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# Integral optimization of spare parts inventories in systems with redundancies

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## Abstract

In this paper, we analyze spare parts supply for a system with a “ $k$ -out-of- $N$ ” redundancy structure for key components, different standby policies (cold, warm and hot standby redundancy) and local spare parts inventories for sub-components. We assume multiple part types (sub-components) that fail randomly with exponentially distributed interfailure times. Due to the standby policies and the limited number of installed components, the total failure rate depends on the state of the system. Replacement times and stock replenishment times are also assumed to be exponentially distributed and depend on the part types. We present an exact method together with a simple and efficient approximation scheme for the evaluation of the system availability given certain stock levels. The proposed approximation is further used in a simple optimization heuristic to demonstrate how the total system costs can be reduced if the redundancy structure is optimized while taking

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into account the local stock of the spare parts. The presented numerical results clearly show the importance of the local inventories with spares even in the systems with redundancies.

*Keywords:* Maintenance, Spare parts, k-out-of-N systems, Availability, Cost optimization

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## 1. Introduction

Maintenance logistics is an important discipline that has received considerable attention both in practice and in the scientific literature. This importance is due to the high investments associated with capital-intensive assets which in turn require a high operational availability (e.g. in high tech production systems), and also due to the role of proper maintenance in preventing environmental damage (as happened in the 2010 BP accident in the Gulf of Mexico) or safety incidents (e.g. medical equipment, aircraft). Unplanned downtime of advanced capital equipment can be extremely expensive; for an average aircraft it is estimated at some \$10,000 per hour and for a high-tech lithography system in the semiconductor industry it may amount up to \$100,000 per hour. Consequently, unplanned downtime should be avoided as much as possible (e.g. by exploiting advanced condition monitoring techniques, built-in redundancy and preventive maintenance policies), and if it occurs, it should be kept as short as possible (by using optimal corrective maintenance policies). The latter implies that malfunctioning parts or components causing the system breakdown are immediately replaced by ready-for-use ones, if they are available on-site.

There are already many scientific studies that investigated different as-

pects of maintenance logistics based on several business cases from different economic sectors: defense (e.g. [Sherbrooke, 1968](#); [Rustenburg et al., 2000](#)), aviation (e.g. [Wu et al., 2004](#)) and railways (e.g. [Diaz and Fu, 1997](#)). Extensive reviews of problems occurring in maintenance logistics can be found in Basten and Van Houtum (2014); Diaz and Fu (1997); Muckstadt (2005); Sherbrooke (2004); Van Horenbeek et al. (2013); Van Houtum and Kranenburg (2015). Many of the existing results concentrate on the availability of parts assuming a highly simplified repair process where the repair facility has ample capacity.

Another common way to increase the reliability of equipment is to introduce redundancy into the system for the most critical components, as is done currently in military, aviation, oil/gas industry and many other applications (Elsayed, 2012). This approach alone (without taking into account the related logistical aspects) can lead to unnecessary costs as demonstrated in this paper.

The model presented in this paper was initiated by a case study at a public organization in Qatar where we analyzed the availability of their central chilling facility with built-in redundancies. The analyzed facility has a system (fleet) of six centrifugal pumps that are compressing the chilling agent for further usage in the central air-conditioning system. These pumps (components) can be switched on or off depending on their availability and current weather conditions; however, only three pumps are required for the full scale operation.

Each of the components (pumps) can fail due to one or another sub-component. In order to repair the failed component, a service engineer takes

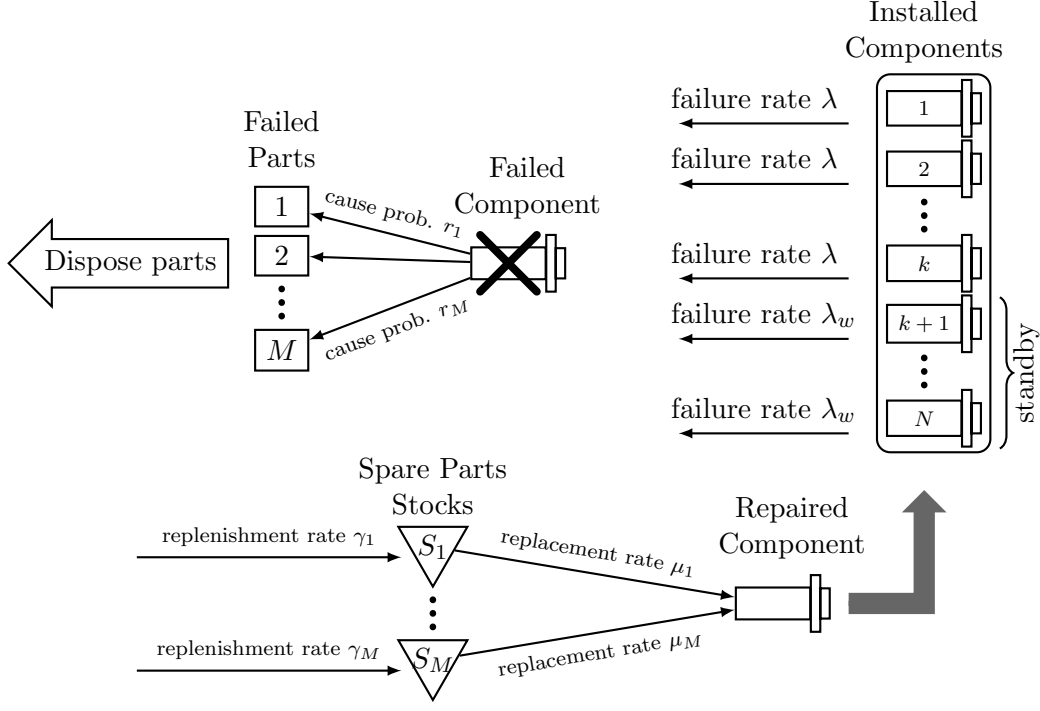


Figure 1: Graphical representation of the failure-repair process in the presented  $k$ -out-of- $N$  system.

a new sub-component (spare part) from available stock and replaces the failed one. The replacement process requires some time during which the pump is not available. The stock of the spare parts is replenished later on from a central stock or from a provider for the spare parts. The replenishment is done based on the  $(S-1, S)$  inventory policy.

Although the facility has almost double capacity due to relatively long replenishment times and no local stocks for the necessary spare parts, the total system availability (at full required capacity) slightly exceeds 92%. This study investigated increasing the system availability by stocking extra spare

parts, possibly in combination with reduction of the system redundancy level.

Similar systems can be encountered in many other situations where a set of heavy machines is required to function with high availability to ensure continuous operation. For example, reliable and consistent power supply is needed in many public and private sectors, such as hospitals, transportation networks, telecommunication, etc. This can be ensured by having available a set of large and expensive components (transformers, generators, switches, etc.) that form a reliable structure at reasonable costs.

Very few existing studies have integrated the logistics aspects and the reliability of the installed base into one model. Among the first ones are Gupta and Sharma (1981) and Fawzi and Hawkes (1991). They presented single item models with redundancy, repair process and inventories for spares. Another earlier study that took into account redundancy is (Sherbrooke, 2004, ch. 6), who optimized spare part inventory levels to attain a target availability of a system with hot standby redundancy at both system and component level and periodic resupply of the spare parts. The arrival process of parts is assumed to be stationary. [Cochran and Lewis \(2002\)](#) modified the model for a small installed base in which the arrival process of parts is significantly influenced by the number of systems being operational. [Godoy et al. \(2013, 2014\)](#) combined reliability aspects with the management of critical spare parts. [Costantino et al. \(2013\)](#) presented how classical spare parts inventory models (VARI-METRIC) are applied to the Italian Air Force.

[De Smidt-Destombes et al. \(2004, 2006, 2007\)](#) considered a  $k$ -out-of- $N$  system of a single component type supported by spare parts for different failure behavior, replacement strategies, and repair capacities. They focused

on relatively large values of  $N$  ( $\geq 100$ ). Selçuk and Ağralı (2013) also indicated the importance of joint spare parts inventory and reliability decisions. In that paper, authors analyzed a system where each component is critical for the functionality of the whole system, and the failure of a component interrupts the operation of the system. In addition, they introduced emergence supply if replacement components are not available in stock.

Chakravarthy and Gómez-Corral (2009) considered a single  $k$ -out-of- $N$  system with spare part delivery times and a single repair man to process failed parts. The spare part inventory level is not a decision variable, and the model is limited to a single part type. Van Jaarsveld and Dekker (2011) used reliability centered maintenance data to optimize spare part inventories for a redundant system. They assumed a positive, deterministic replacement time; also their model is limited to a single part type. Within the same range of single part type models, Xie et al. (2014) considered the joint problem of redundancy allocation and inventory optimization of repairable spare parts for  $k$ -out-of- $N$  hot standby systems in series, with a single server repair shop and strictly positive replacement times for the spare parts, in contrast to other works where the replacement times are zero (or negligible). Jin et al. (2015) considered multiple standby redundant systems with shared standby components and spares. They limited their analysis to a single part and focused on hot standby only. They showed the advantages of sharing redundant components between systems.

De Smidt-Destombes et al. (2011) considered a multi-item, single site spare part optimization problem for  $k$ -out-of- $N$  systems with cold standby redundancy and negligible component replacement times. They minimize

the spare part investment given a target value for the mission reliability with fixed mission length  $T$ . Sahba et al. (2013) studied the impact of dedicated versus pooled spare part inventories as well as dispatching rules for repaired components for multiple k-out-of-n systems consisting of a single component type. Installation times are zero in their model.

Our model is different from the other models described above in three main aspects. First, we allow failures due to multiple component types installed in the system. Most of the papers discussed above (except for Sherbrooke, 2004, ch. 6) assume a single type of installed components, which is not the case in many high-tech systems. The system presented in Sherbrooke (2004) has a specific redundancy type (hot standby) and periodic instead of continuous resupply. Second, we assume that the failures rates depend on the number of the components in the operating and in different standby modes. Most of the other papers (except for Gupta and Sharma 1981; Fawzi and Hawkes 1991; Xie et al. 2014) assume a constant failure rate. We also allow for different (and even mixed) standby modes, which affects the failure rates and complicates the analysis of the model. Finally, we cover strictly positive replacement times of the components received from stock in contrast to other papers with negligible replacement times. This factor has a large influence on the redundancy structure and on the complexity of the analysis, which we will demonstrate in the next sections. Xie et al. (2014) and Van Jaarsveld and Dekker (2011), to our best knowledge, are the only papers that take into account positive replacement times.

Summarizing the previous paragraph, we can say that our paper generalizes previous work by combining the following aspects into a single model:



(i) we cover positive replacement times, as this is an important aspect in the case that was the inspiration for this research (ii) we focus on steady state availability rather than mission reliability as in De Smidt-Destombes et al. (2011), (iii) we include state-dependent demand rates, which is important if the value of  $N$  is relatively small, (iv) we allow for partial system operation if the number of operational systems is positive but less than  $k$ , (v) we cover cold, warm and hot standby redundancy and do not limit the model to a single redundancy type, and (vi) we demonstrate the importance of spare parts stocks in systems with built-in redundancy.

To analyze these aspects we provide: (1) an exact computation method for the availability of a “ $k$ -out-of- $N$ ” system in which each component may consist of multiple part types, (2) a fast and efficient approximation method for cases with a large number of part types, and (3) a simple optimization heuristic that can optimize simultaneously the stock levels and the number of installed components.

The outline of this paper is as follows. We start with the model description and the formulation of is a continuous-time finite-state Markov chain (Section 2). In Section 3, we address the calculation of the system availability for a given number of installed components and given initial stock levels for the spares. Section 4 presents a simple optimization heuristic for the number of the installed components and the initial stock levels of spare parts. In Section 5, we validate our approximations by comparison to results of discrete event simulation. Also, we provide insights in the trade-off between the redundancy level and spare part inventories in presence of significant replacement times. We end this paper with some conclusions and possibilities for further research

in Section 6.

## 2. Model

In the proposed model, we consider a system with  $N$  installed components. In normal operating mode, the system needs  $k$  components running. The remaining installed components are in (cold, warm or hot) standby mode. A component in hot standby mode has the same failure rate as a running component. A cold standby component has failure rate zero, and a warm standby component has failure rate lower than the failure rate of the running components.

Please note here that the components in the cold standby mode can be seen as spares of the highest indenture in the assembly/sub-assembly hierarchy with zero replacement times. However, for the consistency of the presented mathematical model and of the corresponding algorithms, we place them in the same group as the operating components. After all, warm standby and hot standby components do not behave as spare parts.

In the reminder of the paper we will use the following main notation:

$N$  – total number of installed components.

$k$  – minimum required number of operating components.

$\lambda$  – failure rate of the running components and of the components in the hot stand-by mode (Poisson failure process).

$\lambda_W$  – failure rate of the running components and of the components in the warm stand-by mode ( $0 < \lambda_W < \lambda$ , Poisson failure process).

$k_H, k_W, k_C$  – the maximum numbers of hot, warm and cold standby components, respectively (non-negative integers).

$f_j$  – fraction of the capacity available if  $j < k$  components are running.

$M$  – number of part types that can cause failures.

$r_i$  – probability that a part of type- $i$  causes a component failure ( $i = 1, \dots, M$ ),  
 $\sum_{i=1}^M r_i = 1$ .

$\mu_i$  – replacement rate (with exponentially distributed replacement time) of part type  $i$  if a spare is available on stock .

$S_i$  – initial inventory level of part type  $i$ .  $(S_i - 1, S_i)$  inventory policy is assumed.

$\gamma_i$  – replenishment rate (with exponentially distributed replenishment times) for the inventory of part type  $i$ .

$(\mathbf{n}, \mathbf{s})$  – state of the system with  $\mathbf{n} = n_1, \dots, n_M$ , where  $n_i$  denotes the number of failed sub-components/parts of type  $i$ , and  $\mathbf{s} = s_1, \dots, s_M$ , where  $s_i$  denotes the number of type  $i$  parts being replenished (in the pipeline).

$\mathbf{e}_i$  – vector with length  $M$  where entry  $i$  is equal to 1, and all other entries equal to zero. This vector is used for short hand notations of changes in the system state, e.g.  $\mathbf{n} - \mathbf{e}_i \sim (n_1, \dots, n_i - 1, \dots, n_M)$ .

We allow for a combination of standby modes. For example, we may consider a single hot standby component that can be switched to running mode immediately and one or more cold standby modes that require some

set-up costs to change to the hot standby mode. As indicated, we define the maximum number of hot, warm and cold standby components by non-negative integers  $k_H$ ,  $k_W$  and  $k_C$ , respectively. Obviously, we have that  $k + k_H + k_W + k_C = N$ . When  $n < N$  components have failed, we allocate the remaining available to the various modes according to the priorities: (i) running, (ii) hot standby, (iii) warm standby, and (iv) cold standby. As a consequence, the system failure rate when  $n$  components have failed  $\lambda(n)$  can be written as:

$$\lambda(n) = \min\{n, k + k_H\}\lambda + \min\{(n - k - k_H)^+, k_W\}\lambda_W \quad (1)$$

where  $X^+ = \max\{X, 0\}$  for any expression  $X$ .

We assume that there are in total  $M$  different part types (sub-components) that may cause a failure and that can be replaced during component repair. The probability that a part of type- $i$  causes a failure is equal to  $r_i$ , ( $i = 1, \dots, M$ ), with  $\sum_{i=1}^M r_i = 1$ . For ease of notation, we assume that these probabilities are identical for all modes (running, hot/warm standby). For each failure a spare part is required, and if a spare is available on stock the installed component will be repaired in exponential time with rate  $\mu_i$  independently of the other failed components. In fact, this is the replacement time of part  $i$  provided that it is available. After repair, the components are as good as new. The stock of the spare parts is managed according to a base-stock policy. This means that the initial stock level for each part of type- $i$  ( $i = 1, \dots, M$ ) is equal to  $S_i$  and as soon as a spare is taken from the stock, a new item is ordered from an external supplier. The replenishment lead times are independent and exponentially distributed with rate  $\gamma_i$ . It is also known that the spare parts systems (with slow moving items) are typ-

ically insensitive to the shape of the lead time distribution (see [Alfredsson and Verrijdt 1999](#)).

The switching times between the various (standby) modes are assumed to be negligible for the time being. We will show in 3.1 how to deal with significant switch-over times using a simple approximation. The objective is to minimize the costs of the installed base (redundancy level and type), and of the spare parts inventory given a required minimum system availability (either defined as the percentage of time when the system has  $k$  components running, or as the expected fraction of components running).

Please note here, that we can easily cover other objectives or constraints in our optimization. For example, we can use time average system capacity or the system availability and maximize it given a certain budget. The type of the objective or constraints does not affect the methods for the exact or approximate evaluation of the system state probabilities as presented in the next sections. Based on these probabilities we can build several optimization models. In the remainder of this paper, we focus on the analysis of system availability as percentage of time when the system has  $k$  components running.

### *2.1. Equilibrium equations*

The system described above behaves like a Markov process with finite state space. The states of the system can be described by the  $2M$ -dimensional vector  $(\mathbf{n}, \mathbf{s})$ . Here  $\mathbf{n} = (n_1, \dots, n_M)$  indicates the numbers of the components under repair due to a failure of part of type  $i = 1 \dots M$ , and  $\mathbf{s} = (s_1, \dots, s_M)$  indicates the numbers of parts in delivery for each part type  $i$ .

The flow diagram for the case with one type of spare parts and  $N = S = 3$  is shown in Figure 2.

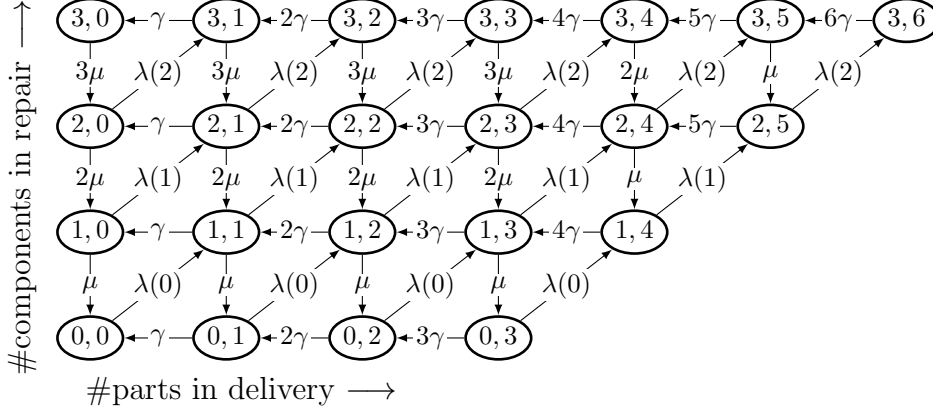


Figure 2: Flow diagram for case with  $M = 1$ ,  $N = S = 3$ .

As shown in this figure, each system state has transition rates that depend on the numbers of the components in repair  $\mathbf{n}$  and the numbers of parts in delivery  $\mathbf{s}$ . We will use notation  $\lambda(n)$  to denote the total failure rate when the system has  $n$  failed components, so  $n = \sum_{i=1}^M n_i$ . The rate  $\lambda(n)$  is computed as shown in (1). Further, we use the notation  $\mathbf{e}_i$  being a vector with length  $M$ , entry  $i$  equal to 1, and all other entries equal to zero. Then we have the following state transitions:

- A state  $(\mathbf{n}, \mathbf{s})$  will be left if (i) a failure occurs, with rate  $\lambda(n)$ , (ii) one of the parts on order arrives, with rate  $s_i \gamma_i$  for part  $i$ , (iii) when an replacement is completed, with rate  $(n_i - (s_i - S_i)^+) \mu_i$  for part  $i$ :  $n_i$  components are down due to part  $i$ ,  $(s_i - S_i)^+$  components are waiting for part  $i$ , and so the rest is being installed.
- A state  $(\mathbf{n}, \mathbf{s})$  will be entered (i) from state  $(\mathbf{n} - \mathbf{e}_i, \mathbf{s} - \mathbf{e}_i)$  upon failure of part  $i$ , which occurs with rate  $\lambda_{n-1} r_i$ , (ii) from state  $(\mathbf{n}, \mathbf{s} + \mathbf{e}_i)$  upon arrival of a part  $i$  from the supplier, with rate  $(s_i + 1) \gamma_i$ , (iii) from

state  $(\mathbf{n} + \mathbf{e}_i, \mathbf{s})$  upon replacement completion of a part  $i$ , with rate  $(n_i + 1 - (s_i - S_i)^+) \mu_i$

This leads to the following set of balance equations:

$$\begin{aligned}
& \left( \lambda(n) + \sum_{i=1}^M (n_i - (s_i - S_i)^+) \mu_i + \sum_{i=1}^M s_i \gamma_i \right) p_{\mathbf{n}, \mathbf{s}} \\
&= \lambda(n-1) \sum_{i=1}^M r_i p_{\mathbf{n}-\mathbf{e}_i, \mathbf{s}-\mathbf{e}_i} \\
&+ \sum_{i=1}^M (n_i + 1 - (s_i - S_i)^+) \mu_i p_{\mathbf{n}+\mathbf{e}_i, \mathbf{s}} + \sum_{i=1}^M (s_i + 1) \gamma_i p_{\mathbf{n}, \mathbf{s}+\mathbf{e}_i},
\end{aligned} \tag{2}$$

where  $\mathbf{n}$  s.t.  $0 \leq \sum_{i=1}^M n_i = n \leq N$  and  $\mathbf{s}$  s.t.  $0 \leq s_i \leq S_i + n_i$ ,

### 3. Evaluation of the system availability

#### 3.1. Exact solution

Since the presented model has a finite state-space, the equilibrium equations can easily be resolved as a system of linear equations using any mathematical or numerical package. Having found the state probabilities, we compute the system availability as follows.

Let us define the set  $\mathcal{U}$  as all the states in which the system is running at full capacity:  $\mathcal{U} = \left\{ (\mathbf{n}, \mathbf{s}) : \sum_{i=1}^M (n_i + s_i - S_i)^+ \leq (N - k) \right\}$ . Here  $(n_i + s_i - S_i)^+$  denote the number of type  $i$  parts in the pipeline (in delivery plus being installed) exceeding the number of spares  $S_i$ . This is the number of standby components consumed by failure of type  $i$ . The total number of

standby components consumed should obviously be not more than  $N - k$ .

Now we can simply state the availability as:

$$A = \sum_{(\mathbf{n}, \mathbf{s}) \in \mathbb{U}} p_{\mathbf{n}, \mathbf{s}} \quad (3)$$

As mentioned before, we can easily deal with other performance indicators. As an example, suppose that the system is running at a fraction  $f_j$  of the capacity if  $j$  out of the required  $k$  components are available ( $f_j$  is non-decreasing in  $j$ , and  $f_k = 1$ ). Let us define the system effectiveness SE as the average production rate of the system. Let us define the set  $\mathcal{U}_j$  as all the states in which the system is running at a fraction  $f_j$  of the capacity ( $1 \leq j \leq k$ ):

$$\mathcal{U}_j = \left\{ (\mathbf{n}, \mathbf{s}) : \sum_{i=1}^M (n_i + s_i - S_i)^+ \leq (N - j) \right\}, \quad 1 \leq j \leq k - 1$$

and

$$\mathcal{U}_k = \left\{ (\mathbf{n}, \mathbf{s}) : \sum_{i=1}^M (n_i + s_i - S_i)^+ \leq (N - k) \right\}.$$

Then we have that  $SE = \sum_{j=1}^k f_j \sum_{(\mathbf{n}, \mathbf{s}) \in \mathcal{U}_j} p_{\mathbf{n}, \mathbf{s}}$ .

Expression (3) is valid if the switching times between standby modes and running mode are negligible. If not, we can use the following approximation. Each time a failure occurs, the system is down during a certain time  $T_H$  ( $T_W, T_C$ ) if the standby component switched into operation is a hot (warm, cold) standby component, with  $0 \leq T_H \leq T_W \leq T_C$ . We assume that the probability that a component fails during switching time (to the running mode as well as between standby modes mutually) is negligible. If



$k_H \geq 1$ , each failure leads to downtime  $T_H$ , and so the availability can be approximated as

$$A \approx \sum_{(\mathbf{n}, \mathbf{s}) \in \mathbb{U}} p_{\mathbf{n}, \mathbf{s}} - \sum_{(\mathbf{n}, \mathbf{s}) \in \mathbb{U}} p_{\mathbf{n}, \mathbf{s}} \lambda(n) T_H = \sum_{(\mathbf{n}, \mathbf{s}) \in \mathbb{U}} (1 - \lambda(n) T_H) p_{\mathbf{n}, \mathbf{s}}.$$

If there are no hot (warm) standby units, the same approximation applies with  $T_H$  replaced by  $T_W$  ( $T_C$ ).

Unfortunately, the size of the linear system (2) grows very fast with the number of part types. The system with zero stocks has  $\binom{N+2M}{2M}$  states, and if the stocks are positive the number of states can be computed as a sum with  $\binom{N+M+1}{M+1}$  terms:

$$\sum_{\substack{n_1, \dots, n_M \\ 0 \leq n_1 + \dots + n_M \leq N}} (S_1 + n_1) \times \dots \times (S_M + n_M),$$

so the number of states grows almost exponentially in  $M$ . Except for small problem instances, exact evaluation is time-consuming and not applicable for optimization procedures.

Separation of the model into a set of equilibrium equations is not possible in the presented case. The repair and replenishment processes for each part will affect the general failure process and the general state of the system. However, the presented system of equilibrium equations is separable when the stock levels are 0 or when the replacement times are negligible. This will be exploited in the developed approximations (Section 3.2) and demonstrated in section with the numerical results (Section 5).

One can also try to exploit the three-diagonal block structure of this linear system, but this does not deliver the necessary improvement in the

computation speed. Therefore, we need a fast approximation technique to evaluate the system performance in terms of the system availability.

### 3.2. Approximate evaluation of the system availability

To evaluate the availability of the system, we only need to know the marginal probabilities of having  $n$  failed components in total in the system. This can be evaluated from the marginal probabilities  $p_{\mathbf{n}}$  to have  $\mathbf{n} = (n_1, \dots, n_M)$  failures of each type in the system.

To evaluate the marginal probabilities  $p_{\mathbf{n}}$ , we observe that the presented repair system behaves similarly to a multi-class Erlang loss system with state dependent arrival and service rates in which the arrival process corresponds to failures and the service process to repairs. The total service time for each part type equals the repair time plus the waiting time for spares. Note that if there is enough stock, then there is no waiting for spares and the sojourn service time equals the repair time.

We can approximate the sojourn service rates by analyzing a single item system. That is, for each part type  $i$ , we solve a single item system (Figure 2), and compute the probabilities. This will give us the service rate  $\alpha_i(n_i, S_i)$ , that is equal to one over sojourn service time, when  $n_i$  parts of type  $i$  are waiting to be repaired given the initial part stock level  $S_i$ . This service rate can be computed as:

$$\alpha_i(n_i, S_i) = \frac{\lambda_{n_i} r_i p_{n_i-1}}{n_i p_{n_i}}. \quad (4)$$

As mentioned above, the service rate  $\alpha_i(n_i, S_i)$  equals  $\mu_i$  when  $S_i \rightarrow \infty$ .

Further, we can use the obtained service rate as state dependent service rate in the proposed Erlang-loss system and compute probabilities  $p_{\mathbf{n}}$  to have  $n = (n_1, \dots, n_M)$  failures in the system due to parts  $1, \dots, M$ .

It has been shown that the multi-item Erlang-loss systems have product form solutions is known ([Kelly, 1986, 1991](#)). Similarly, the probabilities  $p_{\mathbf{n}}$  in our system can be approximated as:

$$p_{\mathbf{n}} = p_{\mathbf{0}} \left( \prod_{i=0}^{n_1+\dots+n_M} \lambda(i) \right) \left( \prod_{j=1}^M \frac{r_j^{n_j}}{n_j! \prod_{k=0}^{n_j} \alpha_j(k, S_j)} \right) \quad (5)$$

or, in other form, these probabilities can be computed iteratively as:

$$p_{\mathbf{n}} = p_{\mathbf{n}-\mathbf{e}_i} \frac{\lambda(n) r_i}{n_i \alpha_i(n_i, S_i)}, \quad \text{for any } i = 1, \dots, M, \quad (6)$$

where  $p_{\mathbf{0}}$  is such that the sum of  $p_{\mathbf{n}}$  is equal to 1. Algorithm 3.1 summarizes the approximation scheme for the marginal probabilities  $p_{\mathbf{n}}$ .

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**Algorithm 3.1** Approximation for the marginal probabilities  $p_{\mathbf{n}}$

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- 1: **for**  $i = 1, \dots, M$  **do**
  - 2:     Solve a single class  $k$ -out-of- $N$  system with parameters  $\lambda_i, \mu_i, \gamma_i$  and  $S_i$  as described in Section 2.1 and 3.1
  - 3:     Using the obtained marginal probabilities  $p_{n_i}$  compute sojourn service rates  $\alpha_i(n_i, S_i)$  as described in (4).
  - 4: **end for**
  - 5: Set  $p_{\mathbf{0}} = 1$ .
  - 6: **for** all  $\mathbf{n}$  such that  $0 < \sum_{m=1}^M n_m \leq N$  **do**
  - 7:     Compute  $p_{\mathbf{n}}$  recursively as shown in (6).
  - 8: **end for**
  - 9: Compute  $p_{\mathbf{0}} = 1 / \sum_{\mathbf{n} \neq \mathbf{0}} p_{\mathbf{n}}$
  - 10: Recompute  $p_{\mathbf{n}} = p_{\mathbf{n}} \cdot p_{\mathbf{0}}$
-

### 3.3. Quality and efficiency of the approximation

To evaluate the quality of the approximation we performed a number of experiments with different values of  $N$ ,  $M$  and  $S_i$ . We could only compare for small values of  $N$  and  $M$  due to the exponential growth of the system of equilibrium equations for the exact method. All the experiments were performed using Python based implementations on a computer with Intel Xeon E5-2697v2 2.70GHz CPU and 64GB RAM.

The input data for these experiments are mainly based on the data used in Section 5; however, we had to modify that input data in order to satisfy the requirement of the experiments presented in this section. To facilitate comparison between scenario's, we scaled the failure rates such that the stock levels  $S = \{1, 2, 1, 2, 1\}$  yield an availability of about 95%–96% for each combination of  $k$ ,  $N$  and  $M$ .

Table 1 contains comparison of the system availability with  $k = 3$  produced by the exact evaluation and by the approximation for different numbers  $N$  and  $M$ . In this table we fixed the stock levels as the first  $M$  elements of vector  $S = (1, 2, 1, 2, 1)$ . The failure rates were set such that the system availability is around 95%–96% (as most practical availability parameter). One of the non-operating components (if not failed) is in warm standby with the failure rate that is twice lowers than of the operating components.

As shown in Table 1, the system size for exact evaluation grows exponentially with the number of the components and part types. This means that the exact evaluation of the system availability becomes impractical for relatively small systems with redundancies. It is also evident (Table 1) that at high levels of availability the presented approximation technique shows quite

Table 1: Comparison of exact evaluation and approximation of the system availability for different numbers of part types ( $S = \{1, 2, 1, 2, 1\}$ ).

N	M	Exact			Approximation		Error
		size	Time (s.)	Avail.(%)	Time (s.)	Avail.(%)	
4	1	20	0.006	96.16	0.004	96.16	0.000
	2	205	0.044	96.12	0.008	96.11	0.006
	3	1134	0.313	95.61	0.014	95.57	0.049
	4	6997	12.551	95.24	0.022	95.18	0.064
	5	27525	720.650	95.06	0.038	94.98	0.077
5	1	27	0.005	96.36	0.006	96.36	0.000
	2	336	0.079	96.26	0.011	96.25	0.014
	3	2212	0.775	95.46	0.022	95.40	0.062
	4	15573	94.636	95.02	0.073	94.94	0.075
	5	69670	14375.885	94.76	0.045	94.67	0.089
6	1	35	0.006	96.84	0.007	96.84	0.000
	2	518	0.114	96.66	0.014	96.64	0.021
	3	3990	1.907	95.56	0.024	95.48	0.073
	4	31695	621.334	95.03	0.073	94.93	0.091
	5	159632	—	—	0.038	94.59	—

good precision where the difference does not exceed 0.1% (for the presented results).

In Table 2, we compare the system availability produced by the exact evaluation and by the approximation for different numbers of part types and increasing stock levels. This system has four components in total, two of them are operating and one component in warm and cold respectively is in standby mode. In this table, the stock levels are set equally for all part types according to the numbers indicated in the second column. The failure rates are fixed for all experiments in order to see how availability changes when spare part stocks increase and how system availability affects the precision of the approximation. The component in warm standby has the failure rate that is twice lower than of the operating components.

We conclude, based on the information in Table 2, that our approximation provides excellent results for different availability levels. We find the least accuracy for availability around 50%–60%, which are not realistic values for most practical settings. In cases where such availability levels are still acceptable (e.g., defense systems), the approximation will underestimate the availability by 1%–2%. In cases with zero stock levels ( $S = 0$ ) our approximation yields exact results (similarly to the single item case), since the sojourn time (replacement + replenishment) does not depend on the number of items in delivery. We observe that the computation times of the exact method explode when the number of the components  $M$  grows, whereas the approximation remains very fast for the same cases.

We further tested the computational performance of our approximation method for larger numbers of the components and part types ( $N = 6, 10, 50, 100$ ,

Table 2: Comparison of exact evaluation and approximation of the system availability for different stock levels ( $N = 4$ ).

M	S	Exact			Approximation		Error
		size	Time (sec.)	Avail.(%)	Time (sec.)	Avail.(%)	
3	0	210	0.047	30.72	0.009	30.72	0.000
	1	833	0.213	57.64	0.012	56.46	1.179
	2	2086	0.743	76.05	0.016	75.24	0.815
	3	4179	2.706	87.12	0.020	86.84	0.276
	4	7322	7.953	93.42	0.024	93.36	0.058
	5	11725	20.530	96.87	0.028	96.86	0.009
4	0	495	0.137	25.14	0.014	25.14	0.000
	1	3649	1.882	53.96	0.019	52.13	1.834
	2	13327	81.206	73.97	0.027	72.75	1.213
	3	35073	852.947	85.86	0.030	85.45	0.408
	4	76111	9174.888	92.63	0.042	92.53	0.103
5	0	1001	0.259	22.99	0.015	22.99	0.000
	1	14002	80.517	52.99	0.060	50.88	2.106
	2	75183	17154.070	73.48	0.043	72.17	1.310
	3	261044	—	—	0.027	85.09	—

Table 3: System sizes ( $|\mathbf{p}| = \binom{N+M}{M}$ ) and computation times of the approximation for different numbers of part types  $M$  and the total number of the components  $N$  (computation time is indicated in seconds).

$M$	$N = 6$		$N = 10$		$N = 50$		$N = 100$	
	size	time	size	time	size	time	size	time
1	7	0.01	11	0.02	51	0.27	101	1.00
3	84	0.02	286	0.05	23426	1.04	176851	6.58
5	462	0.05	3003	0.10	3478761	31.23	96560646	1408.48
10	8008	0.12	184756	0.84	$7.5 \times 10^{10}$	–	$4.7 \times 10^{13}$	–
15	54264	0.32	3268760	11.68	$2.1 \times 10^{14}$	–	$2.4 \times 10^{18}$	–
20	230230	0.95	30045015	102.43	$1.6 \times 10^{17}$	–	$2.9 \times 10^{22}$	–

$M = 1, 3, 5, 10, 15, 20$ ), Table 3). In this table, we present the maximum size of the vector  $\mathbf{p}$  computed in (6) and the needed computational time to evaluate the system availability. As mentioned before, the stock levels do not affect significantly the computational complexity of the approximation, since main complexity is defined by equation (6).

It is easy to see that the presented approximation does not require long computation times when number of the components is relatively small. Nevertheless, the system size ( $|\mathbf{p}|$ ) still grows exponentially with the number of servers and part types, though not as fast as in the exact method. Also, for some cases the size of the vector  $\mathbf{p}$  becomes too large and evaluation of the availability was not possible anymore. On the other hand, the presented model becomes less relevant for the cases with large  $N$ , since in such cases the failure rate becomes relatively constant and we can apply other models such as De Smidt-Destombes et al. (2011).



#### 4. Optimization heuristic

In this section, we present an optimization heuristic to minimize the total costs while the system availability is kept above a certain level. In this heuristic we use a *nested* algorithm where we enumerate over the number of installed components. For each number of installed components, we optimize the stock levels using a simple greedy search that normally shows quite good results, see for example (Sherbrooke, 2004, ch. 2, 3) or Sleptchenko (2002).

The main reason for this nested approach is the very large difference between the prices of the spare parts on one hand, and of the installed components on the other hand (about a factor 300,000). Therefore, installing an additional redundant component has a far larger impact on availability and costs than increasing the stock level of a spare part. As a consequence, we may exceed the target availability significantly in the last iteration and reach a less cost-effective solution. Therefore, we separate the greedy search for the stock levels and optimization of the number of the installed components. In addition to it, the number of the installed components is relatively small in the type of problem we consider, which allows us to check every possible number in the range described below (remarks 1 and 2).

*Remark 1.* The minimum number of the components can be found by solving a multi-class Erlang loss system without stocks and with  $(\mu_1, \dots, \mu_M)$  as service rates.

*Remark 2.* The optimal number of the components does not exceed the level at which the total cost of parts in stock is lower than the cost of an extra component. That is, the maximum impact of adding another redundant component given a certain target availability is that we can remove all spare

part stocks. If the cost of parts in stock is less than the cost of another redundant component, then we should not add this redundant component and we can stop increasing  $N$ .

Using these arguments, we formulate an optimization algorithm (Algorithm 4.1) in which we start with minimum possible number of the components (see remark 1). On each iteration we increase the number of the installed components by 1 and re-optimize the stock levels using the greedy heuristic described in Sherbrooke (2004) or in Sleptchenko (2002). We stop increasing the number of the components when the total cost of parts in stock drops below the cost of an extra component. After the enumeration is stopped we can choose the optimal number of the components from the tested range.

## 5. Numerical experiments

The numerical results presented in this section have two purposes. First, we demonstrate the importance of spare parts stocks even in redundant systems. To do so, we will show (Sections 5.1, 5.2 and 5.3) how system availability and optimal system costs change when we allow on-site storing of the spare parts.

Seconds, we analyze (Section 5.4) how replacement and replenishment times affect the system availability. To do so, we evaluate the system availability for different combinations of stocks, numbers of the components and ratios between replacement and replenishment times.

The analyzed system was shortly described in the introduction and has six components ( $N = 6$ ) with 3 components operating ( $k = 3$ ) and the rest

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**Algorithm 4.1** Optimization Heuristic (cost minimization)

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- 1: Compute  $N^{min}$ , s.t.  $\text{Availability}(N^{min}, \infty) > A^{req}$   $\triangleright$  use remark 1
  - 2: Set  $n = N^{min}$ ,  $\mathbf{S}^n = \infty$
  - 3: **while**  $c^{component} < \sum_{m=1}^M S_i^n \cdot c_i^{part}$  **do**
  - 4:     Set  $\mathbf{S}^n = \mathbf{0}$ ,  $A^n = \text{Availability}(n, \mathbf{0})$
  - 5:     **while**  $A^n < A^{req}$  **do**
  - 6:         Set  $\Delta^{max} = 0$
  - 7:         **for**  $m = 1, \dots, M$  **do**
  - 8:             Set  $\Delta = (\text{Availability}(n, \mathbf{S}^n + \mathbf{e}_i) - \text{Availability}(n, \mathbf{S}^n)) / c_i^{part}$
  - 9:             **if**  $\Delta > \Delta^{max}$  **then**
  - 10:                 Set  $\Delta^{max} = \Delta$ ,  $m^{best} = m$
  - 11:             **end if**
  - 12:         **end for**
  - 13:         Set  $\mathbf{S}^n = \mathbf{S}^n + \mathbf{e}_{m^{best}}$
  - 14:     **end while**
  - 15:     Set  $A^n = \text{Availability}(n, \mathbf{S}^n)$
  - 16:     Set  $C^n = n \cdot c^{component} + \sum_{m=1}^M S_i^n \cdot c_i^{part}$
  - 17:     Set  $n = n + 1$
  - 18: **end while**
  - 19: Choose the minimum cost  $C^n$  such that  $A^n > A^{req}$ .
-

Table 4: Input data for the numerical experiments.

	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10
fail. rate (per year)	1	1	1	0.2	0.2	0.4	0.2	0.2	0.4	1
rep. time (hours)	14	2	8	336	96	4	72	5	168	5
repl. time (days)	84	28	28	112	56	28	28	14	7	$10^{-6}$
Price (1000×USD)	5	1	1	0.01	7	35	5	5	10	0.01

in the cold standby mode.

Each installed component costs 1,500,000USD. The cost parameters as well as the failure rates  $\lambda_i$  and service rates  $\mu_i$  and  $\gamma_i$  are presented in Table 4.

#### *5.1. Impact of the stocks on the availability and the redundancy structure*

In this experiment (Figure 3), we compare system availabilities for different numbers of the components and stock levels. Stock levels here are changed simultaneously for all parts.

Figure 3 shows that the system with three components will never achieve 100% availability, irrespective the stock levels. In fact, the maximum availability for three components in this case is 93.46%. This happens due to the non-zero replacement times during which the system is considered as not available. However, if 90% is already enough, the desired system availability can be achieved just by storing cheaper spare parts on-site. Figure 4 and 5) show that almost 99% availability can be achieved with only one extra component and when the price of the components is high adding extra stock will lead to substantial savings, as shown in the next subsection.

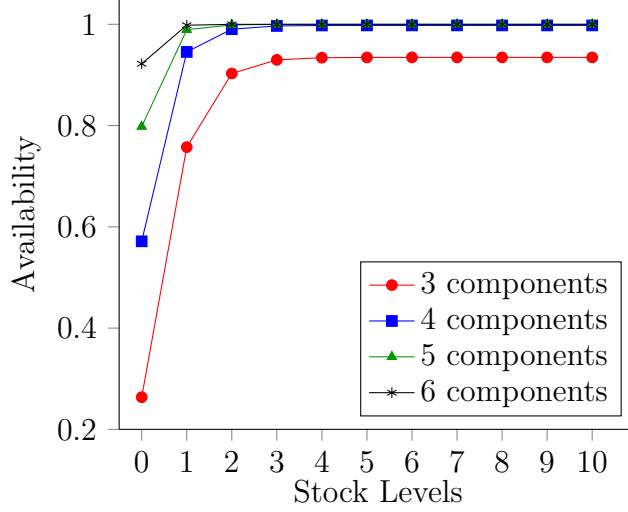


Figure 3: Comparison of the system availability for different stock levels and number of the components (stock levels are equal for all parts).

### 5.2. Impact of the stocks on the availability and the total costs

In this experiment (Figure 4), we compare total system cost (based on the prices from Table 4) and system availability when the number of installed components is fixed and the stock levels are optimized.

In Figure 4, we see that a system without redundancy will never achieve 100% availability; however, already one redundant components can give more than 99% availability. Adding more components will not give much improvement, but will rather lead to much higher total costs. In fact, the current system availability in our case study with six components and no stock (92.2%) can be achieved at half the cost by adding local spare part stocks and removing two redundant components.

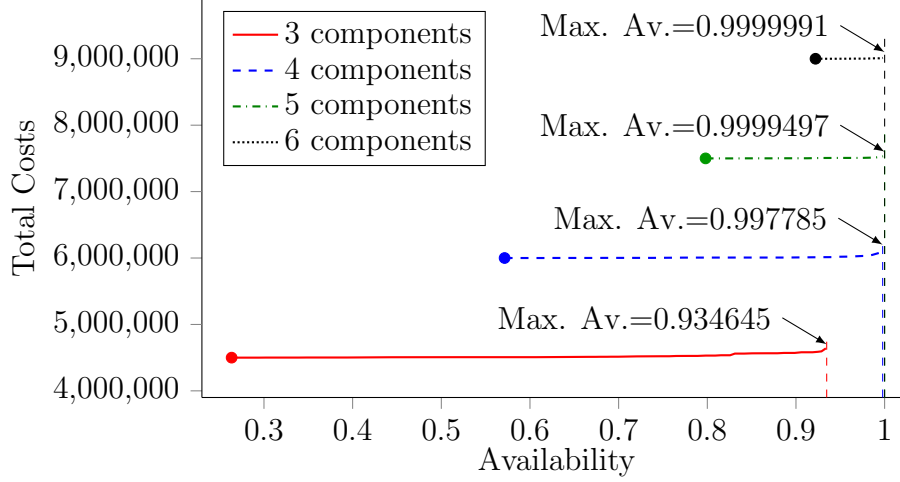


Figure 4: Comparison of the system costs for different number of the components and availability levels.

### 5.3. Impact of component/part price ratio

In the previous subsection we have demonstrated that substantial savings can be achieved by adding spare parts to the system. The level of savings however, strongly depends on the price of the installed components. Therefore, in Figure 5 we perform a similar analysis as in the previous section but for different prices of the installed components. In this experiment, we aggregate all spare part types into one to speed up the calculations. In this aggregated model the price of the spares is the weighted average spare part price with the probabilities  $r_i$  as weights. The aggregated price in this case is \$5073 and the component price is \$1500000 as mentioned before. The failure rate is the total failure rate (5.6 per year), and the replenishment and replacement rates (10.43 and 245.8 per year, correspondingly) are computed from the weighted average replenishment and replacement times.

In figure (Figure 5), we can see that replacement of extra components

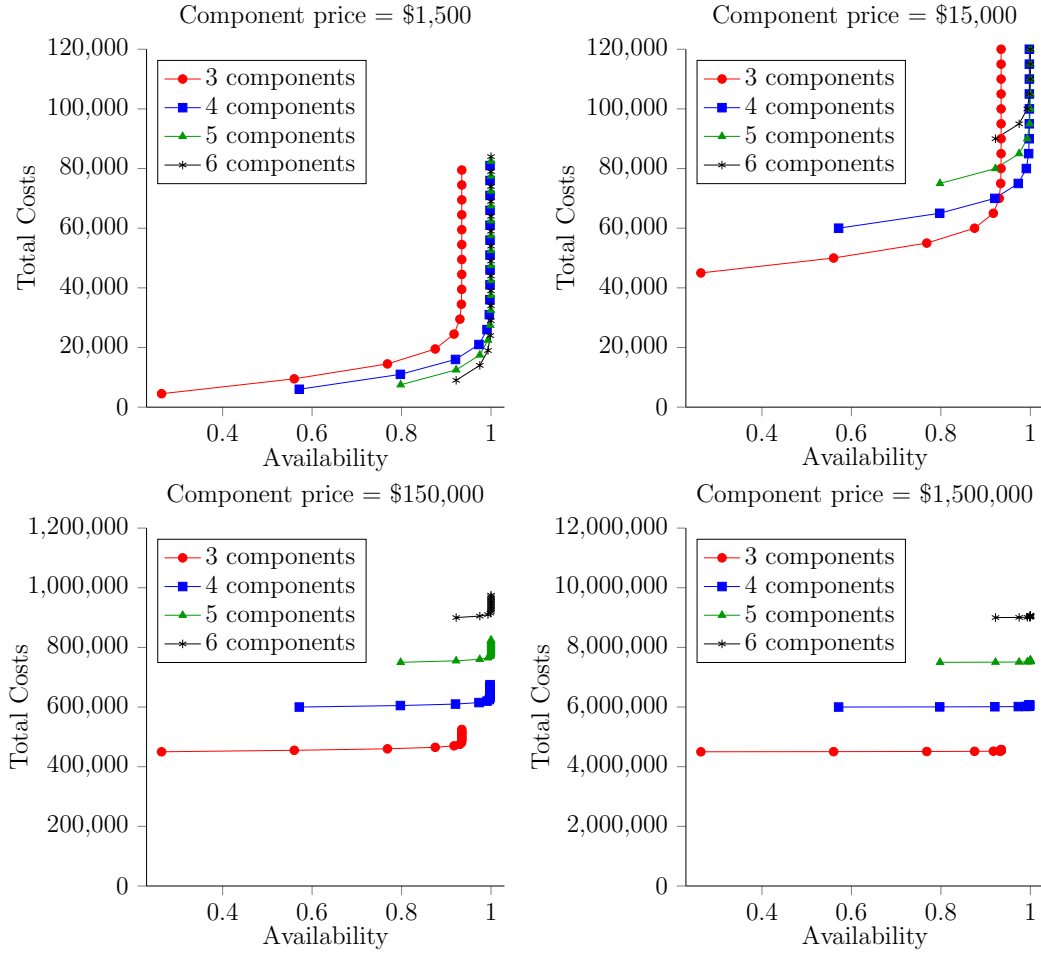


Figure 5: Comparison of the system costs for different number of the components and availability levels.

instead of increasing stock levels pays off only in case when the price of the installed components is lower (upper left plot, recall that the spare part price here is \$5,073), which is quite unrealistic. In all other cases, it is better to increase stock levels first, until the moment when extra spare parts will give only a slight improvement in the availability.

Of course when replacement times are getting longer, component redundancy will become more important. Therefore, in the next section we will demonstrate impact of repair and replenishment rates on the optimal stock and redundancy levels.

#### *5.4. Impact of repair and replenishment rates*

Replacement times play a very important role in the efficiency of redundant systems. In this subsection, we illustrate how availability is influenced by different relations between replacement and replenishment times. In the experiments presented in Table 6 we fixed the sojourn repair time (replenishment + replacement) and varied the percentage of each of these times in the sojourn time. The replacement time is equal to  $\alpha \times$  sojourn time and the replenishment time is equal to  $(1 - \alpha) \times$  sojourn time. Each plot corresponds to a different number of installed components as indicated. The plot lines correspond to different stock levels: the lowest line corresponds to 0 stock and the highest line to the maximum stock level.

Figure 6 shows that when the replacement time is 0 ( $\alpha = 0$ ) the redundancy structure loses its role and it will be cheaper to achieve the target availability by adding extra spares. When the replenishment time is zero ( $\alpha = 1$ ), part stocks are obviously not giving any improvement and the availability can be improved only by adding redundant components. Another



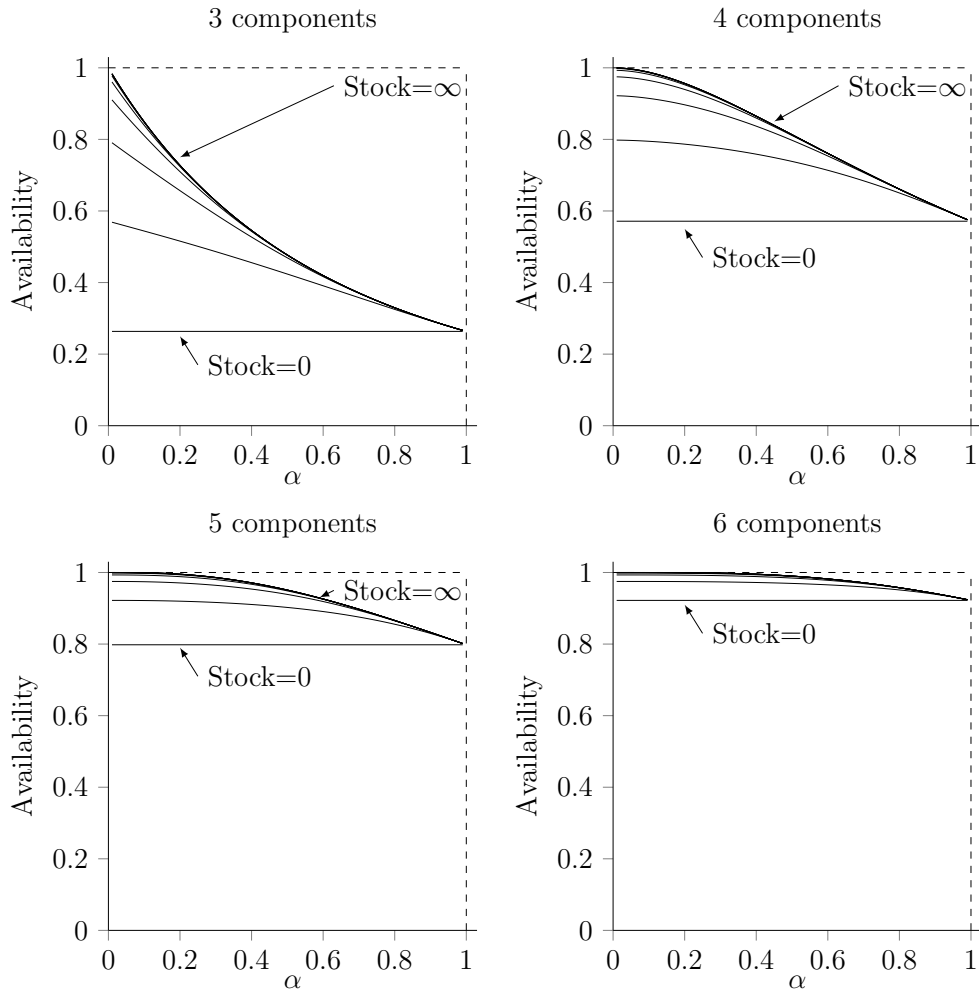


Figure 6: Comparison of availabilities for different repair and replenishment rates (repair + replenishment times = const). Each line corresponds to a certain stock level.

interesting observation that can be made here is that in the situation when the redundancy is relatively low and the replacement times are relatively long ( $> 20\%$  of sojourn time), the replacement times cannot be ignored, as done in many other models existing in the literature.

### *5.5. Non-Poisson failure process and imperfect repair*

The model proposed in this paper assumes that the interfailure times, as well as the repair and replenishment times are exponentially distributed. This allows us to construct relatively simple mathematical models for estimation of the system availability. The failure process of parts with exponentially distributed interfailure times implies, however, that the parts are not subject to wear-out, and that the failed components are always restored to the initial (as-good-as-new) state. The later is not always the case in the real life where machine parts might have a certain lifetime, increasing hazard rates or imperfect repairs ([Zhang and Jardine, 1998](#)). In addition, repair and the replenishment processes often have much lower variance than the assumed exponential distributions, even though the system performance tend to be quite insensitive to the shape of the repair- and replenishment distribution, cf. Alfredsson and Verrijdt (1999).

To analyze the sensitivity of the considered system and of the proposed mathematical models to changes in the variability of the interfailure, replenishment and repair distributions, we performed a number of simulation experiments where we varied coefficients of variation of the corresponding distributions between 0.3 and 2, as shown on Figure 7.

In this figure, the upper plots demonstrate changes in the system availability when the coefficient of variation of the interfailure times for each part

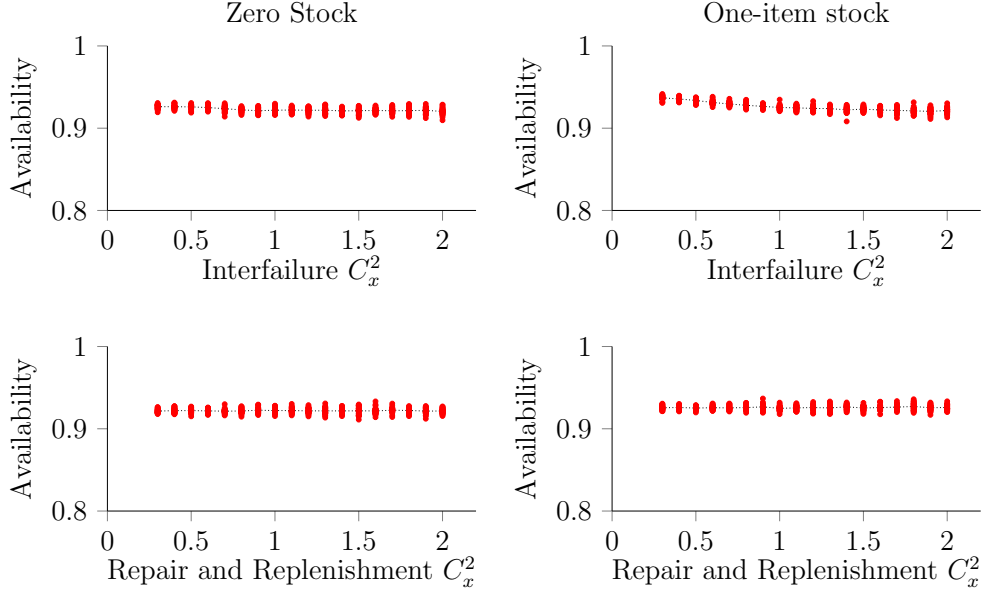


Figure 7: Simulation results for different variability of the interfailure times.

(sub-component) changes from 0.3 to 2. On the upper left plot, we present the system with zero stocks and the input data as provided in Table 4. On the upper right plot, we increase the stock levels for each part to 1, and increase the failure rates by 2.5 in order to have comparable system availabilities.

The lower plots on Figure 7 demonstrate changes in the system availability when the coefficient of variation of both repair and replenishment times change from 0.3 to 2. As before, on the lower left plot is the system with zero stocks and on the lower right plot is the system with stock levels equal 1, and increased (by 2.5) failure rates. Each simulation run had a duration of 1300 time units (years) and for each value of the  $C_x^2$ , 30 simulation runs were performed.

The presented plots clearly indicate that the analyzed system is not very

sensitive to the changes in the variability of the time distributions. When variability of the repair and replenishment times changes (the lower plots), the average system availabilities (dashed lines) change within the range of 0.2%. This is consistent with the findings of [Alfredsson and Verrijdt \(1999\)](#) for non-redundant spare part inventory systems.

When variability of the interfailure times changes (the upper plots), the average system availabilities (dashed lines) change within the range of 0.6% in the case with no stock and within the range of 1.6% when the stocks are present. Moreover, the system availability increases when the variability of the interfailure times goes down. These experiments suggest that we can easily use the presented methods for real life cases with imperfect repairs.

## 6. Conclusion

In this paper, we studied a “ $k$ -out-of- $N$ ” system with different standby modes and positive installation times. Each component in the system consists of multiple part types. Components can be repaired by replacing a failed part by a new part. We developed both an exact method and an approximate approach, and we applied it to a case study. Also, we developed an optimization routine. From our numerical experiments, we draw the following conclusions:

1. The approximate method gives very accurate estimates of the system availability if the availability is relatively high (say  $> 0.9$ ), which is common for redundant systems. The maximum approximation error encountered is about 2%, which occurs for a very low system availability (around 0.5).

2. In our case study it is far cheaper to use spare part inventories to mitigate risk of system failures than to apply redundancy on component level. One or two redundant components may be useful, however, to cover downtime due to positive replacement times of failed parts by spares, several redundant components should be replaced by a set of spare parts in our case study. The total system costs can be reduced by almost 50%.
3. In general, high redundancy levels of components are only efficient and effective if (i) the component is relatively cheap compared to the prices of the parts contained in the component, or (ii) it takes relatively a long time to replace a failed part by a spare.
4. The sensitivity of our results to the distribution of the time to failure is within reasonable bounds, as we have shown using discrete event simulation (subsection 5.5).

Regarding the latter point, it is not straightforward to include non-exponential failure behavior in our model. An option is to model the failure process by a two-stage phase type distribution ([Tijms, 2003](#)) with coefficient of variation less than 1. Such a model is suitable for wear-out processes. In such a model, the replacement of a failed part by a spare will leave the other failure process unaltered, which means that we get a practically relevant minimal repair model ([Zhang and Jardine, 1998](#)). In both models, the exact analysis and the approximation, the state space and complexity of the method will explode much faster than indicated in Table 3; however, we can try to improve our approximate approach to deal with the increasing

complexity of the system. This is an interesting topic for further research.

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