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State synchronization of linear and nonlinear agents in time-varying networks

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SUMMARY

This paper studies state synchronization of homogeneous time-varying networks with diffusive full-state coupling or partial-state coupling. In the case of full-state coupling, linear agents as well as a class of nonlinear time-varying agents are considered. In the case of partial-state coupling, we only consider linear agents, but, in contrast with the literature, we do not require the agents in the network to be minimum phase or at most weakly unstable. In both cases, the network is time-varying in the sense that the network graph switches within an infinite set of graphs with arbitrarily small dwell time. A purely decentralized linear static protocol is designed for agents in the network with full-state coupling. For partial-state coupling, a linear dynamic protocol is designed for agents in the network while using additional communication among controller variables using the same network. In both cases, the design is based on a high-gain methodology. Copyright © 2017 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The problem of synchronization among agents in a multi-agent system has received substantial attention in recent years, because of its potential applications in cooperative control of autonomous vehicles, distributed sensor network, swarming and flocking, and others. The objective of synchronization is to secure an asymptotic agreement on a common state or output trajectory through decentralized control protocols (see [1–4] and references therein).

State synchronization inherently requires homogeneous networks (i.e., agents have identical dynamics). Therefore, in this paper, we focus on homogeneous networks and state synchronization. So far, most work has focused on state synchronization based on diffusive full-state coupling, where the agent dynamics progress from single-integrator and double-integrator dynamics (e.g., [5–9]) to more general dynamics (e.g., [10–13]). State synchronization based on diffusive partial-state coupling has also been considered (e.g., [13–19]).

The extension from fixed networks to time-varying networks is generally carried out in the framework of switching, using the concepts of dwell-time and average dwell-time. A critical assumption in most literature is that the network switches among a finite set of network graphs. For example, see [5], and [9] (full-state coupling) and [17, 18, 20, 21] (partial-state coupling). Also, the time-varying network can be piecewise constant and frequently connected (e.g., [12, 22]), uniformly connected (e.g., [13]), and uniformly connected on average (e.g., [19]) In the case of partial-state coupling,

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restriction is always imposed on the agent dynamics. That is, the poles of agent dynamics should be in the closed left-half complex plane [13, 19] or the zeros of agent dynamics should be in the closed left-half complex plane [20].

Some authors have also studied synchronization in networks with nonlinear agent dynamics (e.g., [23–28]). Explicit controller design for nonlinear networks has, to a large degree, centered on the relatively strict assumption of passivity. Passivity can in some cases be ensured by first applying local prefeedbacks to the system; however, this requires the system to be introspective. The author in [29] addresses the issue of state synchronization for homogeneous networks consisting of SISO, non-introspective agents. However, all the aforementioned nonlinear work consider networks with partial-state coupling and are not applicable to the case of full-state coupling.

1.1. Contribution of this paper

In this paper, we address several challenges in the problem of state synchronization based on diffusive full-state coupling or partial-state coupling:

- We study state synchronization for general agent dynamics with full-state coupling in time-varying networks. Most works in the literature address a finite set of network graphs, while the network in this paper can switch in an infinite set of network graphs that is defined based on some rough information of the graph. Moreover, the dwell-time can be arbitrarily small as long as it does not trigger chattering.
- We study state synchronization for a class of general nonlinear time-varying full-state coupled agents in time-varying networks. In that case, agents are not right-invertible, and previous results mentioned earlier are not applicable.
- We also study state synchronization for general non-minimum-phase partial-state coupled linear agents in time-varying networks. In other words, agents can have both poles and zeros in the open right-half complex plane.

In the case of full-state coupling, a purely decentralized controller, based on a high-gain methodology, is designed for each agent such that all agents achieve state synchronization under any time-varying network that belongs to the set of network graphs. In the case of partial-state coupling, a high-gain observer based controller is designed for each agent, where an additional standard communication channel is used for the exchange of controller states.

1.2. Notations and definitions

Given a matrix $A \in \mathbb{C}^{m\times n}$, A' denotes its conjugate transpose, $\|A\|$ is the induced 2-norm, and $\lambda_i(A)$ denotes its $i'th$ eigenvalue when $m = n$. A square matrix A is said to be Hurwitz stable if all its eigenvalues are in the open left-half complex plane. We denote by blkdiag$\{A_i\}$, a block-diagonal matrix with $A_1, \dots, A_N$ as the diagonal elements, and by col$\{x_i\}$, a column vector with $x_1, \dots, x_N$ stacked together, where the range of index i can be identified from the context. $A \otimes B$ depicts the Kronecker product between A and B. I_n denotes the n-dimensional identity matrix, and 0_n denotes $n \times n$ zero matrix; sometimes, we drop the subscript if the dimension is clear from the context.

A *weighted directed graph* $\mathcal{G}$ is defined by a triple $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, \dots, N\}$ is a node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the adjacency matrix with $a_{ij} > 0$ iff $(i, j) \in \mathcal{E}$. Each pair in $\mathcal{E}$ is called an *edge*. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1, \dots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, \dots, k-1$. A *directed tree* with *root* r is a subset of nodes of the graph $\mathcal{G}$ such that a path exists between r and every other node in this subset. A *directed spanning tree* is a directed tree containing all the nodes of the graph. For a weighted graph $\mathcal{G}$, a matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^{N} a_{ik}, & i = j, \\ -a_{ij}, & i \neq j, \end{cases}$$

is called the *Laplacian matrix* associated with the graph $\mathcal{G}$. In the case where $\mathcal{G}$ has non-negative weights, L has all its eigenvalues in the closed right-half plane and at least one eigenvalue at zero associated with right eigenvector **1**.

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Definition 1
Let $\mathcal{L}_N \subset \mathbb{R}^{N\times N}$ be the family of all possible Laplacian matrices associated with a graph with N agents. We denote by $\mathcal{G}_L$ the graph associated with a Laplacian matrix $L \in \mathcal{L}_N$. Then, a time-varying graph $\mathcal{G}_t$ with N agents is defined by

$$\mathcal{G}_t(t) = \mathcal{G}_{\sigma(t)},$$

where $\sigma : \mathbb{R} \to \mathcal{L}_N$ is a piecewise constant, right-continuous function with minimal dwell-time τ [30], that is, $\sigma(t)$ remains fixed for $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}$ and switches at $t = t_k$, $k = 1, 2, \ldots$ where $t_{k+1} - t_k \geqslant \tau$ for $k = 0, 1, \ldots$. For ease of presentation, we assume $t_0 = 0$.

2. TIME-VARYING NETWORK COMMUNICATION

In this paper, we consider time-varying networks composed of N identical agents of the form

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i + f(x_i, t), \\ y_i &= Cx_i, \end{aligned} \qquad (i = 1, \ldots, N) \tag{1}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, and $y_i \in \mathbb{R}^p$ are the state, input, and output of agent i, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ are constant matrices. The system is either completely linear or A, B, and f are in the so-called strict feedback form (details will be discussed later). The agents have no access to their own states. The only information is from the network, that is, a linear combination of its own output relative to that of other neighboring agents. In particular, each agent $i \in \{1, \ldots, N\}$ has access to the quantity,

$$\zeta_i(t) = \sum_{j=1}^{N} a_{ij}(t)(y_i(t) - y_j(t)), \tag{2}$$

where $a_{ij}(t) \geqslant 0$ and $a_{ii}(t) = 0$, are piecewise constant and right-continuous functions of time t, indicating time-varying communication among agents. This time-varying communication topology of the network can be described by a weighted, time-varying graph $\mathcal{G}_t$ with nodes corresponding to the agents in the network and the weight of edges at time t given by the coefficient $a_{ij}(t)$. Specifically, $a_{ij}(t)$ indicates that at time t there is an edge with weight a_{ij} in the graph from agent j to agent i. The Laplacian matrix associated with $\mathcal{G}_t$ is defined as $L_t = [\ell_{ij}(t)]$. In terms of the coefficients of L_t, ζ_i can be rewritten as

$$\zeta_i(t) = \sum_{j=1}^{N} \ell_{ij}(t) y_j(t). \tag{3}$$

We refer to this network as *partial-state coupling*. The following assumption on the network graph is needed.

Assumption 1
At each time t, the graph $\mathcal{G}_t(t)$ describing the communication topology of the network contains a directed spanning tree.

Based on the aforementioned assumption, it then follows from [9, Lemma 3.3] that the Laplacian matrix L_t at time t has a simple eigenvalue at the origin, with the corresponding right eigenvector $\mathbf{1}$ and all the other eigenvalues are in the open right-half complex plane. Let $\lambda_{t,1}, \ldots, \lambda_{t,N}$ denote the eigenvalues of L_t such that $\lambda_{t,1} = 0$ and $\mathrm{Re}(\lambda_{t,i}) > 0$, $i = 2, \ldots, N$.

Next, we will define a set of time-varying graphs based on some rough information of the graph. Before doing so, we first define a set of fixed graphs, based on which the set of time-varying graphs is defined.

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Definition 2
For any given real numbers $\beta, \gamma > 0$ and a positive integer N, the set $\mathbb{G}^N_{\beta,\gamma}$ is an infinite set of weighted, directed graphs composed of N nodes satisfying the following properties:

- The eigenvalues of the corresponding Laplacian matrix L, denoted by $\lambda_1, \ldots, \lambda_N$, satisfy $\lambda_1 = 0$, $\mathrm{Re}(\lambda_i) > \beta$ for $i = 2, \ldots, N$.
- We have $\|L\| < \gamma$.

Definition 3
For any given real numbers $\beta, \gamma, \tau > 0$ and a positive integer N, the set $\mathbb{G}^{\tau,N}_{\beta,\gamma}$ is the set of all time-varying graphs $\mathcal{G}_t$ for which

$$\mathcal{G}_t(t) = \mathcal{G}_{\sigma(t)} \in \mathbb{G}^N_{\beta,\gamma}$$

for all $t \in \mathbb{R}$, where $\sigma : \mathbb{R} \to \mathcal{L}_N$ is a piecewise constant, right-continuous function with minimal dwell-time τ.

Remark 1
Note that the minimal dwell-time is required to avoid chattering problems. However, it can be arbitrarily small.

Remark 2
If we have a finite set of network graphs each of which contains a directed spanning tree, then there always exists a set of the form $\mathbb{G}^N_{\beta,\gamma}$ for suitable $\gamma, \beta > 0$, and N containing these graphs.

The agents in the network achieve state synchronization if

$$\lim_{t\to\infty} (x_i(t) - x_j(t)) = 0, \tag{4}$$

for all $i, j \in \{1, \ldots, N\}$.

Note that if C has full column rank, without loss of generality $C = I$, and the quantity ζ_i then becomes

$$\zeta_i(t) = \sum_{j=1}^{N} a_{ij}(t)(x_i(t) - x_j(t)) = \sum_{j=1}^{N} \ell_{ij}(t) x_j(t), \tag{5}$$

which means agents have access to the relative state of their neighboring agents in the network. This kind of network is called *full-state coupling*.

3. LINEAR AGENTS WITH FULL-STATE COUPLING

In this section, we consider linear agents with full-state coupling. The agent dynamics are in the form of

$$\dot{x}_i = A x_i + B u_i, \quad (i = 1, \ldots, N). \tag{6}$$

Without loss of generality, we assume that (A, B) is stabilizable, and the matrix B is full column rank.

We then formulate the state synchronization problem for time-varying networks with full-coupled linear agents as follows.

Problem 1
Consider a multi-agent system described by (6) and (5). For any real numbers $\gamma, \beta, \tau > 0$, and a positive integer N that defines a set of time-varying network graphs $\mathbb{G}^{\tau,N}_{\beta,\gamma}$, the *state synchronization* problem with a set of time-varying network graphs $\mathbb{G}^{\tau,N}_{\beta,\gamma}$ is to find, if possible, a linear static protocol of the form,

$$u_i = F\zeta_i, \tag{7}$$

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such that, for any time-varying graph $\mathcal{G}_t \in \mathbb{G}^{\tau,N}_{\beta,\gamma}$ and for all initial conditions of agents, state synchronization among agents is achieved.

3.1. Protocol design

In the case of full-state coupling, the matrix C is an identity matrix that implies agents (A, B, C) are non-right-invertible. Thus, previous results for partial-state coupling cannot be used in this case because they require the system to be right-invertible.

According to [31, Theorem 1], there exists non-singular transformation matrix $T_x \in \mathbb{R}^{n\times n}$ and $T_u \in \mathbb{R}^{m\times m}$ such that $(\hat{A}, \hat{B}) = (T_x^{-1} A T_x, T_x^{-1} B T_u)$. Moreover, $\hat{A}$ and $\hat{B}$ have the following special structure:

$$\hat{A} = \begin{pmatrix} A_{1d} & & \\ & \ddots & \\ & & A_{md} \end{pmatrix} + \begin{pmatrix} B_{1d} & & \\ & \ddots & \\ & & B_{md} \end{pmatrix} E,$$
$$\hat{B} = \begin{pmatrix} B_{1d} & & \\ & \ddots & \\ & & B_{md} \end{pmatrix},$$

where

$$A_{jd} = \begin{pmatrix} 0 & I_{\rho_j-1} \\ 0 & 0 \end{pmatrix}, \quad B_{jd} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad E = \begin{pmatrix} E_{11} & \dots & E_{1m} \\ \vdots & \ddots & \vdots \\ E_{m1} & \dots & E_{mm} \end{pmatrix}.$$

Note that $\sum_{j=1}^m \rho_j = n$. We define

$$A_d = \mathrm{diag}(A_{1d}, \dots, A_{md}) \quad \text{and} \quad B_d = \hat{B} = \mathrm{diag}(B_{1d}, \dots, B_{md}).$$

Define $\hat{F} = \mathrm{diag}(\hat{F}_1, \dots, \hat{F}_m)$ and choose $\hat{F}_j = -B'_{jd} P_j$ where $P_j > 0$ is the unique solution of the following algebraic Riccati equation:

$$A'_{jd} P_j + P_j A_{jd} - 2\beta P_j B_{jd} B'_{jd} P_j + I_{\rho_j} = 0. \tag{8}$$

It is well known that $A_d + \lambda B_d \hat{F}$ is Hurwitz stable for each non-zero eigenvalue λ of the Laplacian matrix L in light of $\mathrm{Re}(\lambda_i) > \beta$ for all $i = 2, \dots, N$.

Let

$$D_\varepsilon = \begin{pmatrix} \varepsilon^{-\rho_1} & & \\ & \ddots & \\ & & \varepsilon^{-\rho_m} \end{pmatrix}, \quad S_\varepsilon = \begin{pmatrix} S_{1\varepsilon} & & \\ & \ddots & \\ & & S_{m\varepsilon} \end{pmatrix}$$

with $S_{j\varepsilon} = \mathrm{diag}(1, \dots, \varepsilon^{\rho_j - 1})$. Choosing $F = T_u D_\varepsilon \hat{F} S_\varepsilon T_x^{-1}$, the static protocol (7) can be designed as

$$u_i = T_u D_\varepsilon \hat{F} S_\varepsilon T_x^{-1} \zeta_i. \tag{9}$$

We state the main result in this section as follows.

Theorem 1
Consider a multi-agent system described by (6) and (5) with (A, B) stabilizable. Let any real numbers $\gamma, \beta, \tau > 0$ and a positive integer N be given, and hence a set of time-varying network graphs $\mathbb{G}^{\tau,N}_{\beta,\gamma}$ be defined.

In that case, the state synchronization problem stated in Problem 1 is solvable. In particular, the protocol (9) with ε sufficiently small achieves state synchronization for any time-varying graph $\mathcal{G}_t \in \mathbb{G}^{\tau,N}_{\beta,\gamma}$.

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Before we prove the aforementioned theorem, we need a preliminary lemma.

Lemma 1
The matrix

$$\tilde{A}_t = (I_{N-1} \otimes A_d) + (U_t \otimes B_d \hat{F})$$

is asymptotically stable for any upper diagonal matrix $U_t \in \mathbb{R}^{(N-1)\times(N-1)}$ with $\|U_t\| < \tilde{\gamma}$ whose eigenvalues satisfy $\mathrm{Re}(\lambda_i) > \beta$ for all $i = 1, \ldots, N-1$. Moreover, there exists $\tilde{P} > 0$ and a small enough $\mu > 0$ such that

$$\tilde{A}_t' \tilde{P} + \tilde{P} \tilde{A}_t \leqslant -\mu \tilde{P} - I \tag{10}$$

is satisfied for all possible upper diagonal matrices U_t.

Proof
If we define

$$\tilde{A}_{t,i} = A_d + \lambda_i B_d \hat{F}$$

and

$$\tilde{B} = B_d \hat{F},$$

then,

$$\tilde{A}_t = \begin{pmatrix} \tilde{A}_{t,1} & \mu_{1,2}\tilde{B} & \cdots & \mu_{1,N-1}\tilde{B} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mu_{N-2,N-1}\tilde{B} \\ 0 & \cdots & 0 & \tilde{A}_{t,N-1} \end{pmatrix},$$

where λ_i is eigenvalues of U_t and $\mu_{i,j} = [U_t]_{ij}$ for $j > i$ is the bounded upper diagonal elements of U. Define

$$P = \mathrm{diag}(P_1, \ldots, P_m)$$

with $P_1, \ldots, P_m$ given in (8). Then, we have

$$\tilde{A}_{t,i}' P + P \tilde{A}_{t,i} \leqslant -I.$$

Define

$$\bar{P}_m = \begin{pmatrix} \alpha^{i_1} P & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha^{i_m} P \end{pmatrix}, \qquad \bar{A}_{t,m} = \begin{pmatrix} \tilde{A}_{t,1} & \mu_{1,2}\tilde{B} & \cdots & \mu_{1,m}\tilde{B} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mu_{m-1,m}\tilde{B} \\ 0 & \cdots & 0 & \tilde{A}_{t,m} \end{pmatrix},$$

for $m = 1, \ldots, N-1$. We will next use a recursive argument. For $m = 1$ and $\alpha^{i_1} = N$, we have

$$\bar{A}_{t,1}' \bar{P}_1 + \bar{P}_1 \bar{A}_{t,1} = \alpha^{i_1} (\tilde{A}_{t,1}' P + P \tilde{A}_{t,1}) < -NI.$$

Assume that for some $m = j$, we have for some $\alpha^{i_1}, \ldots \alpha^{i_j}$ that

$$A_{11} = \bar{A}_{t,j}' \bar{P}_j + \bar{P}_j \bar{A}_{t,j} < -(N-j+1)I \tag{11}$$

We will show that for $m = j+1$ there exists $\alpha^{i_{j+1}}$ such that

$$\bar{A}_{t,j+1}' \bar{P}_{j+1} + \bar{P}_{j+1} \bar{A}_{t,j+1} < -(N-j)I \tag{12}$$

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Note that

$$\bar{A}'_{t,j+1}\bar{P}_{j+1} + \bar{P}_{j+1}\bar{A}_{t,j+1} = \begin{pmatrix} A_{11} & A_{12} \\ A'_{12} & -\alpha^{i_{j+1}} I \end{pmatrix}$$

where A_{11} is defined by (11) while A_{12} is given by

$$A_{12} = \begin{pmatrix} \alpha^{i_1}\mu_{1,N-j}\tilde{B}P \\ \vdots \\ \alpha^{i_j}\mu_{j,N-j}\tilde{B}P \end{pmatrix}$$

Note that the coefficients $\mu_{1,N-j}$ are unknown but bounded because the norm of U is bounded and hence there exists M such that $\|A_{12}\| < M$. Via Schur complement, it is easy to verify that for given bound M there exists $\alpha^{i_{j+1}}$ sufficiently large such that the matrix

$$\begin{pmatrix} A_{11} & A_{12} \\ A'_{12} & -\alpha^{i_j+1} I \end{pmatrix} < -(N-j)I.$$

for all matrices A_{11} and A_{12} such that $A_{11} < -(N-j+1)I$ and $\|A_{12}\| < M$. This guarantees that (12) is satisfied.

Using a recursive argument, we find that there exist $\alpha^{i_1}, \dots, \alpha^{i_{N-1}}$ such that

$$\tilde{A}'_t\bar{P}_{N-1} + \bar{P}_{N-1}\tilde{A}_t \leqslant -2I.$$

because $\tilde{A}_t = \bar{A}_{t,N-1}$. This obviously implies that for μ small enough we have (10) for $\tilde{P} = \bar{P}_{N-1}$. □

Proof of Theorem 1

The closed-loop system of the static protocol (7) and the agent (6) is written as

$$\dot{x}_i = Ax_i + BF\zeta_i.$$

Define $\bar{x}_i = x_N - x_i$ and $\bar{\ell}_{ij}(t) = \ell_{ij}(t) - \ell_{Nj}(t)$ for $i \in \{1, \dots, N-1\}$. Then, we obtain

$$\dot{\bar{x}}_i = A\bar{x}_i + BF\sum_{j=1}^{N-1}\bar{\ell}_{ij}(t)\bar{x}_j \tag{13}$$

for $i = 1, \dots, N-1$. We define $\bar{L}_t \in \mathbb{R}^{(N-1)\times(N-1)}$ such that $[\bar{L}_t]_{ij} = \bar{\ell}_{ij}$. In that case, the eigenvalues of $\bar{L}_t$ are equal to the nonzero eigenvalues of L_t and $\|L_t\| < \gamma$ implies $\|\bar{L}_t\| \leqslant \bar{\gamma} = \sqrt{N}\gamma$.

Next, we will prove that the system (13) is asymptotically stable for any time-varying graph $\mathcal{G}_t \in \mathbb{G}^{\tau,N}_{\beta,\gamma}$, which immediately implies that $\lim_{t\to\infty}(x_i(t) - x_N(t)) = 0$ for $i = 1, \dots, N-1$ under any time-varying graph $\mathcal{G}_t \in \mathbb{G}^{\tau,N}_{\beta,\gamma}$. Let

$$\bar{x} = \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_{N-1} \end{pmatrix}.$$

The error dynamics of the whole network can be written as

$$\dot{\bar{x}} = (I_{N-1} \otimes A)\bar{x} + (\bar{L}_t \otimes BF)\bar{x}. \tag{14}$$

Define $\bar{Q}_t^{-1}\bar{L}_t\bar{Q}_t = \bar{U}_t$, where $\bar{U}_t$ is the Schur form of $\bar{L}_t$. In particular, the diagonal elements of $\bar{U}_t$ are the non-zero eigenvalues of the Laplacian matrix $\bar{L}_t$ at time t while $\|\bar{U}_t\| \leqslant \bar{\gamma}$. Moreover, $\bar{Q}_t$ is unitary. Let $\tilde{x}(t) = (\bar{Q}_t^{-1} \otimes I)\bar{x}(t)$. Then,

$$\dot{\tilde{x}} = (I_{N-1} \otimes A)\tilde{x} + (\bar{U}_t \otimes BF)\tilde{x}. \tag{15}$$

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Now, let $\xi = (I_{N-1} \otimes T_x^{-1})\tilde{x}$. Then,

$$\dot{\xi} = (I_{N-1} \otimes \hat{A})\xi + (\bar{U}_t \otimes T_x^{-1} BFT_x)\xi. \tag{16}$$

Because $B = T_x \hat{B} T_u^{-1}$ and $F = T_u D_\varepsilon \hat{F} S_\varepsilon T_x^{-1}$, we have $T_x^{-1} BFT_x = \hat{B} D_\varepsilon \hat{F} S_\varepsilon$. Moreover, $\hat{A} = A_d + B_d E$ and $B_d = \hat{B}$, and therefore, the dynamics (16) can be rewritten as

$$\dot{\xi} = (I_{N-1} \otimes (A_d + B_d E))\xi + (\bar{U}_t \otimes B_d D_\varepsilon \hat{F} S_\varepsilon)\xi. \tag{17}$$

Let

$$G_\varepsilon = \begin{pmatrix} \varepsilon^{\rho_m-\rho_1} I_{\rho_1} & & & \\ & \varepsilon^{\rho_m-\rho_2} I_{\rho_2} & & \\ & & \ddots & \\ & & & I_{\rho_m} \end{pmatrix}$$

and define $v = (I_{N-1} \otimes S_\varepsilon G_\varepsilon)\xi$. Because $S_{j\varepsilon} A_{jd} S_{j\varepsilon}^{-1} = \varepsilon^{-1} A_{jd}$ and $S_{j\varepsilon} B_{jd} \varepsilon^{\rho_m-\rho_j} = \varepsilon^{\rho_m-1} B_{jd}$, the dynamics of v can be written as

$$\varepsilon\dot{v} = \Big((I_{N-1} \otimes A_d) + (\bar{U}_t \otimes B_d \hat{F})\Big) v + (I_{N-1} \otimes B_d E D_\varepsilon^{-1} S_\varepsilon^{-1})v \tag{18}$$

Note that v experiences discontinuous jumps when the network graph switches. Denote $W_\varepsilon = I_{N-1} \otimes B_d E D_\varepsilon^{-1} S_\varepsilon^{-1}$, which is $O(\varepsilon)$, and let $\tilde{A}_t = (I_{N-1} \otimes A_d) + (\bar{U}_t \otimes B_d \hat{F})$. Then we obtain

$$\varepsilon\dot{v} = \tilde{A}_t v + W_\varepsilon v. \tag{19}$$

Define a Lyapunov function $V = \varepsilon v' \tilde{P} v$. It is easy to find that V also has discontinuous jumps when the network graph changes. The derivative of V is bounded by

$$\begin{aligned}\dot{V} &\leqslant -\mu\varepsilon^{-1} V - \|v\|^2 + 2\mathrm{Re}(v' \tilde{P} W_\varepsilon v) \\ &\leqslant -\mu\varepsilon^{-1} V - \|v\|^2 + \varepsilon r \|v\|^2 \\ &\leqslant -\mu\varepsilon^{-1} V,\end{aligned}$$

for a small enough ε. In the aforementioned second inequality, choose $\varepsilon r \geqslant 2\|\tilde{P} W_\varepsilon\|$.

By integration on both sides, we have

$$V(t_k^-) \leqslant e^{-\mu\varepsilon^{-1}(t_k - t_{k-1})} V(t_{k-1}^+).$$

There is a potential jump in V at time t_{k-1}. However, we have $V(t_{k-1}^+) \leqslant mV(t_{k-1}^-)$, where

$$m = \frac{\lambda_{\max}(\tilde{P})}{\lambda_{\min}(\tilde{P})}.$$

Using the fact that $t_k - t_{k-1} > \tau$, there exists small enough ε such that

$$V(t_k^-) \leqslant e^{-\mu(t_k - t_{k-1})} V(t_{k-1}^-).$$

Combining these time-intervals, we obtain

$$V(t_k^-) \leqslant e^{-\mu t_k} V(0).$$

Assuming $t_{k+1} > t > t_k$, we have

$$V(t) \leqslant m e^{-\mu t} V(0).$$

Hence, $\lim_{t\to\infty} v(t) = 0$ under a time-varying network graph. Because

$$\bar{x}(t) = (\bar{Q}_t \otimes T_x G_\varepsilon^{-1} S_\varepsilon^{-1}) v(t),$$

where $\bar{Q}_t$ is unitary, we obtain that

$$\lim_{t\to\infty} \bar{x}(t) = 0,$$

which proves the result. □

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4. NON-LINEAR TIME-VARYING AGENTS WITH FULL-STATE COUPLING

In this section, we consider nonlinear time-varying agents with full-state coupling. Specifically, we consider agents that can be represented in the canonical form,

$$\dot{x}_i = A_d x_i + \phi(t, x_i) + B_d(u_i + E x_i), \quad (i = 1, \dots, N) \tag{20}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, are states and inputs of agent i. Let the relative degree of the aforementioned agent system (20) be ρ. $A_d \in \mathbb{R}^{\rho m \times \rho m}$ and $B_d \in \mathbb{R}^{\rho m \times m}$ have the following special form:

$$A_d = \begin{pmatrix} 0 & I_m & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_m \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \quad B_d = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_m \end{pmatrix}.$$

We can partition x_i and ϕ as

$$x_i = \begin{pmatrix} x_{i1} \\ \vdots \\ x_{i\rho} \end{pmatrix}, \quad x_{ij} = \begin{pmatrix} x_{ij1} \\ \vdots \\ x_{ijm} \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_\rho \end{pmatrix}, \quad \phi_j = \begin{pmatrix} \phi_{j1} \\ \vdots \\ \phi_{jm} \end{pmatrix}.$$

Then, we assume that the time-varying nonlinearity $\phi(t, x_i)$ satisfies the following assumption.

Assumption 2
Assume that $\phi(t, x_i)$ is continuously differentiable and globally Lipschitz continuous with respect to x_i uniformly in t, and piecewise continuous with respect to t. Moreover, the nonlinearity has the lower-triangular structure

$$\frac{\partial \phi_j(t, x_i)}{\partial x_{ik}} = 0, \quad \forall k > j. \tag{21}$$

Remark 3
This lower-triangular structure for the nonlinearity is the well known strict feedback form, which was widely studied by nonlinear researchers in the early 1990s.

Next, we formulate the state synchronization problem for state-coupled nonlinear time-varying agents:

Problem 2
Consider a multi-agent system described by (5) and (20). For any real numbers $\gamma, \beta, \tau > 0$ and a positive integer N that defines a set of time-varying network graphs $\mathbb{G}_{\beta,\gamma}^{\tau,N}$, the *state synchronization* problem with a set of time-varying network graphs $\mathbb{G}_{\beta,\gamma}^{\tau,N}$ is to find, if possible, a linear static protocol of the form (7) such that, for any time-varying graph $\mathcal{G}_t \in \mathbb{G}_{\beta,\gamma}^{\tau,N}$ and for all the initial conditions of agents, the state synchronization among agents can be achieved.

4.1. *Protocol design*

Let $\varepsilon \in (0, 1]$ be a high-gain parameter, and $S_\varepsilon = \operatorname{diag}(I_m, \dots, \varepsilon^{\rho-1} I_m)$ be the high-gain scaling matrix. Construct the static protocol (7) as

$$u_i = \varepsilon^{-\rho} F S_\varepsilon \zeta_i, \tag{22}$$

where $F = -B_d' P$, and $P = P' > 0$ is the unique solution of the algebraic Riccati equation,

$$P A_d + A_d' P - 2\beta P B_d B_d' P + P + I = 0, \tag{23}$$

where β is the lower bound on the real parts of non-zero eigenvalues of L_t for all time t.

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Theorem 2
Consider a multi-agent system described by (5) and (20) with full-state coupling. Let any real number $\gamma, \beta, \tau > 0$ and a positive integer N be given, and hence, a set of time-varying network graphs $\mathbb{G}_{\beta,\gamma}^{\tau,N}$ be defined.

Under Assumptions 1 and 2, the state synchronization problem is solvable. In particular, there exists an $\varepsilon^* \in (0, 1]$ such that for all $\varepsilon \in (0, \varepsilon^*]$, controller (22) solves the state synchronization problem for any time-varying graph $\mathcal{G}_t \in \mathbb{G}_{\beta,\gamma}^{\tau,N}$.

Proof
For each $i \in \{1, \dots, N\}$, let $\bar{x}_i = x_N - x_i$. The state synchronization is achieved if $\bar{x}_i \to 0$ for all $i \in \{1, \dots, N-1\}$.

By Taylor's theorem, we can write $\phi(t, x_N) - \phi(t, x_i) = \Phi_i(t)\bar{x}_i$, where

$$\Phi_i(t) = \int_0^1 \frac{\partial \phi}{\partial x_i}(t, x_i + p\bar{x}_i)\mathrm{d}p.$$

Because of the Lipschitz property of the nonlinearity, $\Phi_i(t)$ is uniformly bounded, and the lower-triangular structure of the nonlinearity implies that $\Phi_i(t)$ is lower triangular. Now, the dynamics of $\bar{x}_i$ can be written as

$$\dot{\bar{x}}_i = A_d\bar{x}_i + B_d\varepsilon^{-\rho}FS_\varepsilon \sum_{j=1}^{N-1} \bar{\ell}_{ij}(t)\bar{x}_j + B_dE\bar{x}_i + \Phi_i(t)\bar{x}_i,$$

where the equality holds in light of

$$\sum_{j=1}^{N}(\ell_{Nj}(t) - \ell_{ij}(t))x_j = \sum_{j=1}^{N-1} \bar{\ell}_{ij}(t)\bar{x}_j$$

with $\bar{\ell}_{ij}(t) = \ell_{ij}(t) - \ell_{Nj}(t)$. Defining $\xi_i = S_\varepsilon\bar{x}_i$, we have

$$\varepsilon\dot{\xi}_i = A_d\xi_i + B_dF\sum_{j=1}^{N-1} \bar{\ell}_{ij}(t)\xi_j + W_{i\varepsilon}\xi_i,$$

where $W_{i\varepsilon} = \varepsilon^\rho B_dES_\varepsilon^{-1} + \varepsilon S_\varepsilon\Phi_i(t)S_\varepsilon^{-1}$. The first term of $W_{i\varepsilon}$ is obviously $O(\varepsilon)$. The second term is $O(\varepsilon)$ because $\Phi_i(t)$ is lower triangular.

Let $\xi = [\xi_1; \dots; \xi_{N-1}]$ and $\bar{L}_t = [\bar{\ell}_{ij}(t)]$. We obtain

$$\varepsilon\dot{\xi} = ((I_{N-1} \otimes A_d) + (\bar{L}_t \otimes B_dF))\xi + W_\varepsilon\xi,$$

where $W_\varepsilon = \mathrm{diag}(W_{1\varepsilon}, \dots, W_{(N-1)\varepsilon})$. Define Q_t such that $Q_t^{-1}\bar{L}_tQ_t = U_t$, where U_t is the Schur form of $\bar{L}_t$ and Q_t is unitary. Let $\nu = (Q_t \otimes I_{m\rho})\xi$. Then,

$$\varepsilon\dot{\nu} = \hat{A}_t\nu + \hat{W}_{t,\varepsilon}\nu, \tag{24}$$

where $\hat{A}_t = (I_{N-1} \otimes A_d) + (U_t \otimes B_dF)$ and $\hat{W}_{t,\varepsilon} = (Q_t \otimes I_{m\rho})W_\varepsilon(Q_t^{-1} \otimes I_{m\rho})$. We see that ν has discontinuous jumps when the network graph switches. In a similar way, we will first demonstrate that the dynamics (24) is asymptotically stable for a fixed network graph. We also neglect the subscript t for the analysis of the fixed network graph.

Similar to Lemma 1, we can show that

$$\hat{P} = \mathrm{diag}(\alpha^{i_1}P, \alpha^{i_2}P, \dots, \alpha^{i_{N-1}}P)$$

will satisfy

$$\hat{P}\hat{A}_t + \hat{A}_t'\hat{P} \leqslant -\mu\hat{P} - I$$

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for α sufficiently large where $i_1 > i_2 > \ldots > i_{N-1}$. Consider the Lyapunov function $V = \varepsilon \nu' \hat{P} \nu$, for which we have

$$\begin{aligned}\dot{V} &= -\mu\varepsilon^{-1}V - \|\nu\|^2 + 2\mathrm{Re}(\nu'\hat{W}_{t,\varepsilon}\hat{P}\nu)\\ &\leqslant -\mu\varepsilon^{-1}V - \|\nu\|^2 + \varepsilon r_1\|\nu\|^2\\ &\leqslant -\mu\varepsilon^{-1}V,\end{aligned}$$

for a small enough ε, where $\varepsilon r_1 \geqslant 2\|\hat{W}_{t,\varepsilon}\hat{P}\|$.

Following the steps in the proof of Theorem 1, for a small enough ε, we can achieve that $\bar{x}_i \to 0$ for all $i \in \{1, \ldots, N-1\}$ under time-varying graphs. □

4.2. Transforming nonlinear time-varying systems to the canonical form

It is obvious that not any arbitrary nonlinear time-varying multivariable systems can be transformed to the canonical form (20). We will discuss next the transformation of general nonlinear time-varying systems to the canonical form (20).

Consider a general nonlinear time-varying system

$$\dot{\tilde{x}}_i = \tilde{A}\tilde{x}_i + \tilde{B}\tilde{u}_i + \tilde{\phi}(t, \tilde{x}_i), \tag{25}$$

We assume that $(\tilde{A}, \tilde{B})$ is controllable, and $\tilde{B}$ has full column rank. Following the construction in the Appendix, a pre-compensator of the form

$$\dot{x}_{i,c} = A_c x_{i,c} + B_c\bar{u}_i, \qquad \tilde{u}_i = C_c x_{i,c} + D_c\bar{u}_i \tag{26}$$

is designed such that the cascade of the nonlinear system (25), and the pre-compensator (26) is of the form

$$\dot{\bar{x}}_i = \bar{A}\bar{x}_i + \bar{B}\bar{u}_i + \bar{\phi}(t, \bar{x}_i), \tag{27}$$

with all controllability indices of $(\bar{A}, \bar{B})$ equal and denoted by ρ. The following result is a modification of a result from [29].

Theorem 3

Consider the nonlinear time-varying system (27). Assume that $(\bar{A}, \bar{B})$ has all controllability indices equal to ρ; and $\bar{\phi}(t, \bar{x}_i)$ is continuously differentiable and globally Lipschitz continuous with respect to $\bar{x}_i$ uniformly in t, and piecewise continuous with respect to t. Let $\Gamma_x \in \mathbb{R}^{n\times n}$ and $\Gamma_u \in \mathbb{R}^m$ be nonsingular state and input transformations such that the pair $(A, B) = (\Gamma_x^{-1}\bar{A}\Gamma_x, \Gamma_x^{-1}\bar{B}\Gamma_u)$ is in the the short SCB [31], and define $\bar{x}_i = \Gamma_x x_i$ and $\bar{u}_i = \Gamma_u u_i$. Then either

- the system satisfies the canonical form (20) or
- there exists no set of linear, non-singular state, and input transformations that take the system to the canonical form.

Proof

All we have to show is that all transformations that take the linear portion of the system to the short SCB are equivalent with respect to satisfying Assumption 2. Consider therefore the system (20) satisfying Assumption 2, and let (A, B) denote the corresponding linear pair. Let $\breve{\Gamma}_x \in \mathbb{R}^{n\times n}$ and $\breve{\Gamma}_u \in \mathbb{R}^{m\times m}$ denote state and input transformations such that the pair $(\breve{A}, \breve{B}) = (\breve{\Gamma}_x^{-1}A\breve{\Gamma}_x, \breve{\Gamma}_x^{-1}B\breve{\Gamma}_u)$ is also in the short SCB. Define $x_i = \breve{\Gamma}_x\breve{x}_i$ and $u_i = \breve{\Gamma}_u\breve{u}_i$. Then we can write

$$\dot{\breve{x}}_i = A_d\breve{x}_i + \breve{\phi}(t, \breve{x}_i) + B_d(\breve{u}_i + \breve{E}\breve{x}_i),$$

and we need to show that $\breve{\phi}(t, \breve{x}_i)$ satisfies (21). Let

$$\breve{\Gamma}_x = \begin{pmatrix} t_{1,1} & \ldots & t_{1,\rho} \\ \vdots & \ddots & \vdots \\ t_{\rho,1} & \ldots & t_{\rho,\rho} \end{pmatrix},$$

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where $t_{i,j} \in \mathbb{R}^{m\times m}$. Note that $\breve{\Gamma}_x \breve{B} = B\breve{\Gamma}_u$, which implies that $t_{1,\rho} = \ldots = t_{\rho-1,\rho} = 0$. Furthermore, $\breve{\Gamma}_x(A_d + B_d\breve{E}) = (A_d + B_d E)\breve{\Gamma}_x$. Because $\breve{E}$ and E dominate only the last m rows, then $\breve{\Gamma}_x A_d$ equals $A_d\breve{\Gamma}_x$ in terms of the first $(\rho-1)m$ rows. It then follows that $t_{2,1} = \ldots = t_{\rho,1} = 0$ and $t_{i,j} = t_{i+1,j+1}$ for $i, j \in \{1, \rho-1\}$. Together with $t_{1,\rho} = \ldots = t_{\rho-1,\rho} = 0$, we can find that $t_{i,j} = 0$ for $i \neq j$ and $t_{i,i} = t_{j,j}$ for $i, j \in \{1,\ldots,N\}$. Therefore, $\breve{\Gamma}_x = \alpha I$ for some $\alpha \in \mathbb{R}$, which means $\breve{\phi}(t, \breve{x}_i)$ satisfies Assumption 2. □

5. LINEAR AGENTS WITH PARTIAL-STATE COUPLING

In this section, we consider linear agents with partial-state coupling. The agent dynamics are written in the form of

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i, \\ y_i &= Cx_i, \end{aligned} \quad (i = 1,\ldots,N) \tag{28}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, and $y_i \in \mathbb{R}^p$ are the state, input, and output of agent i, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ are constant matrices. Without loss of generality, we assume that matrices B and C are full column rank and row rank, respectively. We make the following standard assumption for the agent dynamics.

Assumption 3
(A, B) is stabilizable and (A, C) observable.

It is worth noting that for partial-coupled agents, we do not impose any constraints on poles and zeros of the agent dynamics. In other words, we allow any general agent dynamics, including non-minimum-phase agents. However, we have to allow the communication among controller states by using the same network. Suppose the state of the controller for agent i is η_i for $i = 1,\ldots,N$. Then, agent i has access to the quantity

$$\hat{\zeta}_i = \sum_{j=1}^{N} \ell_{ij}(t)\eta_j. \tag{29}$$

In this section, we assume that there exists an agent K such that for all time t there exists a directed spanning tree for the graph with root K. Let $\tilde{L}_t$ be the matrix obtained from the Laplacian L_t by deleting the K'th row and column. In that case, K being a root agent guarantees that $\tilde{L}_t$ is invertible. We define the graph set $\mathbb{G}_{\beta,\gamma}^{K,N}$ and $\mathbb{G}_{\beta,\gamma}^{\tau,K,N}$ as before but we assume agent K is a root agent for all time and the eigenvalues $\lambda_1,\ldots,\lambda_{N-1}$ of $\tilde{L}_t$ satisfy $\mathrm{Re}(\lambda_i) \geqslant \beta$ while $\|\tilde{L}_t\| \leqslant \gamma$.

We then formulate the state synchronization problem for time-varying network as follows.

Problem 3
Consider a multi-agent system described by (2) and (28). Suppose agents have access to the quantity (29). For any real numbers $\gamma, \beta, \tau > 0$ and positive integers K and N that defines a set of time-varying network graphs $\mathbb{G}_{\beta,\gamma}^{\tau,K,N}$, the *state synchronization* problem with a set of time-varying network graphs $\mathbb{G}_{\beta,\gamma}^{\tau,K,N}$ is to find, if possible, a linear time-invariant dynamic protocol of the form,

$$\begin{aligned} \dot{x}_{i,c} &= A_{i,c}x_{i,c} + B_{i,c}\mathrm{col}\{\zeta_i, \hat{\zeta}_i\}, \\ u_i &= C_{i,c}x_{i,c}, \end{aligned} \tag{30}$$

where $x_{i,c} \in \mathbb{R}^n$, such that, for any time-varying graph $\mathcal{G}_t \in \mathbb{G}_{\beta,\gamma}^{\tau,K,N}$ and for all the initial conditions of agents, state synchronization among agents can be achieved.

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5.1. Protocol design

The main idea to achieve synchronization is to set the controller of the root agent K to zero (i.e., $u_K = 0$ and $\eta_K = 0$). On the other hand, for all the other agents, we use an identical controller, that is,

$$A_{i,c} = A_c, \quad B_{i,c} = B_c, \quad C_{i,c} = C_c$$

for $i \neq k$. We then design this controller for all the other agents such that their states asymptotically synchronize with the states of root agent K under time-varying networks, that is, $\lim_{t\to\infty}(x_i - x_K) = 0$ for any $\mathcal{G}_t \in \mathbb{G}^{\tau,K,N}_{\beta,\gamma}$.

Let $\bar{x}_i = x_i - x_K$, and $e_i = y_i - y_K$. Then, the dynamics of $\bar{x}_i$ can be written as

$$\begin{aligned} \dot{\bar{x}}_i &= A\bar{x}_i + Bu_i, \\ e_i &= C\bar{x}_i. \end{aligned} \tag{31}$$

Define $\chi_i = T\bar{x}_i$, where

$$T = \begin{pmatrix} C \\ \vdots \\ CA^{n-1} \end{pmatrix}.$$

Note that T is not necessarily a square matrix; however, because of the observability of (A, C), T is injective, which implies that $T'T$ is nonsingular [32]. In terms of χ_i, we can write the equations governing e_i as

$$\begin{aligned} \dot{\chi}_i &= (A_d + \mathcal{L})\chi_i + \mathcal{B}u_i, \quad \chi_i(0) = T\bar{x}_i(0), \\ e_i &= C_d\chi_i, \end{aligned} \tag{32}$$

where $A_d, C_d, \mathcal{L}$, and $\mathcal{B}$ are in a special form

$$A_d = \begin{pmatrix} 0 & I_{p(n-1)} \\ 0 & 0 \end{pmatrix}, \; C_d = \begin{pmatrix} I_p & 0 \end{pmatrix}, \; \mathcal{L} = \begin{pmatrix} 0 \\ L \end{pmatrix}, \quad \mathcal{B} = TB,$$

and where $L = CA^n(TT')^{-1}T'$.

Let $\varepsilon \in (0, 1]$ be a high-gain parameter and define $S_\varepsilon = \operatorname{diag}(I_p, \ldots, I_p\varepsilon^{n-1})$. The high-gain controller for agent $i \in \{1, \ldots, N\}\backslash K$ is designed as

$$\begin{aligned} \dot{\hat{\chi}}_i &= (A_d + \mathcal{L})\hat{\chi}_i + \mathcal{B}u_i + \varepsilon^{-1}S_\varepsilon^{-1}PC_d'(\zeta_i - \hat{\zeta}_i), \\ \eta_i &= C_d\hat{\chi}_i, \\ u_i &= F(T'T)^{-1}T'\hat{\chi}_i, \end{aligned} \tag{33}$$

where

$$\hat{\zeta}_i = \sum_{j=1}^{N} \ell_{ij}(t)\eta_j = \sum_{j=1}^{N} \ell_{ij}(t)\mathcal{C}\hat{\chi}_j$$

with $\eta_K = 0$, F is chosen such that $A + BF$ is Hurwitz, and $P = P' > 0$ is the unique solution of the algebraic Riccati equation,

$$A_dP + PA_d' - 2\beta PC_d'C_dP + I = 0. \tag{34}$$

The main result is given in the following theorem.

Theorem 4

Consider a multi-agent system described by (2) and (28) with partial-state coupling. Suppose the agents have access to the quantity (29). Let any real numbers $\gamma, \beta, \tau > 0$ and positive integers K and N be given, and hence, a set of time-varying network graphs $\mathbb{G}^{\tau,K,N}_{\beta,\gamma}$ be defined.

Under Assumptions 1 and 3, the state synchronization problem stated in Problem 3 is solvable. In particular, controller (33) solves the state synchronization problem under any time-varying graph $\mathcal{G}_t \in \mathbb{G}^{\tau,K,N}_{\beta,\gamma}$.

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Proof
For $i = 1, \ldots, N$ with $i \neq K$, let $\bar{\chi}_i = \chi_i - \hat{\chi}_i$. Then

$$\dot{\bar{\chi}}_i = (A_d + \mathcal{L})\bar{\chi}_i - \varepsilon^{-1} S_\varepsilon^{-1} P C_d' (\zeta_i - \hat{\zeta}_i). \tag{35}$$

Noting that for $i = 1, \ldots, N$, we have $\sum_{j=1}^N \ell_{ij}(t) = 0$, and therefore,

$$\begin{aligned}
\zeta_i - \hat{\zeta}_i &= \sum_{j=1}^N \ell_{ij}(t) y_j - \sum_{j \in \{1,\ldots,N\} \setminus K} \ell_{ij}(t) C_d \hat{\chi}_j \\
&= \sum_{j \in \{1,\ldots,N\} \setminus K} \ell_{ij}(t) C_d \chi_j - \sum_{j \in \{1,\ldots,N\} \setminus K} \ell_{ij}(t) C_d \hat{\chi}_j \\
&= \sum_{j \in \{1,\ldots,N\} \setminus K} \ell_{ij}(t) C_d \bar{\chi}_j.
\end{aligned}$$

Then, dynamics (35) can be rewritten as

$$\dot{\bar{\chi}}_i = A_d \bar{\chi}_i + \mathcal{L}\bar{\chi}_i - \varepsilon^{-1} S_\varepsilon^{-1} P C_d' C_d \sum_{j=1}^N \ell_{ij}(t) \bar{\chi}_j.$$

Define $\xi_i = S_\varepsilon \bar{\chi}_i$. Then, we obtain

$$\varepsilon \dot{\xi}_i = A_d \xi_i + \mathcal{L}_\varepsilon \xi_i - P C_d' C_d \sum_{j \in \{1,\ldots,N\} \setminus K} \ell_{ij}(t) \xi_j,$$

where

$$\mathcal{L}_\varepsilon = \begin{pmatrix} 0 \\ \varepsilon^n L S_\varepsilon^{-1} \end{pmatrix}.$$

Let $\xi = \operatorname{col}\{\xi_i\}$ and $\tilde{\mathcal{L}}_\varepsilon = \operatorname{diag}\{\mathcal{L}_\varepsilon\}$. Then, the dynamics of the complete network becomes

$$\varepsilon \dot{\xi} = [I_{N-1} \otimes A_d + \tilde{\mathcal{L}}_\varepsilon - \tilde{L}_t \otimes P C_d' C_d]\xi, \tag{36}$$

Define $\tilde{Q}_t^{-1} \tilde{L}_t \tilde{Q}_t = \tilde{U}_t$, where $\tilde{U}_t$ is the Schur form of $\tilde{L}_t$ and $\tilde{Q}_t$ is unitary. Let $v = (\tilde{Q}_t^{-1} \otimes I_{pn})\xi$. Then we obtain

$$\varepsilon \dot{v} = (I_{N-1} \otimes A_d)v + W_\varepsilon v - \tilde{U}_t \otimes (P C_d' C_d) v, \tag{37}$$

where

$$W_\varepsilon = (\tilde{Q}_t^{-1} \otimes I_{pn}) \tilde{\mathcal{L}}_\varepsilon (\tilde{Q}_t \otimes I_{pn}).$$

Note that when a switching of the network graph occurs, v will in most cases experiences a discontinuity (because of a sudden change in $\tilde{U}_t$ and $\tilde{Q}_t$). Next we will analyze first the stability of dynamics (37) between the graph switches, that is, for time $t \in [t_{k-1}, t_k)$.

Let $A_{d,t} = I_{N-1} \otimes A_d - U_t \otimes P C_d' C_d$. Similar to Lemma 1, we can show that

$$\tilde{P} = \operatorname{diag}(\alpha^{i_1} P, \alpha^{i_2} P, \ldots, \alpha^{i_{N-1}} P)$$

will satisfy

$$A_{d,t} \tilde{P} + \tilde{P} A_{d,t}' \leqslant -\mu \tilde{P} - I$$

for α sufficiently large where $i_1 > i_2 > \ldots > i_{N-1}$.

Define Lyapunov function $V = \varepsilon v' \tilde{P}^{-1} v$, and we obtain

$$\begin{aligned}\dot{V} &= -\mu\varepsilon^{-1}V - \|\tilde{P}^{-1}v\|^2 + 2\mathrm{Re}(v'\tilde{P}^{-1}W_\varepsilon v)\\ &\leqslant -\mu\varepsilon^{-1}V - \|\tilde{P}^{-1}v\|^2 + \varepsilon r\|\tilde{P}^{-1}v\|^2\\ &\leqslant -\mu\varepsilon^{-1}V,\end{aligned}$$

for a small enough ε. In the second inequality, $\varepsilon r \geqslant 2\|W_\varepsilon \tilde{P}\|$ for suitable r because $\tilde{Q}_t$ is unitary.

Similar to the proof of Theorem 1, for a small enough ε, we can achieve that $\lim_{t\to\infty} V(t) = 0$ under time-varying graphs. Given that Q_t is unitary for any graph in $\mathbb{G}_{\beta,\gamma}^{K,N}$ for any time t, we obtain $\lim_{t\to\infty} \bar{\chi}_i(t) = 0$, that is, $\lim_{t\to\infty}(\chi_i(t) - \hat{\chi}_i(t)) = 0$ under time-varying graphs. Next, we plug the controller input $u_i = F(T'T)^{-1}T'\hat{\chi}_i$ into the dynamics (31). Then, we obtain

$$\begin{aligned}\dot{\bar{x}}_i &= A\bar{x}_i + BF(T'T)^{-1}T'\hat{\chi}_i,\\ &= (A+BF)\bar{x}_i + BF((T'T)^{-1}T'\hat{\chi}_i - \bar{x}_i),\\ &= (A+BF)\bar{x}_i + BF(T'T)^{-1}T'(\hat{\chi}_i - \chi_i),\end{aligned}$$

which is asymptotically stable, because $A + BF$ is Hurwitz and $\lim_{t\to\infty}(\chi_i(t) - \hat{\chi}_i(t)) = 0$. Hence, $\lim_{t\to\infty}(x_i(t) - x_K(t)) = 0$, which proves the result. □

6. EXAMPLE

In this section, we illustrate our results on a time-varying homogeneous network of $N = 6$ agents. The agent model is written as

$$A = \begin{pmatrix} -1 & -1 & -1\\ 2 & 1 & -4\\ 0 & 0 & -2\end{pmatrix}, \quad B = \begin{pmatrix}1\\1\\1\end{pmatrix}.$$

We assume that the communication topology switches among three graphs in a circular manner with dwell-time $\tau = 3$ s, shown in Figure 1. For such a set of graphs, $\beta = 0.1$ and $\gamma = 6$.

By using the state and input transformations

$$T_x = \begin{pmatrix} 1 & -1 & 1\\ 0 & 1 & 1\\ 1 & 0 & 1\end{pmatrix}, \quad T_u = 1,$$

we obtain the agent dynamics in the special structure:

$$A_d = \begin{pmatrix} 0&1&0\\0&0&1\\0&0&0\end{pmatrix}, \quad B_d = \begin{pmatrix}0\\0\\1\end{pmatrix}, \quad E = \begin{pmatrix}-2 & -1 & -2\end{pmatrix}.$$

By using the algebraic Riccati Eq. 8, we easily obtain $\hat{F} = (-2.2361 \quad -6.5161 \quad -8.3762)$. Now choosing the high-gain parameter $\varepsilon = 1$, we obtain the static protocol

$$F = (0.3760 \quad -6.1401 \quad -2.6120).$$

Figure 2 shows that all agents achieve state synchronization under the time-varying communication topologies.

Next, we consider the nonlinear agent model

$$\dot{x} = \begin{pmatrix} 0&1&0\\0&0&1\\0&0&0\end{pmatrix} x + \begin{pmatrix}0\\0\\1\end{pmatrix}(u - 2x_1 - x_2 - 2x_3 + 60\cos(3x_1) + 30\sin(0.1x_2)). \tag{38}$$

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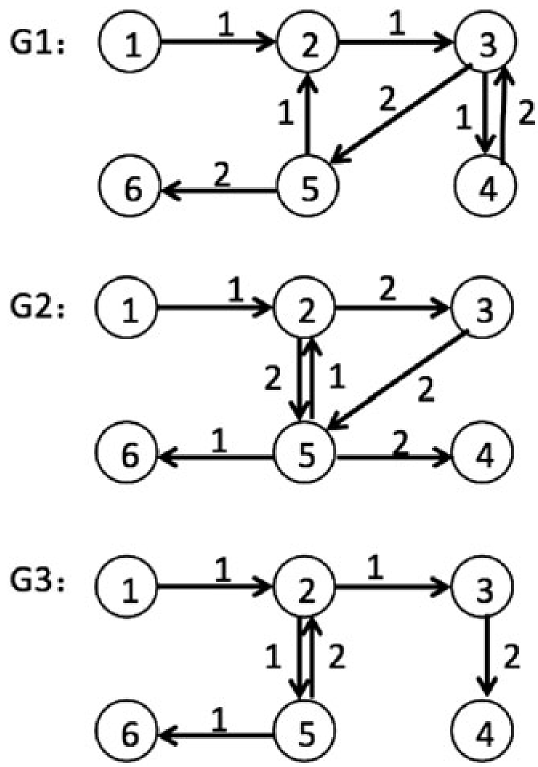

Figure 1. Three communication topologies.

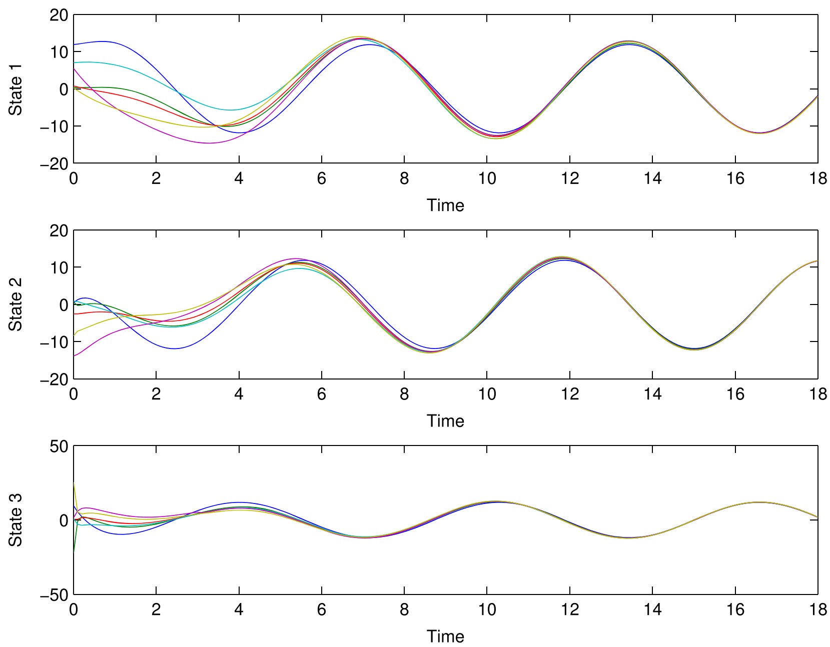

Figure 2. The trajectories of agents' states. [Colour figure can be viewed at wileyonlinelibrary.com]

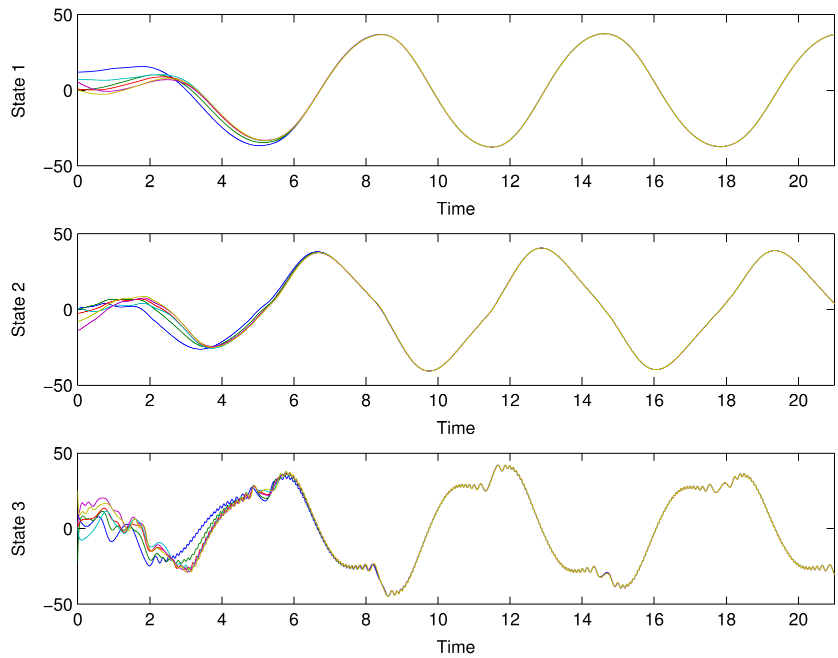

Figure 3. The trajectories of agents' states. [Colour figure can be viewed at wileyonlinelibrary.com]

We use the same $\hat{F}$ as in the aforementioned linear case. Here, we choose the high-gain parameter $\varepsilon = 0.5$. Then the protocol is

$$F = (-17.8885 \quad -26.0644 \quad -16.7524)\,.$$

Figure 3 shows that all agents achieve state synchronization under the time-varying communication topologies.

7. CONCLUSION

In this paper, the state synchronization problem for homogeneous time-varying networks with diffusive full-state coupling and partial-state coupling is solved. For the case of full-state coupling, both linear and nonlinear agents are considered. The essence of the protocol design in this case is to use different time scales such that, by tuning the high-gain parameter ε, the difference between the states of different agents can decay as fast as required. For the case of partial-state coupling, we have so far only dealt with linear agents, but the agents can be general and, for instance, can be non-minimum-phase. The protocol design in this case is based on a high-gain observer with an extra communication channel for controller states. The time-varying network can be switching among an infinite set of graphs with a priori given properties. For a finite set of graphs which each has a directed spanning tree, these required properties are automatically satisfied. Thus, our protocol design can always be applied given a finite set of graphs.

APPENDIX A: PRE-COMPENSATOR DESIGN

In this section, we will construct a pre-compensator (26) for the system

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}, \tag{39}$$

where $x \in \mathbb{R}^{\tilde{n}}$ and $\tilde{u} \in \mathbb{R}^m$, such that the interconnection of the pre-compenstator and system (39) has all controllablity indices the same.

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Assume that $(\tilde{A}, \tilde{B})$ is controllable and that $\tilde{B}$ has full-column rank. According to [31, Theorem 1], there exist nonsingular transformation matrices T_x and T_u such that the transformed system, with $\breve{x} = T_x \tilde{x}$ and $\breve{u} = T_u \tilde{u}$, is in the canonical form (that is the short SCB form),

$$\dot{\breve{x}}_j = A_j \breve{x}_j + B_j \left[\breve{u}_j + \sum_{q=1}^{\rho_j} A_{jq} \breve{x}_q \right], \tag{40}$$

for $j = 1, \ldots, m$ where $\breve{x}_j \in \mathbb{R}^{\rho_j}$, $\breve{u}_j \in \mathbb{R}$, and

$$\breve{x} = \begin{pmatrix} \breve{x}_1 \\ \vdots \\ \breve{x}_m \end{pmatrix}, \quad \breve{u} = \begin{pmatrix} \breve{u}_1 \\ \vdots \\ \breve{u}_m \end{pmatrix}, \quad A_j = \begin{pmatrix} 0 & I_{\rho_j} \\ 0 & 0 \end{pmatrix}, \quad B_j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that $\sum_{j=1}^{m} \rho_j = \tilde{n}$. Next, we will construct a pre-compensator such that all controllability indices are the same. Let $\rho = \max\{\rho_1, \ldots, \rho_m\}$. For each subsystem of dimension ρ_j, we will add $\rho - \rho_j$ integrators before the input $\breve{u}_j$. Thus, the pre-compensator for subsystem j with $\rho_j < \rho$ is

$$\begin{aligned} \dot{x}_{j,c} &= A_{j,c} x_{j,c} + B_{j,c} u_j, \\ \breve{u}_j &= C_{j,c} x \end{aligned} \tag{41}$$

where u_j is the new input, and

$$A_{j,c} = \begin{pmatrix} 0 & I_{\rho-\rho_j} \\ 0 & 0 \end{pmatrix}, \quad B_{j,c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C_{j,c} = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

If $\rho_j = \rho$, we simply set $\breve{u}_j = u_j$.

Next, combine the pre-compensators for $j = 1, \ldots, m$, and use the inverse input transformation T_u^{-1}. Then, we obtain the pre-compensator in the form of (26). Moreover, the interconnection system of the pre-compensator and the system (39) is controllable.

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