the graph obtained from F_1 or F_2 by adding $\frac{n-12}{6} \geq 2$ pendant edges (for a suitable choice of n) at each vertex of degree two of F_1 or F_2. Then it is easy to check that $\delta_2(L(G^*)) = \frac{n}{6} < \frac{2(n+8)}{12}$ and $\delta(L(G^*)) \geq 3$. This example shows that the bound in Theorem 6.5 is asymptotically sharp.

(b) Obviously, Theorem 6.3 cannot imply our Theorems 6.5, 6.6, 6.7, and 6.8. Furthermore, Theorems 6.5, 6.6, 6.7, and 6.8 are improvements of Theorems 5.3 and 5.5, in some sense (if we replace the bound for $d_t(H)$ by the bound for $\delta_t(H)$ in Theorems 5.3 and 5.5, respectively) for $t = 2, 3, 4, 5$, respectively.

6.3 Notation and a technical lemma

In this section, let H be a k-connected claw-free graph of order $n > N(p, \epsilon)$ with $k \in \{2, 3\}$, where $t \geq 1$ and $p \geq t$ are positive integers, and ϵ is a given real number, and $N(p, \epsilon) = \max\{36p^2 - 34p - \epsilon(p+1), 20p^2 - 10p - \epsilon(p+1), (3p+1)(-\epsilon - 4p)\}$. Moreover, we assume that $\delta(H) \geq 3$ and $\delta_t(H) \geq \frac{t(n+\epsilon)}{p}$, and that $cl(H) = L(G)$, and we let G, G_0 and G_0' be the graphs defined in Section 1.4.

For $v \in V(G_0')$, we let $\Gamma_0(v)$ be the collapsible preimage of v in G_0, and we let $\Gamma(v)$ be the preimage of v in G. We also use the following notation.

- $\diamond$ $S_0 = \{v \in V(G_0') \mid v \text{ is a nontrivial vertex in } G_0'\}$;

- $S_t = \{v \in V(S_0) \mid |E(\Gamma(v))| \geq t\}$;
- $S_1 = \{v \in V(S_0) \mid 1 \leq |E(\Gamma(v))| \leq t-1\}$;
- $S^* = S_0 \backslash (S_t \cup S_1)$, the set of vertices $v \in S_0$ with $\Gamma(v) = K_1$ and adjacent to some vertices in $D_2(G)$;
- $V_0 = V(G_0') \setminus (S_t \cup S_1)$, the set of vertices v with $\Gamma(v) = K_1$ in G, which includes S^*;
- $\Phi_0 = G_0'[V_0 \cup S_1]$;
- $E_0 = E(\Phi_0)$ is the set of edges in Φ_0;
- V_E is the set of vertices incident with some edges in E_0;
- $E_R = \cup_{v \in S_1} E(\Gamma(v))$ and $\Phi^* = G[E_0 \cup E_R]$ (and $E_0 \cup E_R = E(\Phi^*)$);
- $U_0 = V_0 \setminus V_E$, and so $V(G_0') = S_t \cup S_1 \cup V_0 = S_t \cup S_1 \cup V_E \cup U_0$.

Since $\overline{\sigma}_2(G) \geq 5$, by the definition of G_0', $D_2(G_0') \subseteq S_t \cup S_1$, U_0 is an independent set, and $N_{G_0'}(x) \subseteq S_t$ for $x \in U_0$. We will use the properties that are stated in the following recent result due to Chen et al. [36].

Lemma 6.9. (Chen et al. [36]). *With the notation defined above, each of the following holds:*

(a) *for each* $v \in S_t$, $|E(\Gamma(v))| \geq \delta_t(H) - d_{G_0'}(v)$;

(b) $|S_t| \leq p/t$. *If* $|S_t| = p/t$, *then* $|E(G_0')| \geq \epsilon + \sum_{v \in S_1} d_{G_0'}(v) + \sum_{v \in V_0} d_{G_0'}(v) + \sum_{v \in S_1} |E(\Gamma(v))|$;

(c) $|E_0| + |E_R| = |E(\Phi^*)| \leq t-1$ *and* $|S_1 \cup V_E| \leq 2|S_1| + |V_E| \leq 2(t-1)$;

(d) $U_0 \leq max\{2, 2|S_t| + 5\} \leq max\{2, 2p/t - 5\}$.

6.4 Proofs of Theorems 6.4, 6.5, 6.6, 6.7, and 6.8

In this section, we will present our proofs of Theorems 6.4, 6.5, 6.6, 6.7, and 6.8.

Proof of Theorem 6.4. Suppose that H is not traceable. Then, in particular, H is not hamiltonian and H is not complete. Let G_0' be the reduction of the core G_0 of G, and let $c = |V(G_0')|$. By Theorem 1.8, $\kappa'(G_0') \geq \kappa'(G_0) \geq k$. Since H is not traceable, by Theorems 1.2 and 1.4, G has no DT. By Theorem 1.9, G_0' has no DT containing all the nontrivial vertices, so, in particular, G_0' has no DCT containing all the nontrivial vertices. By Theorem 6.2, $G_0' \in \mathscr{R}_0(c,k)$, where $c \leq \max\{p/t+2t, 3p/t+2t-7\}$. This completes the proof. $\square$

Proof of Theorem 6.5. This is a special case of Theorem 6.4 with $p = 12$, $t = 2$, $\epsilon = 8$, and $k = 2$. By Theorem 1.3, there is an essentially 2-edge-connected triangle-free graph G such that $cl(H) = L(G)$ and $|E(G)| = |V(H)|$. Since $\delta(cl(H)) \geq \delta(H) \geq 3$, $\overline{\sigma}_2(G) \geq 5$. Let G_0' be the reduction of the core G_0 of G. Suppose that H is not traceable. By Theorem 6.4, $G_0' \in \mathscr{R}_0(c,2)$ and G_0' has no DT containing all the nontrivial vertices. Then $G_0' \neq K_{2,a}$ $(a \geq 2)$; otherwise, G_0' has an ST, a contradiction. By Theorem 2.9, $|V(G_0')| \geq 10$.

Let S_0, S_t, S_1, V_0, Φ_0, E_0, V_E and U_0 be sets relating to G_0' as defined in Section 6.3. Then G_0' does not have a DT containing S_0. By Lemma 6.9(b), $|S_t| \leq p/t = 6$. By Lemma 6.9(c), $2|S_1|+|V_E| \leq 2$, and so $|S_1| \leq 1$. By Lemma 6.9(d), $|U_0| \leq \max\{2, 2p/t-5\} = 7$. Note that $V(G_0') = S_t \cup S_1 \cup V_0 = S_t \cup S_1 \cup V_E \cup U_0$. Then $|V(G_0')| \leq |S_t|+|S_1|+|V_E|+|U_0| \leq 15$. Recall that $d_{G_0'}(v) \geq 3$ for $v \in V_0$. We first prove the following claim and then distinguish the cases that $|S_1| = 0$ and $|S_1| = 1$.

Claim 1. $|S_t| \leq 5$.

Proof. By contradiction, suppose that $|S_t| = 6$. By Lemma 6.9(b), $|E(G_0')| \geq \epsilon+2|S_1|+3|V_0|+|S_1| = 3|S_1|+3|V_0|+\epsilon$. Since $G_0' \neq K_{2,a}$ $(a \geq 2)$, by Theorem 1.5(c), $|E(G_0')| \leq 2|V(G_0')|-5 = 2(|S_t|+|S_1|+|V_0|)-5 = 2|S_1|+2|V_0|+7$. Then $|S_1|+|V_0| \leq 7-\epsilon = -1$, a contradiction. $\square$

Case 1. $|S_1| = 0$.

Then $|S_t|+|S_1| \leq 5$. By Lemma 6.9(c), $|E_0| \leq 1$ and $|V_E| \leq 2$. If $|V(G_0')| \leq 11$, then, by Theorem 2.9, and since G_0' has no ST, $G_0' \in \{F_1, F_2, G_1, G_2, \ldots, G_6\}$. Since $D_2(G_0') \subseteq S_t \cup S_1$, this implies $|S_t|+|S_1| \geq 6$, in contradiction with

$|S_t| + |S_1| \le 5$. Hence, $|V(G_0')| \ge 12$, and so $5 \le |U_0| \le 7$. We treat the three subcases separately.

Suppose first that $|U_0| = 5$. Then $|V(G_0')| = 12$, and so $|S_t| = 5$ and $|V_E| = 2$. Since $G_0' \neq K_{2,a}$ $(a \ge 2)$, by Theorem 1.5(c), $|E(G_0')| \le 2|V(G_0')| - 5 = 19$. By the definitions of E_0 and V_E, $|E_0| = 1$ and $e(V_E, S_t) \ge 4$. Then $|E(G_0')| \ge e(V_E, S_t) + |E_0| + 3|U_0| \ge 20$, contrary to $|E(G_0')| \le 19$.

Next suppose that $|U_0| = 6$. Then $12 \le |V(G_0')| \le 13$. Then $|V_E| \neq 0$. By the definition of V_E, $|V_E| = 2$, and so $|E_0| = 1$. Then $e(V_E, S_t) \ge 4$, and so $|E(G_0')| \ge e(V_E, S_t) + |E_0| + 3|U_0| \ge 23$. Since $G_0' \neq K_{2,a}$ $(a \ge 2)$, by Theorem 1.5(c), $|E(G_0')| \le 2|V(G_0')| - 5 \le 21$, contrary to $|E(G_0')| \ge 23$.

Finally suppose that $|U_0| = 7$. First assume $|V_E| = 0$. Then $|V(G_0')| = 12$, and so $|S_t| = 5$. Since $G_0' \neq K_{2,a}$ $(a \ge 2)$, by Theorem 1.5(c), $|E(G_0')| \le 2|V(G_0')| - 5 = 19$. Then $|E(G_0')| \ge 3|U_0| = 21$, contrary to $|E(G_0')| \le 19$. Hence, $|V_E| \neq 0$. By Lemma 6.9(c), $|V_E| \le 2$. Since $|V_E| \neq 0$, by the definition of V_E, $|V_E| = 2$. Then $12 \le |V(G_0')| \le 14$. Since $G_0' \neq K_{2,a}$ $(a \ge 2)$, by Theorem 1.5(c), $|E(G_0')| \le 2|V(G_0')| - 5 \le 23$. By the definitions of E_0 and V_E, $|E_0| = 1$ and so $e(V_E, S_t) \ge 4$. Then $|E(G_0')| \ge e(V_E, S_t) + |E_0| + 3|U_0| \ge 26$, contrary to $|E(G_0')| \le 23$.

Case 2. $|S_1| = 1$.

Then $|S_t| + |S_1| \le 6$. By Lemma 6.9(c), $|E_0| = |V_E| = 0$. First assume $|V(G_0')| \le 11$. Then, by Theorem 2.9 and since G_0' has no ST, $G_0' \in \{F_1, F_2, G_1, G_2, \ldots, G_6\}$. Since $D_2(G_0') \subseteq S_t \cup S_1$ and using $|S_t| + |S_1| \le 6$, $G_0' \in \{F_1, F_2, G_6\}$. Then, using Claim 1, $E(\Phi_0) = E(G_0'[V_0 \cup S_1]) = E_0 \neq \emptyset$, contrary to $|E_0| = 0$. Hence, $12 \le |V(G_0')| \le 13$, and so $6 \le |U_0| \le 7$. Since $|S_1| = 1$ and using $|E_0| = 0$, we get that $e(S_1, S_t) \ge 2$.

Suppose first that $|U_0| = 6$. Then $|V(G_0')| = 12$ and $|E(G_0')| \ge e(S_1, S_t) + 3|U_0| \ge 20$. Since $G_0' \neq K_{2,a}$ $(a \ge 2)$, by Theorem 1.5(c), $|E(G_0')| \le 2|V(G_0')| - 5 = 19$, contrary to $|E(G_0')| \ge 20$.

Suppose next that $|U_0| = 7$. Then $|V(G_0')| \le 13$ and $|E(G_0')| \ge e(S_1, S_t) + 3|U_0| \ge 23$. Since $G_0' \neq K_{2,a}$ $(a \ge 2)$, by Theorem 1.5(c), $|E(G_0')| \le 2|V(G_0')| - 5 \le 21$, contrary to $|E(G_0')| \ge 23$. This completes the proof. $\square$

Proof of Theorem 6.6. This is a special case of Theorem 6.4 with $p = 15$, $t = 3$, $\epsilon = 6$, and $k = 2$. By Theorem 1.3, there is an essentially 2-edge-connected triangle-free graph G such that $cl(H) = L(G)$ and $|E(G)| = |V(H)|$. Since $\delta(cl(H)) \geq \delta(H) \geq 3$, $\overline{\sigma}_2(G) \geq 5$. Let G_0' be the reduction of the core G_0 of G. Suppose that H is not traceable. By Theorem 6.4, $G_0' \in \mathscr{R}_0(c, 2)$ and G_0' has no DT containing all the nontrivial vertices. Then $G_0' \neq K_{2,a}$ $(a \geq 2)$; otherwise, G_0' has an ST, a contradiction. By Theorem 2.9, $|V(G_0')| \geq 10$.

Let S_0, S_t, S_1, V_0, Φ_0, E_0, V_E, E_R, and U_0 be sets relating to G_0' as defined in Section 6.3. Then G_0' does not have a DT containing S_0. By Lemma 6.9(b), $|S_t| \leq p/t = 5$. By Lemma 6.9(c), $2|S_1| + |V_E| \leq 4$, and so $|S_1| \leq 2$. By Lemma 6.9(d), $|U_0| \leq \max\{2, 2p/t - 5\} = 5$. Note that $V(G_0') = S_t \cup S_1 \cup V_0 = S_t \cup S_1 \cup V_E \cup U_0$. Then $|V(G_0')| \leq |S_t| + |S_1| + |V_E| + |U_0| \leq 14$. Recall that $d_{G_0'}(v) \geq 3$ for $v \in V_0$. We first prove the following two claims and then distinguish the cases that $|S_1| = 0$, $|S_1| = 1$, and $|S_1| = 2$.

Claim 1. If $|S_t| + |S_1| \leq 5$, then $|V(G_0')| \geq 12$.

Proof. By contradiction. Suppose that $|V(G_0')| \leq 11$. Then, by Theorem 2.9 and since G_0' has no ST, $G_0' \in \{F_1, F_2, G_1, G_2, \ldots, G_6\}$. Since $D_2(G_0') \subseteq S_t \cup S_1$, $|S_t| + |S_1| \geq 6$, contrary to $|S_t| + |S_1| \leq 5$. □

Claim 2. $|S_t| \leq 4$.

Proof. By contradiction, suppose that $|S_t| = 5$. By Lemma 6.9(b), $|E(G_0')| \geq \epsilon + 2|S_1| + 3|V_0| + |S_1| = 3|S_1| + 3|V_0| + \epsilon$. Since $G_0' \neq K_{2,a}$ $(a \geq 2)$, by Theorem 1.5(c), $|E(G_0')| \leq 2|V(G_0')| - 5 = 2(|S_t| + |S_1| + |V_0|) - 5 = 2|S_1| + 2|V_0| + 5$. Then $|S_1| + |V_0| \leq 5 - \epsilon = -1$, a contradiction. □

Case 1. $|S_1| = 0$.
Then $|S_t| + |S_1| \leq 4$. By Claim 1, $|V(G_0')| \geq 12$. By Lemma 6.9(c), $|E_0| \leq 2$ and $|V_E| \leq 4$. Then $|U_0| \geq 4$.

Suppose first that $|U_0| = 4$. Then $|V(G_0')| = 12$, and so $|S_t| = 4$ and $|V_E| = 4$. Since $G_0' \neq K_{2,a}$ $(a \geq 2)$, by Theorem 1.5(c), $|E(G_0')| \leq 2|V(G_0')| - 5 = 19$. Then, by the definitions of E_0 and V_E, $|E_0| = 2$, and so $e(V_E, S_t) \geq 2 \times 4 = 8$. Then $|E(G_0')| \geq e(V_E, S_t) + |E_0| + 3|U_0| \geq 22$, contrary to $|E(G_0')| \leq 19$.

Suppose next that $|U_0| = 5$. Then $12 \le |V(G_0')| \le 13$, and so $3 \le |V_E| \le 4$. Since $G_0' \ne K_{2,a}$ $(a \ge 2)$, by Theorem 1.5(c), $|E(G_0')| \le 2|V(G_0')| - 5 \le 21$. Then, by the definitions of E_0 and V_E, $|E_0| = 2$, and either Φ_0 has an induced subgraph isomorphic to $K_{1,2}$ or Φ_0 has an induced subgraph isomorphic to $2K_2$. For the first case, $|E(G_0')| \ge e(V_E, S_t) + |E_0| + 3|U_0| \ge 5 + 2 + 3 \times 5 = 22$. For the second case, $|E(G_0')| \ge e(V_E, S_t) + |E_0| + 3|U_0| \ge 2 \times 4 + 2 + 3 \times 5 = 25$. In both cases we obtain a contradiction with $|E(G_0')| \le 19$.

Case 2. $|S_1| = 1$.
Then $|S_t| + |S_1| \le 5$. By Claim 1, $|V(G_0')| \ge 12$. Since $|S_1| = 1$, $|E_R| \ge 1$. By Lemma 6.9(c), $|E_0| \le 1$ and $|V_E| \le 2$. Then $|V(G_0')| = 12$, and so $|U_0| = 5$, $|S_t| = 4$, $|V_E| = 2$, and $S_1 \cap V_E = \emptyset$. Since $G_0' \ne K_{2,a}$ $(a \ge 2)$, by Theorem 1.5(c), $|E(G_0')| \le 2|V(G_0')| - 5 = 19$. Note that $S_1 \cap V_E = \emptyset$. By the definitions of E_0 and V_E, $|E_0| = 1$, and so $e(V_E, S_t) \ge 4$. Then $|E(G_0')| \ge e(V_E, S_t) + |E_0| + e(S_1, S_t) + 3|U_0| \ge 4 + 1 + 2 + 3 \times 5 = 22$, contrary to $|E(G_0')| \le 19$.

Case 3. $|S_1| = 2$.
Then $|S_t| + |S_1| \le 6$. By Lemma 6.9(c), $|V_E| = 0$, and so $|E_0| = 0$. This implies that $|V(G_0')| \le 11$. By Theorem 2.9 and since G_0' has no ST, $G_0' \in \{F_1, F_2, G_1, G_2, \dots, G_6\}$. Since $D_2(G_0') \subseteq S_t \cup S_1$ and using $|S_t| + |S_1| \le 6$, $G_0' \in \{F_1, F_2, G_6\}$. Since $|S_t| \le 4$, $E(\Phi_0) = E(G_0'[V_0 \cup S_1]) = E_0 \ne \emptyset$, contrary to $|E_0| = 0$. This completes the proof. □

Proof of Theorem 6.7. This is a special case of Theorem 6.4 with $p = 16$, $t = 4$, $\epsilon = 4$, and $k = 2$. By Theorem 1.3, there is an essentially 2-edge-connected triangle-free graph G such that $cl(H) = L(G)$ and $|E(G)| = |V(H)|$. Since $\delta(cl(H)) \ge \delta(H) \ge 3$, $\overline{\sigma}_2(G) \ge 5$. Let G_0' be the reduction of the core G_0 of G. Suppose that H is not traceable. By Theorem 6.4, $G_0' \in \mathscr{R}_0(c, 2)$ and G_0' has no DT containing all the nontrivial vertices. Then $G_0' \ne K_{2,a}$ $(a \ge 2)$; otherwise, G_0' has an ST, a contradiction. By Theorem 2.9, $|V(G_0')| \ge 10$.

Let S_0, S_t, S_1, V_0, V_E, E_R, and U_0 be sets relating to G_0' as defined in Section 6.3. Then G_0' does not have a DT containing S_0. By Lemma 6.9(b), $|S_t| \le p/t = 4$. By Lemma 6.9(c), $2|S_1| + |V_E| \le 6$, and so $|S_1| \le 3$. By Lemma 6.9(d), $|U_0| \le \max\{2, 2p/t - 5\} = 3$. Note that $V(G_0') = S_t \cup S_1 \cup V_0 =$

$S_t \cup S_1 \cup V_E \cup U_0$. Then $|V(G_0')| \le |S_t| + |S_1| + |V_E| + |U_0| \le 13$. Recall that $d_{G_0'}(v) \ge 3$ for $v \in V_0$. We first prove the following claim.

Claim 1. $|S_t| \le 3$.

Proof. By contradiction, suppose that $|S_t| = 4$. By Lemma 6.9(b), $|E(G_0')| \ge \epsilon + 2|S_1| + 3|V_0| + |S_1| = 3|S_1| + 3|V_0| + \epsilon$. Since $G_0' \ne K_{2,a}$ $(a \ge 2)$, by Theorem 1.5(c), $|E(G_0')| \le 2|V(G_0')| - 5 = 2(|S_t| + |S_1| + |V_0|) - 5 = 2|S_1| + 2|V_0| + 3$. Then $|S_1| + |V_0| \le 3 - \epsilon = -1$, a contradiction. □

Using Claim 1, we first assume that $|S_t| \le 2$. Then $|V(G_0')| \le 11$. By Theorem 2.9 and since G_0' has no ST, $G_0' \in \{F_1, F_2, G_1, G_2, \ldots, G_6\}$. Since $D_2(G_0') \subseteq S_t \cup S_1$, $|S_t| + |S_1| \ge 6$. Then $|S_1| \ge 4$, contrary to $|S_1| \le 3$. Hence, $|S_t| = 3$, and so $|V(G_0')| \le 12$.

Suppose first that $|S_1| = 0$. Since $|V(G_0')| \le 12$, $|V_0| \le 9$. Assume that $|V_0| \le 8$. Then $|V(G_0')| \le 11$. By Theorem 2.9 and since G_0' has no ST, $G_0' \in \{F_1, F_2, G_1, G_2, \ldots, G_6\}$. Since $D_2(G_0') \subseteq S_t \cup S_1$, $|S_t| + |S_1| \ge 6$, contrary to $|S_t| + |S_1| = 3$. Hence, $|V_0| = 9$, and so $|V(G_0')| = 12$. This equality holds only if $|V_E| = 6$ and $|U_0| = 3$. Then $|E(G_0')| \le 2|V(G_0')| - 5 = 19$. By Lemma 6.9(c), $|E_0| + |E_R| \le 3$, and so $|E_0| \le 3$. Then, by the definitions of V_E and E_0, $|E_0| = 3$, and so $e(V_E, S_t) \ge 3 \times 4 = 12$. Then $|E(G_0')| \ge e(V_E, S_t) + |E_0| + 3|U_0| \ge 24$, contrary to $|E(G_0')| \le 19$.

Next suppose that $|S_1| = 1$. By Lemma 6.9(c), $|V_E| \le 4$. Then $|V(G_0')| \le |S_t| + |S_1| + |V_E| + |U_0| \le 11$. By Theorem 2.9 and since G_0' has no ST, $G_0' \in \{F_1, F_2, G_1, G_2, \ldots, G_6\}$. Since $D_2(G_0') \subseteq S_t \cup S_1$, $|S_t| + |S_1| \ge 6$, contrary to $|S_t| + |S_1| = 4$.

Now suppose that $|S_1| = 2$. By Lemma 6.9(c), $|V_E| \le 2$. Then $|V(G_0')| \le |S_t| + |S_1| + |V_E| + |U_0| \le 10$. By Theorem 2.9 and since G_0' has no ST, $G_0' \in \{F_1, F_2\}$. Since $D_2(G_0') \subseteq S_t \cup S_1$, $|S_t| + |S_1| \ge 6$, contrary to $|S_t| + |S_1| = 5$.

Finally suppose that $|S_1| = 3$. By Lemma 6.9(c), $|V_E| = 0$. Then $|V(G_0')| \le |S_t| + |S_1| + |V_E| + |U_0| \le 9$, contrary to $|V(G_0')| \ge 10$. This completes the proof. □

Proof of Theorem 6.8. This is a special case of Theorem 6.4 with $p = 15$, $t = 5$, $\epsilon = -5$, and $k = 2$. By Theorem 1.3, there is an essentially 2-edge-connected triangle-free graph G such that $cl(H) = L(G)$ and $|E(G)| = |V(H)|$.

Since $\delta(cl(H)) \geq \delta(H) \geq 3$, $\overline{\sigma}_2(G) \geq 5$. Let G_0' be the reduction of the core G_0 of G. Suppose that H is not traceable. By Theorem 6.4, $G_0' \in \mathscr{R}_0(c,2)$ and G_0' has no DT containing all the nontrivial vertices. Then $G_0' \neq K_{2,a}$ $(a \geq 2)$; otherwise, G_0' has an ST, a contradiction. By Theorem 2.9, $|V(G_0')| \geq 10$.

Let S_0, S_t, S_1, V_0, V_E, and U_0 be sets relating to G_0' as defined in Section 6.3. Then G_0' does not have a DT containing S_0. By Lemma 6.9(b), $|S_t| \leq p/t = 3$. By Lemma 6.9(c), $2|S_1| + |V_E| \leq 8$, and so $|S_1| \leq 4$. By Lemma 6.9(d), $|U_0| \leq \max\{2, 2p/t - 5\} = 2$. Note that $V(G_0') = S_t \cup S_1 \cup V_0 = S_t \cup S_1 \cup V_E \cup U_0$. Then $|V(G_0')| \leq |S_t| + |S_1| + |V_E| + |U_0| \leq 13$. We distinguish the cases that $|S_t| \leq 1$, $|S_t| = 2$, and $|S_t| = 3$.

Suppose first that $|S_t| \leq 1$. Then $|V(G_0')| \leq 11$. By Theorem 2.9 and since G_0' has no ST, $G_0' \in \{F_1, F_2, G_1, G_2, \ldots, G_6\}$. Since $D_2(G_0') \subseteq S_t \cup S_1$, $|S_t| + |S_1| \geq 6$, contrary to $|S_t| + |S_1| \leq 5$.

Next suppose that $|S_t| = 2$. Then $|V(G_0')| \leq 12$. If $|V(G_0')| \leq 11$, then, by Theorem 2.9 and since G_0' has no ST, $G_0' \in \{F_1, F_2, G_1, G_2, \ldots, G_6\}$. Since $D_2(G_0') \subseteq S_t \cup S_1$, $|S_t| + |S_1| \geq 6$. Then $|S_1| = 4$, and so $|V_E| = 0$. Then $|V(G_0')| \leq |S_t| + |S_1| + |V_E| + |U_0| \leq 8$, contrary to $|V(G_0')| \geq 10$. Hence, $|V(G_0')| = 12$, the equality holds only if $|S_1| = 0$, $|V_E| = 8$ and $|U_0| = 2$. Since U_0 is an independent set, and since $N_{G_0'}(x) \subseteq S_t$ for $x \in U_0$, $|S_t| \geq 3$, contrary to $|S_t| = 2$.

Finally suppose that $|S_t| = 3$. Recall that $d_{G_0'}(v) \geq 3$ for $v \in V_0$. By Lemma 6.9(b), $|E(G_0')| \geq \epsilon + 2|S_1| + 3|V_0| + |S_1| = 3|S_1| + 3|V_0| + \epsilon$. Since $G_0' \neq K_{2,a}$ $(a \geq 2)$, by Theorem 1.5(c), $|E(G_0')| \leq 2|V(G_0')| - 5 = 2(|S_t| + |S_1| + |V_0|) - 5 = 2|S_1| + 2|V_0| + 1$. Then $|S_1| + |V_0| \leq 1 - \epsilon = 6$, and so $|V(G_0')| \leq |S_t| + |S_1| + |V_0| \leq 9$, contrary to $|V(G_0')| \geq 10$. This completes the proof. $\square$

Summary

This thesis contains many new contributions to the field of hamiltonian graph theory, a very active subfield of graph theory. In particular, we have obtained new sufficient minimum degree and degree sum conditions to guarantee that the graphs satisfying these conditions, or their line graphs, admit a Hamilton cycle (or a Hamilton path), unless they have a small order or they belong to well-defined classes of exceptional graphs. Here, a Hamilton cycle corresponds to traversing the vertices and edges of the graph in such a way that all their vertices are visited exactly once, and we return to our starting vertex (similarly, a Hamilton path reflects a similar way of traversing the graph, but without the last restriction, so we might terminate at a different vertex).

Degree conditions are the classic approach to hamiltonian problems. All our results are motivated by Dirac's Theorem and Ore's Theorem, that date back to the1950s and 1960s. However, many of our results are based on some very recent results for hamiltonicity of claw-free graphs involving the degree sums of adjacent vertices and other degree and neighborhood conditions. The proofs of our results also rely on some beautiful powerful techniques developed by Catlin and Ryjáček, and recent work due to Chen and his coauthors. With our work we have successfully tried to unify and extend several existing results.

In Chapter 1, we present an introduction to the topics of this thesis together with Ryjáček's closure for claw-free graphs, Catlin's reduction method, and the reduction of the core of a graph. Using these tools, investigating the hamiltonicity or traceability of a claw-free graph H is equivalent to investigating the existence of a dominating closed trail or dominating trail in a

graph G for which the line graph $L(G) = cl(H)$, the closure of H. Specific terminology and notation are also presented in this chapter.

In Chapter 2, we consider the traceability of 2-connected claw-free graphs with specific conditions on degree sums of pairs of adjacent vertices. Our main results imply sharp lower bounds on the minimum degree and the degree sums of adjacent pairs of vertices for the traceability of 2-connected claw-free graphs with sufficiently large order. We also improved some known results. In this chapter, we also characterized the exceptional graphs, not containing a spanning trail, among the 2-edge-connected graphs of order at most 11. We have demonstrated that these exceptional graphs are very useful in dealing with sufficient conditions for traceability of claw-free graphs. These eight exceptional graphs are repeatedly used in Chapters 2, 5 and 6.

In Chapter 3, our results contribute to an old conjecture of Benhocine et al. and the more recent Conjecture 3.1 due to Chen and Lai. Our Theorems 3.9, 3.10 and 3.11 extend several known results. In this chapter, we also characterized other exceptional graphs among the 2-edge-connected triangle-free graphs of order at most 8. These exceptional graphs are very useful in dealing with sufficient conditions for hamiltonicity of claw-free graphs. These twenty-two exceptional graphs are also used in Chapter 4.

In Chapters 4 and 5, we consider sufficient minimum degree and degree sum conditions that imply that graphs admit a Hamilton cycle or a Hamilton path, unless they have a small order or they belong to well-defined classes of exceptional graphs. Our Theorems 4.11, 4.12, 5.3, 5.5 and 5.6 unify and extend several known earlier results. In particular, Theorem 5.6 provides a sharp lower bound on the degree sums of any independent vertex set with cardinality six for the traceability of 2-connected claw-free graph, and, as a corollary, a sharp lower bound on the minimum degree.

In Chapter 6, our results imply that several sufficient neighborhood union conditions force a claw-free graph to be traceable. We also improved some existing results by providing sharp bounds. In particular, the lower bound on the neighborhood union of any pair of vertices for the traceability of 2-connected claw-free graphs in Theorem 6.5 is asymptotically sharp.

Throughout this thesis, we have investigated the existence of Hamilton cycles and Hamilton paths under different types of degree and neighborhood conditions, including minimum degree conditions, minimum degree sum conditions on adjacent pairs of vertices, minimum degree sum conditions over all independent sets of t vertices of a graph, minimum cardinality conditions on the neighborhood union over all independent sets of t vertices of a graph, as well minimum cardinality conditions on the neighborhood union over all t vertex sets of a graph. Despite our new contributions, many problems and conjectures remain unsolved. Specifically, if one would be able to find the smallest 3-edge-connected graphs without a spanning trail, then using a similar approach, one would be able to establish new and sharp sufficient degree and neighborhood conditions for the traceability of 3-connected claw-free graphs.

For our future research, we would like to continue by considering different types of degree and neighborhood conditions related to the existence of disjoint cycles, even factors, as well as supereulerian properties, Hamilton-connectivity, and also continue our work on the hamiltonicity and traceability of graphs.

Samenvatting

Dit proefschrift bevat veel nieuwe bijdragen aan het gebied van de hamiltonse grafentheorie, een zeer actief deelgebied van de grafentheorie. De meeste van die bijdragen zijn gebaseerd op voldoende voorwaarden met betrekking tot de graden van de punten die garanderen dat deze grafen een Hamiltonpad of Hamiltoncykel bevatten, tenzij ze tot zekere welbeschreven klassen van grafen behoren of een klein aantal punten hebben. Zo'n Hamiltonpad of Hamiltoncykel correspondeert met het doorlopen van de punten en lijnen van de graaf op zo'n manier dat alle punten precies één keer doorlopen worden (en men in het beginpunt terugkeert in het geval van een Hamiltoncykel).

Graadvoorwaarden zijn de klassieke benadering binnen dit deelgebied, gemotiveerd door de stellingen van Dirac en van Ore die teruggaan tot de vijftiger en zestiger jaren van de vorige eeuw. Veel van de door ons gepresenteerde resultaten zijn echter gebaseerd op enkele zeer recente resultaten op het gebied van de existentie van Hamiltoncykels in klauw-vrije grafen, wederom met betrekking tot graad- en buurvoorwaarden. De bewijzen voor de gepresenteerde resultaten berusten tevens op twee prachtige reductietechnieken ontwikkeld door Catlin and Ryjáček, alsmede op recent werk van Chen en zijn coauteurs. Met ons gepresenteerde werk hebben we een geslaagde poging gedaan om bekende resultaten onder één noemer te brengen en uit te breiden.

Hoofdstuk 1 bevat een inleiding tot de onderwerpen van dit proefschrift, alsmede beschrijvingen van Ryjáček's afsluiting voor klauw-vrije grafen, de reductiemethode van Catlin, en de reductie van de core van een graaf. Met behulp van deze technieken wordt het onderzoeken van de existentie van Hamiltonpaden en Hamiltoncykels in een klauw-vrije graaf H equivalent met het onderzoeken van de existentie van een dominerende wandeling of gesloten wandeling in een graaf G waarvan de lijngraaf $L(G)$ gelijk is aan de afsluiting van H. Specifieke terminologie en notaties worden ook in dit eerste hoofdstuk gepresenteerd.

In Hoofdstuk 2 beschouwen we specifieke voorwaarden met betrekking tot paren buurpunten voor de existentie van Hamiltonpaden in 2-samenhangende klauw-vrije grafen. Onze hoofdresultaten impliceren een scherpe ondergrens voor de minimale graad en voor de minimale graadsom van buurparen voor deze existentie in grafen met een voldoende groot aantal punten. Hiermee verbeteren we een aantal reeds bekende resultaten. Tevens karakteriseren we de uitzonderingsgrafen die geen opspannende wandeling bevatten, onder alle 2-lijn-samenhangende grafen op ten hoogste elf punten. Deze acht uitzonderingsgrafen worden herhaaldelijk gebruikt in Hoofdstuk 2, 5 en 6.

De resultaten uit Hoofdstuk 3 hebben betrekking op een oud vermoeden van Benhocine et al. en een meer recent vermoeden van Chen en Lai (Conjecture 3.1 in het proefschrift). Onze stellingen Theorems 3.9, 3.10 en 3.11 in dit hoofdstuk zijn uitbreidingen van diverse bekende resultaten. Tevens karakteriseren we nieuwe uitzonderingsgrafen onder de 2-lijn-samenhangende driehoeksvrije grafen op ten hoogste acht punten. Deze uitzonderingsgrafen zijn zeer nuttig bij het bepalen van voldoende voorwaarden voor de existentie van Hamiltoncykels in klauw-vrije grafen. Deze 22 grafen worden ook gebruikt in Hoofdstuk 4.

In Hoofdstuk 4 en 5 beschouwen we voldoende minimale graadvoorwaarden en graadsomvoorwaarden voor de existentie van een Hamiltoncykel of een Hamiltonpad, tenzij de grafen weinig punten hebben of tot een welbeschreven klasse van uitzonderingsgrafen behoren. Onze stellingen Theorems 4.11, 4.12, 5.3, 5.5 en 5.6 unificeren diverse reeds bekende resultaten, en zijn algemener. Onze stelling Theorem 5.6 geeft een scherpe ondergrens voor de graadsom van zes onafhankelijke punten voor de existentie van een Hamiltonpad in een 2-samenhangende klauw-vrije graaf.

Onze resultaten uit Hoofdstuk 6 houden in dat diverse voldoende buurvoorwaarden afdwingen dat een klauw-vrije graaf een Hamiltonpad bevat. Tevens verbeteren we een aantal bestaande resultaten door scherpe grenzen te geven. Zo is de ondergrens voor de buurvereniging van paren punten uit onze stelling Theorem 6.5 asymptotisch scherp.

Dit proefschrift is doorspekt met resultaten die te maken hebben met de existentie van Hamiltoncykels en Hamiltonpaden, met name resultaten die gebaseerd zijn op verschillende typen buur- en graadvoorwaarden, zoals minimale graadvoorwaarden, minimale graadsomvoorwaarden voor buurparen, voor onafhankelijke verzamelingen, en buurverenigingen over alle deelverzamelingen van t punten. Niettegenstaande onze nieuwe bijdragen blijven veel open problemen en vermoedens onopgelost. Eén van die onopgeloste problemen zou opgelost kunnen worden als men in staat was de kleinste 3-lijn-samenhangende grafen te bepalen die geen opspannende wandeling hebben. Dit zou aanleiding geven tot nieuwe en scherpe voldoende graad- en buurvoorwaarden voor de existentie van Hamiltonpaden in 3-samenhangende klauw-vrije grafen.

Voor de toekomst richten we ons op het voortzetten van dit onderzoek naar de relatie tussen verschillende typen graad- en buurvoorwaarden en de existentie van disjuncte cykels, even factoren, zowel als voor supereulerse eigenschappen en Hamilton-samenhang, alsmede een voortzetting van ons werk op het gebied van de existentie van Hamiltoncykels en Hamiltonpaden.

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