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An interdiction game on a queueing network with multiple intruders

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ABSTRACT

Security forces are deployed to protect networks that are threatened by multiple intruders. To select the best deployment strategy, we analyze an interdiction game that considers multiple simultaneous threats. Intruders route through the network as regular customers, while interdictors arrive at specific nodes as negative customers. When an interdictor arrives at a node where an intruder is present, the intruder is removed from the network. Intruders and interdictors compete over the value of this network, which is the throughput of unintercepted intruders. Intruders attempt to maximize this throughput by selecting a fixed route through the network, while the interdictors aim to minimize the throughput selecting their arrival rate at each node. We analyze this game and characterize optimal strategies. For special cases, we obtain explicit formulas to evaluate the optimal strategies and use these to compute optimal strategies for general networks. We also consider the network with probabilistic routing of intruders and show that for this case, the value and optimal strategies of the interdictor of the resulting game remain the same.

1. Introduction

Security plays an important role in society as terrorist actions (Jones (2014)), cyber crime (Ponemon (2014)) and the impact of maritime piracy (Bensassi and Martínez-Zarzoso (2010)) increase. It is therefore not surprising that security problems have been receiving increasing attention, and that some of these problems have been tackled using mathematical modeling (Evers, Barros, and Monsuur (2013), Salmeron, Wood, and Baldick (2004), Washburn and Lee Ewing (2011), Wein and Atkinson (2007)). An important class of security problems is network interdiction. Generally speaking, network interdiction involves two sets of players which compete over the value of the network: the intruder and the interdictor. The intruder tries to optimize the value of the system, for example by (1) computing the shortest path between a source node and a sink node (Bell, Kanturska, Schmöcker, and Fonzone (2008), Fulkerson and Harding (1977), Israeli and Wood (2002)); (2) maximizing the amount of flow through the network (Brown, Carlyle, Salmerón, and Wood (2006), Lim and Smith (2007), Salmeron et al. (2004)); (3) maximizing the probability of completing a route (Dickerson, Simari, Subrahmanian, and Kraus (2010), Monsuur, Janssen, and Jutte (2014), Morton, Pan, and Saeger (2007)), or (4) minimizing the probability of getting caught (Alpern, Morton, and Papadaki (2011), Lin, Atkinson, and Glazebrook (2014), Paruchuri et al. (2008)). The interdictor attempts to intercept the intruder before the goal is achieved.

In the field of network interdiction, a wide variety of models have been proposed. Wollmer (1964) was one of the first authors to consider a network interdiction model on a network defined by a set of arcs and nodes. In this model, the interdictor can remove arcs from a network in order to minimize the maximum flow the intruder can obtain from a source node to a sink node. Several papers generalize this work by accounting for the interdictors resources (Wood (1993)), which they can use to remove arcs from the network. The resource cost for such an action depends on the arc itself. These problems are shown to be NP-complete by Wood (1993), even when the costs are equal for all arcs.

Most of the literature focuses on deterministic network interdiction (Washburn and Wood (1995), Wollmer (1964), Wood (1993)). However, many network properties, such as travel time or detection probability, are uncertain in practice. Cormican, Morton, and Wood (1998) consider a max-flow interdiction model in which interdiction success is a random variable. Moreover, extensions are made in which arc capacities are also considered to be stochastic.

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Early work (Fulkerson and Harding (1977), Wollmer (1964), Wood (1993)) consider network interdiction only from the interdictors point of view. Thus, possible reactions of intruders to the interdiction actions are not taken into account. In order to model intelligent adversaries who know of and react to the strategy of an interdictor, game theoretic models have been developed (Alpern et al. (2011), Tambe (2011), Thomas and Washburn (1991), Washburn and Wood (1995)). Tambe (2011) describes a set of game theory applications for security problems. Washburn and Wood (1995) introduces a two-person zero-sum interdiction game that explicitly model the interaction between interdictors and intruders. Intruders select their path from a source to a sink node and interdictors select an arc to inspect. For each arc, a detection probability is given. By using a minmax formulation, the authors determine probabilistic strategies for both players.

Another type of network interdiction is considered in search models in which the goal of the interdictor is to find a hidden intruder. It is possible that the intruder is hidden at a specific node (Alpern et al. (2011), Lin et al. (2014), Neuts (1963)), or moves through the network (Hespanha, Prandini, and Sastry (2000), Thomas and Washburn (1991)). Neuts (1963) introduces a search game in which the intruder hides in one box, while the interdictor must search in a set of boxes. Thomas and Washburn (1991) consider a dynamic search game in which the intruders move through the network and react to the interdictor in the following way. The interdictor searches for an intruder in a set of cells where the travel time to a new cell depends on the distance. The intruder moves after the interdictor moved, taking into account the interdictors previous location. The interdictor wins if the intruder is found within a certain amount of time, otherwise, the intruder wins. Dynamic programming is used to solve the model, such that a linear programming formulation has to be solved for each state. Wein and Atkinson (2007) combine game theory, dynamic programming and queueing theory to intercept terrorists on their way to the city center. A game theoretic approach is used to determine the sensor configuration and to calculate the detecting probabilities. The outcome of the game then becomes input for the queueing model.

In this paper, we introduce an interdiction game on a queueing network including multiple intruders and interdictors which have stochastic travel or service times. Our model is developed to find an optimal deployment strategy for the interdictors that inspect the network nodes, i.e. which nodes should be inspected more often than others. Intruders enter the network at a certain node modeled as a queue and, after having received service, route through the network to their target node. The routing strategies of the intruders can be modeled in a fixed or probabilistic manner. In the case of fixed routing, upon arrival at the network, intruders select their complete route to the sink node. In the case of probabilistic routing, intruders decide their next step at each node according to a certain probability. At the same time, interdictors inspect nodes of the network to prevent the arrival of intruder at the target nodes. When an interdictor inspects a node in which an intruder is being served, the intruder is removed from the network. In this context, the value of the network can be represented by the throughput of the intruders. Multiple intruders and interdictors compete to maximize and minimize this value respectively.

To model the intruders and interdictors, we use the concept of negative customers, which is introduced by Gelenbe, Glynn, and Sigman (1991). These authors describe a network of single-server queues that includes positive and negative customers. Positive customers join the queue with the intention of getting served and then leave the system. Upon arrival of a negative customer, a positive customer (if present) is removed from the queue. We construct a game on this network to find the optimal deployment strategy for the interdictors. These strategies are reduced to choosing arrival rates for inspecting the nodes of the network. The intruders are modeled as the positive customers of the network, and the interdictors as the negative customers.

Our approach of an interdiction game on a queueing network combines two areas of research: game theory and queueing theory. Game theory is used to model the interaction between the intruder and interdictor. Queueing theory models the dynamic flow and time-dependent interdictions in a stochastic environment. The strategies of the intruders and interdictors influence the queueing system. This approach enables the modeling of the flow of intruders and the timing of the actions of the interdictor. The network itself may represent a region that the intruder is required to traverse before it can reach its destination. The queues then have service times that correspond to the stochastic travel times. Alternatively, routes in the network may represent sequences of tasks an intruder must complete before it is able to reach its target node.

This paper is organized as follows. In the next section, we introduce the problem for fixed routing and analyze the proposed interdiction game on a queueing network. In Section 3 we determine optimal strategies for this game and provide some examples. Next, in Section 4, we discuss the game with probabilistic routing and show that these games are closely related. Finally, in Section 5, we present conclusions and provide directions for future research.

2. Game on a network with negative customers

This section introduces an interdiction game on a queueing network with negative customers and fixed intruder routing. Each node in the network represents a queueing system in which the intruders (positive customers) are served by a single server according to a FIFO service discipline. Intruders enter the network at the source node and travel through the network to the sink node. After service completion at a node, the intruder follows its route to another node in the network. If the intruder is not interdicted at some intermediate node (neither the sink nor the source node), he successfully reaches the sink node. Interdictors (negative customers) arrive at the network nodes to search for intruders. If the interdictor arrives at an empty node, he leaves the network immediately. If an interdictor arrives at a node and finds an intruder being served, then he removes the intruder and leaves the network. Because handling an intruder requires extra effort and time, we assume that only the intruder in service is removed.

The players of the interdiction game, the intruders and the interdictors, are constrained by a budget. This limits the rates with which they arrive at the network: the interdictor has to determine arrival rates at nodes for inspecting the queueing systems and the intruder determines arrival rates at the routes. This repeated interplay results in probabilities of interdiction at nodes and ultimately yield intruder arrival rates at the sink node. The value of the game is therefore defined as the rate of intruders arriving at the sink.

In the following sections, we introduce a network with intruders and interdictors in which fixed routing of intruders is considered. After that, we give the game formulation and prove the existence of optimal strategies.

2.1. Network with fixed routing of intruders

Consider a queueing network with a source node 0, sink node $N+1$ and intermediate nodes $I=\{1,2,\ldots,N\}$, on a connected and directed graph G. Intruders want to travel through the network undetected from source to sink, while interdictors try to intercept them at nodes in I. The source node 0 is linked to a non-empty set $S\subseteq I$ of start nodes, while there is a non-empty set of target nodes $T\subseteq I$ linked to the sink node $N+1$. There is no direct link between the source and sink, but it is possible that $S\cap T\neq\emptyset$. In addition, we assume that each node in S has just one incoming link (from

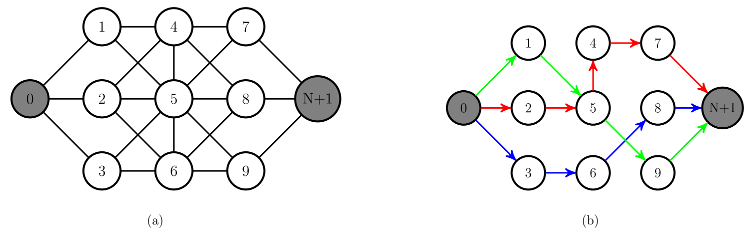


Fig. 1. Example network G with $N = 9$. (a) Underlying network and (b) Network with three routes.

the source); likewise, we assume that each node in T has just one outgoing link (to the sink). An example of such a network is shown in Fig. 1a.

Given this queueing network, we consider the set of all routes from node 0 to node $N+1$, in which a route follows the links in the network. This set may be (countably) infinite, due to cycles in the network. We consider a finite subset K of the set of all routes without cycles. A route $k \in K$ (in which we do not take into account nodes 0 and $N+1$) is given by $r_k = [r(k,1), r(k,2), \ldots, r(k,N_k)]$, where $r(k, s)$ identifies the s-th node on route k and N_k is the length of route k. The set of nodes contained in route k is denoted by I_k. In Fig. 1b an example network with three routes is given.

Intruders arrive at the source of the network according to a Poisson process with rate Λ, and choose route k with probability p_k; i.e. the arrival rate of intruders following route k is given by $\lambda_k = p_k\Lambda$. Therefore, they enter at node $s \in S$ with arrival rate $\lambda_s = \sum_{k\in K, r(k,1)=s} \lambda_k$.

When intruders arrive at node i, they receive service or join the queue. The service time at node i is equal for all intruders and is exponential with rate $\mu_i > 0$. The service time of each node is independent of the service time at other nodes.

Interdictors arrive at the network according to a Poisson process with rate Λ^- and select node i with probability p_i^-, such that they arrive at node $i \in I$ with rate $\lambda_i^- = p_i^-\Lambda^-$. Upon arrival of an interdictor, the intruder in service (if present) is removed from the node. If the interdictor arrives at an empty node, he immediately leaves the network.

Intruders routing through the network leave a node either because of service completion or because of interdiction while being served. Intruders are served at node i with exponential service rate μ_i and interdictors arrive independently according to a Poisson process with rate λ_i^-. This implies that intruders are interdicted with rate λ_i^-. Due to the memoryless property of the exponential distribution, the probability that an intruder leaves node i because of service completion corresponds to the probability that the service is completed before an interdictor arrives at node i:

$$\frac{\mu_i}{\mu_i + \lambda_i^-}, \tag{1}$$

and the probability that the intruder leaves node i (and is removed from the network) due to interdiction equals:

$$\frac{\lambda_i^-}{\mu_i + \lambda_i^-}.$$

These steady state probabilities are independent of the presence of other intruders in the network and of the time the intruders have spent in the queue. Route k is completed if an intruder completed service at each node of the route and reaches the sink node without being interdicted. Therefore, the probability that an intruder actually completes route k is given by:

$$\mathbb{P}(\text{intruder completes route } k) = \prod_{s=1}^{N_k} \frac{\mu_{r(k,s)}}{\mu_{r(k,s)} + \lambda^-_{r(k,s)}}. \tag{2}$$

2.2. Game description

To model the interaction between intruders and interdictors, we create an interdiction game on the queueing network described above. The intruders and interdictors compete over the value of this network, which is the arrival rate of intruders at the sink node, or equivalently, the sum of departure rates at nodes in T. This is a zero-sum game in which the intruders try to maximize their throughput by deciding on their routes, while the interdictors aim at minimizing this throughput by deciding on the inspection rates at nodes in I.

The intruders select their route by choosing λ_k for each route k, constrained by the total arrival rate Λ. Thus, the action set of the intruders given the set of routes K, is given by:

$$A_{intruder} = \left\{ \lambda \,\middle|\, \sum_{k\in K} \lambda_k = \Lambda,\ \lambda_k \geq 0,\ \text{for all } k \in K \right\}, \tag{3}$$

where $\lambda = (\lambda_k : k \in K)$

The interdictors select the inspection rate, which is given by λ_i^- for all $i = 1, \ldots, N$, and the total rate is limited by a nonnegative interdiction budget Λ^-. So the action set of the interdictors is given by:

$$A_{interdictor} = \left\{ \lambda^- \,\middle|\, \sum_{i=1}^{N} \lambda_i^- = \Lambda^-,\ \lambda_i^- \geq 0,\ \text{for all } i = 1, \ldots, N \right\}, \tag{4}$$

where $\lambda^- = (\lambda_1^-, \ldots, \lambda_N^-)$

The payoff function of this game is the throughput (or arrival rate) of the intruders at the sink node, and is obtained by multiplying the arrival rate for each route k by the probability of completing the given route (see Eq. (2)) and summing over all possible routes:

$$v(\lambda, \lambda^-) = \sum_{k\in K} \lambda_k \prod_{s=1}^{N_k} \frac{\mu_{r(k,s)}}{\mu_{r(k,s)} + \lambda^-_{r(k,s)}}. \tag{5}$$

2.3. Game analysis

In this section we analyze the interdiction game and prove the existence of pure optimal strategies.

Strategies for the intruders and interdictors are measures F and G defined for the sets $A_{intruder}$ and $A_{interdictor}$, such that $F(A_{intruder}) = 1$ and $G(A_{interdictor}) = 1$. We define the expected payoff by:

$$\mathbb{E}(v(F,G)) = \int_{A_{intruder}\times A_{interdictor}} v(\lambda, \lambda^-) d(F \times G).$$

A pure strategy for the intruder is a strategy F such that $F(\lambda) = 1$ for a particular $\lambda \in A_{intruder}$. This pure strategy then is denoted by λ, and is chosen with probability one. Likewise, pure strategies for the interdictor are represented by λ^-. The existence of pure strategies can be expressed by the following theorem:

Theorem 1. *Consider the interdiction game on a queueing network. The game has a saddle point λ^* and λ^{-*} in (optimal) pure strategies.*

Moreover, for the interdictor this strategy is unique. The value of the interdiction game is given by:

$$v = \max_{\lambda} \min_{\lambda^-} v(\lambda, \lambda^-) = \min_{\lambda^-} \max_{\lambda} v(\lambda, \lambda^-)$$

Proof. Define the following two values:

$$v_I = \sup_F \inf_G \mathbb{E}(v(F, G)),$$

$$v_{II} = \inf_G \sup_F \mathbb{E}(v(F, G)).$$

The payoff function $v(\lambda, \lambda^-)$ is continuous, and the action sets $A_{intruder}$ and $A_{interdictor}$ are compact. Therefore, sup inf and inf sup may be replaced by max min and min max respectively, and $v_I = v_{II} = v$ and there exist optimal strategies (see Section IV.3 in Owen (1982)).

The existence of optimal pure strategies can be shown through the following function:

$$f(\lambda^-) = \prod_{i=1}^{N} \frac{\mu_i}{\mu_i + \lambda_i^-}.$$

The Hessian $\Delta^2 f(x)$ is positive definite, implying that $f(x)$ is strictly convex. The payoff function $v(\lambda, \lambda^-)$ is therefore strictly convex in λ^- for each λ. Moreover, $v(\lambda, \lambda^-)$ is a linear, and thus concave, function in λ for each λ^-. Thus, both the interdictor and the intruder have an optimal pure strategy and the value is given by v (see Section IV.4.1 in Owen (1982)). Because the payoff function is strictly convex in λ^-, the strategy for the interdictor is unique. □

2.4. Optimization model

Given that optimal pure strategies exist, we formulate a minimization problem to find the optimal strategy of the interdictor. Let K be a fixed, finite set of routes from source to sink through the queueing network. The following optimization problem finds optimal strategies of the intruder and the interdictor:

$$v = \min_{\lambda^-} \max_{\lambda} \sum_{k\in K} \lambda_k \prod_{s=1}^{N_k} \frac{\mu_{r(k,s)}}{\mu_{r(k,s)} + \lambda^-_{r(k,s)}} \tag{6}$$

$$\text{s.t.} \quad \sum_{j=1}^{N} \lambda_j^- = \Lambda^-, \quad \sum_{k=1}^{K} \lambda_k = \Lambda, \tag{7}$$

$$\lambda_i^-, \lambda_k \geq 0, \quad \text{for all } i = 1, \ldots, N, k \in K. \tag{8}$$

Note that the value v is the arrival rate of intruders at the sink node $N+1$. In case $\Lambda = 1$, it also corresponds to the fraction of intruders that reach their destination, and thus the probability of reaching the sink node.

The optimal strategy of the interdictor can be found by solving the optimization problem as described in the next lemma.

Lemma 1. *For the interdiction game on a queueing network, the value of the game and the optimal strategy for the interdictor are found by solving the following convex minimization problem:*

$$v = \min_{\lambda^-} \quad w \tag{9}$$

$$\text{s.t.} \quad \Lambda \prod_{s=1}^{N_k} \frac{\mu_{r(k,s)}}{\mu_{r(k,s)} + \lambda^-_{r(k,s)}} \leq w, \quad \text{for all } k \in K, \tag{10}$$

$$\sum_{j=1}^{N} \lambda_j^- = \Lambda^-, \tag{11}$$

$$\lambda_i^- \geq 0, \quad \text{for all } i = 1, \ldots, N. \tag{12}$$

Proof. The probability of completing route k is given by Eq. (2), so the throughput in the case where the intruder always chooses route k is given by $\Lambda \prod_{s=1}^{N_k} \frac{\mu_{r(k,s)}}{\mu_{r(k,s)}+\lambda^-_{r(k,s)}}$. Given any interdiction strategy λ^-, the worst case for the interdictor is when the intruder chooses to assign his full budget Λ to the set of routes with maximal completion probability. The interdictor tries to minimize this worst case, which can be achieved by solving the non-linear program in Equations (9)–(12). From the proof of Theorem 1, we know that Constraints (10) are convex in λ^-, so Equations (9)–(12) yields a convex optimization problem. □

Depending on the graph structure, the number of constraints in Lemma 1 can grow exponentially. This is certainly the case for a complete graph.

Note that w is the maximum payoff the intruders can obtain for any available route, given the choice of λ^- of the interdictors. In the following section, we solve this model for networks with special structures, such as networks with only parallel or only tandem nodes. These are the networks in which routes do not intersect. Because the payoff-function is continuous in λ^-, the probability of completing a specific route in these networks must be the same for each route. In the next section, we also provide numerical results for networks with a general network structure.

3. Finding optimal strategies

In the previous section, we described an interdiction game in which intruders and interdictors compete over the throughput of the intruders. In this section, we derive analytical expressions and algorithms for finding optimal strategies, for three special cases. In these cases, we let K equal the set of all possible routes. Finally, we use the analytical expressions to speed up the solving process for general networks and provide numerical results.

3.1. Network of parallel nodes

Consider a network of parallel nodes as shown in Fig. 2a. The length of each route k equals one. There are N possible routes such that $r_k = [k]$ for $k = 1, \ldots, N$. The payoff function of the game is given by:

$$v(\lambda, \lambda^-) = \sum_{k=1}^{N} \lambda_k \frac{\mu_k}{\mu_k + \lambda_k^-}.$$

The value and optimal strategies of this game are given in the following theorem:

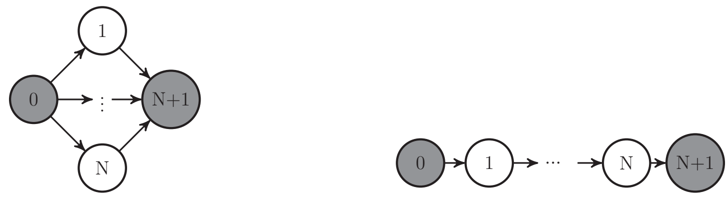
Theorem 2. *Consider the interdiction game on a network of parallel nodes. For the interdictors, the unique optimal strategy λ^{-*} is given by:*

$$\lambda_i^{-*} = \frac{\mu_i}{\sum_{j=1}^{N} \mu_j} \Lambda^-, \quad \text{for all } i = 1, \ldots, N. \tag{13}$$

The value of the game is:

$$v = \frac{\sum_{j=1}^{N} \mu_j}{\sum_{j=1}^{N} \mu_j + \Lambda^-} \Lambda. \tag{14}$$

Proof. According to Theorem 1, there exists an optimal pure strategy and the value is given by $v = \max_{\lambda} \min_{\lambda^-} v(\lambda, \lambda^-)$. Through Lemma 1, we know that optimal strategies for the interdictors can



(a) (b)

Fig. 2. Two networks for which an explicit value of the game can easily be derived. (a) Network of parallel nodes and (b) Network of tandem nodes.

be found by solving:

$$\begin{aligned} \min_{\lambda^-} \quad & w \\ \text{s.t.} \quad & \Lambda \frac{\mu_i}{\mu_i + \lambda_i^-} \le w, \quad \text{for all } i = 1, \ldots, N, \\ & \sum_{j=1}^{N} \lambda_j^- = \Lambda^-, \\ & \lambda_i^- \ge 0, \quad \text{for all } i = 1, \ldots, N. \end{aligned} \tag{15}$$

Given this network of parallel nodes, the interdictor must ensure that the probability of completing a specific route will be the same for each route. Thus, for an optimal λ^{-*}:

$$v = \Lambda \frac{\mu_i}{\mu_i + \lambda_i^{-*}}, \quad \text{for all } i = 1, \ldots, N. \tag{16}$$

By combining Eq. (16) with the interdiction budget constraint $\sum_{j=1}^{N} \lambda_j^- = \Lambda^-$, we obtain the optimal strategy λ^{-*} and the value of the game. □

Eq. (13), shows that inspection rates increase with node service rates.

Given Eq. (14), it follows that the value of the game is dependent only upon the sum of the service rates μ_i and not upon how these rates are assigned to the nodes. Thus, from a game-theoretic point of view, a network of parallel nodes is equivalent to a single queue with service rate equal to the sum of service rates.

3.2. Network of tandem nodes

Consider a network of tandem nodes as shown in Fig. 2b. There is only one route with length N and rate Λ. Therefore, the value of the game only depends on the strategy of the interdictor. The payoff function of the game is given by:

$$v(\lambda^-) = \Lambda \prod_{i=1}^{N} \frac{\mu_i}{\mu_i + \lambda_i^-}. \tag{17}$$

For technical purposes, we introduce a relaxation of the optimization model described in Section 2.4. In this model, only the budget Constraint (7) is taken into account, relaxing the non-negativity Constraints (8). The value and optimal solutions of this relaxation model with the Objective function (17) are given by the following lemma:

Lemma 2. *Consider the relaxation problem on a network of tandem nodes. The optimal solution λ^{-*} is given by:*

$$\lambda_i^{-*} = \frac{\Lambda^- + \sum_{j=1}^{N} \mu_j}{N} - \mu_i, \quad \textit{for all } i = 1, \ldots, N, \tag{18}$$

and the value of this relaxation is:

$$v^r = \Lambda \prod_{i=1}^{N} \frac{N\mu_i}{\sum_{j=1}^{N} \mu_j + \Lambda^-}. \tag{19}$$

Moreover, if $\frac{\Lambda^- + \sum_{j=1}^{N} \mu_j}{N} \ge \max_j \mu_j$, the optimal solution and the value of the relaxation problem are equal to the optimal strategies and the value of the original interdiction game.

Proof. The value v^r of the relaxation can be found by solving the following optimization problem:

$$\begin{aligned} v^r = \min_{\lambda^-} \quad & \Lambda \textstyle\prod_{i=1}^{N} \frac{\mu_i}{\mu_i + \lambda_i^-}, \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^{N} \lambda_i^- = \Lambda^-. \end{aligned}$$

In order to derive v^r, we use a Lagrangian approach. The Lagrangian of this problem is given by:

$$L(\lambda^-, \psi) = \Lambda \prod_{i=1}^{N} \frac{\mu_i}{\mu_i + \lambda_i^-} + \psi \left(\sum_{j=1}^{N} \lambda_j^- - \Lambda^- \right).$$

Taking the partial derivatives with respect to λ_i^- and ψ, and rewriting, enables the calculation of the optimal solution of the relaxation. If $\frac{\Lambda^- + \sum_{j=1}^{N} \mu_j}{N} \ge \max_j \mu_j$, then $\lambda_i^{-*} \ge 0$ for all $i = 1, \ldots, N$. In that case, it is also a feasible solution to the original game and v^r is an upper bound for the value v of the original game. Because there are fewer constraints in the relaxation model, v^r also gives a lower bound for v. Combining the lower and upper bound, gives $v^r = v$ and the resulting solution is also an optimal strategy for the original game. □

Eq. (18) shows that the inspection rate increases as the service rate decreases, contrary to the case for parallel nodes. This equation also suggests that if the service rate of a particular node i is very high, it is optimal to set $\lambda_i^- = 0$ beforehand. To be more precise, suppose that $\frac{\Lambda^- + \sum_{j=1}^{N} \mu_j}{N} < \max_j \mu_j$. Then there is a node i such that $\lambda_i^- < 0$ and the value of the relaxation does not correspond to the value of the original game. To find a feasible solution for the original interdiction game, we introduce an algorithm that, starting with the solution of the relaxation, sequentially removes nodes for which $\lambda_i^- < 0$. In every step of the algorithm, the state space is reduced by adjusting the value of λ_i^- that violates Constraints (8). By using this relaxation and iterative approach, we eventually find the optimal pure strategy for the interdictor for the original game.

Algorithm 1.
Let I' be a subset of the set I, and $N' = |I'|$.

1. Set $I' = I$, and $N' = |I'|$.
2. Calculate for all $i \in I'$:

$$\lambda_i^- = \frac{\Lambda^- + \sum_{j \in I'} \mu_j}{N'} - \mu_i. \tag{20}$$

If $\lambda_i^- \ge 0$ for all $i \in I'$: STOP, λ^- is given by Eq. (20) and the value of the game is given by:

$$v = \Lambda \prod_{i \in I'} \frac{N' \mu_i}{\sum_{j \in I'} \mu_j + \Lambda^-}.$$

Else: Go to next step

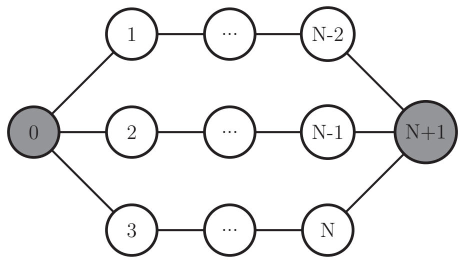


Fig. 3. Network of parallel tandem nodes: the number of nodes per route may differ.

3. For all i such that $\lambda_i^- < 0$: Set $\lambda_i^- = 0$, remove i from I' and update N'. Return to step 2.

Theorem 3. *Algorithm 1 finds the optimal strategy for the interdictors and the value of the interdiction game on a network of tandem nodes.*

The proof of Theorem 3 can be found in Appendix A.

3.3. Networks without intersecting routes

In this section, we consider networks in which the set of routes K is restricted to routes that do not intersect. An example of such a network with three routes is shown in Fig. 3. Consider a network of N nodes with routes K that do not intersect, in which route k consists of N_k nodes. The value function of this game is given by:

$$v(\lambda, \lambda^-) = \sum_{k\in K} \lambda_k \prod_{s=1}^{N_k} \frac{\mu_{r(k,s)}}{\mu_{r(k,s)} + \lambda^-_{r(k,s)}}. \tag{21}$$

As before, we first consider the relaxation model such that $\lambda_i^- < 0$ is allowed. Λ_k^- is defined as the interdiction budget assigned by the interdictor to route k:

$$\Lambda_k^- = \sum_{s=1}^{N_k} \lambda^-_{r(k,s)}, \quad \text{for all } k \in K,$$
$$\Lambda^- = \sum_{k\in K} \Lambda_k^-. \tag{22}$$

The optimal solution and the value of this relaxation are given by the following lemma:

Lemma 3. *Consider the relaxation problem with Objective function (21) on a network without intersecting routes. The value v^r of this model can then be found by solving:*

$$\Lambda^- + \sum_{i=1}^{N} \mu_i = \sum_{k\in K} N_k \sqrt[N_k]{\frac{\Lambda \prod_{s=1}^{N_k} \mu_{r(k,s)}}{v^r}}.$$

Moreover, the budget assigned to route k in the optimal solution, is given by:

$$\Lambda_k^{-*} = N_k \sqrt[N_k]{\frac{\Lambda \prod_{s=1}^{N_k} \mu_{r(k,s)}}{v^r}} - \sum_{t=1}^{N_k} \mu_{r(k,t)}.$$

Proof. If we knew the interdiction budget Λ_k^-, Lemma 2 could be used to obtain the value of the relaxation, and its optimal budget assignment to individual nodes on route k. The throughput of intruders over route k is:

$$v_k^r = \lambda_k \prod_{s=1}^{N_k} \frac{N_k \mu_{r(k,s)}}{\sum_{t=1}^{N_k} \mu_{r(k,t)} + \Lambda_k^-}. \tag{23}$$

Therefore, similar to the approach followed in Lemma 1, the optimal solution and value v^r in this relaxation can be found by solving:

$$\min_{\Lambda_k^-} \; w$$
$$\text{s.t.} \;\; \Lambda \prod_{s=1}^{N_k} \frac{N_k \mu_{r(k,s)}}{\sum_{t=1}^{N_k} \mu_{r(k,t)} + \Lambda_k^-} \le w, \quad \text{for all } k \in K \tag{24}$$
$$\sum_{k\in K} \Lambda_k^- = \Lambda^-.$$

Solving Eq. (24) yields the optimal strategy Λ^{-*} for the relaxation. As routes do not intersect, for an optimal Λ_k^{-*}:

$$v^r = \Lambda \prod_{s=1}^{N_k} \frac{N_k \mu_{r(k,s)}}{\sum_{t=1}^{N_k} \mu_{r(k,t)} + \Lambda_k^{-*}}, \quad \text{for all } k \in K, \tag{25}$$

implying:

$$\Lambda_k^{-*} = N_k \sqrt[N_k]{\frac{\Lambda \prod_{s=1}^{N_k} \mu_{r(k,s)}}{v^r}} - \sum_{t=1}^{N_k} \mu_{r(k,t)}. \tag{26}$$

Combining Eq. (26) and Eq. (22) yields:

$$\Lambda^- + \sum_{i=1}^{N} \mu_i = \sum_{k\in K} N_k \sqrt[N_k]{\frac{\Lambda \prod_{s=1}^{N_k} \mu_{r(k,s)}}{v^r}}. \tag{27}$$

The value v^r can be found by solving Eq. (27) iteratively. □

The optimal strategy is one in which the probability of completing a particular route, is the same for each possible route. It may happen that for some route k, $\frac{\Lambda_k^- + \sum_{s=1}^{N_k} \mu_j}{N} < \max_{j\in r_k} \mu_j$, in which case the value of the relaxation model is not necessarily equal to the value of the original game with inequality constraints. Therefore, we introduce an algorithm to find a feasible solution. The core of this algorithm is similar to Algorithm 1: set λ_i^- to zero if it violates the inequality constraints and recalculate optimal strategies for the relaxation without these nodes.

Algorithm 2.
Let I' be a subset of the set I, let $I_k = \{i \in I | i \in r_k\}$ and I'_k a subset of I_k. Moreover, let $N' = |I'|$ and $N'_k = |I'_k|$.

1. Set $I' = I$, $N' = |I'|$, and $I'_k = I_k$, $N'_k = |I'_k|$ for all $k \in K$.
2. Obtain v^r from:
$$\Lambda^- + \sum_{i\in I'} \mu_i = \sum_{k\in K} \sqrt[N'_k]{\frac{\prod_{i\in I'_k} N'_k \mu_i}{v^r}}.$$
3. For all $k \in K$, let:
$$\Lambda_k^- = N'_k \sqrt[N'_k]{\frac{\Lambda \prod_{i\in I'_k} \mu_i}{v^r}} - \sum_{i\in I'_k} \mu_i.$$
4. For all $k \in K$ and for all $i \in I'_k$, let:
$$\lambda_i^- = \frac{\Lambda_k^- + \sum_{j\in I'_k} \mu_j}{N'_k} - \mu_i. \tag{28}$$
If $\lambda_i^- > 0$ for all $k = 1, \ldots, K$ and for all $i \in I'_k$: STOP, λ^- is given by Eq. (28) and the value of the game is given by v^r.
Else: Go to the next step
5. For all $k \in K$ and for all $i \in I'_k$:
If $\lambda_i^- \le 0$ and $\mu_i = \max_{j\in I'_k} \mu_j$ $(i \in I'_k)$: Set $\lambda_i^- = 0$ and remove i from I' and I'_k. Then, go back to Step 2.

Theorem 4. *Algorithm 2 finds the optimal strategy for the interdictors and the value of the interdiction game on a network of parallel tandem nodes without intersections.*

The proof of Theorem 4 can be found in Appendix A.

Remark 1. The algorithm can be more efficient by replacing Step 5 of the algorithm with:

- For all $k \in K$ and for all $i \in I'_k$:
  If $\lambda_i^- < 0$: Set $\lambda_i^{-*} = 0$ and remove i from I' and I'_k. Then, go back to Step 2.

Due to its length, a proof that the adjusted algorithm also finds an optimal solution is omitted.

3.4. General network

In the previous sections, we obtained analytical expressions and algorithms to find optimal strategies for special networks, which do not contain intersecting routes. In this section, we discuss the general network case.

The optimal strategy for the general network case is obtained using Lemma 1. The previously introduced results can be used to speed up the process of solving general networks. In particular, utilizing Lemma 2 may decrease the number of general network variables with equal service rates in the following way. Each route can be split into a set of intersection nodes I_k^I (nodes that are also part of another route) and, between these intersection nodes, segments of tandem nodes $I_k^T = I_k \setminus I_k^I$. Constraints (10) in Lemma 1 can then be rewritten as follows:

$$\Lambda \prod_{i \in I_k^I} \frac{\mu_i}{\mu_i + \lambda_i^-} \prod_{i \in I_k^T} \frac{\mu_i}{\mu_i + \lambda_i^-} \leq w, \quad \text{for all } k \in K.$$

Given the interdiction rates λ^- and a route k, the order of the nodes in this route has no impact on the game value. Therefore, route k can be seen as a sequence of intersection nodes I_k^I and one separate tandem queue with nodes I_k^T. Let $\tilde{\Lambda}_k^-$ be the total budget that is assigned to the tandem nodes in route k ($\tilde{\Lambda}_k^- = \sum_{i \in I_k^T} \mu_i$). If $\tilde{\Lambda}_k^-$ is known, it is optimal to divide this budget over the nodes using Lemma 2, as this can be seen as a separate tandem queue. So, by Lemma 2, the constraints can be replaced with the following constraints:

$$\Lambda \prod_{i \in I_k^I} \frac{\mu_i}{\mu_i + \lambda_i^-} \prod_{i \in I_k^T} \frac{|I_k^T| \mu_i}{\sum_{j \in I_k^T} \mu_j + \tilde{\Lambda}_k^-} \leq w, \quad \text{for all } k \in K. \tag{29}$$

Remark 2. Lemma 2 gives a value for the relaxation, which equals the value of the original game only if no negative interdiction rate is assigned to one of the nodes. This is always the case if all nodes have an equal service rate because nodes with equal service rates always have the same interdiction rate. Constraints (29) can also be used to solve networks with unequal service rates. Then, by analogy with Algorithm 2, the nodes with a negative interdiction rate can be removed from the network and the resulting non-linear program must be solved again.

3.5. Numerical examples

We have developed an interdiction game with intruders and interdictors and derived optimal strategies. In this section, we first consider the computational efforts of our algorithms. Then, we present two illustrative examples.

3.5.1. Computational efforts

This section explores the computational efforts required to obtain the optimal strategy. To this end, Table 1 below presents the running times for randomly generated networks both for a direct implementation of Lemma 1 and invoking the structural results of Section 3.4 based on Lemma 2. For the results of Table 1, we constructed random routes in a network whose underlying graph is complete, and all nodes have service rate one. For each case, we generated ten random instances and show the average values and 95%-confidence intervals in Table 1. The average length of the routes equals the square root of the number of nodes, and in general it holds that if the number of routes is small, the number of intersection nodes is also small.

To find optimal strategies, we used CVX 2.1, a package for solving convex programs (CVX Research (2016), Grant and Boyd (2008)), in Matlab version R2014b MATLAB (2014) on an Intel(R) Core(TM) i7 CPU, 2.4 gigahertz, 8 gigabytes of RAM. To this end, we reformulated Constraints (10) such that they comply with the ruleset of Disciplined Convex Programming (DCP) (Grant, Boyd, and Ye (2006)) as follows:

$$\Lambda \prod_{s=1}^{N_k} \mu_{r(k,s)} \prod_{s=1}^{N_k} \left(\mu_{r(k,s)} + \lambda_{r(k,s)}^-\right)^{-1} \leq w,$$

which can be rewritten as:

$$C \prod_{s=1}^{N_k} \nu_{r(k,s)}^{-1} \leq w,$$

where $C = \Lambda \prod_{s=1}^{N_k} \mu_{r(k,s)}$ is a constant and $\nu_{r(k,s)} = \mu_{r(k,s)} + \lambda_{r(k,s)}^-$. Invoking the function *prod_inv* from the CVX library, the reformulation of the convex program of Lemma 1 meets the requirements of the DCP ruleset (Grant et al. (2006)). With this formulation, CVX finds the optimal solution of the problem.

From Table 1, we observe that the running time for networks of reasonable size remains acceptable for practical purposes. The network structure exploited in Lemma 2 considerably reduces the running time for networks containing a relatively low number of routes.

3.5.2. Networks of parallel and tandem nodes

First, we compare a network of parallel nodes with a network of tandem nodes. Both networks consist of ten nodes with service rate one. The results are shown in Fig. 4a. For a network with tandem nodes, the throughput is much lower than for the network with parallel nodes. This is an intuitive result because the intruder must be served at all nodes within a tandem node network, while in the network with parallel nodes, intruders are only required to complete service at one node.

Second, we investigate whether it is better to design a network with one node or with multiple nodes, i.e. the optimal locations for protection against intruders. In a network with parallel nodes, we see that the value of the game increases in the number of nodes because intruders can choose between multiple paths (see Theorem 2). Therefore, in order to obtain the same value in a network with multiple nodes, the service rate must be smaller in proportion to the number of nodes, e.q., the services rate must be halved if the number of nodes is doubled.

Now, consider a tandem network in which the intruders are required to complete service at all nodes. We compare one and two node cases. In the two node case the intruder is served twice as fast. Fig. 4b shows that for a low interdiction budget, it is better to have one node, while for a high interdiction budget, most intruders are intercepted if multiple nodes are considered. These examples not only illustrate that our model can be used to determine optimal deployment strategies of the interdictors, but they may also help in the design of an effective network topology.

3.5.3. General network

Consider the network in Fig. 5 with six intersecting routes $r_1, r_2, \ldots, r_6$. These routes have six intersection nodes $i_1, i_2, \ldots, i_6$ and 35 tandem nodes. For each node, the service rate equals one. We solved this model in Matlab for different values of Λ^-. The total arrival rate of the intruder Λ equals one. The value ν and optimal strategies λ^- and $\tilde{\Lambda}^-$ for the interdictor are shown in Table 2. The rates for all intersection nodes are given by $\lambda_{i_1}^-, \ldots, \lambda_{i_6}^-$ and

Table 1
Running times for solving Lemma 1 with and without implementation of Lemma 2.

# Nodes	# Routes	Λ	Running time after implementation of Lemma 2 (seconds)	Running time without implementation of Lemma 2 (seconds)	Game value
1000	10	5	2.44 ($\pm$ 0.13)	4.03 ($\pm$ 0.16)	0.381 ($\pm$ 0.020)
1000	50	5	13.66 ($\pm$ 2.99)	14.07 ($\pm$ 2.81)	0.694 ($\pm$ 0.024)
1000	100	5	27.39 ($\pm$ 1.21)	27.41 ($\pm$ 1.32)	0.777 ($\pm$ 0.010)
5000	10	10	4.17 ($\pm$ 0.46)	12.53 ($\pm$ 0.62)	0.199 ($\pm$ 0.025)
5000	50	10	25.13 ($\pm$ 1.10)	36.11 ($\pm$ 1.47)	0.563 ($\pm$ 0.018)
5000	100	10	66.95 ($\pm$ 5.10)	72.94 ($\pm$ 4.50)	0.680 ($\pm$ 0.014)
25000	10	20	8.81 ($\pm$ 1.79)	63.52 ($\pm$ 2.16)	0.058 ($\pm$ 0.018)
25000	50	20	55.66 ($\pm$ 3.97)	121.96 ($\pm$ 5.08)	0.385 ($\pm$ 0.015)
25000	100	20	273.56 ($\pm$ 19.42)	553.85 ($\pm$ 62.16)	0.536 ($\pm$ 0.009)

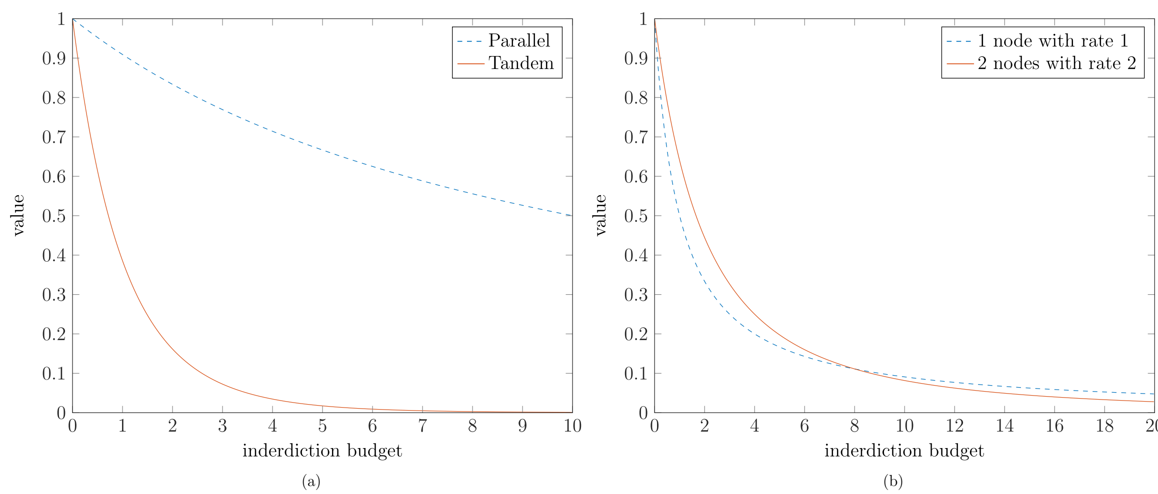


Fig. 4. Illustrative examples. (a) Compare parallel and tandem nodes and (b) Different network design.

Table 2
Strategies interdictor for the general network of Fig. 5.

v	$\Lambda^- = 0.5$ 0.8512	$\Lambda^- = 1$ 0.7321	$\Lambda^- = 5$ 0.2818	$\Lambda^- = 10$ 0.1162	$\Lambda^- = 50$ 0.0012
$\lambda^-_{i_1}$	–	–	0.158 (3.2%)	0.684 (6.8%)	2.175 (4.4%)
$\lambda^-_{i_2}$	0.070 (14.1%)	0.141 (14.1%)	0.497 (9.9%)	0.832 (8.3%)	2.778 (5.6%)
$\lambda^-_{i_3}$	0.098 (19.5%)	0.197 (19.7%)	1.047 (20.9%)	1.789 (17.9%)	4.361 (8.7%)
$\lambda^-_{i_4}$	0.098 (19.5%)	0.197 (19.7%)	0.779 (15.6%)	1.184 (11.8%)	2.839 (5.7%)
$\lambda^-_{i_5}$	0.070 (14.1%)	0.141 (14.1%)	0.722 (14.4%)	1.098 (11.0%)	2.734 (5.5%)
$\lambda^-_{i_6}$	0.070 (14.1%)	0.141 (14.1%)	0.734 (14.7%)	1.290 (12.9%)	3.110 (6.2%)
$\tilde{\Lambda}^-_{r_1}$	–	–	–	–	3.193 (6.4%)
$\tilde{\Lambda}^-_{r_2}$	–	–	–	0.110 (1.1%)	3.738 (7.5%)
$\tilde{\Lambda}^-_{r_3}$	–	–	0.295 (5.9%)	0.828 (8.3%)	6.238 (12.5%)
$\tilde{\Lambda}^-_{r_4}$	–	–	–	0.304 (3.0%)	4.769 (9.5%)
$\tilde{\Lambda}^-_{r_5}$	–	–	0.007 (0.1%)	0.400 (4.0%)	5.537 (11.1%)
$\tilde{\Lambda}^-_{r_6}$	0.094 (18.8%)	0.182 (18.2%)	0.761 (15.2%)	1.481 (14.8%)	8.539 (17.1%)

$\tilde{\Lambda}^-_{r_1}, \ldots, \tilde{\Lambda}^-_{r_6}$ are the rates for all tandem nodes of one route. The results are summarized in Table 2.

We would expect that the interdiction budget is evenly spread over the routes to make sure that the maximum completion probability is minimal. Table 2 shows the expected spread of interdiction budget over the routes. For example in the last case ($\Lambda^- = 50$), all routes get around 24% of the total budget. From Lemma 2, we expect that nodes in shorter routes (routes 3, 5 and 6) would have higher interdiction rates than nodes along longer routes. This can also be seen in Table 2. Table 2 also shows that if the interdiction budget Λ^- is low, most budget is assigned to the intersection nodes because multiple routes can be protected simultaneously from these nodes. However, if the total interdiction budget increases, more budget remains for the tandem nodes. Moreover, more budget is assigned to intersection nodes where more routes intersect, such as i_3, because more routes can be protected from the same point. Also, routes with a small number of intersection nodes, such as r_6, have more budget allocated on the tandem nodes to ensure that these routes are sufficiently protected. In this example, the total route budget is almost the same for each route. This doesn't have to be the case if the lengths of all routes are very different or the service rates are unequal.

4. Probabilistic routing of intruders

In Section 2, we described an interdiction game on a network with fixed routing of intruders. In that game, intruders select their route upon arrival at the network by choosing from a fixed set of

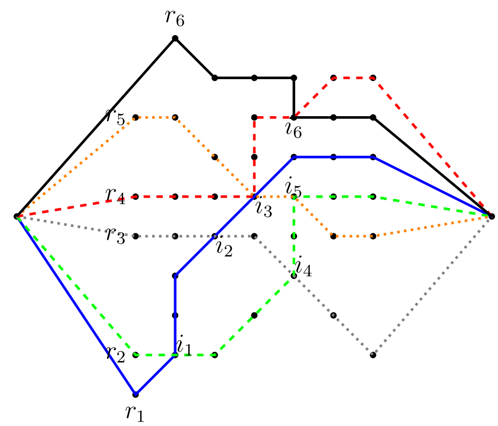


Fig. 5. Example of a general network.

routes. We can also model probabilistic routing of the intruders. In this case, intruders decide their next step at each node according to a certain probability. In this section, we describe the game with probabilistic routing of intruders and show that the results coincide with those for fixed routing of intruders.

4.1. Network with probabilistic routing of intruders

Consider a network, similar to the network of Section 2.1, but now with probabilistic routing of the intruders. Intruders arrive at the network according to a Poisson process with rate Λ and route through the network using a probability matrix $P=(p_{i,j})$, $i,j\in\{0,1,\dots,N,N+1\}$ where $p_{i,j}$ is the probability of routing to node j after service completion at node i. This probability $p_{i,j}$ is only allowed to be positive if there is a link between node i and node j in the queueing network; the set of all possible links is given by E. Intruders arrive at node i, $i\in S$, with probability $p_{0,i}$, so the arrival rate at node i is given by $\lambda_i=p_{0,i}\Lambda$. If $i\notin S$, $\lambda_i=0$. As $i\in T$ has just one outgoing arc (to $N+1$), the probability of leaving the network after service completion at node $i\in T$ is given by $p_{i,N+1}=1$. Note that a P matrix may introduce routes with an arbitrary number of cycles.

Let R be the (possibly infinite) set of all possible finite routes through the network, in which $r(k,s)$ is the sth node of route $k\in R$ and N_k is the length of route k (in which 0 and $N+1$ are not accounted for). We let $r(k,N_k+1)=N+1$. Then, given matrix P, the probability that route k is chosen by the intruder equals:

$$\mathbb{P}(\text{route } k \text{ is chosen}) = p_{0,r(k,1)}\prod_{s=1}^{N_k-1} p_{r(k,s),r(k,s+1)}. \tag{30}$$

The probability that intruders leave node i because they finished service is given by Eq. (1) and the probability that route k is actually completed without interdiction is given by Eq. (2).

4.2. Game description

Consider the interdiction game with the probabilistic routing of intruders. Instead of intruders selecting arrival rates λ_k for route k, intruders select a routing matrix P. Therefore, the action set of the intruders (Eq. (3)) is replaced by:

$$\bar{A}_{intruder} = \left\{P \middle| \sum_{j=1}^{N+1} p_{i,j}=1 \text{ for all } i=0,\dots,N;\ p_{i,j}\ge 0 \right.$$
$$\left. \text{if } (i,j)\in E,\ p_{i,j}=0 \text{ if } (i,j)\in\bar{E} \right\},$$

The interdictors action set remains the same as in the fixed routing scenario (Eq. (4)). The payoff function is replaced by the corresponding payoff function, which defines the arrival rate of intruders at node $N+1$:

$$\bar{v}(P,\lambda^-) = \sum_{k\in R}\lambda_{r(k,1)}\prod_{s=1}^{N_k} p_{r(k,s),r(k,s+1)}\frac{\mu_{r(k,s)}}{\mu_{r(k,s)}+\lambda^-_{r(k,s)}}, \tag{31}$$

with $\lambda_{r(k,1)}=p_{0,r(k,1)}\Lambda$.

4.3. Relation between optimal strategies

In Section 3, we described methods to find optimal strategies for the interdiction game on a network with fixed routing of intruders. In this section, we discuss the relationship between the optimal strategies for a network with probabilistic routing of intruders. We first show that for each network with probabilistic routing, there exists a network with fixed routing of intruders such that the average arrival rates are equal and vice versa.

Lemma 4. *Take λ^- fixed. For every network with probabilistic routing of intruders and given λ, there exists a network with fixed routing of intruders, such that the average arrival rate at each node is the same in both networks. Furthermore, for every network with fixed routing of intruders and given λ, there exists a network with probabilistic routing of intruders, such that the average arrival rate at each node is the same in both networks.*

The proof of Lemma 4 can be found in Appendix A.

We use Lemma 4 to prove that optimal strategies also exist in the case that intruders use probabilistic routing. Consider a network with N intermediate nodes, a source node and a sink node. Moreover, let F_{total} be the finite set of all possible fixed routes without cycles between the source node 0 and the sink node $N+1$. For that case, an optimal strategy for the intruders and interdictors can be calculated by the optimization model in Section 2.4. These strategies are given by λ^* and λ^{-*} and the optimal value is given by v. We show that the value of the game with probabilistic routing of intruders exists and is the same as the value of the game with fixed routing of intruders. Moreover, optimal strategies of the interdictor are the same for both games.

Theorem 5. *Consider the interdiction game on a queueing network with probabilistic routing of intruders. There exist optimal strategies P^* and λ^{-*} and the value of the game with probabilistic routing of intruders equals the value v of the game with fixed routing on F_{total}. Moreover, the strategy of the interdictor is also optimal for the game with fixed routing.*

Proof. Take an arbitrary routing matrix P that describes a strategy of intruders for a network with probabilistic routing of intruders. Suppose that the interdictor chooses the arrival rates according to the optimal strategy λ^{-*} of the game with fixed routing of intruders on F_{total}. By Lemma 4 and given λ^{-*}, we can construct a network with a set of fixed routes $\bar{F}$ and strategy $\bar{\lambda}$ such that the average arrival rate at each node is the same for the network with probabilistic routing and fixed routing of intruders. Because the payoff of both games (Eqs. (5) and (31)) is given by the arrival rate at the sink node, it follows that:

$$v(\bar{\lambda},\lambda^{-*}) = \bar{v}(P,\lambda^{-*}). \tag{32}$$

The set of fixed routes $\bar{F}$, derived from probabilistic routing may be infinite. This is due to the fact that probabilistic routing may induce cyclic paths. We show that for our model with fixed routing, cyclic routes can be eliminated. To this end, suppose that the intruder assigns a positive arrival rate to a cyclic route k: $\lambda_k>0$. By arbitrarily eliminating detours in the cyclic route, we obtain a non-cyclic route $\bar{k}$ such that $\mathbb{P}(\text{route } \bar{k} \text{ is completed}) \ge$

$\mathbb{P}$(route k is completed) (by Eq. (2)). Transferring the rate λ_k to $\lambda_{\bar{k}}$ results in an improved strategy for the intruder.

So, let $\bar{F}'$ be the set of routes derived from P, with all cyclic routes eliminated and let $\bar{\lambda}'$ be the corresponding improved strategy for the intruder, so:

$$v(\bar{\lambda}, \lambda^{-*}) \leq v(\bar{\lambda}', \lambda^{-*}). \tag{33}$$

Also, because λ^* is the optimal strategy of the intruder for the case that all possible fixed routes without cycles are allowed, it follows that

$$v(\bar{\lambda}', \lambda^{-*}) \leq v(\lambda^*, \lambda^{-*}) = v. \tag{34}$$

Combining Eqs. (32)–(34) yields:

$$\bar{v}(P, \lambda^{-*}) \leq v, \quad \text{for all } P. \tag{35}$$

We now complete the proof by showing that there exists a P^* such that $\bar{v}(P^*, \lambda^-) \geq v$, for all λ^-. Given optimal strategies λ^* and λ^{-*} from the game with fixed routing, a routing matrix P^* can be constructed according to Lemma 4. Because the average arrival rates are the same, the average arrival rates at the sink node are also equal and the values of the payoff functions of both the game with probabilistic routing and the game with fixed routing are equal. Therefore:

$$\bar{v}(P^*, \lambda^{-*}) = v(\lambda^*, \lambda^{-*}) = v. \tag{36}$$

Consider an arbitrary strategy λ^- for the interdictor. Using the same argument, we know that:

$$\bar{v}(P^*, \lambda^-) = v(\lambda^*, \lambda^-) \geq v, \tag{37}$$

as λ^* is optimal for the intruder.

Combining (Eq. 35) and (Eq. 37) proves that the value exists and is given by v. Moreover, the optimal strategy of the interdictor remains the same. □

Remark 3. For a network with probabilistic routing, the construction of the network with fixed routing as in Lemma 4 also ensures that the probability of following a specific route is the same for both networks. This means that for every network with probabilistic routing of intruders, there exists an equivalent network with fixed routing of intruders.

However, the reverse is not necessarily true. In fact, consider a network with fixed routing of intruders; it is not always possible to construct an equivalent network with probabilistic routing of intruders as the next example shows. Consider a network with two routes and one intersection node. If two routes intersect, there may exist more routes in the network with probabilistic routing than in the original network with fixed routing, due to combinations of the original routes.

Although it is not always possible to create a probabilistic network that is equivalent to the network with fixed routing, one can introduce multiple intruders to ensure that the probabilistic network contains the same routes.

The question now becomes: how many intruders types are necessary to construct a network with probabilistic routing which is equivalent to the network with fixed routing? Below we describe how to find an upper bound for the number of types we need.

Consider a network with fixed routing and K possible routes. To find an upper bound for the number of customer types, we construct a graph G, that has K nodes. All nodes correspond to a route, and two nodes are considered to be connected if the corresponding routes intersect in the network with fixed routing. An upper bound for the number of types equals the chromatic number $\chi(G)$. The nodes that do have the same type in the vertex coloring do not intersect, so they are allowed to have the same intruder type in the probabilistic network. This upper bound cannot be improved for the general case. However, fewer types will be enough in many specific situations, such as when routes only intersect at their last node.

Remark 4. Note that the network with probabilistic routing is known to have a product-form solution (Gelenbe (1991)). We do not need this to calculate the intruders' probability of completing a specific route because we can rely on the fact that intruders are only removable from the queue if they are in service and that interdictors arrive at the network according to an independent Poisson process.

5. Concluding remarks

In this paper, we have described an interdiction game on a network with intruders and interdictors. The interdiction game is played within a queueing network where intruders are the regular customers and the interdictors are the negative customers. For the case that the routing of the intruders is considered fixed, we designed a network game that has optimal pure strategies and we found analytical expressions for special cases, such as networks with only tandem or parallel queues. Also, for a network without intersecting nodes, we introduced an algorithm to find optimal strategies for the interdictors.

For general networks, we showed that the analytical results can be used to speed up finding optimal strategies, by dividing the network into a set of intersection nodes and separate tandem nodes. Moreover, if there is a subnetwork of the network, that only consists of parallel routes which do not intersect, then the optimal strategy of the interdictor is such that the completion probability is the same for each of these routes. Also, if the network contains a part that only consists of tandem nodes without intersections, the nodes with a lower service rate must be inspected more often.

In this paper, we also considered modeling the routing of intruders in a probabilistic manner. We showed that in this case, optimal strategies for interdictors and intruders also exist. Moreover, the value and optimal strategies of the interdictor of this game equal the value and optimal strategies of the interdictor of the corresponding game with fixed routing of intruders. So, the intruders cannot improve their strategy by deciding to use a probabilistic routing strategy.

There are several possible extension of our model. Instead of modeling interdictors that arrive at a specific node for inspection and then leave the network, it could be more realistic in some cases to model routing of interdictors. In this approach, interdictors not only inspect the nodes, but also route through the network. Another possible extension concerns each node as a single server queue with exponential service time. For further research, it would be interesting to study different types of queues, for example with multiple servers or a different service discipline.

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Appendix A. Proofs

Proof of Theorem 3. Rewrite the optimization problem for the original game as:

$$\begin{aligned} \min_{\lambda^-} \ & \Lambda \prod_{i=1}^N \frac{\mu_i}{\mu_i + \lambda_i^-} \\ \text{s.t.} \ & \sum_{i=1}^N \lambda_i^- - \Lambda^- = 0, \\ & -\lambda_i^- \quad \leq 0, \quad \text{for all } i = 1, \dots, N. \end{aligned}$$

The KKT-conditions can be used to prove optimality of a solution in a non-linear program (see Section 4.3 in Bazaraa, Sherali, & Shetty (1993)). In order to prove optimality, we show that the

KKT-conditions hold for the solution λ^{-*} obtained by Algorithm 1 . Thus, α and β must be found such that:

$$\frac{-1}{\mu_i+\lambda_i^{-*}}v(\lambda^{-*})+\alpha-\beta_i=0, \quad i=1,\dots,N, \tag{A.1}$$

$$\sum_{j=1}^{N}\lambda_j^{-*}-\Lambda^{-}=0, \tag{A.2}$$

$$\beta_i\lambda_i^{-*}=0, \quad i=1,\dots,N, \tag{A.3}$$

$$\beta_i\geq 0, \quad i=1,\dots,N. \tag{A.4}$$

Let $I'=\{i\in I|\lambda_i^{-*}>0\}$ and let $N'=|I'|$. The equality condition Eq. (A.2) holds by construction of the algorithm:

$$\sum_{i=1}^{N}\lambda_i^{-*}=\sum_{i\in I'}\left(\frac{\sum_{j\in I'}\mu_j+\Lambda^{-}}{N'}-\mu_i\right)$$
$$=N'\frac{\sum_{j\in I'}\mu_j+\Lambda^{-}}{N'}-\sum_{i\in I'}\mu_i=\Lambda^{-}$$

Moreover, from (A.3), we know that $\beta_i=0$, for all $i\in I'$, so (A.1) gives:

$$\frac{-1}{\mu_i+\lambda_i^{-*}}v(\lambda^{-*})+\alpha=0, \quad \text{for all } i\in I'.$$

Therefore, using Eq. (20):

$$\alpha=\frac{1}{\mu_i+\lambda_i^{-*}}v(\lambda^{-*})=\frac{N'}{\sum_{j\in I'}\mu_j+\Lambda^{-}}v(\lambda^{-*}).$$

If $\lambda_i^{-*}=0$ (for all $i\in I\backslash I'$), (A.1) gives:

$$\frac{-1}{\mu_i}v(\lambda^{-*})+\frac{N'}{\sum_{j\in I'}\mu_j+\Lambda}v(\lambda^{-*})-\beta_i=0,$$

$$\beta_i=\left(\frac{N'}{\sum_{j\in I'}\mu_j+\Lambda^{-}}-\frac{1}{\mu_i}\right)v(\lambda^{-*}).$$

By proving that $\beta_i\geq 0$ for all i, the proof that the KKT-conditions hold is completed

Note that the value of the function $v(\lambda^{-*})$ is positive by definition, Moreover, by construction of the algorithm, we know for any $i\in I\backslash I'$:

$$\frac{\sum_{j\in I'\cup\{i\}}\mu_j+\Lambda^{-}}{N'+1}\leq\mu_i,$$
$$\frac{\sum_{j\in I'}\mu_j+\Lambda^{-}}{N'+1}\leq\frac{N'}{N'+1}\mu_i,$$
$$\frac{N'}{\sum_{j\in I'}\mu_j+\Lambda^{-}}-\frac{1}{\mu_i}\geq 0.$$

Therefore, $\beta_i\geq 0$, for all $i\in I$ and the KKT-conditions hold for the solution found by Algorithm 1. Furthermore, because of the convexity of the value function and linearity of the constraints, the KKT-conditions are sufficient, which completes the proof. □

Proof of Theorem 4.. The value of route k is defined as the payoff of the game if the intruders choose to use only route k, so $\lambda_k=\Lambda$. In an optimal solution, the value for each route should be equal. If this is not the case, the strategy of the interdictor can be improved by shifting the arrival rate λ^{-*}, such that more interdiction budget is assigned to the route of minimal detection probability. To prove optimality of the algorithm, (1) the value over each route must be equal, (2) the algorithm must find a feasible solution, and (3) the arrival rates that are set to zero by the algorithm, must also be zero in the optimal solution.

The last condition is necessary because of the following. If for a certain node the interdictor's arrival rate is set to zero in the optimal solution, then the probability of service completion at this node equals one. In essence, this node has no impact on the total throughput of the intruders. Therefore, ignoring these nodes in the optimization model, which is done for the relaxation in the last step of the algorithm, gives the same solution. If solving this relaxation without these nodes also yields a feasible solution, the solution of the relaxation also is a solution for the original game.

The first condition holds because of construction of the algorithm: the optimal strategy is calculated under this assumption (by Eq. (25)). The second condition holds because the algorithm stops if all arrival rates are larger than or equal to zero. We will prove that the arrival rates that are set to zero by the algorithm, are also zero in the optimal solution. Let I be the set of all nodes and let I_k be the set of nodes that are in route k. Moreover, let $I'=\{i\in I|\lambda_i^{-}>0, \text{ by algorithm}\}$ and let $I^{OPT}=\{i\in I|\lambda_i^{-*}>0, \text{ in optimal solution}\}$. We must therefore prove that $I'=I^{OPT}$. Let v' be the value found by the algorithm, let v^{OPT} be the value of the original game and let v^t be the value of the relaxation calculated during iteration t in step 2 of the algorithm.

We first prove that $I'\supseteq I^{OPT}$. Take $i\notin I'$ such that i is removed during the first iteration and $\mu_i=\max_{j\in I_k}\mu_j$ for some k. Because v^1 is the value for the relaxation without any inequality constraints, we know that $v^{OPT}\geq v^1$. Let Λ_k^{-1} be the budget assigned to route k during the first iteration and let Λ_k^{-OPT} be the budget assigned to route k in the optimal solution. Consider two cases:

1. $\Lambda_k^{-1}\geq\Lambda_k^{-OPT}$:
   Thus, in the optimal solution, route k receives a smaller or equal amount of budget. The arrival rate λ_i^{-} is obtained from Eq. (28) and is increasing in Λ_k^{-}. Since it is optimal to use Algorithm 1 to assign the budget to the nodes of one tandem, the same formula must hold for λ_i^{-*} in the optimal solution. So if $\Lambda_k^{-1}\geq\Lambda_k^{-OPT}$, λ_i^{-*} would also be zero in the optimal solution and therefore, $i\notin I^{OPT}$.
2. $\Lambda_k^{-1}<\Lambda_k^{-OPT}$:
   Assume that $i\in I^{OPT}$, so $\lambda_i^{-*}>0$. Because of maximality of μ_i, all $j\in I_k$ are in I^{OPT}. Because for all $j\in I_k$, $\lambda_i^{-*}>0$ and the value of the game equals the value for each route (Lemma 1), v^{OPT} is given by Eq. (23) with $\Lambda_k^{-}=\Lambda_k^{-OPT}$ and $\lambda_k=\Lambda$. Moreover, during the first iteration, $I'=I$, so v^1 is given by Eq. (23) with $\Lambda_k^{-}=\Lambda_k^{-1}$. Eq. (23) is decreasing in Λ_k^{-} and therefore, $v^{OPT}<v^1$, but this contradicts with $v^{OPT}\geq v^1$. So, our assumption is not correct and $i\notin I^{OPT}$.

Therefore, if $i\notin I'$ such that i is removed during the first iteration, it follows that $i\notin I^{OPT}$.

Now assume that for all $i\notin I'$ that are removed until iteration $t-1$, $i\notin I^{OPT}$. Take $j\notin I'$ such that j is removed during iteration t ($t>1$). Let $\bar{I}=\{i\in I|i \text{ not removed before iteration } t\}$ and $\bar{I}^{OPT}=\{i\in\bar{I}|\lambda_i^{-*}>0, \text{ in optimal solution}\}$. By induction, we know for all $i\in I\setminus\bar{I}$ that $i\notin I^{OPT}$. So, running the algorithm with $\bar{I}$ gives the same solution as running the algorithm with I. By this logic, the above argument can be used to prove that if $i\notin I'$ such that i is removed during the t-th iteration and $\mu_i=\max_{j\in I_k}\mu_j$ for some k, then $i\notin I^{OPT}$. In general, if $i\notin I'$ then $i\notin I^{OPT}$ and $I'\supseteq I^{OPT}$.

We now prove $I'\subseteq I^{OPT}$ by contradiction. Assume that $I'\not\subseteq I^{OPT}$. Because we already proved $I'\supseteq I^{OPT}$, it follows that $I'\supset I^{OPT}$. The values v' and v^{OPT} are the solutions of the optimization problem (Lemma 1) under the additional constraints $\lambda_i^{-}=0$, for all $i\notin I'$ and $\lambda_i^{-}=0$, for all $i\notin I^{OPT}$ respectively. If $I'\supset I^{OPT}$, there exists at least one $i'\in I'$ such that $i'\notin I^{OPT}$. Therefore, $\lambda_{i'}^{-}>0$ for the first case ($\lambda_i^{-}>0$, for all $i\notin I'$), but $\lambda_{i'}^{-}=0$ for the second case ($\lambda_i^{-}=0$, for all $i\notin I^{OPT}$). This results in a worse solution for the second case and

$v' < v^{OPT}$. This also contradicts with optimality of v^{OPT}, thus the assumption was incorrect and $I' \subseteq I^{OPT}$.

Combining $I' \supseteq I^{OPT}$ and $I' \subseteq I^{OPT}$ gives $I' = I^{OPT}$, which completes the proof. □

Proof of Lemma 4. Consider a network with probabilistic routing of intruders with arrival rates of intruders λ_i and interdictors λ_i^-, and service rates μ_i for all $i \in I$. R is the set of all possible routes induced by P (R may be infinite). Now, we construct a network with fixed routing of intruders with arrival rate of intruders $\bar{\lambda}_k$ for a route k. Define for each route $k \in R$ the arrival rate of intruders following fixed routing by:

$$\bar{\lambda}_k = \lambda_{r(k,1)} p_{r(k,1),r(k,2)} p_{r(k,2),r(k,3)}, \ldots, p_{r(k,N_k),N+1}. \tag{A.5}$$

This is the arrival rate at the first node multiplied by the probability of following this route, given P. For the network with probabilistic routing, the mean arrival rate α_i at node i is given by the traffic equations:

$$\alpha_i = \lambda_i + \sum_j \alpha_j p_{j,i} \frac{\mu_j}{\lambda_j^- + \mu_j}. \tag{A.6}$$

For the network with fixed routing, the mean arrival rates are defined by:

$$\bar{\alpha}_i = \sum_{k \in R} \sum_{s=1}^{N_k} a_i(k,s),$$

where

$$a_i(k,s) = \begin{cases} \bar{\lambda}_k \prod_{t=1}^{s-1} \dfrac{\mu_{r(k,t)}}{\mu_{r(k,t)} + \lambda^-_{r(k,t)}}, & \text{if } r(k,s) = i, \\ 0, & \text{otherwise.} \end{cases}$$

Substituting the definitions (A.5) and (A.6), and rearranging terms yields:

$$\alpha_i = \lambda_i + \sum_j \lambda_j p_{j,i} \frac{\mu_j}{\lambda_j^- + \mu_j} + \sum_j \sum_h \lambda_h p_{h,j} p_{j,i} \frac{\mu_j}{\lambda_j^- + \mu_j} \frac{\mu_h}{\lambda_h^- + \mu_h} + \cdots.$$

The same expression can be found for $\bar{\alpha}_i$ by rewriting the traffic equations for the network with fixed routing and substituting the definition of $a_i(k, s)$. Thus, by construction, the average arrival rates at each node of the network with intruders following fixed routing equal the average arrival rates of the network with intruders following probabilistic routing.

Now consider a network with fixed routing of intruders with rates $\bar{\lambda}_k$ and λ^-. We can construct a network with probabilistic routing of intruders such that the average arrival rates are equal. For every route k, we have $r(k,1), r(k,2), \ldots, r(k,N_k)$ and arrival rate $\bar{\lambda}_k$ at node $r(k, 1)$. The probability $p_{i,j}$ that an intruder is going to node j after completing service in node i can be calculated by dividing the flow from i to j by the total flow out of i:

$$p_{i,j} = \frac{\sum_{k=1}^{K} \sum_{s=1}^{N_k-1} \bar{b}_{i,j}(k,s)}{\sum_{k=1}^{K} \sum_{s=1}^{N_k} \bar{a}_i(k,s)},$$

where:

$$\bar{a}_i(k,s) = \begin{cases} \bar{\lambda}_k, & \text{if } r(k,s) = i, \\ 0, & \text{otherwise,} \end{cases}$$

$$\bar{b}_{i,j}(k,s) = \begin{cases} \bar{\lambda}_k, & \text{if } r(k,s) = i \text{ and } r(k,s+1) = j, \\ 0, & \text{otherwise.} \end{cases}$$

Also, the arrival rates at node i for the network with probabilistic routing are given by:

$$\lambda_i = \sum_{k=1}^{K} \bar{a}_i(k,1).$$

Now, we can readily show that given λ^- the average arrival rates at each node of the network with fixed routing equal the average arrival rates of the network with probabilistic routing. □

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