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H_∞ & H_2 almost state synchronization with full-state coupling for homogeneous multi-agent systems

Anton A. Stoorvogel[1], Ali Saberi[2], Meirong Zhang[2] and Francesco Acciani[1]

Abstract— **This paper studies the H_∞ and H_2 almost state synchronization problem for homogeneous multi-agent systems with general linear agents affected by external disturbances and a directed communication topology. Agents are connected via diffusive full-state coupling, i.e. agents are coupled through states. A necessary and sufficient condition is developed for the solvability of the H_∞ and H_2 almost state synchronization problem. Moreover, a family of static protocols are developed such that the impact of disturbances on the network disagreement dynamics, expressed in terms of the H_∞ and H_2 norm of the corresponding closed-loop transfer function, is reduced to any arbitrarily small value. The protocol design is based on two methods: algebraic Riccati equation (ARE) or asymptotic time-scale eigenstructure assignment (ATEA).**

I. Introduction

Over the past decade, the synchronization problem of a multi-agent system (MAS) has received substantial attention because of its potential applications in cooperative control of autonomous vehicles, distributed sensor network, swarming and flocking and others. The objective of synchronization is to secure an asymptotic agreement on a common state or output trajectory through decentralized control protocols (see [1], [4], [11], [22] and references therein).

State synchronization inherently requires homogeneous MAS (i.e. agents have identical dynamics). Therefore, in this paper we focus on homogeneous MAS. So far most work has focused on state synchronization based on diffusive full-state coupling, where the agent dynamics progress from single- and double-integrator dynamics (e.g. [5], [6], [8], [9], [10]) to more general dynamics (e.g. [15], [19], [21], [23]). State synchronization based on diffusive partial-state coupling has also been considered (e.g. [2], [15], [16], [17], [20]).

Most research works have focused on the idealized case where the agents are not affected by external disturbances. In the literature where external disturbances are considered, γ-suboptimal H_∞ design is developed for MAS to achieve H_∞ norm from an external disturbance to the synchronization error among agents less to a priori given γ. In particular, [2], [25] considered the H_∞ norm from an external disturbance to the output error among agents. [14] considered the H_∞ norm from an external disturbance to the state error among agents, whereas [3] tries to obtain an H_∞ norm from a disturbance to the average of the states in a network of single or double integrators. By contrast, [7] introduced the notion of H_∞ *almost synchronization* for homogeneous MAS, where the goal is to reduce the H_∞ norm from an external disturbance to the synchronization *error*, to any arbitrary desired level. This work is extended later in [24]. However, in these works, H_∞ *almost output synchronization* is achieved.

In this paper, we will study H_∞ *almost state synchronization* for a MAS with full-state coupling. We will also study H_2 *almost state synchronization*, since it is closely related to the problems of H_∞ almost state synchronization. In H_∞ we look at the worst case disturbance with the only constraints being the power, while in H_2 we only consider white noise disturbances which is a more restrictive class. Our contribution in this paper in three-folded.

- We obtain necessary and sufficient conditions for H_∞ and H_2 almost state synchronization for a MAS in the presence of external disturbances.
- We develop a protocol design for H_∞ and H_2 almost state synchronization based on an algebraic Riccati equation (ARE) method
- We develop a protocol design for H_∞ and H_2 almost state synchronization based on an asymptotic time-scale eigenstructure assignment (ATEA) method

A. Notations and definitions

A *weighted directed graph* $\mathcal{G}$ is defined by a triple $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, \ldots, N\}$ is a node set, $\mathcal{E}$ is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the weighting matrix, and $a_{ij} > 0$ iff $(i, j) \in \mathcal{E}$ which denotes an *edge* from node j to node i. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1, \ldots, i_k\}$ such that $(i_{j+1}, i_j) \in \mathcal{E}$ for $j = 1, \ldots, k-1$. A *directed tree* with *root* r is a subset of nodes of the graph $\mathcal{G}$ such that a path exists between r and every other node in this subset. A *directed spanning tree* is a directed tree containing all the nodes of the graph. For a weighted graph $\mathcal{G}$, a matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^{N} a_{ik}, & i = j, \\ -a_{ij}, & i \neq j, \end{cases}$$

is called the *Laplacian matrix* associated with the graph $\mathcal{G}$. In the case where $\mathcal{G}$ has non-negative weights, L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $\mathbf{1}$. A specific class of graphs is presented below:

Definition 1 *For any given $\alpha \geq \beta > 0$, let $\mathbb{G}^N_{\alpha,\beta}$ denote the set of directed graphs with N nodes that contain a directed*

[1]Anton A. Stoorvogel and Francesco Acciani are with Department of Electrical Engineering, Mathematics and Computer Science, University of Twente, P.O. Box 217, Enschede, The Netherlands A.A.Stoorvogel@utwente.nl, F.Acciani@utwente.nl

[2]Ali Saberi and Meirong Zhang are with School of Electrical Engineering and Computer Science, Washington State University, Pullman,WA, USA saberi@eecs.wsu.edu, meirong.zhang@wsu.edu

spanning tree and for which the corresponding Laplacian matrix L satisfies $\|L\| < \alpha$ while its nonzero eigenvalues have a real part larger than or equal to β.

II. Problem formulation

Consider a MAS composed of N identical linear time-invariant agents of the form,

$$\dot{x}_i = Ax_i + Bu_i + E\omega_i, \qquad (i = 1, \ldots, N) \tag{1}$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$ are respectively the state and input vectors of agent i, and $\omega_i \in \mathbb{R}^q$ are the external disturbances.

The communication network provides each agent with a linear combination of its own states relative to that of other neighboring agents. In particular, each agent $i \in \{1, \ldots, N\}$ has access to the quantity,

$$\zeta_i(t) = \sum_{j=1}^{N} a_{ij}(x_i(t) - x_j(t)) = \sum_{j=1}^{N} \ell_{ij} x_j(t), \tag{2}$$

where the weighting matrix $A = [a_{ij}]$ or the Laplacian matrix $L = [\ell_{ij}]$ describe the communication among agents. These matrices can be connected to an associated graph $\mathcal{G}$. If the graph describing the communication topology of the network contains a directed spanning tree, then it follows from [10, Lemma 3.3] that the Laplacian matrix L has a simple eigenvalue at the origin, with the corresponding right eigenvector $\mathbf{1}$ and all the other eigenvalues are in the open right-half complex plane. Let $\lambda_1, \ldots, \lambda_N$ denote the eigenvalues of L such that $\lambda_1 = 0$ and $\mathrm{Re}(\lambda_i) > 0$, $i = 2, \ldots, N$.

Let N be any agent and define $\bar{x}_i = x_N - x_i$ and

$$\bar{x} = \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_{N-1} \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_N \end{pmatrix}.$$

Obviously, synchronization is achieved if $\bar{x}(t) \to 0$ or, equivalently,

$$\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0, \quad \forall i, \in \{1, \ldots, N-1\}. \tag{3}$$

We define the following transfer function with the appropriate dimension:

$$\bar{x} = T_{\omega\bar{x}}(s)\omega. \tag{4}$$

We formulate below two problems for a network with full-state coupling with either H_2 or H_∞ almost synchronization.

Problem 1 *Consider a MAS described by* (1) *and* (2). *Let* $\mathbf{G}$ *be a given set of graphs such that* $\mathbf{G} \subseteq \mathbb{G}^N$. *The* $\boldsymbol{H_\infty}$ ***almost state synchronization*** *problem via full-state coupling (in short H_∞-ASSFS) with a set of network graphs* $\mathbf{G}$ *is to find, if possible, a linear static protocol parameterized in terms of a parameter ε of the form,*

$$u_i = F(\varepsilon)\zeta_i, \tag{5}$$

such that, for any given real number $r > 0$, there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$ and for any graph $\mathcal{G} \in \mathbf{G}$, (3) *is satisfied for all initial conditions in the absence of disturbances and the closed loop transfer matrix $T_{\omega\bar{x}}$ satisfies*

$$\|T_{\omega\bar{x}}\|_\infty < r. \tag{6}$$

Problem 2 *Consider a MAS described by* (1) *and* (2). *Let* $\mathbf{G}$ *be a given set of graphs such that* $\mathbf{G} \subseteq \mathbb{G}^N$. *The* $\boldsymbol{H_2}$ ***almost state synchronization*** *problem via full-state coupling (in short H_2-ASSFS) with a set of network graphs* $\mathbf{G}$ *is to find, if possible, a linear static protocol parameterized in terms of a parameter ε of the form* (5) *such that, for any given real number $r > 0$, there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$ and for any graph $\mathcal{G} \in \mathbf{G}$,* (3) *is satisfied for all initial conditions in the absence of disturbances and the closed loop transfer matrix $T_{\omega\bar{x}}$ satisfies*

$$\|T_{\omega\bar{x}}\|_2 < r. \tag{7}$$

Note that the problems of H_∞ almost state synchronization and H_2 almost state synchronization are closely related.

III. Main results

In this section, we establish a connection between the almost state synchronization among agents in the network and a robust H_∞ or H_2 almost disturbance decoupling problem via state feedback with internal stability (in short H_∞ or H_2-ADDPSS). Then, we use this connection to derive the necessary and sufficient condition and design appropriate protocols.

A. Necessary and sufficient condition for H_∞-ASSFS

The MAS system described by (1) and (2) after implementing the linear static protocol (5) is described by

$$\dot{x}_i = Ax_i + BF(\varepsilon)\zeta_i + E\omega_i,$$

for $i = 1, \ldots, N$. Let

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}, \quad \omega = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_N \end{pmatrix}.$$

Then, the overall dynamics of the N agents can be written as

$$\dot{x} = (I_N \otimes A + L \otimes BF(\varepsilon))x + (I_N \otimes E)\omega. \tag{8}$$

We define the robust H_∞-ADDPSS with bounded input as follows. Given $\Lambda \subset \mathbb{C}$, there should exist $M > 0$ such that for any given real number $r > 0$, we can find a parameterized controller

$$u = F(\varepsilon)x \tag{9}$$

for the following subsystem,

$$\dot{x} = Ax + \lambda Bu + Bw, \tag{10}$$

such that for any $\lambda \in \Lambda$ the following hold:

1) The interconnection of the systems (10) and (9) is internally stable;
2) The resulting closed-loop transfer function T_{wx} from w to x has an H_∞ norm less than r.

3) The resulting closed-loop transfer function T_{wu} from w to u has an H_∞ norm less than M.

In the above, Λ denotes all possible locations for the nonzero eigenvalues of the Laplacian matrix L when the graph varies over the set $\mathbf{G}$. It is also important to note that M is independent of the choice for r.

Theorem 1 *Let* $\mathbf{G}$ *be a set of graphs such that the associated Laplacian matrices are uniformly bounded and let* Λ *consist of all possible nonzero eigenvalues of Laplacian matrices associated with graphs in* $\mathbf{G}$.

(Necessity) The H_∞*-ASSFS for the MAS described by* (1) *and* (2) *given* $\mathbf{G}$ *is solvable by a parameterized protocol* $u_i = F(\varepsilon)\zeta_i$ *only if*

$$\operatorname{im} E \subset \operatorname{im} B. \tag{11}$$

(Sufficiency) The H_∞*-ASSFS for the MAS described by* (1) *and* (2) *given* $\mathbf{G}$ *is solved by a parameterized protocol* $u_i = F(\varepsilon)\zeta_i$ *if the robust* H_∞*-ADDPSS with bounded input for the system* (10) *with* $\lambda \in \Lambda$ *is solved by the parameterized controller* $u = F(\varepsilon)x$.

Proof: Note that L has eigenvalue 0 with associated right eigenvector $\mathbf{1}$. Let

$$L = TS_LT^{-1}, \tag{12}$$

with T unitary and S_L the upper-triangular Schur form associated to the Laplacian matrix L such that $S_L(1,1) = 0$. Let

$$\eta := (T^{-1} \otimes I_n)x = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_N \end{pmatrix}, \qquad \bar{\omega} = (T^{-1} \otimes I)\omega = \begin{pmatrix} \bar{\omega}_1 \\ \vdots \\ \bar{\omega}_N \end{pmatrix}$$

where $\eta_i \in \mathbb{C}^n$ and $\bar{\omega}_i \in \mathbb{C}^q$. In the new coordinates, the dynamics of η can be written as

$$\dot{\eta}(t) = (I_N \otimes A + S_L \otimes BF(\varepsilon))\eta + (T^{-1} \otimes E)\omega, \tag{13}$$

which is rewritten as

$$\begin{aligned} \dot{\eta}_1 &= A\eta_1 + \sum_{j=2}^{N} s_{1j}BF(\varepsilon)\eta_j + E\bar{\omega}_1, \\ \dot{\eta}_i &= (A + \lambda_i BF(\varepsilon))\eta_i + \sum_{j=i+1}^{N} s_{ij}BF(\varepsilon)\eta_j + E\bar{\omega}_i, \\ \dot{\eta}_N &= (A + \lambda_N BF(\varepsilon))\eta_N + E\bar{\omega}_N, \end{aligned} \tag{14}$$

for $i \in \{2, \ldots, N-1\}$ where $S_L = [s_{ij}]$. The first column of T is an eigenvector of L associated to eigenvalue 0 with length 1, i.e. it is equal to $\pm\mathbf{1}/\sqrt{N}$. Using this we obtain:

$$\begin{aligned} \bar{x} &= \left(\begin{pmatrix} -1 & 0 & \cdots & 0 & 1 \\ 0 & -1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix} \otimes I_n \right) (T \otimes I_n)\eta \\ &= \left(\begin{pmatrix} 0 & V \end{pmatrix} \otimes I_n \right) \eta, \end{aligned}$$

for some suitably chosen matrix V. Therefore, we have

$$\bar{x} = (V \otimes I_n) \begin{pmatrix} \eta_2 \\ \vdots \\ \eta_N \end{pmatrix}. \tag{15}$$

Note that since T is unitary, also the matrix T^{-1} is unitary and the matrix V is uniformly bounded. Therefore the H_∞ norm of the transfer matrix from ω to $\bar{x}$ can be made arbitrarily small if and only if the H_∞ norm of the transfer matrix from $\bar{\omega}$ to η can be made arbitrarily small.

In order for the H_∞ norm from $\bar{\omega}$ to η to be arbitrarily small we need the H_∞ norm from $\bar{\omega}_N$ to η_N to be arbitrarily small. From classical results (see [13], [18]) on H_∞ almost disturbance decoupling we find that this is only possible if (11) is satisfied. Now suppose $u = F(\varepsilon)x$ solves the simultaneous H_∞-ADDPSS of (10) and assume (11) is satisfied. We show next that $u_i = F(\varepsilon)\zeta_i$ solves the H_∞-ASSFS for the MAS described by (1) and (2). Let X be such that $E = BX$.

The fact that $u = F(\varepsilon)x$ solves the simultaneous H_∞-ADDPSS of (10) implies that for small ε we have that $A + \lambda BF(\varepsilon)$ is asymptotically stable for all $\lambda \in \Lambda$. In particular, $A + \lambda_i BF(\varepsilon)$ is asymptotically stable for $i = 2, \ldots, N$ which guarantees that $\eta_i \to 0$ for $i = 2, \ldots, N$ for zero disturbances and all initial conditions. Therefore we have state synchronization.

Next, we are going to show that for any $\bar{r} > 0$, we can choose ε sufficiently small such that the transfer matrix from $\bar{\omega}$ to η_i is less than $\bar{r}$ for $i = 2, \ldots, N$. This guarantees that we can achieve (6) for any $r > 0$. We have for a given M and arbitrary small $\tilde{r}$ that for ε small enough that:

$$\begin{aligned} T^{\lambda}_{wx}(s) &= (sI - A - \lambda BF(\varepsilon))^{-1}B, \\ T^{\lambda}_{wu}(s) &= F(\varepsilon)(sI - A - \lambda BF(\varepsilon))^{-1}B. \end{aligned}$$

satisfies

$$\|T^{\lambda}_{wx}\|_\infty < \tilde{r}, \qquad \|T^{\lambda}_{wu}\|_\infty < M$$

for all $\lambda \in \Lambda$. Denote $\nu_i = F(\varepsilon)\eta_i$.

When $i = N$, it is easy to find that,

$$T_{\bar{\omega}\eta_N} = T^{\lambda_N}_{wx} \begin{pmatrix} 0 & \cdots & 0 & X \end{pmatrix}$$

and hence

$$\|T_{\bar{\omega}\eta_j}\|_\infty < \bar{r}, \qquad \|T_{\bar{\omega}\nu_j}\|_\infty < \bar{M}_j \tag{16}$$

for $j = N$ provided

$$\|X\|\tilde{r} < \bar{r}, \qquad \|X\|M < \bar{M}_N. \tag{17}$$

Recall that we can make $\tilde{r}$ arbitrarily small preserving the bound M. Assume (16) holds for $j = i+1, \ldots, N$. We have:

$$T_{\bar{\omega}\eta_i}(s) = T^{\lambda_i}_{wx}(s) \left[e_i \otimes X + \sum_{j=i+1}^{N} s_{ij}T_{\bar{\omega}\nu_j}(s) \right]$$

where e_i is a row vector with elements equal to zero except for the ith component which is equal to 1. Since

$$\left\| e_i \otimes X + \sum_{j=i+1}^{N} s_{ij}T_{\bar{\omega}\nu_j} \right\|_\infty < \|X\| + \sum_{j=i+1}^{N} |s_{ij}|\bar{M}_j$$

we find (16) for $j = i$ provided:

$$\left(\|X\| + \sum_{j=i+1}^{N} |s_{ij}|\bar{M}_j\right)\tilde{r} < \bar{r}, \left(\|X\| + \sum_{j=i+1}^{N} |s_{ij}|\bar{M}_j\right)\tilde{M} < \bar{M}_i. \tag{18}$$

Note that s_{ij} depends on the graph in $\mathbb{G}$ but since the Laplacian matrices associated to graphs in $\mathbb{G}$ are uniformly bounded we find that also the s_{ij} are uniformly bounded. In this way we can recursively obtain the bounds in (16) for $j = 2, \ldots, N$ provided we choose ε sufficiently small such that the corresponding $\tilde{r}$ satisfies (17) and (18) for $i = 2, \ldots, N-1$. Hence, we can choose ε sufficiently small such that the transfer matrix from $\bar{\omega}$ to η_i is less than $\bar{r}$ for $i = 2, \ldots, N$. As noted before this guarantees that we can achieve (6) for any $r > 0$. ■

For the case when the set of graph $\mathbf{G}$ equals $\mathbb{G}^N_{\alpha,\beta}$ with given $\alpha, \beta > 0$, we develop necessary and sufficient conditions for the solvability of the H_∞-ASSFS for MAS as follows:

Theorem 2 *Consider a MAS described by* (1) *and* (2) *with an associated graph in* $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$. *The* H_∞*-ASSFS is solvable if and only if* (11) *is satisfied and* (A, B) *is stabilizable.*

Proof: We have already noted before that (11) is actually a necessary condition for H_∞-ASSFS. Sufficiency is a direct result of Theorems 5 or Theorem 7 for H_∞-ASSFS. ■

B. Necessary and sufficient condition for H_2-ASSFS

We define the robust H_2-ADDPSS with bounded input as follows. Given $\Lambda \subset \mathbb{C}$, there should exist $M > 0$ such that for any given real number $r > 0$, we can find a parameterized controller (9) for the following subsystem, (10) such that the following holds for any $\lambda \in \Lambda$:

1) The interconnection of the systems (10) and (9) is internally stable;
2) The resulting closed-loop transfer function T_{wx} from w to x has an H_2 norm less than r.
3) The resulting closed-loop transfer function T_{wu} from w to u has an H_∞ norm less than M.

In the above, Λ denotes all possible locations for the nonzero eigenvalues of the Laplacian matrix L when the graph varies over the set $\mathbf{G}$. It is also important to note that M is independent of the choice for r. Note that we need to address two aspects in our controller: H_2 disturbance rejection and robust stabilization (because of the uncertain Laplacian). The latter translates in the H_∞ norm constraint from w to u.

Theorem 3 *Let* $\mathbf{G}$ *be a set of graphs such that the associated Laplacian matrices are uniformly bounded and let* Λ *consist of all possible nonzero eigenvalues of Laplacian matrices associated with graphs in* $\mathbf{G}$.

(Necessity) The H_2*-ASSFS for the MAS described by* (1) *and* (2) *given* $\mathbf{G}$ *is solvable by a parameterized protocol* $u_i = F(\varepsilon)\zeta_i$ *only if* (11) *is satisfied.*

(Sufficiency) The H_2*-ASSFS for the MAS described by* (1) *and* (2) *given* $\mathbf{G}$ *is solvable by a parameterized protocol* $u_i = F(\varepsilon)\zeta_i$ *if the robust* H_2*-ADDPSS with bounded input for the system* (10) *with* $\lambda \in \Lambda$ *is solved by the parameterized controller* $u = F(\varepsilon)x$.

Proof: The proof is similar to the proof of Theorem 1 except that we require the H_2 norm from $\bar{\omega}$ to η_j arbitrarily small while we keep the H_∞ norm from $\bar{\omega}$ to ν_j bounded. ■

If $\mathbf{G}$ equals $\mathbb{G}^N_{\alpha,\beta}$ for certain $\alpha, \beta > 0$ then we have necessary and sufficient conditions:

Theorem 4 *Consider a MAS described by* (1) *and* (2) *with an associated graph in* $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$. *The* H_2*-ASSFS is solvable if and only if* (11) *is satisfied and* (A, B) *is stabilizable .*

Proof: We have already noted before that (11) is actually a necessary condition for H_2-ASSFS. Clearly, also (A, B) is stabilizable is a necessary condition. Sufficiency for H_2-ASSFS, is a direct result of either Theorem 6 or Theorem 8. ■

C. Protocol design for H_∞-ASSFS and H_2-ASSFS

We present below two protocol design methods for both H_∞-ASSFS and H_2-ASSFS problems. One relies on an algebraic Riccati equation (ARE), and the other is based on an asymptotic time-scale eigenstructure assignment (ATEA) method.

1) ARE-based method: Using an algebraic Riccati equation, we can design a suitable protocol provided (A, B) is stabilizable. Consider a set of graphs $\mathbb{G}^N_{\alpha,\beta}$. We design a protocol,

$$u_i = \rho F \zeta_i, \tag{19}$$

where $\rho = \frac{1}{\varepsilon}$ and $F = -B'P$ with P being the unique solution of the continuous-time algebraic Riccati equation

$$A'P + PA - 2\beta PBB'P + I = 0, \tag{20}$$

The main result regarding H_∞-ASSFS is stated as follows.

Theorem 5 *Consider a MAS described by* (1) *and* (2) *such that* (11) *is satisfied.*

If (A, B) *is stabilizable then the* H_∞*-ASSFS stated in Problem 1 with* $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ *is solvable. In particular, for any given real number* $r > 0$, *there exists a* ε^*, *such that for any* $\varepsilon \in (0, \varepsilon^*)$, *the protocol* (19) *achieves state synchronization and an* H_∞ *norm from* ω *to* $x_i - x_j$ *less than* r *for any* $i, j \in 1, \ldots, N$ *and for any graph* $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$.

Proof: Using Theorem 1, we know that we only need to verify that $u = \rho F x$ solves the robust H_∞-ADDPSS with bounded input for the system (10) with $\lambda \in \Lambda$. Given $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$, we know that $\lambda \in \Lambda$ implies $\mathrm{Re}\,\lambda \geq \beta$. Consider the interconnection of (10) and $u = \rho F x$. We define $V(x) = x'Px$ and we obtain:

$$\begin{aligned}
\dot{V} &= x'(A - \rho\lambda BB'P)'Px + w'B'Px \\
&\qquad + x'P(A - \rho\lambda BB'P)x + x'PBw \\
&= x'PBB'Px - x'x - 2\rho\beta x'PBB'Px + 2x'PBw \\
&\leq (1 - \tfrac{\beta}{\varepsilon})x'PBB'Px - x'x + \tfrac{\varepsilon}{\beta}w'w \\
&\leq -\tfrac{\beta}{2}\varepsilon u'u - x'x + \tfrac{\varepsilon}{\beta}w'w
\end{aligned}$$

which implies that the system is asymptotically stable and the H_∞ norm of the transfer function from w to x is less that ε/β while the H_∞ norm of the transfer function from w to u is less that $2/\beta^2$. Therefore, $u = \rho F x$ solves the robust H_∞-ADDPSS with bounded input for the system (10) as required. ■

The main result regarding H_2-ASSFS is stated as follows.

Theorem 6 *Consider a MAS described by* (1) *and* (2) *such that* (11) *is satisfied.*

If (A, B) *is stabilizable then the* H_2*-ASSFS stated in Problem 2 with* $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ *is solvable. In particular, for any given real number* $r > 0$*, there exists a* ε^**, such that for any* $\varepsilon \in (0, \varepsilon^*)$*, the protocol* (19) *achieves state synchronization and an* H_∞ *norm from* ω *to* $x_i - x_j$ *less than* r *for any* $i, j \in 1, \ldots, N$ *and for any graph* $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$.

Proof: Using Theorem 3, we know that we only need to verify that $u = \rho F x$ solves the robust H_2-ADDPSS with bounded input for the system (10) with $\lambda \in \Lambda$. We use the same feedback as in the proof of Theorem 5. In the proof of Theorem 5 it is already shown that the closed loop system is asymptotically stable and the H_∞ norm of the transfer function from w to u is bounded. The only remaining part of the proof is to show that the H_2 norm from w to x can be made arbitrarily small. It is easy to see that we have:

$$(A - \rho\lambda BB'P)'P + P(A - \rho\lambda BB'P) + \rho\beta PBB'P \le 0$$

for large ρ. But then we have:

$$Q_\varepsilon(A - \rho\lambda BB'P)' + (A - \rho\lambda BB'P)Q_\varepsilon + BB' \le 0$$

for $Q_\varepsilon = \varepsilon\beta^{-1}P^{-1}$. It can be shown that this yields that we can make the H_2 norm from w to x arbitrarily small by choosing a sufficiently small ε. ■

2) ATEA-based method: The ATEA-based design is basically a method of time-scale structure assignment in linear multivariable systems by high-gain feedback. For here, it is sufficient to note that there exists non-singular transformation matrix $T_x \in \mathbb{R}^{n\times n}$ (See [12, Theorem 1]) such that

$$\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = T_x x, \tag{21}$$

and the dynamics of $\hat{x}$ is represented as

$$\begin{aligned} \dot{\hat{x}}_1 &= \bar{A}_{11}\hat{x}_1 + \bar{A}_{12}\hat{x}_2, \\ \dot{\hat{x}}_2 &= \bar{A}_{21}\hat{x}_1 + \bar{A}_{22}\hat{x}_2 + \lambda\bar{B}u + \bar{B}\omega, \end{aligned} \tag{22}$$

with $\bar{B}$ invertible. (A, B) is stabilizable implies that $(\bar{A}_{11}, \bar{A}_{12})$ is stabilizable. Choose F_1 such that $\bar{A}_{11} + \bar{A}_{12}F_1$ is asymptotically stable. In that case a suitable protocol for (1) is

$$u_i = F_\varepsilon \zeta_i, \tag{23}$$

where F_ε is designed as

$$F_\varepsilon = \frac{1}{\varepsilon}\bar{B}^{-1}\begin{pmatrix} F_1 & -I \end{pmatrix} T_x \tag{24}$$

The main result regarding H_∞-ASSFS is stated as follows. The result is basically the same as Theorem 5 except for a different design protocol.

Theorem 7 *Consider a MAS described by* (1) *and* (2) *such that* (11) *is satisfied.*

If (A, B) *is stabilizable then the* H_∞*-ASSFS stated in Problem 1 with* $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ *is solvable. In particular, for any given real number* $r > 0$*, there exists a* ε^**, such that for any* $\varepsilon \in (0, \varepsilon^*)$*, the protocol* (23) *achieves state synchronization and an* H_∞ *norm from* ω *to* $x_i - x_j$ *less than* r *for any* $i, j \in 1, \ldots, N$ *and for any graph* $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$.

Proof: Similarly to the proof of Theorem 5, we only need to establish that $u = F_\varepsilon x$ solves the robust H_∞-ADDPSS with bounded input for the system (10) with $\lambda \in \Lambda$. Given $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$, we know that $\lambda \in \Lambda$ implies $\operatorname{Re}\lambda \ge \beta$.

After a basis transformation, the interconnection of the interconnection of (10) and $u = F_\varepsilon x$ is equal to the interconnection of (22) and (23). We obtain:

$$\begin{aligned} \dot{\hat{x}}_1 &= \bar{A}_{11}\hat{x}_1 + \bar{A}_{12}\hat{x}_2, \\ \varepsilon\dot{\hat{x}}_2 &= (\varepsilon\bar{A}_{21} + \lambda F_1)\hat{x}_1 + (\varepsilon\bar{A}_{22} - \lambda I)\hat{x}_2 + \varepsilon\bar{B}w. \end{aligned} \tag{25}$$

Define $\tilde{x}_1 = \hat{x}_1$, $\tilde{x}_2 = \hat{x}_2 - F_1\hat{x}_1$. Then we can write this system (25) in the form:

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{A}_{11}\tilde{x}_1 + \tilde{A}_{12}\tilde{x}_2, \\ \varepsilon\dot{\tilde{x}}_2 &= \varepsilon\tilde{A}_{21}\tilde{x}_1 + (\varepsilon\tilde{A}_{22} - \lambda I)\tilde{x}_2 + \varepsilon\bar{B}w, \end{aligned} \tag{26}$$

where

$$\begin{aligned} &\tilde{A}_{11} = \bar{A}_{11} + \bar{A}_{12}F_1, \quad \tilde{A}_{12} = \bar{A}_{12}, \\ &\tilde{A}_{21} = \bar{A}_{21} - F_1\bar{A}_{11} + \bar{A}_{22} - F_1\bar{A}_{12}, \quad \tilde{A}_{22} = \bar{A}_{22} - F_1\bar{A}_{12}. \end{aligned}$$

In the absence of the external disturbances, the above system (26) is asymptotically stable for small enough ε.

Since $\tilde{A}_{11} = \bar{A}_{11} + \bar{A}_{12}F_1$ is Hurwitz stable, there exists $P > 0$ such that the Lyapunov equation $P\tilde{A}_{11} + \tilde{A}'_{11}P = -I$ holds. For the dynamics $\tilde{x}_1$, we define a Lyapunov function $V_1 = \tilde{x}'_1 P\tilde{x}_1$. Then the derivative of V_1 can be bounded

$$\begin{aligned} \dot{V}_1 &\le -\|\tilde{x}_1\|^2 + \tilde{x}'_2\tilde{A}'_{12}P\tilde{x}_1 + \tilde{x}'_1 P\tilde{A}_{12}\tilde{x}_2 \\ &\le -\|\tilde{x}_1\|^2 + 2\operatorname{Re}(\tilde{x}'_1 P\tilde{A}_{12}\tilde{x}_2) \\ &\le -\|\tilde{x}_1\|^2 + r_1\|\tilde{x}_1\|\|\tilde{x}_2\|, \end{aligned}$$

where $2\|P\tilde{A}_{12}\| \le r_1$. Now define a Lyapunov function $V_2 = \varepsilon\tilde{x}'_2\tilde{x}_2$ for the dynamics $\tilde{x}_2$, where $d_2 > 0$ is to be selected. The derivative of V_2 can then also be bounded.

$$\begin{aligned} \dot{V}_2 &\le -2\operatorname{Re}(\lambda)\|\tilde{x}_2\|^2 + 2\varepsilon\operatorname{Re}(\tilde{x}'_2\tilde{A}_{21}\tilde{x}_1) \\ &\qquad + 2\varepsilon\tilde{x}'_2\tilde{A}_{22}\tilde{x}_2 + 2\varepsilon\operatorname{Re}(\tilde{x}'_2\bar{B}w) \\ &\le -\beta\|\tilde{x}_2\|^2 + \varepsilon r_2\|\tilde{x}_1\|\|\tilde{x}_2\| + \varepsilon r_4\|\omega\|\|\tilde{x}_2\| \end{aligned}$$

for a small enough ε, where we choose that $2\|\tilde{A}_{21}\| \le r_2$, $2\|\tilde{A}_{22}\| \le r_3$, and $2\|\bar{B}\| \le r_4$.

Let $V = V_1 + \gamma V_2$ for some $\gamma > 0$. Then, we have

$$\begin{aligned} \dot{V} \le -\|\tilde{x}_1\|^2 + r_1\|\tilde{x}_1\|\|\tilde{x}_2\| - \gamma\beta\|\tilde{x}_2\|^2 \\ + \varepsilon\gamma r_2\|\tilde{x}_1\|\|\tilde{x}_2\| + \varepsilon\gamma r_4\|\omega\|\|\tilde{x}_2\|. \end{aligned}$$

We have that

$$\begin{aligned} r_1\|\tilde{x}_1\|\|\tilde{x}_2\| &\le r_1^2\|\tilde{x}_2\|^2 + \tfrac{1}{4}\|\tilde{x}_1\|^2, \\ \varepsilon\gamma r_2\|\tilde{x}_1\|\|\tilde{x}_2\| &\le \varepsilon^2\gamma^2 r_2^2\|\tilde{x}_1\|^2 + \tfrac{1}{4}\|\tilde{x}_2\|^2, \\ \varepsilon\gamma r_4\|\omega\|\|\tilde{x}_2\| &\le \varepsilon^2\gamma^2 r_4^2\|w\|^2 + \tfrac{1}{4}\|\tilde{x}_2\|^2. \end{aligned}$$

Now we choose γ such that $\gamma\beta = 1 + r_1^2$ and $r_5 = \gamma r_4$. Then, we obtain

$$\begin{aligned}\dot{V} &\leq -\tfrac{1}{2}\|\tilde{x}_1\|^2 - \tfrac{1}{2}\|\tilde{x}_2\|^2 + \varepsilon^2 r_5^2\|\omega\|^2 \\ &\leq -\tfrac{1}{2}\|\tilde{x}\|^2 + \varepsilon^2 r_5^2\|w\|^2,\end{aligned}$$

for a small enough ε. This yields that $\|T_{w\tilde{x}}\|_\infty < 2\varepsilon r_5$, which immediately leads to $\|T_{wx}\|_\infty < r$ for any real number $r > 0$ as long as we choose ε small enough. On the other hand:

$$T_{wu}(s) = -\tfrac{1}{\varepsilon}\begin{pmatrix}0 & \bar{B}^{-1}\end{pmatrix} T_{w\tilde{x}}(s)$$

and hence:

$$\|T_{wu}\|_\infty \leq \|\bar{B}^{-1}\| r_5.$$

Therefore, $u = F_\varepsilon x$ solves the robust H_∞-ADDPSS with bounded input for the system (10) as required. ■

The main result regarding H_∞-ASSFS is stated as follows.

Theorem 8 *Consider a MAS described by* (1) *and* (2) *such that* (11) *is satisfied.*

If (A, B) *is stabilizable then the* H_2*-ASSFS stated in Problem 1 with* $\mathbf{G} = \mathbb{G}^N_{\alpha,\beta}$ *is solvable. In particular, for any given real number* $r > 0$*, there exists a* ε^**, such that for any* $\varepsilon \in (0, \varepsilon^*)$*, the protocol* (23) *achieves state synchronization and an* H_2 *norm from* ω *to* $x_i - x_j$ *less than* r *for any* $i, j \in 1, \ldots, N$ *and for any graph* $\mathcal{G} \in \mathbb{G}^N_{\alpha,\beta}$.

Proof: Using Theorem 3, we know that we only need to verify that the feedback solves the robust H_2-ADDPSS with bounded input for the system (10) with $\lambda \in \Lambda$. We use the same feedback as in the proof of Theorem 7 where it is already shown that the closed loop system is asymptotically stable and the H_∞ norm from w to u is bounded. The only remaining part of the proof is to show that the H_2 norm from w to x can be made arbitrarily small. This clearly is equivalent to showing that the system (26) has an arbitrary small H_2 norm from w to $\tilde{x}_1$ and $\tilde{x}_2$ for sufficiently small ε. Choose Q such that

$$Q\tilde{A}_{11}' + \tilde{A}_{11}Q = -I$$

In that case we have:

$$\begin{aligned}A_{cl}\begin{pmatrix}\sqrt{\varepsilon}Q & 0\\ 0 & \sqrt{\varepsilon}I\end{pmatrix} + \begin{pmatrix}\sqrt{\varepsilon}Q & 0\\ 0 & \sqrt{\varepsilon}I\end{pmatrix}A_{cl}' + \begin{pmatrix}0 & 0\\ 0 & \bar{B}\bar{B}'\end{pmatrix} \\ \leq \begin{pmatrix}\sqrt{\varepsilon} & \sqrt{\varepsilon}(\tilde{A}_{12} + Q\tilde{A}_{21}')\\ \sqrt{\varepsilon}(\tilde{A}_{12}' + \tilde{A}_{21}Q) & -\frac{\beta}{\sqrt{\varepsilon}}I\end{pmatrix}\end{aligned}$$

for sufficiently small ε where:

$$A_{cl} = \begin{pmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ \tilde{A}_{21} & \tilde{A}_{22} - \frac{\lambda}{\varepsilon}I\end{pmatrix}$$

and we used that $\lambda + \lambda' \geq 2\beta$. We then obtain for sufficiently small ε that:

$$A_{cl}\begin{pmatrix}\sqrt{\varepsilon}Q & 0\\ 0 & \sqrt{\varepsilon}I\end{pmatrix} + \begin{pmatrix}\sqrt{\varepsilon}Q & 0\\ 0 & \sqrt{\varepsilon}I\end{pmatrix}A_{cl}' + \begin{pmatrix}0 & 0\\ 0 & \bar{B}\bar{B}'\end{pmatrix} \leq 0$$

This implies that we can make the H_2 norm from w to x arbitrarily small by choosing sufficiently small ε. ■

References

[1] H. Bai, M. Arcak, and J. Wen. *Cooperative control design: a systematic, passivity-based approach*. Communications and Control Engineering. Springer Verlag, 2011.

[2] Z. Li, Z. Duan, G. Chen, and L. Huang. Consensus of multi-agent systems and synchronization of complex networks: A unified viewpoint. *IEEE Trans. Circ. & Syst.-I Regular papers*, 57(1):213–224, 2010.

[3] P. Lin and Y. Jia. Robust H_∞ consensus analysis of a class of second-order multi-agent systems with uncertainty. *IET Control Theory and Applications*, 4(3):487–498, 2010.

[4] M. Mesbahi and M. Egerstedt. *Graph theoretic methods in multiagent networks*. Princeton University Press, Princeton, 2010.

[5] R. Olfati-Saber, J.A. Fax, and R.M. Murray. Consensus and cooperation in networked multi-agent systems. *Proc. of the IEEE*, 95(1):215–233, 2007.

[6] R. Olfati-Saber and R.M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans. Aut. Contr.*, 49(9):1520–1533, 2004.

[7] E. Peymani, H.F. Grip, and A. Saberi. Homogeneous networks of non-introspective agents under external disturbances - H_∞ almost synchronization. *Automatica*, 52:363–372, 2015.

[8] W. Ren. On consensus algorithms for double-integrator dynamics. *IEEE Trans. Aut. Contr.*, 53(6):1503–1509, 2008.

[9] W. Ren and E. Atkins. Distributed multi-vehicle coordinate control via local information. *Int. J. Robust & Nonlinear Control*, 17(10-11):1002–1033, 2007.

[10] W. Ren and R.W. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Trans. Aut. Contr.*, 50(5):655–661, 2005.

[11] W. Ren and Y.C. Cao. *Distributed Coordination of Multi-agent Networks*. Communications and Control Engineering. Springer-Verlag, London, 2011.

[12] A. Saberi. Decentralization of large-scale systems: a new canonical form for linear multivariable systems. *IEEE Trans. Aut. Contr.*, 30(11):1120–1123, 1985.

[13] A. Saberi, Z. Lin, and A.A. Stoorvogel. H_2 and H_∞ almost disturbance decoupling problem with internal stability. *Int. J. Robust & Nonlinear Control*, 6(8):789–803, 1996.

[14] I. Saboori and K. Khorasani. H_∞ consensus achievement of multi-agent systems with directed and switching topology networks. *IEEE Trans. Aut. Contr.*, 59(11):3104–3109, 2014.

[15] L. Scardovi and R. Sepulchre. Synchronization in networks of identical linear systems. *Automatica*, 45(11):2557–2562, 2009.

[16] J.H. Seo, J. Back, H. Kim, and H. Shim. Output feedback consensus for high-order linear systems having uniform ranks under switching topology. *IET Control Theory and Applications*, 6(8):1118–1124, 2012.

[17] Y. Su and J. Huang. Stability of a class of linear switching systems with applications to two consensus problem. *IEEE Trans. Aut. Contr.*, 57(6):1420–1430, 2012.

[18] H.L. Trentelman. *Almost invariant subspaces and high gain feedback*, volume 29 of *CWI Tracts*. Amsterdam, 1986.

[19] S.E. Tuna. LQR-based coupling gain for synchronization of linear systems. Available: arXiv:0801.3390v1, 2008.

[20] S.E. Tuna. Conditions for synchronizability in arrays of coupled linear systems. *IEEE Trans. Aut. Contr.*, 55(10):2416–2420, 2009.

[21] P. Wieland, J.S. Kim, and F. Allgöwer. On topology and dynamics of consensus among linear high-order agents. *International Journal of Systems Science*, 42(10):1831–1842, 2011.

[22] C.W. Wu. *Synchronization in complex networks of nonlinear dynamical systems*. World Scientific Publishing Company, Singapore, 2007.

[23] T. Yang, S. Roy, Y. Wan, and A. Saberi. Constructing consensus controllers for networks with identical general linear agents. *Int. J. Robust & Nonlinear Control*, 21(11):1237–1256, 2011.

[24] M. Zhang, A. Saberi, A. A. Stoorvogel, and P. Sannuti. Almost regulated output synchronization for heterogeneous time-varying networks of non-introspective agents and without exchange of controller states. In *American Control Conference*, pages 2735–2740, Chicago, IL, 2015.

[25] Y. Zhao, Z. Duan, G. Wen, and G. Chen. Distributed H_∞ consensus of multi-agent systems: a performance region-based approach. *Int. J. Contr.*, 85(3):332–341, 2015.