On the Logic of Choice*

Berend Roorda† Reinoud Joosten†

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Abstract

We describe an axiomatic framework for standard preference orderings, entirely based on the principle that values should be in the range of their updates. The corresponding fixed point update rule is proposed as the unifying update formula without the Sure Thing Principle.

We show how this rule is compatible with the non-consequentialist aspects of conditional choice implied by plan consistency.

This puts the normative content of behavioral models in quite a different perspective. We therefore conclude by indicating how the principles of Cumulative Prospect Theory apply to consistent choice.

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†Faculty of Behavioural, Management and Social Sciences, Department of Industrial Engineering and Business Information Systems, University of Twente, the Netherlands. Email corresponding author: b.roorda@utwente.nl.

1 Introduction

Models are reference points in thinking, where paradoxes invite to think further. It goes without saying that Expected Utility (EU) is the reference point of unrivaled importance in Decision Theory. Famous paradoxes, in particular those of Allais and Ellsberg, have directed the attention to structural aspects of human decision making that deviate from it. Whereas these paradoxes have ignited many developments in the field of behavioral modeling, which by now has left EU as intermediate station since long, they were less successful in pointing a direction in which the quest for normative modeling could proceed. Many contributors to behavioral modeling emphasize that their findings contrast with rational decision making. To quote Tversky and Kahneman (1992), "Prospect Theory has departed from the tradition that assumes the rationality of economic agents". It has become common nowadays to attribute the divergence from normative models primarily to bounded rationality. Quite strikingly, this boundedness turns out to have systematic effects, already in the simplest experiments. Is it really bounded rationality, or is rationality bounded?

One of the factors that hinders the development of normative modeling, in our opinion, is the common interpretation of a mathematical preference ordering, under which $f \succ g$ entails willingness to accept f in exchange for g. In particular, *indifference means interchangeability.*

This aspect is essential in the classic Dutch book arguments that lead to EU as the only normative model for complete preference orderings. Furthermore, it makes completeness such a strong assumption, that its validity in normative modeling is debatable. In the words of Binmore (2017), it means "... always be ready to take one side or the other of any bet. [...] One cannot but wonder whether there has ever been a financier or industrialist who felt subject to such a constraint!". This is one of the reasons why Gilboa (2015) distinguishes between subjective and objective rationality, where completeness is a requirement in the first and declined for the latter. With or without completeness, however, the aforementioned interpretation is mostly taken for granted in the analysis of rational choice. The following small paradox zooms in on this point.

> A strictly risk-averse decision maker (DM) cannot set a suitable price for the symmetric lottery e with outcome ± 1 dollar with equal probability.

Indeed, a zero price is in conflict with her risk attitude, but a negative price for e is inconsistent: it means a positive price for $-e$, 'the other side of the bet'. However, the both sides, $\pm e$, are identical lotteries, hence should have the same price.

We take the paradox as an indication to retrace our steps, and to revise the interpretation of $f \succ g$ well before reaching the more complex paradoxes of Allais and Ellsberg. Assume, for the sake of argument, that the DM sets prices of lotteries equal to their certainty equivalent (ceq), according to a complete preference ordering $\preceq$, with outcome, say, minus 20 cents for e. So she charges 20 cents for taking the long position e, as well as for taking the short position $-e$. In brief, she assigns to e a long-ceq -0.20 and a short-ceq $+0.20$, or, in one word, a 'thick' value $[-0.20, 0.20]$. This restores the symmetry, and preserves completeness.

We believe that there is a rock-solid consistency in this thickness, as effect of the same risk aversion working in opposite directions for opposite directions of trade. This direction is not a property of e itself, which justifies two value aspects for the same thing. Dutch book opportunities against the DM have no chance, since she buys and sells at more favorable prices than expected value.

The symmetric lottery is just an icebreaker to crack an overly tight link between conditional value and conditional choice. When e is a sub-lottery of a lottery f, there is no compelling reason anymore to assume that the lower value of f depends on no other aspect of e than its lower value. The update rule produces just lower and upper conditional values, but conditional choice depends on more.

This brings us to the heart of the matter, the paradox of conditional thinking:

mutually exclusive events separate possibilities, not considerations.

When casting a die, for instance, risk emerges from states in parallel (ex ante), but is non-existent per state separately (ex post). That there is no risk anymore ex post, is not a reason to exclude risk aversion. Risk aversion is a well-accepted consideration that cannot be understood per possible outcome.

The key question is: should conditional choice be understood per condition? We combine two viewpoints in the literature that seem to exclude each other: *yes*, the update of a preference ordering should be consequentialist, like the initial preference is, and *no*, plan consistency outside the STP requires that ex post conditional choice depends on bygone exposure.

The backbone of our framework is a consequentialist update rule, entirely based on the principle that values should be in the range of their conditional versions (sequential consistency, axiom S1). Existence and uniqueness of such an update is characterized by two static axioms, which are equivalent for upper and lower values. We call it *the* (central) update of an initial preference. It is produced by a simple fixed point rule, also known as Pires' rule (Pires 2002).

On the other hand, plan consistency requires full alignment of forward and backward reasoning. We argue, however, that this puts no further restriction on preference orderings, but only on their interpretation: the ex ante replacement value of a sub-act should be its anticipated value ex post, and hence govern conditional choice. This defines *embedded* updates, the 'ribs' of our framework, so to speak. These 'side-updates' correspond to the non-consequentialist update principle proposed in Machina and Schmeidler (1992).

The reconciliation comes from the observation that the central update only determines the 'willingness to conditionally pay', while the side updates determine the 'conditional willingness to pay'. The fixed point update rule pins down when they amount to the same - where the ribs are attached to the backbone. We illustrate this rationalization of dynamic choice by the Allais and Ellsberg paradoxes.

The interpretation gives rise to a rigorous reassessment of the normative content of mainstream descriptive models. To make a start, we interpret the axioms for capacities, and indicate how the principles of Rank Dependent Utility (RDU) and Cumulative Prospect Theory (CPT) apply to consistent choice.

This paper is organized as follows. Section 2 describes the mathematical setting and scope (axioms A1-4). The three axioms for updating (axioms S1-3), and the fixed point update rule, are in Section 3. Section 4 is devoted to axiom S4 and absence of arbitrage. Dynamic consistency is discussed in Section 5. In Section 6 we analyse the relationship with RDU and CPT. A discussion of related literature is largely postponed to Section 7, and conclusions follow in Section 8.

2 Scope and notation

We consider acts of the form $f : \Omega \to X$, with Ω a finite outcome space, and X a finite interval $[w, b] \subseteq \mathbb{R}$ of monetary outcomes, with standard cases $[0, 1]$ and $[-1, 1]$. The set of all acts is denoted by $\mathcal{A}$. The interval $[\min f, \max f]$ is denoted as range(f). If an act f has $f(\omega) = c \in X$ on Ω, it is called a constant (act), and then we use the symbol c also for f. An act is also called a lottery when an externally given probability measure on Ω is specified.

Our scope is the class $\mathcal{P}$ of preference orderings satisfying the usual basic axioms.

Definition 2.1 $\mathcal{P}$ is the class of preference orderings $\preceq \subset \mathcal{A} \times \mathcal{A}$ that satisfy

A1 *(Weak order)* $\preceq$ is complete and transitive.

A2 *(Monotonicity in final outcomes)* If $f(\omega) \leq g(\omega)$ on Ω, then $f \preceq g$.

A3 *(Strict monotonicity for constants)* For $c, d \in X$: $c < d$ implies $c \prec d$.

A4 *(Continuity)* For all $f \in \mathcal{A}$, the upper set $\{g \in \mathcal{A} \,|\, g \succeq f\}$ and the lower set $\{g \in \mathcal{A} \,|\, g \preceq f\}$ are closed.

Orderings in $\mathcal{P}$ are called *regular*. The equivalence $f \sim c$ if and only if $V(f) = c$ defines a one-to-one correspondence between $\mathcal{P}$ and the class of value functions $V : \mathcal{A} \to X$ that are continuous, monotone, and normalized, i.e., have $V(c) = c$. This V is called the (normalized) *value function* of $\preceq$, or the *certainty equivalence function* of $\preceq$. Proofs of these elementary facts are left to the reader. We will often simply refer to the certainty equivalent (also ceq) of an act as its value.

Updates are defined with respect to a state space S, identified with a partitioning of Ω. The sub-act of an act $f \in \mathcal{A}$ in $s \in S$ is denoted as f_s, and $\mathcal{A}_s$ denotes the set of all sub-acts in state s. A (state) update of $\preceq$ in s is a preference ordering on $\mathcal{A}_s$, denoted as $\preceq_s$. For the vectors $(\mathcal{A}_s)_{s \in S}$ and $(\preceq_s)_{s \in S}$, we use the notation $\mathcal{A}_1$ and $\preceq_1$, but $(f_s)_{s \in S}$ is simply identified with f. The (vector of) preference ordering(s) $\preceq_1$ is referred to as a (vector) update of $\preceq$. The definition of regularity extends to updates in the obvious way. We write $f_s h$ for the result of pasting sub-act f_s in state s into act h.

$\mathcal{S}$ is the subclass of $\mathcal{P}$ with well-defined sequentially consistent updates with respect to a given state space S. Notation related to comparison with specific models is introduced where it is used.

3 Updating without the STP

The following axiom is the cornerstone of our framework. Axioms apply to $f, g \in \mathcal{A}$ and $c, d \in X$.

S1 *(Sequential Consistency)* If $f_s \sim_s c$ on S, then $f \sim c$.

The notion of sequential consistency has been developed and analyzed in a longstanding research line on risk measures and valuations, see Section 7.1. It is equivalent, in $\mathcal{P}$, to the condition

$$c \preceq_1 f \preceq_1 d \quad \Rightarrow \quad c \preceq f \preceq d. \tag{3.1}$$

In other words, *values should be in the range of their updates.* This replaces the common notion of monotonicity,

$$f \preceq_s g \text{ on } S \quad \Rightarrow \quad f \preceq g. \tag{3.2}$$

Our object of study is the following class.

Definition 3.1 $\mathcal{S}$ is the subclass of preferences in $\mathcal{P}$ with unique regular sequentially consistent updates.

The class $\mathcal{S}$ is characterized by the following static axioms.

S2 *(Equal Level Principle)* If $f_s c \sim c \in \text{range}(f_s)$ on S, then $f \sim c$.

S3 *(c-Sensitivity)* If $f_s c \sim c$, then $f_s d \succ d$ for $d < c$ and $f_s d \prec d$ for $d > c$, for all $s \in S$ and $c, d \in \text{range}\, f_s$.

Axiom S2 is the weakening of the STP that characterizes existence of consistent updates, under the sensitivity condition in axiom S3 that guarantees their uniqueness.

As shown in the theorem below, updating in $\mathcal{S}$ amounts to the following mechanism, which we call *fixed point updating (fpu)*:

$$f_s \sim_s c \quad :\Leftrightarrow \quad f_s c \sim c \text{ with } c \in \text{range}(f_s). \tag{3.3}$$

We call $\preceq_s$ a *fixed point update* of $\preceq$ (in state s) if it satisfies the forward implication in (3.3); it satisfies (3.3) if and only if it is the unique one.

Theorem 1 *A preference ordering* $\preceq$ *in* $\mathcal{P}$ *has unique fixed point updates* $\preceq_s$ *on* S *if and only if* $\preceq$ *satisfies axiom S3, and then* $\preceq_s$ *is given by (3.3), and regular. The (vector) update* $(\preceq_s)_{s\in S} =: \preceq_1$ *is then sequentially consistent (axiom S1) if and only if* $\preceq$ *also satisfies axiom S2, otherwise* $\preceq$ *has no regular sequentially consistent update.*

Proof

We first prove that (3.3) defines a unique update $\preceq_s$ under axiom S3, for each $s \in S$. Let V denote the (normalized) value function of $\preceq$. Consider, for given $f_s \in \mathcal{A}_s$, the mapping $\rho : c \mapsto V(f_s c)$ on the domain range$(f_s) =: [l, r]$. Since V is continuous and monotone, ρ is continuous, $\rho(l) \geq l$ and $\rho(r) \leq r$. So ρ has a fixed point c' on this domain, i.e., there exists c' satisfying the right-hand side (rhs) of (3.3). Axiom S3 guarantees that such c' is unique, and hence that $\preceq_s$ is uniquely determined by (3.3). This means that $\preceq_1$ is indeed unambiguously defined by (3.3).

This proves the if-part of the first claim of the theorem. The only if-part is obvious from the formulation of S3.

Regularity of $\preceq_s$, under axiom S3, follows straightforwardly from regularity of $\preceq$. In particular, $\preceq_s$ is continuous, because for a series $f_s^k \to f_s$ in $\mathcal{A}_s$, with c_k the unique solution of the rhs of (3.3) for f_s^k, any converging subseries $(c_k)_{k\in\mathcal{I}\subset\mathbb{N}} \to c'$ yields $V(f_s c') = c' \in$ range(f_s), by continuity of V; so c' must be the unique solution of the rhs in (3.3), and hence the full series $(c_k)_{k\in\mathbb{N}}$ is converging to c'.

To see that $\preceq_1$ defined by (3.3) is sequentially consistent if $\preceq$ satisfies axiom S2, consider $f \in \mathcal{A}$ with $f \sim_1 c$. Then (3.3) implies that for all $s \in S$, $f_s c \sim c$ with $c \in$ range(f_s), and by axiom S2, $f \sim c$, so that axiom S1 follows.

It remains to show, under axiom S3, that if $\preceq$ has a regular sequentially consistent update $\preceq_1$, then $\preceq$ must satisfy axiom S2. Let an act $f \in \mathcal{A}$ be given with $f_s c \sim c$ and $c \in$ range(f_s) for all $s \in S$. We have to prove that $f \sim c$. Consider an $s \in S$. As $\preceq_1$ is regular, there exists $c' \in$ range(f_s) such that $f_s \sim_s c'$, and hence $f_s c' \sim_1 c'$.

But then $f_s c' \sim c'$ by axiom S1, while also $f_s c \sim c$ by assumption, and axiom S3 implies that $c' = c$. Since $s \in S$ was arbitrary, $f_s \sim_s c$ for all $s \in S$, and, again by axiom S1, indeed $f \sim c$. □

So $\mathcal{S}$ is the class of regular preferences that satisfy axiom S2 and S3. In this sense, these axioms are equivalent to axiom S1. It may be illuminating to compare the implications of axiom S2 and the STP for a strictly monotone preference ordering $\preceq$ on acts with three outcomes, (x, y, z), and two states (s, s'), corresponding to resp. the first two outcomes and the third. The STP requires that $(x, y, z) \sim (c, c, z)$ either for all z or none. Axiom S2 amounts to the implication that if $(x, y, c) \sim c$ and $(c, c, z) \sim c$, then $(x, y, z) \sim c$, which is void, since $z = c$ when $(c, c, z) \sim c$. Note that axiom S3 is satisfied, for instance, when the induced value function $V(x, y, z)$ has both the third partial derivative strictly bounded by 1, as well as the sum of the first two.

We conclude this section by a remark on compatibility of updates. The notation and preceding results generalize from states $s \in S$ to events E in a partition of S in the obvious way. In particular, under the analogues of axiom S2 and S3, the consistent update $\preceq_E$ is then determined by (3.3), with s replaced by E. This satisfies a compatibility property, called commutativity in Gilboa and Schmeidler (1989), which requires that $\preceq_s$ can also be obtained as the update of $\preceq_E$ with $s \in E$.

4 Twin preferences and absence of arbitrage

As explained in the introduction, $\preceq$ only compares taking long positions. The fourth axiom addresses the relation with the preference ordering for taking short positions, denoted by $\preceq^*$, with value function V^*. We first assume it is another externally given ordering in $\mathcal{S}$, but we explain later on how it can also be derived from $\preceq$. Our normative claim hence does not rely on the introduction of a second ordering. For reasons explained below, we call $\preceq^*$ a twin preference.

S4 *(Twin order)* If $f \sim c$, then $f \succeq^* c$.

It states that the willingness to obtain something cannot be less than the willingness to keep it. Intuitively speaking, it would be absurd to buy something for 100 that one finds only worth 80 to keep. In other words, value cannot have negative thickness. The reason to allow it to be strictly positive, has been given in the introduction.

Now it is relatively easy to cope with arbitrage opportunities in our framework. In this context, we interpret $V(f)$ as the price the DM is willing to pay for f, expressed in units of a certain outcome 1. Axiom A2 already rules out the most direct form of arbitrage: the DM paying a positive amount for an act with only non-positive outcomes. Depending on the context, also series of acts $f_1, \ldots, f_K$ have to be excluded that have the same net effect, i.e., with non-positive sum yet positive sum of values. For $K = 2$ this amounts to excluding *round trip arbitrage*, with f_1 and f_2 two opposite positions, and this is precisely what axiom already S4 does. A rigorous way to exclude arbitrage, for all K, is to impose

$$V \leq L \leq V^*, \tag{4.1}$$

with L a linear operator that is arbitrage-free when values are thin. This is a standard assumption in bid-ask price modeling, cf. Section 7.3. The DM then always trades at more favorable prices than the arbitrage-free ones. Since it involves unlimited asset aggregation, tuned to price setting in competitive markets, we consider it too restrictive to include as standard axiom in our framework, in which $\preceq$ in principle applies to single acts, not their sum. We remark that in the examples on the paradoxes, (4.1) is met.

4.1 Twins by reflection

Twin preferences $\preceq^*$ can also be seen as a logical implication of $\preceq$, when a short position is identified with a long position in the opposite outcomes,

$$f \preceq^* g \;\Leftrightarrow\; -g \preceq -f. \tag{4.2}$$

We call this the *reflection principle.* Of particular interest is the case with symmetric outcome range $X = [-b, b]$, since then also $\preceq^*$ is a preference ordering on $\mathcal{A}$. As $\preceq^*$ shares many properties with $\preceq$, we call them twin preferences. None of the axioms A1-4 and S1-3 can tell the difference. Correspondingly, we call $V^*(f) = -V(-f)$ the twin value of f induced by $\preceq$. Notice that $V^*(c) = c = V(c)$.

Axiom S4 now becomes a static axiom for $\preceq$.

S4′ *(Twin order under reflection)* If $f \sim c$, then $-f \preceq -c$, i.e., $f \succeq^* c$.

This allows us to refer to V and V^* as resp. the lower and upper value induced by $\preceq$, following the terminology in e.g. Walley (1991).

5 Dynamic consistency

The challenge of updating without the Sure Thing Principle is the interpretation of pairs of acts f, g, of the form $f_s h$, $g_s h$, for which

$$f_s h \succ g_s h \quad \text{yet} \quad f_s \prec_s g_s. \tag{5.1}$$

This suggests that the planned choice for f_s will not be kept in state s if there is a free option to switch to g_s. It is inevitable that such pairs exist, under consequentialist updating, and we agree that such a predictable switch would be inconsistent. However, we deny that (5.1) implies a change of plans. We call it a reversal, rather than a dynamic inconsistency, since we defend it as normative: that what reverses does not reverse plans.

To summarize a long quest for a fifth axiom on dynamic choice consistency: we probably searched for nothing. We believe that no additional axiom is needed, as the key is in the semantics, not in the syntax.

We take starting point in a preference ordering in $\mathcal{S}$, represented by V, which is interpreted as the DM's willingness to pay (WTP), and first take a closer look at the interpretation of V_s. According to the fpu, $V_s(f_s)$ is defined as the constant d for which $V(f_s d) = d$, which means that the DM is willing to pay d for $f_s d$. This can also be interpreted as the willingness to pay d for f_s only in case s obtains, since otherwise he gets back exactly what he is willing to pay. The fpu hence identifies $V_s(f_s)$ with the *willingness to conditionally pay (WTCP)* for f_s in case s obtains.

This gives V_s in s exactly the interpretation that V has initially: it governs the 'fresh' choice between taking a long position in f_s or g_s, as if s is the starting point. It is not independent of consequences outside s by assumption, but by definition, in the same way as the WTCP for $f_s h$ is independent of h.[1]

Furthermore, axiom S2 identifies d also with DM's WTP for $f_s h$ in case $V_1(h) = d$. We call such h a *neutral embedding.* Notice that the reversal (5.1) is impossible when h is the constant $V_s(f_s)$ (or $V_s(g_s)$, or some value in between).

In a similar way, the conditional twin preference V_s^* is related to the willingness to conditionally accept (namely a payment as compensation) for taking a 'fresh' short position. The central updates V_s and V_s^* govern conditional choice in s *when there has been no bygone exposure*, or when it was is neutral.

The bite is in the third category: *non-neutrally embedded choice*, with $V_1(h)$ not equal to d. Then a reversal (5.1) is quite possible. However, as argued in Machina (1989), this does not resemble a conflict between ex ante and ex post conditional

[1] In fact, we could also have taken starting point in WTCP as definition of updating, and then derive the fpu as a consequence. This again confirms the fpu as fundamental update principle, independently of axiom S1.

choice in s, but rather an influence of the counterfactual exposure h on both.

Alignment of ex ante and ex post conditional choice is straighforwardly achieved by the update principle, mentioned in e.g. Machina and Schmeidler (1992),

$$g_s \preceq_s^h f_s \text{ if and only if } g_s h \preceq g_s h, \tag{5.2}$$

where $\preceq_s^h$ denotes the ex post preference ordering in state s. As a rational DM can choose a risk aversion level for a static bet, so he can choose how ex post values in s depend on the bygone embedding h.

Our contribution to this standard idea is to attach this principle to consequentialist updating when h is neutral. Concretely, with $\preceq_s^h$ represented by V_s^h, denoting the DM's conditional willingness to pay (CWTP), axiom S2 implies that

$$V_s^h(f_s) = V_s(f_s) \text{ when } V_1(h) = V_s(f_s). \tag{5.3}$$

In words, CWTP may depend on bygone exposure, but should amount to WTCP when that exposure is neutral.

To further enhance the comparison between V_s^h and V_s, let us rephrase the principle (5.2) in terms of replacement values.

Definition 5.1 (Replacement value) For given V, $r \in X$ is called a replacement value of sub-act $f_s \in \mathcal{A}_s$ embedded in act $h \in \mathcal{A}$, when $f_s h \sim r_s h$.

For the sake of argument, we adopt the state sensitivity condition

$$(r+\delta)_s h \succ r_s h \quad (\delta > 0, r, r+\delta \in X, h \in \mathcal{A}), \tag{5.4}$$

so that replacement values are unique, and the replacement operator, denoted by R_s^h, is well defined.[2] It is easily verified that R_s^h represents a regular preference

[2]In general, f_s can be assigned an interval of replacement values $\{r \mid f_s h \sim r_s h\}$, and our line of reasoning can be adjusted in an obvious way.

ordering on $\mathcal{A}_s$, namely the one characterized by the equivalence (5.2). In other words, (5.2) is the requirement that

$$V_s^h = R_s^h. \tag{5.5}$$

This complies with (5.3), since the replacement value $R_s^h(f_s) =: r$ can only be different from its ceq $V_s(f_s) =: d$ when h is not neutral, and f_s not constant. Non-consequentialism is hence restricted to the influence of non-neutral bygone exposure on the valuation of still several possible consequences.

Notice that the equality (5.5) does not pose additional restrictions on the class $\mathcal{S}$, but only on the interpretation of the induced replacement values. We call it free induction, also because it is liberated from the STP that chains the ex post value $V_s^h(f_s) =: v$ to the long-ceq d.

Definition 5.2 (Free Induction) The principle of *free induction* requires that ex ante replacement values are interpreted as anticipated ex post conditional values, and vice versa.

To summarize, we relax the STP requirement $d = r = v$, by imposing the first equality only for the case with $V_1(h) = d$, which renders the second equality a matter of interpretation. This way we combine the central update V_s, which is by definition consequentialist, with the side-updates V_s^h, which by definition guarantee plan consistency, without compromise. We believe they reinforce eachother. The simplicity of the fixed point update that defines V_s can only be appreciated in combination with V_s^h. Conversely, the non-consequentialist nature of V_s^h is better understood with V_s as anchor point.

We illustrate these ideas by the Allais and Ellsberg paradoxes.

5.1 Application to the Ellsberg paradox

In the Ellsberg paradox (Ellsberg, 1961), a ball is drawn from an urn with 90 balls, 30 of which are red, and 60 black or yellow, in unknown proportion. In line with Ellsberg's exposition, we assume that the DM has a preference ordering represented by a value function V of the form

$$V = \rho V^{\text{est}} + (1-\rho)V^{\text{min}}, \tag{5.6}$$

with V^{est} the expected value under symmetric assumptions, V^{min} the expected value under the worst possible distribution of yellow and black balls, and ρ the 'degree of confidence'. We choose, somewhat arbitrarily, $\rho = 2/3$, so that

$$V(r,b,y) = \min\{r/3 + 2b/9 + 4y/9, r/3 + 4b/9 + 2y/9\} \tag{5.7}$$

$$= \mu - |b-y|/3 \text{ with } \mu := (r+b+y)/3. \tag{5.8}$$

By the reflection principle, $V^*(r,b,y) = \mu + |b-y|/3$, so the exposure to ambiguity, $|b-y|$, determines the thickness of value. All axioms A1-4′ and S1-3, are satisfied, and also (4.1). The paradox concerns choice sets of the form

$$\mathcal{C}_y = \{(1,0,y),(0,1,y)\}.$$

Contrary to the STP, we have for $y=0$ and $y=1$,

$$V(1,0,0) = 1/3 \quad V(0,1,0) = 2/9$$
$$V(1,0,1) = 5/9 \quad V(0,1,1) = 2/3.$$

The state space $\{E, \bar{E}\}$ is the partitioning of the outcome space in 'not yellow, yellow' (here we follow the notational convention to denote state s as event E).

The DM's willingness to conditionally pay for (r,b,y) in E, is the value d for which $V(r,b,d) = d$ - we could also write $d = V(r,b,\text{-})$, to stress the independency of y. This is the fixed point update rule that determines the central update V_E, and yields

$$V_E(1,0) = 3/7 \quad V_E(0,1) = 2/5.$$

The DM's conditional willingness to pay for (r, b, y) in E, is the value r' for which $V(r', r', y) = V(r, b, y)$. This is the definition of the replacement value $R_E^y(r, b)$, and yields,

$$\begin{aligned} R_E^0(1,0) = 3/5 &\leq R_E^0(0,1) = 2/5 \\ R_E^1(1,0) = 3/7 &\leq R_E^1(0,1) = 4/7. \end{aligned}$$

Now it is either way: there was an agreement on the prize y before the truth value of E is known, and the DM chooses on the basis of R_E^y when E obtains, in line with free induction, or there was no such agreement, and the DM's criterion is V_E, in fact also in line with free induction, since $V_E(r, b) = V(r, b, \text{-})$. In either case, the DM sticks to his initial plan.

We conclude that there is no inconsistency of choice: the DM sticks to his plan, all values are in the range of their updates, Dutch book arguments have no chance in view of (4.1), and all replacement values and certainty equivalents derive from one and the same value function V.

Nevertheless, there are still several questions to be addressed.

5.2 Additional topics

Firstly, should updates be in the same class as initial preferences? The example suggests that this is a reasonable condition for the central update, but less so for the side-updates: they are different by nature. The central update takes the form

$$V_E(r, b) = \min\{3r/5 + 2b/5, 3r/7 + 4b/7\}, \tag{5.9}$$

which coincides with full Bayesian updating of the 'priors' in the expression (5.7) of V. This keeps the central update in the class MEU. However, there are many ways to express V_E in the form (5.6). A subtle point is the choice of V_E^{est}, since the symmetry between black and yellow is broken by the information that E obtains. Therefore, the DM can give V_E^{est} more weight on black than red, lower ρ, and also measure the ambiguity exposure $V^{\min}$ asymmetrically in red and black. Note that this also means that the replacement value 4/7 is no proof of a switch to ambiguity loving, since V_E^{est} can be given more weight on black by the DM than 4/7. Our approach, however, provides a shortcut to updated values, avoiding the complicated route via updating taste and belief.

Keeping track of beliefs and taste is even more complicated for replacement values, since the dependency on y induces an additional subjective degree of freedom. The replacement values of (r, b) in E for $r < b$ are given by

$$R_E^y(r,b) = \begin{cases} d & y \leq d \\ 5d/7 + 2y/7 & d \leq y \leq b \\ d^* & y \geq b \end{cases},$$

where d^* denotes the upper conditional value $V^*(r, b)$. For $r > b$,

$$R_E^y(r,b) = \begin{cases} d^* & y \leq b \\ 7d/5 - 2y/5 & b \leq y \leq d \\ d & y \geq d \end{cases}$$

This no longer belongs to MEU, as function of r, b, y, which in Hanany and Klibanoff (2007) is seen as a reason to exclude (5.2) as update principle. In our interpretation, however, there is no reason to impose such a closedness property. Ribs are less like V than the backbone. More fundamentally, V applies to acts as one whole, whereas R_E^y carves out a part of its domain, and the class to which V belongs need not be closed under this form of dis-aggregation.

Another question we like to address, concerns a methodological issue. If non-neutral embedding influences conditional valuation, should it not also be taken into account in unconditional valuation? Indeed, this could be done, and, in fact, perhaps has been done by the DM. The considerations to take $\rho = 2/3$ in V may very well be influenced by, say, the fact he obtained the right on choosing one of the bets in

$\mathcal{C}_y$ as a compensation for bygone exposure to the possibility of a loss. As observed in Machina (1989), this, in turn, raises the issue how far we should go back in time, once non-consequentialism is allowed in updating. In principle, one could say, until a neutral starting point is found, whatever that may be. Note, however, that our notion of neutral embedding *given* V does not require neutrality *of* V.

Thirdly, in view of our emphasis on the principle that values should be in the range of their updates, should this not also hold in terms of replacement values? After all, according to free induction, the DM perceives an act $f \in \mathcal{A}$ as something of value $R_s^f(f)$ when s obtains, and it would be counterintuitive if $V(f)$ is not in the range of the entries in the vector $R_1^f(f)$.

Indeed, but, roughly speaking, this is already guaranteed by axiom S2. More precisely, when V satisfies this axiom also for the binary state spaces $\{s, \bar{s}\}$, with $\bar{s}$ the complement of s in S. In fact this implies the somewhat stronger property, that central and embedded updates are always at the same side of initial value in each state. Recall that we also assume (5.4) to ensure uniqueness of replacement values.

Lemma 5.3 *Under the assumptions above, the differences* $R_s^f(f_s) - V(f)$ *and* $V_s(f_s) - V(f)$ *cannot have opposite signs.*

PROOF Assume $V_s(f_s) =: d > c := V(f)$. Define $r := R_s^f(f_s)$, so $r_s f \sim c$. By axiom S2 for $\{s, \bar{s}\}$, then $V_{\bar{s}}(f_{\bar{s}}) =: d' \leq c$. Again by axiom S2, and (5.4), if $d' < c$, then $r > c$, and if $d' = c$, then $r = c$. So $d > c$ implies $r \geq c$. Similarly, $d < c$ implies $r \leq c$. Finally, when $d = c$, then also $d' = c$, and hence $r = c$. □

So bygone exposure never turns a sub-act that is 'good' on its own (with $V_1(f_s) > V(f)$) into a 'bad' one (with $R_1^f(f_s) < V(f)$), but only affects its degree of goodness.

Finally, what is the role of ambiguity, as compared to risk? The Ellsberg paradox makes clear that bygone exposure can be objectively relevant for conditional choice: it matters, *even ex post*, whether $(0, 1)$ completes an unambiguous bet or not. Hence it is problematic *not* to discriminate between R_E^y and V_E, at least in principle.

How then about the Allais paradox, where probabilities are known? What if the urn contains 10 red, 1 black and 89 yellow balls, and we have to justify

$$(1,1,1) \succ (5,0,1)$$
$$(5,0,0) \succ (1,1,0)?$$

When E obtains, it remains to compare the sure amount 1 with the lottery (5,0), with known probability 10/11 on outcome 5. The lottery is completely independent of the prize y on yellow.

However, even ex post there still is subjective interaction, due to e.g. borne risk, and regret, as argued in Machina (1989). In the Allais paradox, the effect of conservatism takes the form of a subjective risk premium, which may be non-consequentialist due to subjective interaction effects. This leaves ample room for accommodating the Allais preferences. In fact, virtually all examples in the literature satisfy axioms S2 and S3, since they are hardly restrictive for lotteries with three outcomes, cf. the introduction. Many of them also satisfy (4.1). For instance, one can take worst expected value of outcomes for probabilities that are up- or down-scaled by at most factor 10 (or factor 2.5, combined with utility function $u(x) = x^{1/4}$).

In absence of Dutch book arguments, again due to (4.1), there is no objection to choose the risk premium on the sub-lottery (5,0) depending on y. What remains is the perfect illustration by the Allais paradox how intuitive it is to do so.

6 RDU and CPT in the class $\mathcal{S}$

The question now arises how large the gap is between the class $\mathcal{S}$ and mainstream descriptive models. We concentrate on the role of rank-dependent preference orderings, with value functions defined in terms of non-additive probabilities, or *capacities*. The corresponding class is known under the names Rank Dependent (Expected) Utility (RDU) and Choquet Expected Utility (CEU) (Quiggin, 1982; Schmeidler, 1989), and is a cornerstone of CPT (Tversky and Kahneman, 1992).

6.1 Some basic examples and constructions

We first formulate some preliminary results regarding axioms S1-3. For $V = E^Q$, the expected value under a probability measure Q on Ω, axiom S3 requires $Q(s) > 0$ on S, and then axiom S2 is always satisfied. Updating is Bayesian.

It may be noted here that axiom S3 reduces to the more straightforward sensitivity condition

$$(f + \delta)_s 0 \succ f_s 0 \quad \text{for (some) } \delta > 0, \tag{6.1}$$

for the class of translation invariant preferences, i.e., with

$$V(f + c) = V(f) + c. \tag{6.2}$$

Characterizations for the class of expected utility, $V = u^{-1}E^Q(u \circ f)$, are directly obtained from the following result.

Lemma 6.1 *All axioms A1-4 and S1-3 are invariant under a strictly increasing utility transformation of X, and fixed point updating commutes with such a transformation.*

Proof Let u be a such a utility function, and define $\hat{V} := u^{-1}V(u \circ \cdot)$, so that $V = u\hat{V}(u^{-1} \circ \cdot)$. Then $\hat{V}(f_s c) = c$ if and only if $V(\hat{f}_s \hat{c}) = \hat{c}$ for $\hat{f}_s = u \circ f_s$ and $\hat{c} = u(c)$. All axioms A1-4, S1-3 pose the same condition on V as on $\hat{V}$, and $\hat{V}_s(f_s) = V(\hat{f}_s)$. □

To provide some intuition for the type of restrictions that axiom S2 imposes, we consider the minimum and maximum of two value functions.

Lemma 6.2 *Let V and V' be two preference orderings in $\mathcal{V}$. If V and V' satisfy axiom S3, then also $V \wedge V'$ and $V \vee V'$. Their fixed point update in state s is resp. $V_s \wedge V_s'$ and $V_s \vee V_s'$.*

Proof Consider $V(f_s c) = c$, $V'(f_s c') = c'$, with $c' \leq c$, and define $\hat{V} := V \wedge V'$ (the case with $c' \geq c$ and/or $\hat{V} := V \vee V'$ is similar). Then $\hat{V}(f_s c') = c'$. By axiom S3 for V', $V'(f_s d) < d$ for $d > c'$, hence also $\hat{V}(f_s d) < d$ in that case. For $d < c'$, both $V'(f_s d) > d$ and $V(f_s d) > d$, by axiom S3, and hence $\hat{V}(f_s d) > d$. It follows that $\hat{V}$ satisfies axiom S3, and that $\hat{V}_s = V_s \wedge V_s'$. □

So fixed point updating per state commutes with taking the maximum or minimum. However, axiom S2 constitutes a substantial restriction to ensure that the vector of these updates satisfies axiom S1. To illustrate the issue, consider $S = \{s, \bar{s}\}$ and f with

$$V_s(f_s c) = c, V_{\bar{s}}(f_{\bar{s}} c) > c, V_s'(f_s c) > c, V_{\bar{s}}'(f_{\bar{s}} c) = c.$$

Then $\hat{V}_1(f) = c$, with $\hat{V} = V \wedge V'$, but under suitable sensitivity conditions both $V(f) > c$ and $V'(f) > c$, hence also $\hat{V}(f) > c$: a sequential inconsistency. The point is that the entry-wise minimum of V_1 is not achievable as a joint minimum of V_1, which is what axiom S2 has to exclude. This issue has been extensively analysed for classes of concave value functions, see Section 7.2.

6.2 RDU in the class $\mathcal{S}$

A capacity ν is characterized by the properties $\nu(\Omega) = 1$, $\nu(\emptyset) = 0$, and $\nu(A) \leq \nu(B)$ when $A \subset B$. For a capacity ν, and an act $f = (x_1, \ldots, x_n)$, now with indices rearranged so that $x_1 \leq x_2 \leq \ldots x_N$, define

$$\nu \cdot f := \pi_1 x_1 + \pi_2 x_2 + \cdots + \pi_N x_N, \text{ with } \pi_j := \nu(x \geq x_j) - \nu(x > x_j) \quad (6.3)$$

RDU is the class representable by V of the form $\nu \cdot u \circ f$, with u a strictly monotone utility function. In view of Lemma 6.1, we can ignore utility transformations in the characterizations of axioms S1-3. We first characterize axiom S3.

Lemma 6.3 *A capacity* ν *on* Ω *satisfies S3 if and only if*

$$\nu(A \cap s) + \bar{\nu}(\bar{A} \cap s) > 0 \text{ on } S \quad (A \subset \Omega) \tag{6.4}$$

Proof Without loss of generality we can assume $0, 1 \in X$. Take $A \subset \Omega$, and consider $f = 1_A$. The fpu criterion (3.3) for this f in state s is

$$\nu(A \cap s) + \nu((A \cap s \cup \bar{s}) - \nu(A \cap s))\, c = c,$$

which can be rewritten as

$$\nu(A \cap s) = (\nu(A \cap s) + \bar{\nu}(\bar{A} \cap s))\, c. \tag{6.5}$$

Since axiom S3 requires a unique solution for c, it implies (6.4). To derive sufficiency of (6.4), assume, contrary to axiom S3, $f_s c \sim c$ and $f_s d \sim d$ for some $f_s \in \mathcal{A}_s$ and $c > d$. By translation invariance (6.1), then $\nu \cdot (f - c')_s 0$ for all $c' \in [c, d]$, i.e.,

$$\nu \cdot (f - c)_s 0 = 0 = \nu \cdot (f - c - \delta)_s 0 \quad (\delta \in [0, d - c]).$$

For $\delta > 0$ sufficiently small, both acts are comonotone, and hence for $A := \{f_s > c\} \subset s$, $\delta(\nu(A) + \bar{\nu}(\mathcal{A} \cap s)) = 0$, so (6.4) does not hold. □

It follows from (6.5) that for binary acts f of the form 1_A, the fpu V_s must coincide with the conditional capacity ν_s defined by

$$\nu_s(A \cap s) = \frac{\nu(A \cap s)}{\nu(A \cap s) + \bar{\nu}(\bar{A} \cap s)} \quad (A \subset \Omega). \tag{6.6}$$

So if the update of ν in s is a capacity, then it is ν_s. As shown in Horie (2013), however, this is generally not the case, even when ν is convex. To see why, we give an expression for the fixed point update.

Lemma 6.4 *A capacity ν on Ω that satisfies S3, has fixed point update V_s in state s given by*

$$V_s(f_s) = \frac{\nu \cdot (1_A f)_s 0 + \bar{\nu} \cdot (1_{\bar{A}} f)_s 0}{\nu(A) + \bar{\nu}(\bar{A} \cap s)}$$

for $A := \{f_s \succ V_s(f_s)\}$.

Proof The fpu rule (3.3) defines $V_s(f_s)$ as c being the unique solution of $(f-c)_s 0 \sim 0$, hence of the equality

$$\nu \cdot ((1_A f - c)_s 0) + \bar{\nu} \cdot ((1_{\bar{A}} f - c)_s 0) = 0.$$

This can be rewritten as

$$\nu \cdot ((1_A f)_s 0) + \bar{\nu} \cdot ((1_{\bar{A}} f)_s 0) = (\nu(A) + \bar{\nu}(\bar{A} \cap s))\ c,$$

and the formula for $V_s(f_s)$ follows. □

So this is the only update compatible with the rule that values should be in the range of their updates. The reason that it is generally not a capacity, is that comonotone sub-acts f_s, f'_s need not have comonotone neutral embeddings $f_s c$, $f'_s c'$, since the rank of c in $f_s c$ need not be the same as the rank of c' in $f'_s c'$. This suggest to weaken the notion of comonotonicity, by imposing the property for pairs $(f, V(f)), (f', V(f'))$.

Concerning axiom S2, we focus on the simplest case for which it poses a substantial restriction: acts with four outcomes, and state-independent capacity ν. Since $\nu(A)$ only depends on $\#A$, it can be represented by the three parameters (ν_1, ν_2, ν_3) for events consisting of resp. 1, 2 and 3 elements of Ω. The corresponding weights are given by $\pi_1 = \nu_1$, $\pi_2 = \nu_2 - \nu_1$, $\pi_3 = \nu_3 - \nu_2$, and $\pi_4 = 1 - \nu_3$. The axioms S2 and S3 now apply to all partitions S of Ω.

Proposition 6.5 *A state-invariant capacity $\nu = (\nu_1, \nu_2, \nu_3)$, applying to the case $\#\Omega = 4$, satisfies axiom S3 if and only if $\nu_1 > 0$ or $\nu_3 < 1$. Then it satisfies axiom*

S2 if and only if there exist $\alpha, \beta \in [0,1]$ *such that*

$$(\nu_1, \nu_2, \nu_3) = (\alpha\beta, \alpha, \alpha + (1-\alpha)(1-\beta)).$$

Equivalently,

$$(\pi_1, \ldots, \pi_4) = (\alpha\beta, \alpha(1-\beta), (1-\alpha)(1-\beta), (1-\alpha)\beta).$$

The capacity ν *is convex if and only if* $\alpha, \beta \leq 1/2$*, and satisfies axiom S4*$'$ *if and only if* $\alpha \leq 1/2$*. The conjugate* $\bar{\nu}$ *corresponds to* $(1-\alpha), (1-\beta)$*.*

Proof The first claim directly follows from the previous lemma. Axiom S2 only poses a restriction for S consisting of two states s, s', each with two outcomes. In obvious notation, consider $f = (x, y, x', y')$ with $x \geq y$ and $x' \geq y'$. Axiom S2 requires that when $(x, y, 0, 0) \sim 0$ and $(0, 0, x', y') \sim 0$, then $f \sim 0$. The premise implies $\pi_1 x + \pi_4 y = 0$ and $\pi_1 x' + \pi_4 y' = 0$, so $(x, y) = \lambda(\pi_4, -\pi_1)$ and $(x', y') = \lambda'(\pi_4, -\pi_1)$ for $\lambda, \lambda' \geq 0$. For $\lambda < \lambda'$, $V(f) = \nu \cdot f = \pi_2 x + \pi_3 y + \pi_1 x' + \pi_4 y'$, so axiom S2 requires $\pi_1\pi_2 = \pi_3\pi_4$. The other cases, $\lambda = \lambda'$ and $\lambda > \lambda'$ do not lead to other restrictions, and it follows that the equality for π characterizes axiom S2. We can write $(\pi_1, \pi_4) = \beta(\alpha, 1-\alpha)$, and the claim follows.

For the claim on convexity of ν, i.e., $\nu(A) + \nu(B) \leq \nu(A \cup B) + \nu(A \cap B)$, consider (i) A, B disjoint singletons and (ii) disjoint pairs. For (i), convexity implies that $2\alpha\beta \leq \alpha$, and for (ii) that $2\alpha \leq 1$. So it is necessary for convexity that $\alpha, \beta \leq 1/2$, and sufficiency can be verified straightforwardly. Finally, the claim on S4$'$ follows from the fact that the twin V^* is represented by $\bar{\nu}$.

□

Some remarks are in order here. The update V_s is indeed the conditional capacity ν_s in (6.6), but only if axiom S2 holds true. It is remarkable that axiom S2 coincides with the condition that updates remain capacities. Secondly, the combination of state independence, RDU and axiom S2 turns out to be overly restrictive for $N > 4$;

it can be shown that it only tolerates expected utility under the uniform distribution. Relaxing state independence leads to combinatorial complexities that we leave as topic of further research.

Thirdly, there is an aspect of non-recursiveness in the example that deserves more attention. Recursiveness is only imposed by axiom S2 for (x, y, x', y') flat, i.e, with both sub-acts of the same value. In RDU, this implies recursiveness also for any nested sub-acts, i.e., with $[x, y] \subset [x', y']$ or vice versa. However, for other orderings, such as $x' > x > y' > y$, or $x' > y' > x > y$, no longer the aspect $\pi_1 x + \pi_4 y$ matters in $V(f)$, but $\pi_2 x + \pi_4 y$, or $\pi_3 x + \pi_4 y$. So different contexts (x', y') make that different aspects of (x, y) matter in valuation, rather than that the weight of of its ceq $\pi_1 x + \pi_4 y$ changes (although mathematically it can always be expressed that way).

6.3 Gains, losses, and the reflection principle

To prepare for the comparison with CPT, we have to pay more attention to the reflection principle (4.2). Since 0 now is an intrinsic reference point, we consider a symmetric outcome range, $[-1, 1]$, with zero the boundary between losses and gains.

Let V^+ and V^- denote the restriction of V to the domain of respectively gains and losses, and V^{+-} the restriction to mixes, i.e., acts with mixed signs. Furthermore, $f^+ := f \vee 0$ and $f^- := f \wedge 0$ denote the nonnegative and non-positive part of f. We take starting point in V^+ satisfying axioms S1-3, and sketch the set of extensions to V admitted by axioms S1-4.

The first step concerns the choice of V^-. One possibility is to choose $V^-(f^-) = -V^+(-f^-)$. Then $V^*(f^+) = V(f^+)$, so this choice of V^- leads to thin values for gains (and also for losses). This is in line with the principle that indifference, $f \sim c$, means interchangeability, $-f \sim -c$, which is only the limiting case of what axiom S4$'$ allows. Any $(V^*)^+ \geq V^+$ that satisfies axiom S1-3 is allowed, and this determines

V^- by $V^-(f^-) = -(V^*)^+(-f^-)$.

A remark on utility transformations is in order here. They now act on gains and losses, and satisfy $u(-1) = -1$, $u(0) = 0$, and $u(1) = 1$. By Lemma 6.1, they leave axioms S1-3 invariant for V^+ and V^- separately. However, care has to be taken with regards to the reflection principle, since it can now be applied to the original units, or the transformed ones, and this difference is relevant for axiom S4. We stick to the reflection principle in the original, monetary units. In extensions to non-monetary settings, the reflection principle would involve a suitable choice of utils so that (4.2) makes sense for $u \circ f$.

The second step concerns the extension to mixes, completing the domain of V. We sketch the picture for acts with only two outcomes, $N = 2$. In view of the comparison with CPT, we emphasize the choice of zero level curves L_0 of V. L_0 can be chosen, for instance, on the basis of gain-loss separability, also called double matching in CPT, which requires that $V(f)$ is a function of only $V(f^+)$ and $V(f^-)$. The choice of L_0 can then be parametrized by a monotone loss-utility function z on $[0, 1]$ that determines L_0 as the solution of

$$V(f^+) = z(-V(f^-)). \tag{6.7}$$

This also determines the twin zero-level curve L_0^* of V^*, as $\{f \mid V(-f^-) = zV(f^+)\}$. Axiom S4′ requires that L_0^* is nowhere above L_0, which implies $z(\ell) \geq \ell$. In other words, within an act, losses require compensation by gains of higher value.

Once L_0 has been determined, the level curves in V^+ and V^- can be extended to mixes in still many ways. For both parts, axiom S1-3 apply separately, but axiom S4′ compares opposite level curves at c and $-c$. The idea is depicted in Figure 1.

This sketch for $N = 2$ suffices for the comparison with CPT below. In higher dimensions, the space of gain-loss connectors V^{+-} expands rapidly. Under gain-loss separability, however, the 2D picture remains representative for choosing V^{+-}.

In brief, consistently extending V^+ on $[0, 1]$ to V on $[-1, 1]$ amounts to choosing value thickness $(V^*)^+ - V^+$, then choose the zero-level curves of V, which amounts to choosing loss-utility z if gain-loss separability is imposed, and subsequently complete V by choosing level curves of mixes with either positive or negative value.

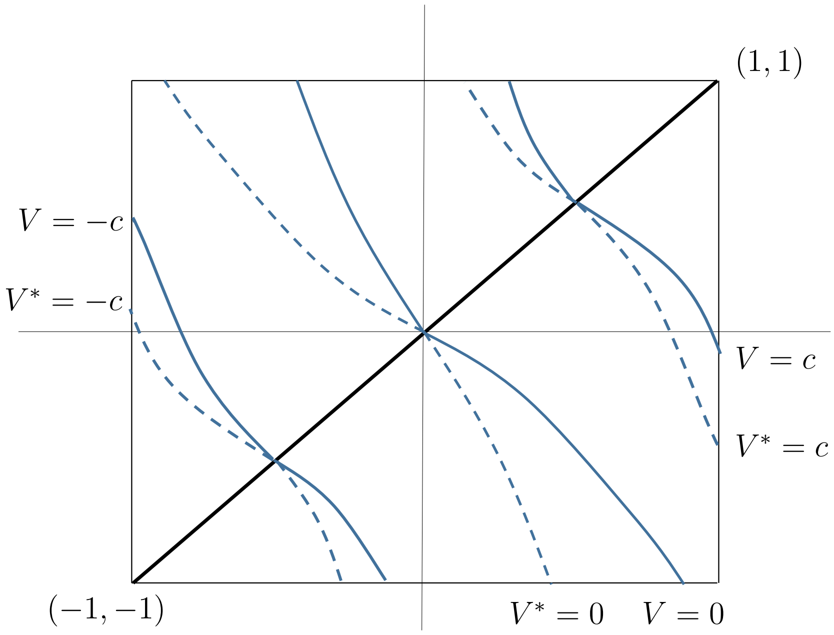

Figure 1: Sketch for $N = 2$. Dashed level curves of V^* are the reflection in the origin of the solid level curves of V.

6.4 The empirical CPT model and the class $\mathcal{S}$

Our aim is to compare the principles of CPT with the axioms for $\mathcal{S}$, and in particular to identify in what way the estimated descriptive model deviates from the class $\mathcal{S}$.

A CPT value function takes the form $\hat{V}(f) = \hat{V}(f^+) + \hat{V}(f^-)$ with

$$\hat{V}(f^+) = \nu^+ \cdot u^+ \circ f^+, \quad \hat{V}^-(f^-) = \nu^- \cdot u^- \circ f^-. \tag{6.8}$$

The descriptive model, estimated in (Tversky and Kahneman, 1992), has

$$u^+(x) = x^\alpha \quad u^-(x) = -\lambda(-x)^\beta, \quad \alpha = \beta = 0.88, \lambda = 2.25$$

The capacities are specified in terms of probability weighting functions,

$$w^+(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}},\ w^-(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}},\ \gamma = 0.61, \delta = 0.69,$$

by the law-invariant rule

$$\nu^+(A) = w^+(P(A)), \quad \nu^+(A) = w^+(P(A)) \qquad (A \subset \Omega),$$

where P is a given probability distribution on Ω. The corresponding certainty equivalent function is $u^{-1}\hat{V}(f)$, i.e.,

$$V(f^+) = {u^+}^{-1}\hat{V}(f^+), \quad V(f^-) = {u^-}^{-1}\hat{V}(f^-). \tag{6.9}$$

The question arises how this estimated model relates to $\mathcal{S}$. To start with, axiom S2 and S3 pose restrictions on ν^+ and ν^- separately, given a choice for the state space S. The criterion (6.4) for axiom S3 is met, assuming non-zero probabilities for all outcomes. Concerning axiom S2, recall that it is not restrictive for $N \leq 3$, as explained in Section 3. Since the experiments for the CPT only involve lotteries with two outcomes, they cannot provide direct evidence against axiom S3. We leave the analysis of CPT for the case $N \geq 4$ as a topic of future research. Points of considerations are the role of law invariance, and the choice of state spaces S for which axiom S2 is imposed.

Next we investigate the implications of axiom S4′, first for the relationship between V^+ and V^-. To this end, we compare $V(f)$ and $V^*(f)$, for f of the form $\mu 1_A$ with $P(A) =: p$ and $\mu \in (0, 1]$:

$$V(f) = {u^+}^{-1}\hat{V}(f) = \mu(w^+(p))^{1/\alpha} \tag{6.10}$$

$$V^*(f) = {u^-}^{-1}\hat{V}(-f) = \mu(w^-(p))^{1/\beta}. \tag{6.11}$$

This reduces to the simple condition $w^+ \leq w^-$, since $\alpha = \beta$. It is remarkable that this is indeed the case, approximately, cf. (Tversky and Kahneman, 1992, Figure

3). It should be noted here that the slight violation of axiom S4$'$, for probabilities below 0.24, may be due to the chosen functional form of the weights.

Finally, we consider the implications of axiom S4$'$ for V^{+-}. It is only here that the loss aversion parameter λ becomes relevant. We now consider f of the form $f = \mu 1_A - \mu' 1_B$, with $\mu, \mu' \in (0, 1]$, $P(A) := p$, $p(B) := p'$, in line with the CPT experiments. At the zero level curves for V, we have gain/loss ratio

$$\frac{\mu}{\mu'} = \left(\frac{\lambda w^-(p')}{w^+(p)}\right)^{1/\alpha},$$

whereas at the zero level curves of V^*,

$$\frac{\mu}{\mu'} = \left(\frac{w^+(p')}{\lambda w^-(p)}\right)^{1/\alpha},$$

where we used that $\alpha = \beta$. The corresponding loss-utility function z, defined by (6.7), is simply $z(\ell) = \lambda \ell$. This is clearly in line with the requirement $z(\ell) \geq \ell$ imposed by axiom S4$'$. We interpret the high value of λ, 2.25, as a strong indication that the thickness of value the axiom allows for, is essential in bridging the gap with descriptive models.

The choice of V^{+-} in CPT, given $V(f^+)$ and $V(f^-)$, is characterized by the combination of double matching and comonotonicity, which amounts to sign-comonotonic tradeoff consistency (SCTC), cf. also Wakker and Tversky (1993). Our analysis indicates that these principles are compatible with the axioms for $\mathcal{S}$, but far from indispensable. We consider further exploration of the boundaries of $\mathcal{S}$ in more advanced versions of CPT, and in the variety of other behavioral models that do not rely on comonotonicity and double matching, important topics of future research.

Summarizing, we find no indications that the principles of CPT are in conflict with the axioms of $\mathcal{S}$, and find strong confirmation of the distinction we make between upper and lower values in the empirical findings underlying CPT.

7 Related literature

We first explain the background of this paper. Then we discuss related literature on two central topics: updating, and the distinction between buying and selling prices that we based on the reflection principle.

7.1 Background

The central notion in our framework, sequential consistency, has been introduced in Roorda and Schumacher (2007), and further developed in mathematically more advanced settings in Roorda and Schumacher (2013, 2016) (henceforth RS07; RS13; RS16). A common assumption in these papers is that value functions V are translation invariant (6.2). Sequential consistency then amounts to the criterion in axiom S1 restricted to $c = 0$. These papers describe axiomatic frameworks with sequential consistency as basis for unique updating, applicable to both a pricing context, with $V(f)$ the bid price of f, as well as a regulatory context, with $E^P(f) - V(f)$ a required capital buffer against extreme losses.[3] The fixed point update rule (3.3) closely relates to the conditionally consistent updating rule in RS07 and the refinement update introduced in RS13. Compatibility for iterated updating is addressed in (RS13, Prop. 4.6) and (RS16, Prop. 6.7). It turned out that the fixed point update rule already was present in the literature on preference orderings, as discussed below.

[3]There is a continuous spectrum from pricing with small risk premiums (close to expected values) to so-called risk measures (much closer to worst-case), which explains that axiomatic frameworks in both domains are strikingly similar. For instance, the seminal paper Artzner et al. (1999) advocates coherent risk measures, which corresponds to MEU with linear utility $u(x) = x$ and a negative sign convention. The class of monetary risk measures have V translation invariant and monotone, its subclass of convex risk measures has $-V$ (ordinarily) convex, see Föllmer and Schied (2011).

For a general introduction to sequential consistency, and other forms of non-recursive, so-called *weak time consistency* concepts we refer to the aforementioned papers and the references therein. In a multiple prior setting, for instance, it means the relaxation of the rectangularity condition in Epstein and Schneider (2003), into a 'junctedness' condition that only requires that conditional probabilities from S occur in *some* probability measure towards S. That recursiveness is problematic in a regulatory context has been signaled in RS07, Ex. 8.8. Similar, yet less pronounced, concerns about recursive pricing have been indicated in RS13, Ex. 3.9. The observed preference reversals described in these examples gave rise to further investigation at the level of complete preference orderings, of which the current paper is a reflection.

As compared to this previous work, the scope of the current paper is expanded significantly, by relaxing translation invariance and the implicit assumption of linear utility. Most aspects of the normative interpretation we present are entirely new, including the connection with the paradoxes. In RS16 the role of sequential consistency in reconciling long and short term modeling has been emphasized. This has led to a model for tuned risk aversion, in which risk premiums interact across time steps (Roorda and Joosten, 2015). These topics are not addressed in this paper.

7.2 On the fixed point update rule

The fixed point update rule (3.3) (fpu) is not new. It is essentially the same as the notion of *conditional ceq consistency* in Eichberger et al. (2007), building on Pires (2002, Axiom 9), which is the forward implication in (3.3). In Siniscalchi (2001) it appears as constant-act dynamic consistency, and is interpreted as fixed point criterion. Its close connection with the Generalized Bayesian Rule in Walley (1991) and the Full Bayesian Updating Rule in Jaffray (1994) is well understood for the Gilboa-Schmeidler framework with Multiple Priors, also called Maxmin Expected Utility (MEU), see Pires (2002). In Eichberger et al. (2007) this connection is addressed at the level of capacities, and applied to the class of Choquet Expected Utility (CEU), also where it lies outside MEU (see also Horie (2013) for a correction). In line with the latter reference, our findings question the idea that updating can be well-understood within the class of capacities.

Our main intended contribution concerning the fpu is to establish it as a fundamental principle of updating outside the STP. To our knowledge, the underpinning of the rule, by axiom S1 and by the notion of WTCP, both without any reference to probability or utility, is new. Furthermore, in contrast to Eichberger et al. (2007) and many other references, we claim that it does not lead to dynamic inconsistencies, since it is fully compatible with the non-consequentialist update rule (5.2) in Machina and Schmeidler (1992). Conversely, we are not aware of other frameworks that accept the latter rule without giving up consequentialism for the 'true' central update.

The dynamic consistency principle proposed in Hanany and Klibanoff (2007, 2009), also requires that updated preferences must support any ex-ante optimally chosen plan given the choice set. They reject (5.2), however, because it lacks the closedness property that updates are in the same class as initial preferences. We have argued why we do not require this property for side-updates.

The main alternative to this forward-oriented approach is the principle of *consistent planning* (Strotz, 1955; Siniscalchi, 2011) and *behavioral consistency* (Karni and Safra, 1990), also phrased as the slogan that 'bygones are bygones'. We recognize this idea, by leaving room for (consequentialist) backward induction when states are reached without a preceding agreement, hence without real bygone exposure. We have argued that otherwise the influence of bygone exposure is quite intuitive, and does not lead to Dutch book opportunities, nor to plan inconsistency.

The aforementioned approaches have controversial aspects, such as the dependency of updates on the choice set. Our update rule, however, remains incredibly simple, despite the doubts that have been expressed that such a universal update principle can exist outside expected utility models (Wakker, 2010; Machina and Viscusi, 2013).

Concerning its generality, we have shown that the fpu applies to all regular preference orderings that satisfy the sensitivity condition of axiom S3, and that any alternative rule brings a conflict with axiom S1. The rule abstracts from convexity and concavity, risk and ambiguity, probability and utility, taste and belief. So, when the rule is adopted that values should be in the range of their updates, the fpu provides the only candidate for consistent updating for virtually all known classes of complete preference orderings.

7.3 On the reflection principle and bid-ask prices

One of the cornerstones of our framework is the observation that acts have not just one value, but (at least) two. This is well recognized in the literature, in several ways, under the heading of e.g. first order risk aversion, endowment effect, the WTP-WTA bias, and the law of two prices. An overview is beyond our scope, and we only discuss a few representative examples, with a focus on the given interpretation of bid-ask spreads.

The reflection principle is a standard way to relate bid and ask prices, in particular in monetary settings. It is used for instance in *conic finance*, introduced in Madan and Cherny (2010) as a new way to model markets with bid-ask spreads.[4] Also the way in which loss aversion and probability weighting induce bid-ask spreads, as described in (Wakker, 2010, Ex. 6.6.1 and 9.3.2),[5] is in line with the reflection

[4]The term conic refers to cones as the acceptance set of so-called coherent risk measures, which have been introduced by Artzner et al. (1999).

[5]Loss aversion in CPT is a form of first-order risk aversion, which corresponds to utility functions with a kink in a reference point, commonly at zero, cf. (Segal and Spivak, 1990).

principle. In fact, the findings in Birnbaum and Stegner (1979) already point in this direction. It links bid-ask spreads to so-called *configural weighting*, in the context of estimating used car prices. They find that "Judges instructed to take the buyer's point of view gave greater weight to the lower estimate, whereas judges who identified with the seller placed a greater weight on the high estimate," and they emphasize the point that it is the same cautiousness that results in different prices for opposite directions of trade.

To discern two prices for the same thing is hence by no means new, nor to base it on reflection, but we think that the consequences for interpretation and rationalization have not yet been fully recognized. Firstly, in conic finance, recursiveness is still commonly imposed in bid and ask prices separately (Madan, 2016), so that the induced (dynamic) preference reversals are avoided. However, as we have indicated in RS13, Ex. 3.9, this may lead to a market in which round trip costs can always be avoided. An exception is Bielecki et al. (2013), which applies a notion of weak time consistency (their definition D7) to conic finance that corresponds to sequential consistency; the special role of the fpu is however not addressed, nor the idea of a joint recursion involving bid and ask prices.

Secondly, common terms in the empirical literature, like *WTP-WTA bias* (Machina and Viscusi, 2013, Chapter 4), *endowment effect* (Kahneman et al., 1990), (static) *preference reversal* (Karni and Safra, 1987), *failure of procedure invariance* (Tversky and Thaler, 1990) indicate that two values for the same thing is primarily viewed as irrational by nature.

Thirdly, our 'twin view' emphasizes the importance of keeping 'looking through' sub-acts, rather than perceiving them as represented by just one 'thin' value and then concentrate on the weight of that value. This focus on weights is also the primary perspective in Buchak's extensive plea for adopting risk dependent value weights in normative models (Buchak, 2013). We do agree that such weights are ‘configural’, ‘risk weighted’, but we go one step further, and say that there are also more types of value in play. Our interpretation is not so much that a configuration influences the weight of a sub-act’s (long)-ceq in total value, but rather that it shifts the focus to another aspect of the sub-act, as illustrated for RDU in Section 6.2. This is a subtle difference, since mathematically both interpretations are valid, but nevertheless evoke different pictures. Another difference with the arguments in Buchak for relaxing the STP is that we do not invoke the idea that revealing a state can be ‘misleading’.

8 Conclusions

We gave three reasons to adopt the fixed point rule as universal principle of updating: the axiom of sequential consistency, the interpretation of willingness to conditionally pay, and the fact that it combines with free induction, which governs the non-consequentialist aspects of consistent choice outside the STP. We found no clear contradiction between our framework and the classic behavioral findings reflected in the CPT model. This indicates that there is possibly more sense in intuitive decision making than generally believed.

Several important themes fell largely out of the scope of this paper. Our framework is about the syntax of choice, rather than the semantics. This syntax does not discern belief from taste, leaving the deeper question at this point open, how value is synthesized from these two aspects. Nevertheless, our results provide an anchor point, as a kind of shortcut towards a normative structure in which updating *given* V is straightforward.

We took our starting point in completeness of preference orderings. Although its restrictiveness is softened by the thickness of value to a great extent, it may be problematic in applications to larger, complex worlds. An interesting connection can be made with how Bewley's incomplete preferences are used in Gilboa (2015) to model objective rationality, by observing that twin preferences also induce an incomplete preference ordering, that compares the lower value of an act with the higher value of another.

There is also the fundamental problem of defining and interpreting states, in applications beyond the level of dice and urns. We followed the standard that Ω specifies all possible states, and that all risk and uncertainty is resolved when ω is known. Both assumptions have problematic aspects, and in particular their combination. A problem of an a priori fixed Ω is that in reality information not just reduces the number of possibilities, but also can expand the world in directions we are simply not aware of (see e.g. Karni and Vierø (2015)). To assume that all risk resolves in Ω leads to the anomaly of ending up in overspecified states. We refer to Gilboa et al. (2018) for a recent account on this topic.

This issue clearly goes beyond our scope, but there is one remark to make. In analogy with an observation in RS16, we can extend our framework by treating Ω at the same footing as the intermediate states S, and allow also for thickness of value of $f(\omega)$, to reflect unmodeled risk after Ω. In this way models carve out a part of reality with a richer interface with the outside world, rather than starting to incorporate the entire world in the model.

We conclude with a word on the epistemological aspect of choice. This is particularly relevant in game theoretic applications, in which opponents play their part in risk and uncertainty. It is clear that the distinction between fresh and embedded choices questions the validity of subgame perfectness under our interpretation. Our findings suggest that the idea of thickness of value loosens the chains of backward induction, and gives more room for undetermined epistemological aspects of a game under the assumption of rationality. Still many paradoxes pose a challenge, in particular that of the centipede game, and the related surprise examination paradox, see e.g. Binmore (1997). These are clear invitations to think further.

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