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A FREQUENCY DOMAIN SUBSPACE ALGORITHM FOR MIXED CAUSAL, ANTI-CAUSAL LTI SYSTEMS

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Abstract: The paper extends the subspace identification method to estimate state-space models from frequency response function (FRF) samples, proposed by McKelvey *et al.* (1996) for mixed causal/anti-causal systems, and shows that other frequency domain subspace algorithms can be extended similarly. The method is demonstrated by simulation experiments.

Keywords: frequency domain identification, subspace method, descriptor system, Kronecker canonical form, state-space model

1. INTRODUCTION

Subspace identification methods are powerful methods in identifying linear multi variable systems. This is because these methods are based on numerically reliable algorithms as SVD and QR-decomposition and directly yield a state-space model. Advantages of a state-space model over a transfer function model are e.g. that a resonance mode of the system, which is observed at multiple outputs, is modeled only once and its numerical sensitivity to round-off errors is in general significantly smaller (see e.g. Gevers and Li (1993)). Further, most modern control methods are based on state-space models.

Originally, subspace identification methods were based on time-domain measured input/output data (Verhaegen, 1994; Overschee and Moor, 1996). Not much time later, subspace identification methods were proposed which are based on frequency-domain data: Fourier transformed input/output data (McKelvey, 1997) or FRF samples McKelvey *et al.* (1996); see Pintelon (2002) for new results on consistency and convergence. Using frequency domain data the number of samples may be significantly reduced, especially in case of systems with (many) widely separated resonance modes (e.g. stiff systems and systems with a high number of resonances like acoustical systems) and a non-uniform frequency grid can be exploited to accurately model the system at specific frequencies (see Pintelon and Schoukens (2001) for more details on system identification in the frequency domain).

Using frequency domain subspace identification methods systems with anti-causal/unstable modes can be identified too, which is an advantage over time-domain subspace algorithms. However, to study the causal and/or anti-causal behavior, additional post-processing has to be a applied to separate the causal and anti-causal modes. This separation is also necessary, when only a model of the causal part of the system has to be used for control/filter design, such as in the Causal Wiener filter, see e.g. (Fraanje *et al.*, 2001).

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This paper proposes a frequency domain subspace method to identify a state-space model of the causal/stable part and the anti-causal/unstable part of a system *directly*. The method is obtained by adjusting *Algorithm 2* from McKelvey (1997) using techniques from Verhaegen (1996), who proposed a *time-domain* algorithm to identify mixed causal, anti-causal systems. We will base our method for mixed causal, anti-causal systems on FRF samples, however the same reasoning can be used to extend other subspace methods based on e.g. discrete Fourier transforms (DFT's) of input and output data as discussed by McKelvey (1997).

Problems with identification of anti-causal (or unstable) systems do arise e.g. in closed-loop identification of a stabilized unstable system, in discretized systems with fractional delay (Laakso *et al.*, 1996), in estimating the inverse of non-minimum phase systems (usually due to time delay) or direct estimation of a deconvolution filter. Furthermore, this problem arises in the frequency domain implementation of a method we proposed to estimate the Causal Wiener filter (Fraanje *et al.*, 2001).

The paper is organized as follows. Section 2 describes the problem of frequency domain subspace algorithm for mixed causal, anti-causal systems in more detail. Section 3 derives a solution of the problem based on FRF samples. Section 4 illustrates the method by simulation examples.

2. PROBLEM DESCRIPTION

Consider the following discrete time mixed causal, anti-causal state-space system

$$x^c(k+1) = A^c x^c(k) + B^c u(k) \quad \text{(causal)} \qquad (1)$$
$$x^{ac}(k-1) = A^{ac} x^{ac}(k) + B^{ac} u(k) \quad \text{(anti-causal)} \qquad (2)$$
$$y(k) = C^c x^c(k) + C^{ac} x^{ac}(k) + Du(k) \qquad (3)$$

with $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^l$, $x^c(k) \in \mathbb{R}^{n_c}$ and $x^{ac}(k) \in \mathbb{R}^{n_{ac}}$ and A^c, B^c, C^c, A^{ac}, B^{ac}, C^{ac} and D of appropriate dimensions. Equation (1) models the causal and (2) the anti-causal dynamics. The order of the system is given by $n = n_c + n_{ac}$. Furthermore, the eigenvalues of A^c and A^{ac} are inside the unit-circle and we assume that the state-space description is minimal, i.e. there are no unobservable or uncontrollable modes. This class of systems is a special case of descriptor systems, described in the so-called Kronecker canonical form (Verhaegen, 1996). The transfer function of the system is given by

$$G(z) = \underbrace{\sum_{i=-\infty}^{1} C^{ac} A^{ac(1-i)} B^{ac} z^{-i} +}_{anti-causal}$$
$$\underbrace{+ D + \sum_{i=1}^{\infty} C^c A^{c(i-1)} B^c z^{-i}}_{causal} \qquad (4)$$

and clearly consists of a causal and an anti-causal part. The FRF of the system is given by

$$G(e^{j\omega}) = D + C^c(e^{j\omega} I_{n_c} - A^c)^{-1} B^c +$$
$$+ C^{ac}(e^{-j\omega} I_{n_{ac}} - A^{ac})^{-1} B^{ac} \qquad (5)$$

The problem is to estimate the matrices A^c, B^c, C^c (up to a similarity transformation T_c), A^{ac}, B^{ac}, C^{ac} (up to a similarity transformation T_{ac}) and D using M noise corrupted estimates of the frequency response

$$G_k = G(e^{j\omega_k}) + n_k, \quad k = 1, 2, \cdots, M \qquad (6)$$

at a given but arbitrary number of distinct frequencies ω_k (cf. McKelvey *et al.* (1996)).

Note, that if $\omega_k = \pi k/M$, $k = 1, \cdots, M$ (uniformly spaced frequencies) the impulse response coefficients g_i of $G(z) = \sum_{i=-\infty}^{\infty} g_i z^{-i}$ can be calculated by the two-sided inverse discrete Fourier transform

$$g_i = \frac{1}{2M} \sum_{k=1-M}^{M} G_k e^{j2\pi ik/2M}, \quad i = 1-M, \cdots, M$$

with $G_{-k} = G_k^*$, $(k = 1, \cdots, M-1)$. Then, D can be set to $D = g_0$. A_T^c, B_T^c and C_T^c can be calculated from g_i $(i = 1, \cdots, M)$ by well known realization algorithms (e.g. Kung (1978)). Dually A_T^{ac}, B_T^{ac} and C_T^{ac} can be calculated from g_{-i} $(i = 1, \cdots, M)$ by these same realization algorithms. This is basically an extension of *Algorithm 1* of McKelvey *et al.* (1996) for mixed causal, anti-causal systems. The following Section derives an alternative algorithm, which can also be used for non-uniformly spaced frequencies, which is basically an extension of *Algorithm 2* of McKelvey *et al.* (1996).

3. DERIVATION OF THE ALGORITHM

First consider the DFT of (1)–(3), where we shifted equation (2) $i-1$ samples forward in time

$$e^{j\omega} X^c(\omega) = A^c X^c(\omega) + B^c U(\omega) \qquad (7)$$
$$e^{-j\omega} X^{ac,i}(\omega) = A^{ac} X^{ac,i}(\omega) + B^{ac} e^{j(i-1)\omega} U(\omega) \qquad (8)$$
$$Y(\omega) = C^c X^c(\omega) + C^{ac} e^{-j(i-1)\omega} X^{ac,i}(\omega) +$$
$$+ DU(\omega) \qquad (9)$$

where $X^c(\omega)$, $X^{ac,i}(\omega)$, $U(\omega)$ and $Y(\omega)$ denoted the DFT of $x^c(k)$, $x^{ac}(k+i-1)$, $u(k)$ and $y(k)$ respectively and let $i > n$.

As in (McKelvey *et al.*, 1996), let $X_i^c(\omega)$ the resulting state transform when $U(\omega) = e_r$, with e_r the unit-vector with 1 on the r^{th} position. $X_r^{ac,i}(\omega)$ is defined similarly. By defining the compound state matrix

$$X_C^c(\omega) = [X_1^c(\omega)\, X_2^c(\omega)\, \cdots\, X_m^c(\omega)]$$

and $X_C^{ac,i}$ similarly, the transfer function can be implicitly described as

$$G(e^{j\omega}) = C^c X_C^c(\omega) + C^{ac} e^{-j(i-1)\omega} X_C^{ac,i}(\omega) + D \tag{10}$$

with

$$e^{j\omega} X_C^c(\omega) = A^c X_C^c(\omega) + B^c \tag{11}$$

$$e^{-j\omega} X_C^{ac,i}(\omega) = A^{ac} X_C^{ac,i}(\omega) + B^{ac} e^{j(i-1)\omega} \tag{12}$$

By iterative substituting the state-equations we obtain the relation

$$\begin{bmatrix} G(e^{j\omega}) \\ e^{j\omega}G(e^{j\omega}) \\ \vdots \\ e^{j(i-2)\omega}G(e^{j\omega}) \\ e^{j(i-1)\omega}G(e^{j\omega}) \end{bmatrix} = \mathcal{O}_i \begin{bmatrix} X_C^c(\omega) \\ X_C^{ac,i}(\omega) \end{bmatrix} + \Gamma_i \begin{bmatrix} I_m \\ e^{j\omega} I_m \\ \vdots \\ e^{j(i-2)\omega} I_m \\ e^{j(i-1)\omega} I_m \end{bmatrix} \tag{13}$$

with

$$\mathcal{O}_i = \begin{bmatrix} C^c & C^{ac}A^{ac(i-1)} \\ C^cA^c & C^{ac}A^{ac(i-2)} \\ \vdots & \vdots \\ C^cA^{c(i-2)} & C^{ac}A^{ac} \\ C^cA^{c(i-1)} & C^{ac} \end{bmatrix} \tag{14}$$

$$= [\,\mathcal{O}_i^c\ \mathcal{O}_i^{ac}\,] \tag{15}$$

and the following Toeplitz matrix filled with impulse response coefficients

$$\Gamma_i = \begin{bmatrix} D & C^{ac}B^{ac} & \cdots & C^{ac}A^{ac(i-2)}B^{ac} \\ C^cB^c & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & C^{ac}B^{ac} \\ C^cA^{c(i-2)}B^c & \cdots & C^cB^c & D \end{bmatrix}$$

By repeating (13) for all ω_k ($k = 1, \cdots, M$) and using the frequency response estimates G_k, the range space of $\mathcal{O}_i$ (and thus also the order n of the system) can be determined by a QR factorization and an SVD as explained by McKelvey *et al.* (1996). If the covariance function $E[n_k n_s^H] = R_k\delta_{ks}$ of the noise n_k is known, a weighting can be performed to compensate for n_k.

Let U_n be such that its columns span the range space of $\mathcal{O}_i$. Then, we look for an invertible $n \times n$ transformation matrix P such that

$$U_nP = [\,\mathcal{O}_i^c|\mathcal{O}_i^{ac}\,] \left[\begin{array}{c|c} T_c & 0 \\ \hline 0 & T_{ac} \end{array}\right] \tag{16}$$

This problem to separate U_n in a causal part fully determined by the pair (A^c, C^c) and an anti-causal part fully determined by the pair (A^{ac}, C^{ac}) is exactly the problem which also appears in mixed causal, anti-causal subspace identification using time-domain data, and is solved e.g. by Verhaegen (1996).

Verhaegen (1996) calculates the matrix P and n_c, n_{ac}. Then from the first l rows in $U_n' = U_nP$, C_T^c and C_T^{ac} can be picked up:

$$C_T^c = U_n'(1:l, 1:n_c), \tag{17}$$

$$C_T^{ac} = U_n'(1:l, n_c+1:n_c+n_{ac}) \tag{18}$$

and A_T^c, A_T^{ac} can be calculated by solving

$$U_n'(1:(i-1)l, 1:n_c)A_T^c = U_n'(l+1:il, 1:n_c) \tag{19}$$

and

$$\begin{aligned} U_n'(1:(i-1)l, n_c+1:n_c+n_{ac})A_T^{ac} &= \\ = U_n'(l+1:il, n_c+1:n_c+n_{ac}) \end{aligned} \tag{20}$$

If A_T^c, C_T^c, A_T^{ac}, C_T^{ac} are calculated, B_T^c, B_T^{ac} and D can be calculated by solving a least squares problem using the samples (6), because B_T^c, B_T^{ac} and D appear linearly in $G(e^{j\omega})$, cf. (5). Again a weighting using the noise covariance matrices R_k can be used to compensate for n_k (McKelvey *et al.*, 1996).

Let us summarize the method in the following *Algorithm 2 (Causal/Anti-causal)*.

Algorithm 2 (C/AC):

(1) Given: Samples G_k of the frequency response, and the covariance R_k at frequency ω_k for $k = 1, \cdots, M$.
(2) Calculate the estimate U_n of the extended observability matrix $\mathcal{O}_i$ as in *Algorithm 2* in (McKelvey *et al.*, 1996).
(3) Calculate the matrix P, which separates the causal and anti-causal part in U_n, and the orders of the causal and anti-causal part n_c and n_{ac} respectively as in Section 3.3 and 3.5 in (Verhaegen, 1996).
(4) Calculate $U_n' = U_nP$ and select C_T^c and C_T^{ac} according to (17) and (18) respectively. Further solve A_T^c and A_T^{ac} from (19) and (20) respectively.
(5) Solve B_T^c, B_T^{ac} and D from:

$$\begin{aligned} (B_T^c, B_T^{ac}, D) = \arg\min \sum_{k=1}^{M} ||R_k^{-1/2}\cdot \\ \Big(G_k - D - C_T^c(e^{j\omega_k}I_{n_c} - A_T^c)^{-1}B_T^c + \\ -C_T^{ac}(e^{-j\omega_k}I_{n_{ac}} - A_T^{ac})^{-1}B_T^{ac}\Big)||_F^2 \end{aligned}$$

We remark, that solving the least squares problem for B_T^c, B_T^{ac} and D might be ill-conditioned, especially if the system has poles close to the unit circle (McKelvey *et al.*, 1996). Regularization with a small $\epsilon > 0$ parameter can improve the conditioning of the least squares problem, at the expense of a small bias, see e.g. Golub and van Loan (1996).

4. SIMULATION EXAMPLES

We will consider two simulation examples: the identification of an academic system and of an acoustic system, both with a stable and an unstable resonance modes.

4.1 *Academic example*

The discrete time academic system has a stable mode at 100Hz and an unstable mode at 300Hz, the sampling rate is $F_s = 1000$Hz, and its transfer function is given by

$$G(z) = \frac{(z - 0.9e^{-j2\pi 0.15})(z - 0.9e^{+j2\pi 0.15})}{(z - 0.95e^{-j2\pi 0.1})(z - 0.95e^{+j2\pi 0.1})} \cdot \frac{(z - 0.5e^{-j2\pi 0.4})(z - 0.5e^{+j2\pi 0.4})}{(z - 1.1e^{-j2\pi 0.3})(z - 1.1e^{+j2\pi 0.3})}$$

The system was excited with a Schroeder multi sine, with frequencies ranging from .5Hz to 500Hz in steps of $\Delta F = .5Hz$. The number of samples in one block to estimate the frequency response was chosen to be 2000 ($= F_s/\Delta F$) such that leakage due to Fourier transforming a finite block of samples is prevented (Pintelon and Schoukens, 2001). The measured output was corrupted with unit variance Gaussian white noise $v(k)$ filtered by $H(z)$

$$y(k) = G(z)u(k) + H(z)v(k)$$

with $H(z)$ given by

$$H(z) = \frac{0.3z^2}{(z - 0.95e^{-j2\pi 0.1})(z - 0.95e^{+j2\pi 0.1})} \cdot \frac{z^2}{(z - 0.91e^{-j2\pi 0.3})(z - 0.91e^{+j2\pi 0.3})}$$

The output data was generated by splitting G in a stable/causal part and an unstable/anti-causal part. The latter was simulated by filtering anti-causally, to prevent the output from exploding.

Based on 100 blocks, three methods were used to calculate the state-space model of G:

PO-MOESP: Time-domain PO-MOESP (Verhaegen, 1996);

A2: *Algorithm 2 (C/AC)* without knowledge of R_k;

A2wi: *Algorithm 2(C/AC)* with estimated R_k.

The i parameter of (13) was chosen to be $i = 10$. To make a fair comparison, for PO-MOESP the 100 blocks each of 2000 samples were used to average the output to reduce. In A2 and A2wi, the 100 blocks we used to average the estimated frequency responses, and in A2wi also to calculated the variance R_k ($k = 1, \cdots, 2000$).

Each experiment of 100 blocks was repeated 1000 times. Figure 4.1 shows the real frequency response of G, the average of the frequency response error made by PO-MOESP, A2 and A2wi. From this Figure, we clearly see that the stable as well as the unstable mode are accurately modeled by all three methods. Further, we infer that by taking the covariance information R_k of the noise into account, the model is more accurately estimated, as was also concluded in (McKelvey *et al.*, 1996) for the causal method. Finally, we infer that on the average using the causal/anti-causal frequency domain with covariance information, a more accurate model was estimated then by using the causal/anti-causal time domain PO-MOESP method.

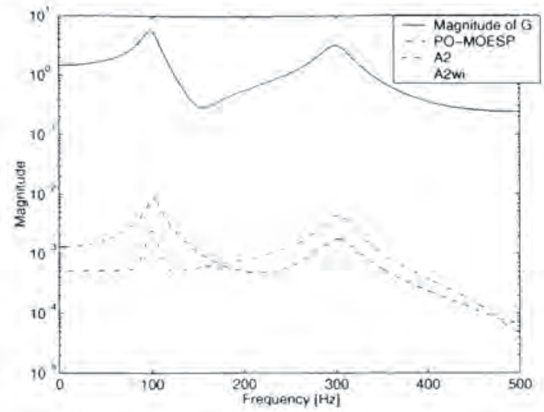

Fig. 1. Magnitude of G, and the frequency response estimation errors obtained by using 100 blocks of 2000 samples, which were averaged over 1000 experiments.

4.2 *Acoustic system*

The acoustic system to be identified is a transfer function in an acoustical duct, which contains unstable modes due to the inversion of delays between actuators and sensors which are not collocated, for more details we refer to Fraanje *et al.* (2001). The sampling rate is again 1000Hz and the frequency response of the real system G and the noise coloring H is shown in Figure 2. The unstable poles of the system are given in Table 1.

Table 1. Frequency and magnitude of unstable poles.

Frequency	Magnitude	Frequency	Magnitude
0	1.64	$\pm$ 171	1.31
0	3.65	$\pm$ 278	1.41
0	8.21	$\pm$ 388	1.46
$\pm$ 32.1	1.01	500	1.59
$\pm$ 33.5	1.08	500	2.44
$\pm$ 76.0	1.04	500	12.6
$\pm$ 91.7	1.00		

The excitation signal was chosen the same as in the previous example, a Schroeder multi sine with frequencies ranging from 0.5Hz to 500Hz with steps of 0.5Hz. Each of the 100 blocks consists of 2000 samples. The results of the three methods, PO-MOESP, A2 and A2wi, were averaged over 200 experiments and the i parameter of (13) was chosen to be $i = 100$. We note, that the least squares problem to solve B_T^c, B_T^{ac} and D in A2 and A2wi was ill conditioned, due to poles close to the unit circle.

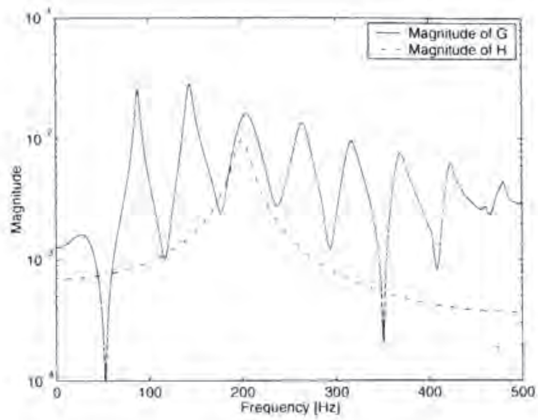

Fig. 2. Magnitude of the real system G and the noise model H.

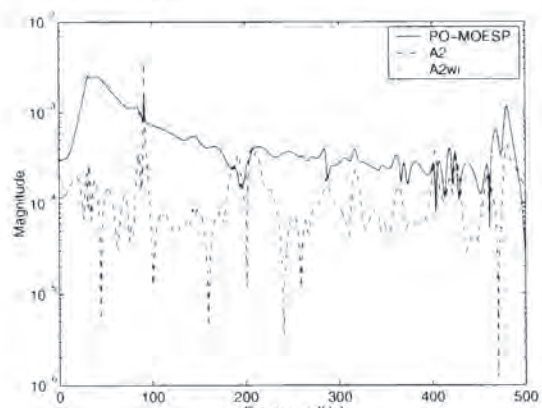

Fig. 3. Magnitude of the frequency response estimation errors obtained by the 44^{th} order models identified by using 100 blocks of 2000 samples, which were averaged over 200 experiments.

Figure 3 shows the average frequency response estimation error when the order of the state-space model was chosen to be 44, which is the order of the real system G. We observe, that again A2wi gives the smallest estimation error, but the difference with A2 is not that large as in the previous academic example. The causal, anti-causal PO-MOESP method gives less accurate models, which is currently under study. Finally, Figure 4 gives the average estimation error, which is defined as (McKelvey *et al.*, 1996):

$$||\widehat{G} - G||_2 = \sqrt{\frac{1}{M}\sum_{k=1}^{M}|\widehat{G}(e^{j\omega_k}) - G(e^{j\omega_k})|^2}$$

for different model orders. From the Figure, we infer that for orders above 32 A2wi yields the best result, closely followed by A2. It is remarkable that for orders between 20 and 30, A2wi yields significantly less accurate results, whereas A2 gives reasonable good results for these orders.

Though, some questions remain on the precise interpretation of the simulation results, the simulation experiments show that the extension of frequency domain subspace identification methods for mixed causal, anti-causal systems was successful.

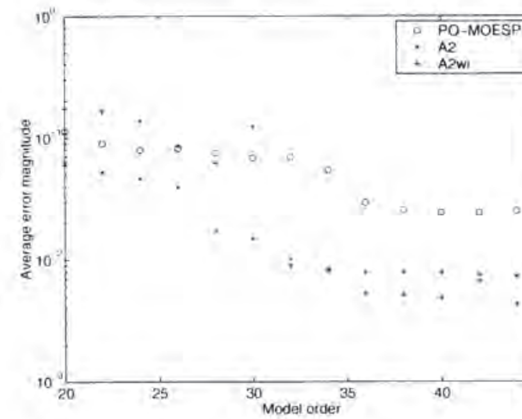

Fig. 4. Estimation error obtained by using 100 blocks of 2000 samples, averaged over 200 experiments.

5. CONCLUSIONS

It has been shown how subspace identification methods based on frequency domain data, can be adjusted to estimate a state-space model which models the causal and anti-causal part of the system separately. The crucial step in extending the frequency domain methods is to split up the extended observability matrix in a part due to causal modes and a part due to anti-causal modes. The two simulation experiments demonstrated that with the derived mixed causal, anti-causal subspace identification algorithm using FRF samples the causal and anti-causal modes of the systems could be accurately identified. We also observed, that including noise weighting to compensate for noise on the FRF samples, the obtained model error was better than using time-domain PO-MOESP for mixed causal, anti-causal systems.

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