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The Boundedly Rational User Equilibrium: A parametric analysis with application to the Network Design Problem

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ABSTRACT

In this paper, we study a static traffic assignment that accounts for the boundedly rational route choice behavior of travelers. This assignment induces uncertainties to the ex-ante evaluation of a policy measure: the boundedly rational assignment is non-unique and the indifference band is an uncertain parameter. We consider two different ways to model the optimization problem that finds the best and worst-performing Boundedly Rational User Equilibrium with respect to the total travel time (Best/Worst-case BRUE). The first is the so-called branch approach, the second is a bilevel model. The latter approach is better suited to exploit techniques from parametric optimization and enables us, e.g., to prove the continuity of the optimal value function corresponding to the Best/Worst-case BRUE with respect to perturbations in the indifference band. We report on some numerical experiments. In addition, we extend our results to the Network Design Problem: we prove the existence of a second-best toll pricing scheme under bounded rationality.

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1. Introduction

Traffic assignment models often presume perfect rationality in route choice decision making: travelers are selfish, fully informed, and can perfectly assess the consequences of choosing an alternative (Conlisk, 1996; Simon, 1997; Vreeswijk et al., 2013a). The corresponding concept in traffic networks is the *Wardrop equilibrium*: a network state (i.e., distribution of traffic) in which no traveler marginally benefits by a unilateral switch in routes. Empirical studies (e.g., Ciscal-Terry et al., 2016 and Zhu and Levinson, 2012) however suggest that the economic assumptions of Wardrop are debatable. For instance, in the study by Zhu and Levinson (2012) only 34% of all travelers followed the shortest-time path. Since the actual equilibrium will not be a Wardrop equilibrium (van Essen et al., 2016), it is naive to evaluate the performance of policies under such strict economic assumptions. Indeed, real-world application of measures based on Wardrop's condition may show undesirable results.

Ample studies proposed extensions of the Wardrop equilibrium to incorporate the imperfect information and/or behavior of travelers. Most notably is the Stochastic User Equilibrium in Daganzo and Sheffi (1977) with a random term for user's inaccurate perception. More recently, Xu et al. (2011) proposed a static traffic assignment based on prospect theory, in

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which drivers compare their travel time to a reference travel time. These approaches have a more complicated mathematical structure and cause additional computational challenges compared to Wardrop's equilibrium (Sun et al., 2016).

Boundedly Rational User Equilibrium.

We study a static traffic assignment model that incorporates a realistic view on decision making. To this end, we adopt the notion of bounded rationality from Mahmassani and Chang (1987). Boundedly rational travelers make *suboptimal* choices and an intervention leads to a *Boundedly Rational User Equilibrium* (BRUE): a network state in which a unilateral switch in paths does not lead to a travel time improvement of more than an *indifference band*. In Zhu and Levinson (2012), 90% of the trips can be explained by this notion.

The BRUE has, compared to Wardrop's equilibrium, two main sources of uncertainty. Both uncertainties are significant in ex-ante evaluation setting, where a policy measure is evaluated based on drivers' predicted behavioral response (Sun et al., 2016). First: bounded rationality in the traffic assignment leads to a set of possible (link) flow patterns, i.e., the BRUE is generally non-unique. Secondly, the indifference band is a context-dependent parameter, difficult to capture and therefore subject to uncertainty (Vreeswijk et al., 2013b).

Regarding the first source of uncertainty, it is impossible to enumerate all possible behavioral responses to an intervention and therefore we have to make an assumption on the realized BRUE flow. However, this may lead to adverse effects. For example, if we naively assume that the best-possible BRUE flow with respect to the total travel time is realized in practice and thereby ignore all other possible realizations, we might draw wrong conclusions (Lou et al., 2010). In this paper, we therefore investigate the extremes of possible network performances assuming bounded rationality: an indication of what a policy could achieve under uncertainty. In fact, we consider the best and worst-performing BRUE with respect to total travel time (Best/Worst-case BRUE). The performance of the physically realized BRUE flow lies within the range defined by the total travel time of the Best and Worst-case BRUE respectively. Note that the Best-case BRUE concept makes perfectly sense in a situation where all travelers follow a reasonable route advice of an authority.

Regarding the second source of uncertainty, several studies attempted to calibrate the values of the indifference band (see for an overview: Di et al., 2013 and Di and Liu, 2016). The indifference band is however a context-dependent (Vreeswijk et al., 2013b), high-dimensional parameter and the estimated parameter is unlikely to perfectly mirror the real-world indifference band. We explicitly account for this uncertainty and approach the Best/Worst-case BRUE problem using sensitivity analysis. We evaluate what happens with these traffic states and corresponding system performances under perturbations in the indifference band. Specifically, we study (global) continuity of the feasible set, optimal value function, and optimal solution set as a function of the indifference band. For instance, given a policy measure, sensitivity analysis of the optimal value function indicates whether the range of possible performances in real life substantially differs from the modeled range.

The Network Design Problem with boundedly rational travelers.

The uncertainties of the BRUE particularly apply to the *Network Design Problem* (NDP). The NDP asks for improvements of network settings (e.g., tolls, road capacity) so that a transportation system performs optimally (Abdulaal and LeBlanc, 1979) and the standard approach is to evaluate a set of configurations based on a traffic assignment model to account for the expected behavioral response (Brands and van Berkum, 2014; Sun et al., 2016). Existing studies on the NDP (see for an overview e.g. Farahani et al., 2013) often integrate the naive Wardrop equilibrium to predict the behavioral response.

Since it is unknown which BRUE flow distribution arises for a single configuration, it is unclear which policy measure optimizes the system objective (Ban et al., 2009). Ban et al. (2009) and Lou et al. (2010) suggest therefore to apply, depending on the attitude of the network designer towards this uncertainty, those settings which minimize the best or worst-case travel time. Given the indifference band, optimal settings from the NDP are likely to be optimal under a small perturbation in this band as well provided that the optimal value function is continuous. In this perspective, the network settings turn out to be robust: the optimal configuration is locally not sensitive to changes in the parameter.

We consider an application of the NDP in a BRUE context: the second-best toll pricing problem. We study sensitivity of the performance of the Best/Worst-case BRUE with respect to changes in the toll. Although this analysis does not directly lead to an algorithm to derive such a tolling scheme, it suits the purpose to identify the mathematical properties of this problem (i.e., existence of an optimal toll setting).

Towards solving the Best/Worst-case BRUE.

Essential for the ex-ante evaluation is an efficient method to solve the Best/Worst-case BRUE problem for a fixed indifference band. Lou et al. (2010) formulated the more restrictive link-based BRUE to apply an earlier developed algorithm. Di et al. (2013) (and subsequent studies in Di et al., 2014 and Di et al., 2016) used a mathematical program with equilibrium constraint that derive all possible boundedly rational path flows for a given band. The mentioned studies require complex and computationally expensive algorithms, solely useful for assignments that are small in terms of network size and number of routes. In particular in the context where designers evaluate multiple configurations on large network instances, we need an efficient approach to calculate the range of possible performances for the BRUE (Sun et al., 2016).

Although a comparative analysis of different algorithms to solve the Best/Worst-case BRUE is outside the scope of the paper, we contribute to future algorithms by means of an in-depth mathematical analysis of the problem. We show that we can reformulate the Best/Worst-case BRUE as a bilevel optimization problem (*cf.* Section 3.2) and the bilevel structure allows a new research direction for algorithm development. Intuitively, the bilevel structure relates this problem to the well-known *Continuous Network Design Problem* (CNDP). (Note that the CNDP determines the optimal network settings, while the bilevel Best/Worst-case BRUE evaluates system performance for a single configuration). Sensitivity analysis of the lower level in the CNDP was successfully used in algorithms (e.g., Friesz et al., 1990 and Josefsson and Patriksson, 2007). We also perform sensitivity analysis with respect to the lower-level problem in the Best/Worst-case BRUE problem and identify the difficulties that arise in applying the CNDP algorithms in the BRUE context.

Related literature.

To the best of our knowledge, our study is the first that approaches the BRUE problem under realistic assumptions with respect to the travel time function and the indifference band. Some authors investigated the mathematical structure of the boundedly rational traffic assignment but assume the indifference band to be fixed and simplify the problem. Lou et al. (2010) showed that the Best/Worst-case BRUE problem can be modeled as a mathematical program with complementarity constraints. They found that the feasible set that corresponds to this problem is not convex and violates a constraint qualification. Di et al. (2013) indicated that in a linear latency context the feasible set of BRUE flows is a union of polyhedra but not necessarily a polyhedron itself. In Han et al. (2015), the mathematical properties of the boundedly rational assignment in the selection of departure times and routes were considered. Although the results in Han et al. (2015) particularly apply to the static assignment, the study mainly focused on finding a BRUE flow distribution in a dynamic context. Di et al. (2016) considered the NDP with boundedly rational travelers. They investigated the Best/Worst-case BRUE problem under a perturbation in the second-best toll vector but the study is limited to linear latencies while our results apply to more general cost functions. For day-to-day dynamic processes under the bounded rationality assumptions we refer to, e.g., Di et al. (2015) and Ye and Yang (2017).

Summarizing, the boundedly rational traffic assignment allows authorities to perform ex-ante evaluation under realistic conditions. Our study assesses the theoretical and practical consequences of a BRUE assignment. Our setting assumes travelers to be boundedly rational rather than perfectly rational utility maximizers and adopts a realistic travel time function. We present an in-depth mathematical analysis of the structure of the Best/Worst-case BRUE problem in dependence of the (uncertain) indifference band and discuss practical implications of this model.

The paper is organized as follows. We give a formal description of the (static) BRUE assignment in Section 2. In Section 3, we give the definitions of continuity of a set and a function and discuss two reformulations of the BRUE assignment problem. In Section 4, we study continuity of the feasible set, optimal value function and optimal solution set of the Best/Worst-case BRUE problem with respect to changes in the indifference band. In Section 5, we show an application of the NDP with boundedly rational travelers. In fact, we study the continuity of the optimal value function of the Best/Worst-case BRUE with respect to perturbations in the indifference band and the toll vector. The main results of our paper are illustrated by two examples in Section 6.

2. Problem formulation

2.1. Static traffic assignment

We study the static traffic assignment with fixed demand. Given is a directed traffic network $G = (V, E)$, with V being the set of nodes and E is a set of directed edges (roads, links, or arcs) $e = (i, j)$, with $i, j \in V$. The network includes a set of *origin-destination pairs* (OD pairs) $\mathcal{K} \subseteq V \times V$, with static demand $d_k > 0$, $k \in \mathcal{K}$. Each OD pair $k \in \mathcal{K}$ is referred to as *commodity k* and is connected by the set $\mathcal{P}_k$ of simple directed paths. The set $\mathcal{P}$ of all paths in the network is the union of the path sets per commodity: $\mathcal{P} = \cup_{k\in\mathcal{K}}\mathcal{P}_k$.

A feasible *traffic flow* or *flow* for given demand $d \in \mathbb{R}_+^{|\mathcal{K}|}$ (we denote by $|.|$ the cardinality of a set) is a pair of vectors $(f, x) \in \mathbb{R}^{|\mathcal{P}|} \times \mathbb{R}^{|E|} = (f_p, p \in \mathcal{P}; x_e, e \in E)$ so that

$$(f, x) \in \mathcal{F}_0 := \left\{ \ (f, x) \in \mathbb{R}^{|\mathcal{P}|} \times \mathbb{R}^{|E|} \ \middle| \ \Lambda f = d, x = \Delta f, f \geq 0 \ \right\}.$$

Here, matrix $\Lambda \in \mathbb{R}^{|\mathcal{K}|\times|\mathcal{P}|}$ is the *OD-path incidence matrix* in which $\Lambda_{kp} = 1$ if $p \in \mathcal{P}_k$ and $\Lambda_{kp} = 0$ otherwise. $\Delta \in \mathbb{R}^{|E|\times|\mathcal{P}|}$ denotes the *link-path incidence matrix*: $\Delta_{ep} = 1$ if edge e is in route p and $\Delta_{ep} = 0$ otherwise.

Each link $e \in E$ in the network has a flow-dependent *travel time, latency* or *cost* $l_e(x)$. The cost of a route $c_p(f)$, $p \in \mathcal{P}$, is the sum of travel costs of all edges in that path: $c_p(f) = \sum_{e\in p} l_e(x)$.

2.2. The Boundedly Rational User Equilibrium

Wardrop (1952) formulated two criteria to determine the distribution of flow over a traffic network. Wardrop's first principle assumes travelers to be perfectly rational in making route choice decisions: users maximize own utility by considering and evaluating the consequences of all possible alternatives (Conlisk, 1996; Vreeswijk et al., 2013a). The resulting traffic flow pattern, under the assumptions of this behavior in a static environment, is a traffic state in which no traveler can unilaterally change routes to decrease its travel time. In other words, at a *Perfectly Rational User Equilibrium* (PRUE) - also called Wardrop equilibrium - all *flow-carrying paths* (i.e. $p \in \mathcal{P} : f_p > 0$) among a commodity experience equal, in fact minimum, cost.

Definition 1 (PRUE)**.** A traffic flow $(f, x) \in \mathcal{F}_0$ with corresponding cost vector $c(f)$ is said to be a Perfectly Rational User Equilibrium (PRUE) if for all $k \in \mathcal{K}$ the following condition holds for all $p, q \in \mathcal{P}_k$:

$$f_p > 0 \Rightarrow \begin{cases} c_p(f) = c_q(f) & \text{if } f_q > 0; \\ c_p(f) \leq c_q(f) & \text{if } f_q = 0. \end{cases} \tag{1}$$

Empirical studies suggest that the assumptions of perfect rationality in route choice decision making are, from a behavioral perspective, naive. In other words, a traffic flow distribution as in (1) does not arise in practice. The boundedly rational equilibrium condition in (2) states that travelers choose satisfactory routes: a unilateral switch in routes does not lead to a travel time improvement of more than an indifference band (Mahmassani and Chang, 1987).

Definition 2 (BRUE)**.** For indifference band $\varepsilon \in \mathbb{R}_+^{|\mathcal{K}|}$, a traffic flow $(f, x) \in \mathcal{F}_0$ with corresponding path costs $c(f)$ is called a Boundedly Rational User Equilibrium (BRUE) if for all $k \in \mathcal{K}$ the following condition is satisfied for all $p \in \mathcal{P}_k$:

$$f_p > 0 \Rightarrow c_p(f) \leq \min_{q \in \mathcal{P}_k} c_q(f) + \varepsilon_k. \tag{2}$$

BRUE condition (2) was first discussed by Mahmassani and Chang (1987), and was, among others, formalized by Di et al. (2013) and Lou et al. (2010). Condition (2) formulates a range of allowed travel times for a user by introducing indifference band $\varepsilon \in \mathbb{R}_+^{|\mathcal{K}|}$. This clearly contrasts PRUE condition (1) that only contains a single value of allowed travel times for each OD pair. The BRUE flow distribution, i.e. $(f, x) \in \mathcal{F}_0$ that satisfies (2), is a traffic state in which the travel time of any flow-carrying path is within the formulated range.

2.3. The Best and Worst-case BRUE

Authorities are concerned with the impact of measures on the performance of transportation systems. Lou et al. (2010) and Mahmassani and Chang (1987) highlighted that generally multiple BRUE flow distributions exist. We assume that without any additional information all BRUE for a given indifference band (and network configuration) are equally likely to be realized in practice and therefore we assess the distributions with best and worst-case performance with respect to the system objective. Following our discussion, the Best/Worst-case BRUE flow is a solution of the program

$$\min / \max_{(f,x) \in \mathcal{F}_0} s(x) \quad \text{s.t.} \quad (f, x) \text{ satisfies } (2). \tag{3}$$

In our case, the performance function $s(x)$ in (3) is the total travel time, $s(x) := \Sigma_{e \in E} x_e l_e(x_e)$. The upcoming sections particularly focus on the Best-case BRUE problem, although an analogous analysis holds for the Worst-case BRUE problem (see Remark 1 in Section 4.2).

3. Best-case BRUE

We consider the Best-case BRUE problem:

$$\mathcal{Q}(\varepsilon) \qquad \min_{(f,x)} \; s(x) \quad \text{s.t.} \quad (f, x) \in \mathcal{F}(\varepsilon).$$

$\mathcal{F}(\varepsilon)$ is the set of feasible BRUE flow distributions for indifference band ε:

$$\mathcal{F}(\varepsilon) = \left\{ \; (f, x) \in \mathbb{R}^{|\mathcal{P}|} \times \mathbb{R}^{|E|} \;\middle|\; \begin{array}{l} (f, x) \in \mathcal{F}_0 \\ (f, x) \text{ satisfies } (2) \end{array} \right\}.$$

It easily follows (see Lou et al., 2010) that $\mathcal{F}(\varepsilon)$ is equivalent to a set given by complementarity constraints:

$$\mathcal{F}(\varepsilon) = \left\{ \; (f, x) \in \mathbb{R}^{|\mathcal{P}|} \times \mathbb{R}^{|E|} \;\middle|\; \begin{array}{l} (f, x) \in \mathcal{F}_0 \\ f_p \cdot (c_p(f) - \min_{q \in \mathcal{P}_k} c_q(f) - \varepsilon_k) \leq 0 \quad \forall p \in \mathcal{P}_k, k \in \mathcal{K} \end{array} \right\}.$$

The complementarity constraints make that $\mathcal{F}(\varepsilon)$ lacks favorable mathematical properties. The feasible set $\mathcal{F}(\varepsilon)$ is not convex (see for an example Lou et al., 2010 and Section 6), for a feasible point $(f, x) \in \mathcal{F}(\varepsilon)$ a regularity condition (e.g. Mangasarian–Fromovitz constraint qualification (MFCQ)) does not necessarily hold, and many local minimizers can coexist. Hence, standard optimization tools do not directly apply.

We introduce some notations:

$$\mathcal{S}(\varepsilon) = \{(f, x) \mid (f, x) \text{ is a global minimizer of } \mathcal{Q}(\varepsilon)\};$$
$$v(\varepsilon) = \min_{(f,x) \in \mathcal{F}(\varepsilon)} s(x).$$

We evaluate the impact of a change in ε on the feasible set $\mathcal{F}(\varepsilon)$, optimal value function $v(\varepsilon)$ and optimal solution set $\mathcal{S}(\varepsilon)$ by means of parametric analysis. By that we mean that we evaluate the behavior of the mentioned function and sets if the indifference band ε is subject to a small perturbation. Intuitively, for given design settings we consider the impact of a changing ε on the traffic state (optimal solution set) and corresponding total travel time (optimal value function). We underline that we have no information with respect to the realized BRUE and that we only consider the Best/Worst-case BRUE for design purposes.

The sensitivity analysis is practically and theoretically relevant if ε does not perfectly mirror the real-world indifference band (see Section 1). Moreover, we can estimate best-case performance for a given indifference band and network setting as follows. The Best-case BRUE obviously reduces to solving for the PRUE if $\varepsilon = 0$ (Di et al., 2013; Lou et al., 2010), and to the system-optimal assignment if ε is sufficiently large. An assessment of the rate of change of the optimal value functions of these problems if ε is perturbed provides an estimate of the best-case performance for the indifference band under consideration.

In this study, we distinguish single- and multi-valued functions. A multi-valued function F from $X \subseteq \mathbb{R}^n$ into $Y \subseteq R^m$ assigns to each $x \in X$ a (possibly empty) subset $F(x)$ of Y. For single-valued functions: $m = 1$ and $|F(x)| = 1$.

We answer the question whether the (single- and multi-valued) functions change continuously with ε. A (multi-valued) function is said to be continuous if a small perturbation in the parameter leads to a small change in the output (Lu and Nie, 2010). Such a behavior is also called *stable* (Lu and Nie, 2010; Smith, 1979). We introduce the definitions (Bank et al., 1983) where $\|.\|$ denotes the Euclidean norm.

Definition 3. A single-valued function $v(\varepsilon)$ is said to be

(a) *upper semicontinuous* (usc) at ε^0 if for any $\tau > 0$ there exists $\delta > 0$ such that

$$v(\varepsilon) \leq v(\varepsilon^0) + \tau, \qquad \text{for all } \|\varepsilon - \varepsilon^0\| < \delta;$$

(b) *lower semicontinuous* (lsc) at ε^0 if for any $\tau > 0$ there exists $\delta > 0$ such that

$$v(\varepsilon) \geq v(\varepsilon^0) - \tau, \qquad \text{for all } \|\varepsilon - \varepsilon^0\| < \delta.$$

Intuitively, a single-valued function is upper (lower) semicontinuous when a small perturbation of ε^0 does not lead to a substantially greater (smaller) value. A single-valued function $v(\varepsilon)$ is said to be continuous at ε^0 if it is both upper and lower semicontinuous at ε^0.

We introduce the neighborhoods $U_\delta(\varepsilon^0) = \{\varepsilon \mid \|\varepsilon - \varepsilon^0\| < \delta\}$ $(\delta > 0)$ and $U_\tau(F(\varepsilon^0)) = \{x \mid \text{there exists some } x^0 \in F(\varepsilon^0) \text{ such that } \|x - x^0\| < \tau\}$ $(\tau > 0)$.

Definition 4. A multi-valued function $F(\varepsilon)$ is said to be

(a) *closed* at ε^0 if for any sequences ε^l, x^l, $l \in \mathbb{N}$, with $\varepsilon^l \to \varepsilon^0$, $x^l \in F(\varepsilon^l)$, the relation $x^l \to x^0$ implies $x^0 \in F(\varepsilon^0)$;
(b) *upper semicontinuous* (usc) at ε^0 if for each $\tau > 0$ exists a $\delta > 0$ so that the following condition holds:

$$F(\varepsilon) \subseteq U_\tau(F(\varepsilon^0)), \qquad \text{for all } \varepsilon \in U_\delta(\varepsilon^0);$$

(c) *lower semicontinuous* (lsc) at ε^0 if for each $\tau > 0$ exists a $\delta > 0$ so that the following condition holds:

$$F(\varepsilon^0) \subseteq U_\tau(F(\varepsilon)), \qquad \text{for all } \varepsilon \in U_\delta(\varepsilon^0).$$

An upper (lower) semicontinuous multifunction assures that a set does not explode (implode) after a small change in the parameter ε^0. A multi-valued function $F(\varepsilon)$ is said to be continuous at ε^0 if it is both usc and lsc at ε^0. It turns out that, in the compact spaces we consider, closedness and upper semicontinuity of a function $F(\varepsilon)$ are easily satisfied. Lower semicontinuity of $F(\varepsilon)$ is less easily guaranteed.

As mentioned, we define $s(x)$ to be the total travel time in the network. Under the following assumption (i.e. Assumption 1), $s(x)$ is strictly convex with respect to link flow x. Note that Assumption 1 is not necessarily a very strong one and applies to many cost functions, including the well-known Bureau of Public Roads-function (Bureau of Public Roads, 1964).

Assumption 1. We assume that the travel time function $l_e(x)$ is separable, i.e. $l_e(x) = l_e(x_e)$, and the functions $l_e(x_e)$ are continuous, convex and strictly monotone increasing: $l_e(x_e) < l_e(x_e^0)$ provided $x_e < x_e^0$, for all $e \in E$.

$\mathcal{F}(\varepsilon)$ has a complex structure. To facilitate parametric analysis, we consider two reformulations of the problem $\mathcal{Q}(\varepsilon)$: a branch approach based on a selection of the path sets $P \subseteq \mathcal{P}$ (Section 3.1), and an alternative bilevel (BL) approach (Section 3.2). We discuss both approaches and use techniques from parametric optimization to study the impact of a perturbation in ε on the feasible set, optimal value function and optimal solution set (for details on parametric optimization we refer to Bank et al., 1983).

3.1. Branch approach

The branch approach decomposes the complementarity constraint problem $\mathcal{Q}(\varepsilon)$ into easier subproblems. Each branch considers a subset of paths and assumes that all the demand is distributed over the routes in this subset while satisfying (2).

For a given subset of paths $P \subseteq \mathcal{P}$ and indifference band $\varepsilon \geq 0$, with $P_k := \mathcal{P}_k \cap P$, $k \in \mathcal{K}$, the feasible set of the branch approach is given by (note the similarities with Eq. (23) in Di et al., 2013)

$$\mathcal{F}_P(\varepsilon) = \left\{ \quad (f,x) \in \mathbb{R}^{|\mathcal{P}|} \times \mathbb{R}^{|E|} \;\middle|\; \begin{array}{ll} (f,x) \in \mathcal{F}_0 & \\ f_p = 0 & \forall p \notin P \\ c_p(f) - \min_{q \in \mathcal{P}_k} c_q(f) - \varepsilon_k \leq 0 & \forall p \in P_k, k \in \mathcal{K} \end{array} \right\}.$$

The branch approach considers the following subproblem:

$$\mathcal{Q}_P(\varepsilon) \qquad \min_{(f,x)} \; s(x) \quad \text{s.t.} \quad (f,x) \in \mathcal{F}_P(\varepsilon).$$

The optimal value function $v_P(\varepsilon)$ and optimal solution set $\mathcal{S}_P(\varepsilon)$ of problem $\mathcal{Q}_P(\varepsilon)$ are given by:

$v_P(\varepsilon) = \min_{(f,x) \in \mathcal{F}_P(\varepsilon)} s(x)$;
$\mathcal{S}_P(\varepsilon) = \{(f,x) \mid (f,x) \text{ is a global minimizer of } \mathcal{Q}_P(\varepsilon)\}$.

$\mathcal{Q}(\varepsilon)$ relates to $\mathcal{Q}_P(\varepsilon)$. We have

$$\mathcal{F}(\varepsilon) = \bigcup_{P \subseteq \mathcal{P}} \mathcal{F}_P(\varepsilon), \qquad \text{and} \qquad v(\varepsilon) = \min_{P \subseteq \mathcal{P}} v_P(\varepsilon).$$

So, we can solve $\mathcal{Q}(\varepsilon)$ using the following bilevel program with upper level variable P:

$$\mathcal{Q}(\varepsilon) \qquad \min_{(P;f,x)} s(x) \quad \text{s.t.} \quad (f,x) \text{ solves } \mathcal{Q}_P(\varepsilon). \tag{4}$$

Note that the unknown $P \subseteq \mathcal{P}$ is a discrete variable.

We summarize the mathematical properties of problem $\mathcal{Q}_P(\varepsilon)$, given $P \subseteq \mathcal{P}$ and assuming $\mathcal{F}_P(\varepsilon) \neq \emptyset$. Obviously, $\mathcal{F}_P(\varepsilon) \subseteq \mathcal{F}(\varepsilon)$ for any $P \subseteq \mathcal{P}$. Furthermore, $\mathcal{Q}_P(\varepsilon)$ represents a (standard) parametric constrained program. We distinguish properties under affine linear and more general cost functions.

Affine linear cost functions.

Assume that the latency functions are affine linear under Assumption 1, i.e. $l_e(x_e) = a_e x_e + b_e$ where $a_e > 0$ and $b_e \in \mathbb{R}_+$. The feasible set of problem $\mathcal{Q}_P(\varepsilon)$ is then defined by linear (in)equalities and $\mathcal{F}_P(\varepsilon)$ is thus a closed polyhedron. $\mathcal{Q}_P(\varepsilon)$ is a standard convex optimization problem. Since the objective function $s(x)$ is strictly convex on $\mathbb{R}^{|E|}$, $\mathcal{Q}_P(\varepsilon)$ has a unique global minimizer with respect to x: the projection $\mathcal{S}_P^x(\varepsilon)$ of $\mathcal{S}_P(\varepsilon)$ onto the x-space is a singleton (this does not apply to $\mathcal{S}_P(\varepsilon)$).

In Eikenbroek (2016) it has been shown that $\mathcal{F}_P(\varepsilon)$ is (globally) Lipschitz continuous with respect to ε. Moreover, $v_P(\varepsilon)$ is a convex, piecewise quadratic and continuous function in ε. The mapping $\mathcal{S}_P^x(\varepsilon)$ is continuous and piecewise linear in ε (this does not necessarily apply to $\mathcal{S}_P(\varepsilon)$).

General cost functions.

Assume that the latency functions are not affine linear under Assumption 1. Then, for given ε, $\mathcal{F}_P(\varepsilon)$ is defined by continuous functions and, hence, $\mathcal{F}_P(\varepsilon)$ is a closed and bounded (compact) set but possibly not convex and not connected. $\mathcal{Q}_P(\varepsilon)$ has a minimum (i.e. $\mathcal{S}_P(\varepsilon) \neq \emptyset$) but possibly multiple minimizers coexist.

The feasible set $\mathcal{F}_P(\varepsilon)$ is a closed mapping and usc at any $\varepsilon \geq 0$. It is not directly clear whether $\mathcal{F}_P(\varepsilon)$ is lsc. It follows that the optimal value function $v_P(\varepsilon)$ is lsc but it is not clear whether $v_P(\varepsilon)$ is a continuous function in ε.

If the latency functions in the network are affine linear, $\mathcal{F}(\varepsilon)$ is a union of closed polyhedra but not necessarily a polyhedron itself (see also Di et al., 2013). Although under affine linear latencies $\mathcal{F}_P(\varepsilon)$ is continuous with respect to its ε-domain, that does not imply that $\mathcal{F}(\varepsilon)$ is continuous in ε. Similar claims hold with respect to $v(\varepsilon)$ and $\mathcal{S}(\varepsilon)$.

3.2. Bilevel approach

In this subsection, we formulate an alternative BL model equivalent to $\mathcal{Q}(\varepsilon)$. For this purpose, we introduce parameter $\rho \in \mathbb{R}^{|\mathcal{P}|}$ and

$$E(\varepsilon) = \left\{\rho \in \mathbb{R}^{|\mathcal{P}|} \mid 0 \leq \rho \leq \Lambda^T \varepsilon\right\} = \left\{\rho \in \mathbb{R}^{|\mathcal{P}|} \mid 0 \leq \rho_p \leq \varepsilon_k, \forall p \in \mathcal{P}_k, k \in \mathcal{K}\right\}.$$

Let us introduce a parametric problem to which we refer in the remainder of this paper as the lower-level problem:

$$q(\rho) \qquad \min_{(f,x)} z(\rho, f, x) = z_0(x) - \rho^T f \quad \text{s.t.} \quad (f,x) \in \mathcal{F}_0,$$

where $z_0(x) = \sum_{e \in E} \int_0^{x_e} l_e(\omega) d\omega$. Under Assumption 1, the objective function $z(\rho, f, x)$ is continuous in (ρ, f, x) and convex in (f, x) for fixed ρ. We observe that for $\rho = 0$, optimization problem $q(\rho)$ reduces to Beckmann's formulation (Beckmann et al., 1956) to find a PRUE flow distribution. So, for fixed ρ the program $q(\rho)$ can be solved, e.g., by the convex combination algorithm. Note that the (necessary and sufficient) Karush–Kuhn–Tucker (KKT) optimality conditions that correspond to $q(\rho)$ are equivalent to the nonlinear complementarity (NCP) formulation of the BRUE problem given in Di et al. (2013). This NCP formulation also applies if we relax Assumption 1 and allows non-separable cost functions.

We introduce some corresponding notations.

$\phi(\rho) = \min_{(f,x) \in \mathcal{F}_0} z(\rho, f, x)$;

$\psi(\rho) = \{(f, x) \mid (f, x) \text{ is a global minimizer of } q(\rho)\}$;
$\psi^x(\rho) = \{x \mid \text{there exists } f \text{ so that } (f, x) \in \psi(\rho)\}$.

The approach relies on the following fact, based on a proposition in Di et al. (2013).

Proposition 1.

$$(f, x) \in \mathcal{F}(\varepsilon) \Leftrightarrow \textit{ there is some } \rho \in E(\varepsilon) \textit{ with } (f, x) \in \psi(\rho).$$

Proof. The proof is based on the proof in Di et al. (2013). We consider the system of (necessary and sufficient) KKT optimality conditions that correspond to $q(\rho)$. We introduce Lagrange multiplier vector $(\beta, \lambda, \gamma) \in \mathbb{R}^{|E|} \times \mathbb{R}^{|\mathcal{K}|} \times \mathbb{R}^{|\mathcal{P}|}$. Any $(f, x) \in \mathcal{F}_0$ is a global optimal solution (i.e. $(f, x) \in \psi(\rho)$) if and only if (f, x) satisfies the following system with $\gamma \geq 0$:

$$\begin{array}{rr} l(x) - \beta = 0 & f^T\gamma = 0 \\ \Delta^T\beta - \gamma - \Lambda^T\lambda - \rho = 0 & (f, x) \in \mathcal{F}_0 \end{array} \tag{5}$$

We substitute $\beta_e = l_e(x_e)$ for all $e \in E$. The remainder of the proof is similar to the proof of Proposition 2.2 in Di et al. (2013). □

Proposition 1 allows us to equivalently model $\mathcal{Q}(\varepsilon)$ as a BL problem, with upper level variable ρ:

$$\mathcal{Q}(\varepsilon) \qquad \min_{(\rho, f, x)} s(x) \quad \text{s.t.} \quad \begin{array}{l} \rho \in E(\varepsilon); \\ (f, x) \in \psi(\rho). \end{array} \tag{6}$$

We notice the following properties:

- Based on Proposition 1, the feasible set $\mathcal{F}(\varepsilon)$, $\varepsilon \geq 0$, is equivalently defined as
$$\mathcal{F}(\varepsilon) = \bigcup_{\rho \in E(\varepsilon)} \psi(\rho); \tag{7}$$
- The feasible set that corresponds to the problem defined in (6) is defined on the space $\mathbb{R}^{|\mathcal{P}|} \times \mathbb{R}^{|\mathcal{P}|} \times \mathbb{R}^{|E|}$ while the feasible set $\mathcal{F}(\varepsilon)$ is defined on $\mathbb{R}^{|\mathcal{P}|} \times \mathbb{R}^{|E|}$;
- The mapping $\psi : E(\varepsilon) \to \mathbb{R}^{|\mathcal{P}|} \times \mathbb{R}^{|E|}$ is not injective, i.e. different $\rho^1 \neq \rho^2$ may have a common solution $(f^0, x^0) \in \psi(\rho^1) \cap \psi(\rho^2)$.

We discussed, for the purpose of parametric analysis, two reformulations for $\mathcal{Q}(\varepsilon)$ given by (4) and (6). Di et al. (2016) used the first approach to analyze $\mathcal{Q}(\varepsilon)$. We use formulation (6) in the remainder of this paper to study the behavior of problem $\mathcal{Q}(\varepsilon)$ in ε. (6) is a bilevel program. However, for fixed (upper-level parameter) ρ the lower-level problem $q(\rho)$ is a standard convex program and this convex structure eases the parametric analysis. In comparison, the lower-level problem $\mathcal{Q}_P(\varepsilon)$ in (4) has a non-convex feasible set for (upper-level) parameter P.

3.3. Solving the Best/Worst-case BRUE

We add a short discussion on the problem of computing the Best/Worst-case BRUE for a given indifference band and network configuration. We need a more efficient method to find BRUE traffic flows, as stressed in Sun et al. (2016). Both the branch approach (Section 3.1) and the BL approach (Section 3.2) can be used to solve $\mathcal{Q}(\varepsilon)$ for a fixed ε. Although a comprehensive analysis of algorithms that solve $\mathcal{Q}(\varepsilon)$ is beyond the purpose of this paper, we discuss both approaches with respect to finding the Best-case BRUE. We do not elaborate on the Worst-case BRUE, a program which has a convex objective function to maximize and, even in simple cases, may possess multiple local minimizers (Benson, 1995).

The difficulty for the branch approach in (4) lies in finding the subsets of paths $P \subseteq \mathcal{P}$ for which $\mathcal{F}_P(\varepsilon)$ is nonempty. Although only a finite number of subproblems can occur, still $2^{|\mathcal{P}|}$ choices for P exist. Di et al. (2013) proposed a sequence of mathematical programs with equilibrium constraints to find these path sets. In general networks, we could construct a path set by using heuristics or intuitive rules: for instance the k-shortest path set or the set of shortest paths in the user equilibrium (Eikenbroek et al., 2016). However, while solving the NDP, the branching process will be repeated for different settings under consideration. An advantage (of the branch approach) is that standard nonlinear programming algorithms apply to solve $\mathcal{Q}_P(\varepsilon)$.

The BL reformulation in (6) avoids the ‘combinatorial curse’ of (4). Recall that for a fixed $\rho \in E(\varepsilon)$, $q(\rho)$ allows us to easily calculate an arbitrary (i.e., not necessarily the best or worst) BRUE flow using, e.g., the convex combinations method. However, it is difficult to find $\rho \in E(\varepsilon)$ for which the x-part of $(f, x) \in \psi(\rho)$ minimizes $s(x)$.

An often efficient method to find a global minimizer of $\mathcal{Q}(\varepsilon)$ in a linear latency context in a finite number of iterations is a *branch-and-bound* scheme. This method applies to the branch approach in (4) as well as the BL model in (6). We show the details if we apply this method to the BL reformulation.

The scheme replaces the bilevel problem with a single-level reformulation of $\mathcal{Q}(\varepsilon)$ and therefore replaces $q(\rho)$ by its (necessary and sufficient) KKT optimality conditions, i.e.,

$$\min s(x) \quad \text{s.t.} \quad (\rho, f, x, \beta, \lambda, \gamma) \text{ satisfies } (5). \tag{8}$$

We cite Bard and Moore (1990) and Faigle et al. (2013) to elaborate on the branch-and-bound scheme. The branch-and-bound scheme suppresses the complementarity constraint $f^T\gamma = 0$ in (8) by $f_p = 0,\ p \in P,\ \gamma_p = 0,\ p \in R$, for $P, R \subseteq \mathcal{P}$. The resulting subproblem of (8) is a convex optimization problem.

This scheme starts by solving for the system optimum, i.e., the subproblem with $P = R = \emptyset$. For each subproblem, we *branch* on (one) $p \in \mathcal{P}$ for which $f_p\gamma_p \neq 0$: we generate two subproblems, one with $P = P \cup \{p\}, R = R$, and one with $P = P, R = R \cup \{p\}$. In this way, we construct a whole *tree* of subproblems of (8). If for a particular subproblem the complementarity constraint is satisfied, there is no need to explore the *sub-tree* that is rooted at this subproblem. If we obtain a solution of a subproblem that is more than the objective value of some known feasible solution of (8), we remove the sub-tree rooted at this subproblem. Hereby, we solve the combinatorial problem in (8).

The BL structure in (6) has not been applied in solving the BRUE problem. This structure shows similarities with the mathematical structure of the CNDP. Intuitively, algorithms designed for the CNDP possibly apply to $\mathcal{Q}(\varepsilon)$ as well. These methods use either differentiability of the lower-level solution or differentiability of the lower-level optimal value function with respect to the upper-level variable (ρ). In the remainder, we elaborate on these aspects in our context and thereby illustrate the difficulties in applying these algorithms to solve the Best/Worst-case BRUE for a given indifference band.

We emphasize that for both approaches we need to identify (a subset of) the path set $\mathcal{P}$ beforehand. Lou et al. (2010) overcome this issue and formulated the more restrictive link-based BRUE. They proposed an algorithm that solves a sequence of non-linear programs to find a local minimizer of the Best/Worst-case link-based BRUE.

4. Parametric analysis

We are mainly concerned with the impact of a varying ε on the optimal value function $v(\varepsilon)$ (see Sections 1 and 3). To this end, we need to study the continuity of the feasible set $\mathcal{F}(\varepsilon)$ in ε. In addition, we shortly discuss the continuity of the optimal solution set $\mathcal{S}(\varepsilon)$. The union of the solution sets $\psi(\rho)$ of the lower-level problem $q(\rho)$ serves as feasible set for the BL problem $\mathcal{Q}(\varepsilon)$. Therefore, to study the behavior of $\mathcal{Q}(\varepsilon)$ in dependence of ε, we need to study the continuity of lower-level problem $q(\rho)$ with respect to perturbations in ρ. We underline that the results of this section with respect to $v(\varepsilon)$, $\mathcal{F}(\varepsilon)$, and $\mathcal{S}(\varepsilon)$ are independent of the reformulation (see Sections 3.1 and 3.2). We solely chose the BL approach since it is more appropriate to apply parametric analysis.

4.1. Behavior of the lower-level problem

We study continuity of the feasible set, optimal value function and optimal solution set of the lower-level problem $q(\rho)$ with respect to perturbations in ρ. Lemma 2 is about existence and uniqueness of a solution of problem $q(\rho)$.

Lemma 2. *For any fixed $\rho \geq 0$:*

(i) *$\psi(\rho)$ is nonempty;*
(ii) *the x-part of a solution $(f, x) \in \psi(\rho)$ is uniquely determined, $x = x(\rho)$, as well as $w_\rho = \rho^T f$.*

Proof. The proof of (i) follows from Weierstrass' Theorem: $\mathcal{F}_0$ is a compact set and $z(\rho, f, x)$ is a continuous function. The proof of (ii) follows from the strict monotonicity of the latency function $l(x)$ with respect to x, see Smith (1979) for a proof. □

For any $\rho \geq 0$, the solution part $x(\rho)$ is a single-valued function with respect to ρ. This holds as well for w_ρ. We emphasize that there may exist several route flows f that correspond to the same x-part of a solution $(f, x) \in \psi(\rho)$.

Lemma 3. *The value function $\phi(\rho)$ is concave and (globally) Lipschitz continuous in ρ.*

Proof. Since $\mathcal{F}_0$ is compact set, with $L := \max_{(f,x)\in\mathcal{F}_0} \|f\|$ for all $\rho^1, \rho^2 \in E(\varepsilon)$ and $(f, x) \in \mathcal{F}_0$ the following holds:

$$\left|z(\rho^1, f, x) - z(\rho^2, f, x)\right| = \left|(\rho^1 - \rho^2)^T f\right| \leq L\|\rho^1 - \rho^2\|.$$

So with solution $(f^i, x^i) \in \psi(\rho^i) \subset \mathcal{F}_0,\ i = 1, 2,$ we find

$$\phi(\rho^1) - \phi(\rho^2) = z(\rho^1, f^1, x^1) - z(\rho^2, f^2, x^2) \leq z(\rho^1, f^2, x^2) - z(\rho^2, f^2, x^2) \leq L\|\rho^1 - \rho^2\|$$

Interchanging ρ^1 and ρ^2 yields the Lipschitz condition

$$\left|\phi(\rho^1) - \phi(\rho^2)\right| \leq L\|\rho^1 - \rho^2\|.$$

The concavity proof is trivial and therefore skipped. □

The behavior of optimal value function $\phi(\rho)$ in ρ is of interest for algorithms that solve $\mathcal{Q}(\varepsilon)$. For instance, to apply a penalty function approach as the augmented Lagrangian algorithm (Meng et al., 2001), the value function $\phi(\rho)$ should be continuously differentiable. The result of Lemma 3 is however solely a sufficient condition for $\phi(\rho)$ to be directionally differentiable in ρ along any $\rho' \in E(\varepsilon)$.

The upcoming lemma requires continuity of $\phi(\rho)$ to prove usc of $\psi(\rho)$.

Lemma 4. *The set-valued function $\psi(\rho)$ is closed and usc in ρ.*

Proof. ($\psi(\rho)$ is closed at ρ^0). Let be given sequences $\rho^l \to \rho^0$, $y^l \to y^0$, $l \in \mathbb{N}$, with $y^l := (f^l, x^l) \in \psi(\rho^l)$. We have to show that $y^0 \in \psi(\rho^0)$. Let us assume for the sake of contradiction that $y^0 \notin \psi(\rho^0)$. Then $z(\rho^0, y^0) > \phi(\rho^0)$ must hold. It follows that

$$\phi(\rho^l) = z(\rho^l, y^l) \to z(\rho^0, y^0) > \phi(\rho^0),$$

in contradiction to the continuity of $\phi(\rho)$.

($\psi(\rho)$ is usc at ρ^0). Assume that $\psi(\rho)$ is not usc at ρ^0. Then exists sequences $\rho^l \to \rho^0$, $y^l \in \psi(\rho^l)$ and some $\tau > 0$ so that

$$\|y^l - y^0\| \geq \tau \quad \text{for all } l, \text{ and all } y^0 \in \psi(\rho^0).$$

By compactness we have (for a subsequence) $y^l \to y^1$ and by closedness of $\psi(\rho)$ it follows that $y^1 \in \psi(\rho^0)$. Contradiction. □

Recall that by Lemma 2 the part $x = x(\rho)$ of a solution (f, x) of $q(\rho)$ is uniquely determined. So the projection $\psi^x(\rho)$ of $\psi(\rho)$ is a singleton, $\psi^x(\rho) = \{x(\rho)\}$. We now show that the function $x(\rho)$ (and thus $\psi^x(\rho)$) is continuous in ρ. The proof of Theorem 5 uses the fact that a projection mapping preserves compactness (see, e.g., Kuratowski, 1958).

Theorem 5. *The function $x(\rho)$ is continuous (usc and lsc) in ρ.*

Proof. Consider a sequence $\rho^l \to \rho^0$, $l \in \mathbb{N}$, and corresponding $x(\rho^l)$. For each ρ^l exists (possibly multiple) f^l such that $y^l = (f^l, x(\rho^l)) \in \psi(\rho^l)$ (Lemma 2). Since y^l are contained in a compact set, we can assume (by taking a subsequence) $y^l \to y^0$. By closedness of $\psi(\rho)$ at ρ^0 it follows that $y^0 = (f^0, x^0) \in \psi(\rho^0)$. By uniqueness with respect to x (Lemma 2) we get $x^0 = x(\rho^0)$. □

We proved that the x-part of $(f, x) \in \psi(\rho)$ behaves continuously in ρ. In fact, $x(\rho)$ is piecewise linear in ρ if we consider affine linear cost functions under Assumption 1. Algorithms for the CNDP under economic assumptions (e.g. Friesz et al., 1990) often use differentiability of the optimal solution of the lower-level problem. We emphasize that differentiability of $x(\rho)$ with respect to ρ does not follow from Theorem 5.

Although we showed that $x(\rho)$ is continuous in ρ, that does not imply that the f-part behaves continuously in ρ as well. Let us denote the route flow set corresponding to an optimal link-flow solution $x(\rho)$ as

$$H_0(\rho) = \left\{ \ f \in \mathbb{R}^{|\mathcal{P}|} \ \middle| \ \Lambda f = d, \Delta f = x(\rho), f \geq 0, \rho^T f = w_\rho \ \right\}. \tag{9}$$

Note that in general $H_0(\rho)$ (for fixed ρ) need not be a singleton. There may exist a whole set of route flows $f \in H_0(\rho)$ that correspond to the single link flow distribution $x(\rho)$. This is similar to the fact that also for a PRUE flow (f^{PRUE}, x^{PRUE}), the f-part does not need to be uniquely determined (see Bar-Gera, 2006, Borchers et al., 2015 and Lu and Nie, 2010).

Even worse, the set $H_0(\rho)$ and thus the solution set $\psi(\rho)$ is usc but does not need to be lsc in general. Solely under the strong and unnatural condition that the matrix

$$\begin{bmatrix} \Lambda \\ \Delta \\ \rho^T \end{bmatrix}$$

has full rank, for ρ near ρ^0, the sets $\psi(\rho)$ are singletons and thus $\psi(\rho)$ is lsc at ρ^0. More precisely, by considering the definition of $H_0(\rho)$ in (9) this set may not be lsc at a point ρ^0 where the following relation holds:

$$\operatorname{rank}\begin{bmatrix} \Lambda \\ \Delta \\ (\rho^0)^T \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \Lambda \\ \Delta \end{bmatrix}.$$

In other words, although $x(\rho)$ is continuous in ρ, a gradual change in ρ may lead to a drastic change in the corresponding route flows.

4.2. Behavior of the bilevel problem

We use formulation (6) of problem $\mathcal{Q}(\varepsilon)$ to evaluate its behavior with respect to ε. It is natural to consider the projection of the feasible set corresponding to $\mathcal{Q}(\varepsilon)$ onto the (ρ, x)-space since the objective function $s(x)$ solely depends on the link flow and the optimal link flow $x(\rho)$ is continuous in ρ (see Theorem 5). Hence, we reformulate (6) as

$$\mathcal{Q}(\varepsilon) \qquad \min_{(\rho,x)} s(x) \quad \text{s.t.} \qquad (\rho, x) \in \mathcal{F}^{\rho,x}(\varepsilon) := \left\{ \ (\rho, x) \ \middle| \ \begin{array}{l} \rho \in E(\varepsilon) \\ x = x(\rho) \end{array} \right\}.$$

We study existence of an optimal solution, i.e., is the solution set $\mathcal{S}^{\rho,x}(\varepsilon)$ of $\mathcal{Q}(\varepsilon)$ nonempty for any $\varepsilon \geq 0$? We assure existence of a Best-case BRUE flow distribution under the same mild assumptions (Beckmann et al., 1956) used to assure existence of a PRUE flow distribution.

Before we continue, we emphasize that for $\varepsilon^1 \geq \varepsilon^0 \geq 0$ we have $\mathcal{F}(\varepsilon^0) \subseteq \mathcal{F}(\varepsilon^1)$, since for any $\rho^0 \in E(\varepsilon^0) \Rightarrow \rho^0 \in E(\varepsilon^1)$. Thus $(\rho^0, x^0) \in \mathcal{F}^{\rho,x}(\varepsilon^0) \Rightarrow (\rho^0, x^0) \in \mathcal{F}^{\rho,x}(\varepsilon^1)$ and we conclude

$$v(\varepsilon^0) \geq v(\varepsilon^1), \quad \text{for any } \varepsilon^1, \varepsilon^0 : \varepsilon^1 \geq \varepsilon^0.$$

It directly follows that the Best-case BRUE optimal value function $v(\varepsilon)$ for any $\varepsilon \geq 0$ is a lower bound on the PRUE total travel time $v(0)$. We underline that the total travel time for the system-optimal flow distribution is a lower bound on the Best-case BRUE travel time.

Lemma 6. *For any $\varepsilon \geq 0$, the minimum objective value $v(\varepsilon)$ of $\mathcal{Q}(\varepsilon)$ is attained, i.e., there exists $(\rho, x(\rho)) \in \mathcal{F}^{\rho,x}(\varepsilon)$ such that $v(\varepsilon) = s(x(\rho))$.*

Proof. Recall that the function $x(\rho)$ is continuous on $E(\varepsilon)$. So, the program $\mathcal{Q}(\varepsilon)$ can be written as

$$\min_{\rho} s(x(\rho)) \quad \text{s.t} \quad \rho \in E(\varepsilon)$$

and by the Weierstrass Theorem, the continuous function $s(x(\rho))$ attains its minimum on the compact set $E(\varepsilon)$. □

Lemma 7. *The set-valued function $\mathcal{F}^{\rho,x}(\varepsilon)$ is a closed and continuous mapping at all $\varepsilon \geq 0$.*

Proof. ($\mathcal{F}^{\rho,x}(\varepsilon)$ is closed and usc at $\varepsilon^0 \geq 0$). To show closedness at ε^0 we consider sequences $\varepsilon^l \to \varepsilon^0$, and $(\rho^l, x(\rho^l)) \in \mathcal{F}^{\rho,x}(\varepsilon^l)$ satisfying $(\rho^l, x(\rho^l)) \to (\rho^0, x^0)$. By continuity of the mapping $E(\varepsilon)$ we must have $\rho^0 \in E(\varepsilon^0)$ and by continuity of $x(\rho)$ we find

$$x^0 = \lim_{l \to \infty} x(\rho^l) = x(\rho^0)$$

and thus $(\rho^0, x(\rho^0)) \in \mathcal{F}^{\rho,x}(\varepsilon^0)$. Using the closedness we then can prove that $\mathcal{F}^{\rho,x}(\varepsilon)$ is usc, by following the arguments in the proof of Lemma 4.

($\mathcal{F}^{\rho,x}(\varepsilon)$ is lsc at $\varepsilon^0 \geq 0$). Suppose now that $\mathcal{F}^{\rho,x}(\varepsilon)$ is not lsc at $\varepsilon^0 \geq 0$. Then there exist $(\rho^0, x(\rho^0)) \in \mathcal{F}^{\rho,x}(\varepsilon^0)$, a sequence $\varepsilon^l \to \varepsilon^0$ and some $\tau > 0$ such that

$$\|(\rho^l, x(\rho^l)) - (\rho^0, x(\rho^0))\| \geq \tau, \qquad \text{for any sequence } (\rho^l, x(\rho^l)) \in \mathcal{F}^{\rho,x}(\varepsilon^l).$$

However, by continuity of $E(\varepsilon)$ for each ρ^0, ε^0, $\rho^0 \in E(\varepsilon^0)$ and $\varepsilon^l \to \varepsilon^0$ there exists a sequence $\rho^l \in E(\varepsilon^l)$ such that $\rho^l \to \rho^0$ (take ρ^l with $\rho^l_p = \min\{\rho^0_p, \Lambda^T \varepsilon^l\}$, $p \in \mathcal{P}_k$). By continuity of $x(\rho)$ it follows that $x(\rho^l) \to x(\rho^0)$, a contradiction. □

Continuity of the feasible set $\mathcal{F}^{\rho,x}(\varepsilon)$ highly depends on Assumption 1. Otherwise, we can only guarantee $\mathcal{F}^{\rho,x}(\varepsilon)$ to be usc. Continuity of $\mathcal{F}^{\rho,x}(\varepsilon)$ in ε turns out to be sufficient for continuity of the optimal value function $v(\varepsilon)$ of $\mathcal{Q}(\varepsilon)$ with respect to ε.

Theorem 8 is a main result of our paper. The optimal value function $v(\varepsilon)$ is continuous in ε, which means that a slight change in ε does not lead to a drastic change in the performance of the Best/Worst-case BRUE. From a designer's perspective, the range of performances in real life for given network settings does not substantially differ from the modeled range of performances for given settings. This result is relevant since the indifference band is difficult to estimate.

Theorem 8. *The value function $v(\varepsilon)$ is continuous (lsc and usc) in ε.*

Proof. ($v(\varepsilon)$ is lsc at $\varepsilon^0 \geq 0$). Given $\varepsilon^0 \geq 0$ and sequence $\varepsilon^l \to \varepsilon^0$, take a corresponding $(\rho^l, x(\rho^l)) \in \mathcal{F}^{\rho,x}(\varepsilon^l)$. We can assume for a subsequence that $(\rho^l, x(\rho^l)) \to (\rho^0, x(\rho^0))$. By closedness of the mapping $\mathcal{F}^{\rho,x}(\varepsilon)$ at all $\varepsilon \geq 0$ it follows that $(\rho^0, x^0) \in \mathcal{F}^{\rho,x}(\varepsilon^0)$ and thus

$$v(\varepsilon^0) \leq s(x^0) = \lim_{l \to \infty} s(x(\rho^l)) = \lim_{l \to \infty} v(\varepsilon^l).$$

($v(\varepsilon)$ is usc at $\varepsilon^0 \geq 0$). Since $\mathcal{F}^{\rho,x}(\varepsilon)$ is continuous, for any $(\rho^0, x^0) \in \mathcal{F}^{\rho,x}(\varepsilon^0)$ and sequence $\varepsilon^l \to \varepsilon^0$ exists a sequence $(\rho^l, x^l) \to (\rho^0, x^0)$ with $(\rho^l, x^l) \in \mathcal{F}^{\rho,x}(\varepsilon^l)$. Take (ρ^0, x^0) so that $s(x^0) = v(\varepsilon^0)$. We have for the mentioned sequence that for any $\tau > 0$ exists $L \in \mathbb{N}$ such that

$$v(\varepsilon^l) \leq s(x^l) \leq s(x^0) + \tau = v(\varepsilon^0) + \tau, \quad \text{for all } l \geq L.$$

So, $v(\varepsilon)$ is usc and, hence, continuous in ε. □

Since multiple minimizers may exist that yield the same optimal value, we can generally not guarantee continuity of the optimal solution set $\mathcal{S}(\varepsilon)$ and $\mathcal{S}^x(\varepsilon)$. However, we can guarantee that $\mathcal{S}^x(\varepsilon)$ (and $\mathcal{S}(\varepsilon)$) is usc in ε. So, small perturbations $\varepsilon \approx \varepsilon^0$ of ϵ^0 may only lead to an implosion of optimal link flows (i.e., the set $\mathcal{S}^x(\varepsilon)$ may become substantially smaller compared to $\mathcal{S}^x(\varepsilon^0)$).

Lemma 9. *The set-valued function $\mathcal{S}^x(\varepsilon)$ is usc in ε.*

Proof. We easily find that $\mathcal{S}^x(\varepsilon)$ is a closed mapping. Let us assume to the contrary that $\mathcal{S}^x(\varepsilon)$ is not usc at ε^0. So there exists a sequence $\varepsilon^l \to \varepsilon^0$, $x^l \in \mathcal{S}^x(\varepsilon)$, and $x^l \to x^0$ with $x^0 \notin \mathcal{S}^x(\varepsilon^0)$. Since $x^0 \notin \mathcal{S}^x(\varepsilon^0)$ we have that $s(x^0) > v(\varepsilon^0)$. So,

$$\lim_{l \to \infty} v(\varepsilon^l) = \lim_{l \to \infty} s(x^l) = s(x^0) > v(\varepsilon^0).$$

This contradicts the continuity of $v(\varepsilon)$ at ε^0. □

Remark 1 (Worst-case BRUE)**.** The previous analysis and results also apply to the Worst-case BRUE problem

$$\max_{(\rho,x)} s(x) \quad \text{s.t.} \quad \begin{array}{l} \rho \in E(\varepsilon), \\ (f,x) \in \psi(\rho). \end{array} \tag{10}$$

This problem maximizes a convex objective function. Notice that the lower-level problem remains a convex optimization problem for each $\rho \in E(\varepsilon)$, i.e., $\psi(\rho)$ is a convex set for each ρ. It follows directly that, for the Worst-case BRUE problem (10), the function $x(\rho)$ is continuous in ρ (Theorem 5).

We can write problem (10) equivalently as

$$\max_{\rho} s(x(\rho)) \quad \text{s.t.} \quad \rho \in E(\varepsilon). \tag{11}$$

By the Weierstrass Theorem, the continuous function $s(x(\rho))$ attains its maximum on the compact set $E(\varepsilon)$ (Lemma 6). Furthermore, Lemma 7 concerns the feasible set, is thus independent of the objective function and, hence, the projection of the feasible set (that corresponds to (10)) onto the (ρ, x)-space is continuous in ε. The optimal value function of (10) is continuous in ε (Theorem 8 solely depends on the continuity of the objective function). The results from Lemma 9 and Theorem 10 also apply to the optimal solution set and feasible set, respectively, of problem (10).

This section provided a main result of our study. We proved that the optimal value function is continuous in ε. Hence, the range of possible performances for given network settings in real life does not substantially differ from the modeled range of performances defined by the Best/Worst-case BRUE.

In the following, we shortly discuss connectedness of $\mathcal{F}(\varepsilon)$ for an arbitrary ε. Although not directly related to our study, connectedness is of interest to study BRUE flows in a day-to-day context (see Di et al., 2015). For general latency functions, Han et al. (2015) showed that under some extra assumptions $\mathcal{F}(\varepsilon)$ consists of a union of connected sets. But this does not mean that $\mathcal{F}(\varepsilon)$ is connected. Di et al. (2015) claimed that $\mathcal{F}(\varepsilon)$ is connected in case the latency functions in the network are affine linear. We now give a proof for general latency functions satisfying Assumption 1.

Theorem 10. *$\mathcal{F}(\varepsilon)$ is a connected set for each $\varepsilon \geq 0$.*

Proof. We use arguments taken from Zgurovsky et al. (2010). Because $E(\varepsilon)$ is a compact and convex set for all $\varepsilon \geq 0$, it is also connected for each $\varepsilon \geq 0$. Recall $\mathcal{F}(\varepsilon) = \cup_{\rho \in E(\varepsilon)} \psi(\rho)$ (see (7)) and suppose, for the sake of contradiction, that $\mathcal{F}(\varepsilon)$ is not connected. Under this assumption, there are two relatively open sets A, B so that $\mathcal{F}(\varepsilon) = A \cup B$ and $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ ($\bar{A}$ ($\bar{B}$) denotes the complement of A (B)). Denote by $\rho(A)$, $\rho(B)$ the following projections:

$$\rho(A) = \{\rho \in E(\varepsilon) \mid \psi(\rho) \subseteq A\};$$
$$\rho(B) = \{\rho \in E(\varepsilon) \mid \psi(\rho) \subseteq B\}.$$

We may assume that both $\rho(A)$ and $\rho(B)$ are nonempty. Furthermore, $\rho(A) \cap \rho(B) = \emptyset$, otherwise exists $\rho^0 \in \rho(A) \cap \rho(B)$ and $\psi(\rho^0) \in A \cap B$, contradicting $\bar{A} \cap B = \emptyset$. So $\rho(A) \cup \rho(B) = E(\varepsilon)$.

By usc of $\psi(\rho)$ at a $\rho^0 \in E(\varepsilon)$, for each $\tau_{\rho^0} > 0$ there exists $\delta_{\rho^0} > 0$ so that

$$\psi(\rho) \subseteq U_{\tau_{\rho^0}}(\psi(\rho^0)), \quad \text{for all } \rho \in U_{\delta_{\rho^0}}(\rho^0).$$

For any $\rho \in \rho(A)$ take τ_ρ and corresponding δ_ρ so that $U_{\tau_\rho}(\psi(\rho)) \subseteq A$. We have that

$$\bigcup_{\rho \in \rho(A)} U_{\delta_\rho}(\rho) \cap E(\varepsilon) = \rho(A).$$

The union of the open balls $\cup_{\rho \in \rho(A)} U_{\delta_\rho}(\rho)$ is open. So we find that $\rho(A)$ is relatively open in $E(\varepsilon)$. Similarly, $\rho(B)$ is relatively open in $E(\varepsilon)$. Then, $E(\varepsilon)$ can be partitioned into two relatively open sets in $E(\varepsilon)$. This contradicts the connectedness of $E(\varepsilon)$. □

5. Second-best toll pricing

In previous sections, we considered the Best/Worst-case BRUE for a given network configuration. In what follows, we consider the context in which we determine the optimal network setting with respect to the Best/Worst-case BRUE (i.e., NDP with boundedly rational travelers).

We discuss the toll pricing problem within the concept of bounded rationality which is a special case of the NDP. The Best/Worst-case BRUE serves here as a subproblem. Similarly, the PRUE was used as a subproblem for the NDP under economic assumptions (Friesz et al. (1990); Josefsson and Patriksson (2007); Yang (1997)). In this section, by applying the results above, we are able to sharpen the linear latency results by Di et al. (2016) on second-best toll pricing in the BRUE context. We use the continuity of the optimal value function to prove existence of a toll setting scheme which minimizes the Best/Worst-case BRUE for a fixed ε under general cost functions that satisfy Assumption 1.

We consider the conventional toll pricing problem under the assumption of perfectly rational behavior in route choice (see e.g. Yang and Huang, 2005). The optimal toll vector $\alpha \in \mathbb{R}_+^{|E|}$ is a solution of the BL problem

$$\min_{\alpha \in C} s(x) \quad \text{s.t.} \quad (f, x) \in \mathcal{F}_0 \text{ satisfies (1) w.r.t. } l_e(x_e, \alpha_e) := l_e(x_e) + \alpha_e, \tag{12}$$

where C is the set of allowed toll vectors:

$$C := \left\{ \alpha \in \mathbb{R}_+^{|E|} \;\middle|\; 0 \le \alpha \le \alpha^{\max} \right\}.$$

Here we consider the models under Assumption 1 and we assume that $l(x)$ is differentiable. The simplest case occurs when $\alpha^{\max} = \infty$. Then a comparison of the KKT conditions of Beckmann's problem to compute the PRUE (f^{PRUE}, x^{PRUE}) and the KKT conditions of the system optimum (f^{so}, x^{so}), leads to the formula $\alpha_e^{so} = x_e^{so}(\partial l_e(x_e^{so})/\partial x_e)$, for all $e \in E$, as the first-best toll. This means that for $\alpha = \alpha^{so}$ the PRUE with respect to $l_e(x_e, \alpha_e^{so})$ coincides with the system optimum with respect to $l_e(x_e)$. Under additional constraints on α_e (e.g., some of the links cannot be tolled) the toll pricing problem in (12) becomes a BL problem. The solution α^* of (12) is then called the second-best toll.

The problem in (12) is relatively easy. Any toll α leads to a unique PRUE and the effect of this measure is thus uniquely determined. Under bounded rationality, however, the response to an intervention is subject to uncertainty. Similar as in previous sections, our toll pricing scheme requires an assumption on the realized BRUE flow. Again, we consider the best and worst-case with respect to the total travel time, since we have no additional information about the BRUE that realizes in practice.

Di et al. (2016) studied the pricing problem for the BRUE and considered two design strategies of an authority under uncertainty:

(RP) : A toll pricing scheme is said to be *risk-prone* (RP) if we assume that for a given toll vector α the Best-case BRUE with respect to a given ε is realized among all allowable BRUE flows;

(RA) : A toll pricing scheme is said to be *risk-averse* (RA) if we assume that for a given toll vector α the Worst-case BRUE with respect to a given ε is realized among all allowable BRUE flows;

The corresponding RP/RA tolling problem for indifference band ε are given by

$$\text{RP}(\varepsilon) \qquad \min_{\alpha \in C} \min_{(f,x)} s(x) \quad \text{s.t.} \quad (f, x) \in \mathcal{F}(\varepsilon, \alpha);$$

$$\text{RA}(\varepsilon) \qquad \min_{\alpha \in C} \max_{(f,x)} s(x) \quad \text{s.t.} \quad (f, x) \in \mathcal{F}(\varepsilon, \alpha).$$

$\mathcal{F}(\varepsilon, \alpha)$ is the set of feasible BRUE flow distribution under toll α for a given ε:

$$\mathcal{F}(\varepsilon, \alpha) = \left\{ (f, x) \in \mathbb{R}^{|\mathcal{P}|} \times \mathbb{R}^{|E|} \;\middle|\; \begin{array}{l} (f, x) \in \mathcal{F}_0 \\ (f, x) \text{ satisfies (2) w.r.t. } l_e(x_e, \alpha_e) \end{array} \right\}.$$

We introduce the following value function with respect to RP(ε)

$$v(\varepsilon, \alpha) = \min_{(f,x) \in \mathcal{F}(\varepsilon,\alpha)} s(x).$$

Clearly, in the case that there are no restrictions on α, i.e. $\alpha^{\max} = \infty$, the first-best toll α^{so} leads to system-optimal flow (f^{so}, x^{so}) as solution of RP(ε), for any $\varepsilon \ge 0$. This is, of course, not true for RA(ε). We emphasize that other optimal toll vectors for RP(ε) may exist.

We now consider the Best-case BRUE under (second-best) toll α as a BL model. Let us introduce a parametric lower-level problem

$$q(\rho, \alpha) \qquad \min_{(f,x)} z(\rho, \alpha, f, x) = z_0(x) - \rho^T f + \alpha^T x \quad \text{s.t. } (f, x) \in \mathcal{F}_0.$$

The KKT optimality conditions for $q(\rho, \alpha)$ reveal that a flow (f, x) is a BRUE flow with respect to $l_e(x_e, \alpha_e) = l_e(x_e) + \alpha_e$ if and only if there exists $\rho \in E(\varepsilon)$ so that (f, x) is a solution to $q(\rho, \alpha)$. We introduce notation that correspond to problem $q(\rho, \alpha)$:

$$\phi(\rho, \alpha) = \min_{(f,x) \in \mathcal{F}_0} z(\rho, \alpha, f, x);$$
$$\psi(\rho, \alpha) = \{(f, x) \mid (f, x) \text{ is a global minimizer of } q(\rho, \alpha)\};$$
$$\psi^x(\rho, \alpha) = \{x \mid \text{there exists } f \text{ so that } (f, x) \in \psi(\rho, \alpha)\}.$$

We consider the behavior of these functions in dependence of toll vector (parameter) α. A similar analysis as in previous sections shows the following properties:

- For fixed ρ, α, $\psi(\rho, \alpha)$ is nonempty and the link part $x = x(\rho, \alpha)$ of a solution $(f, x) \in \psi(\rho, \alpha)$ is uniquely determined (Lemma 2);
- The value function $\phi(\rho, \alpha)$ is concave and Lipschitz continuous in (ρ, α) (Lemma 3);
- The set-valued function $\psi(\rho, \alpha)$ is closed and usc in (ρ, α) (Lemma 4);

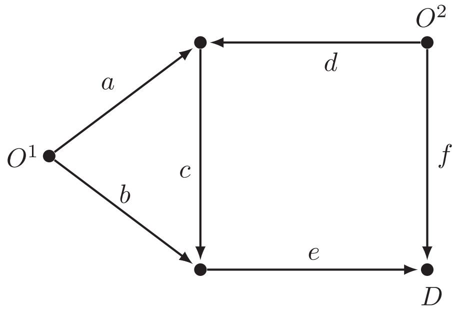

Fig. 1. Traffic network.

- The function $x(\rho, \alpha)$ is continuous in (ρ, α) (Theorem 5);
- $\mathcal{F}^{\rho,x}(\varepsilon, \alpha)$ (the projection of $\mathcal{F}(\varepsilon, \alpha)$ onto the (ρ, x)-space) is a continuous mapping for all $\varepsilon \geq 0$, $\alpha \in C$ (Lemma 7).

The latter result is stronger than Proposition 3.5 in Di et al. (2016), which shows upper semicontinuity (under linear latencies) of the mapping $\mathcal{F}(\varepsilon, \alpha)$ in α. Here, we only assume that the latency functions $l_e(x_e)$ satisfy the condition in Assumption 1. Lemma 11 says that $v(\varepsilon, \alpha)$ is continuous in ε and α. Hence, if we apply a tolling scheme which is close to α (e.g. due to rounding off) the range of performances in real life does not significantly differ from the modeled range of performances. This result holds for both the RP and RA setting. This result is stronger than Proposition 5.5 in Di et al. (2016), which shows upper semicontinuity of $v(\varepsilon, \alpha)$ in α for the Worst-case BRUE under linear latencies.

Lemma 11. *The value function $v(\varepsilon, \alpha)$ is continuous in (ε, α).*

Proof. See the proof corresponding to Theorem 8. □

We find the main result of this section, namely that there exists a RP (and RA) toll vector $\alpha \in C$ which is optimal for a given indifference band ε. Moreover, since $v(\varepsilon, \alpha)$ is continuous, this RP (RA) toll is also likely to be optimal from a designer's perspective if ε is slightly perturbed. Under Assumption 1, Theorem 12 is stronger than the result from Section 5 in Di et al. (2016), which proves existence of such a toll vector for the RP setting with linear latencies.

Theorem 12. *For any $\varepsilon \geq 0$, there exists a toll vector $\alpha \in C$ so that the minimum objective value of problem RP(ε) (RA(ε)) is attained.*

Proof. To prove the theorem we only have to observe that for fixed $\varepsilon \geq 0$ the optimal toll vector α_ε with respect to RP(ε) is given as solution of the program

$$\min_{\alpha \in C} v(\varepsilon, \alpha).$$

This solution exists as a minimizer of a continuous function on the compact set C according to Weierstrass. □

Again, we emphasize that we used the BL structure in this section for the sake of parametric analysis. In fact, the results with respect to $\mathcal{F}(\varepsilon, \alpha)$ and $v(\varepsilon, \alpha)$ are independent of reformulation (4) or (6).

This section showed that there exists a toll vector which is optimal under the BRUE conditions for strategies RP and RA. We emphasize that although a globally optimal toll vector for both strategies RP/RA exist, algorithms are only likely to find a local optimum (*cf.* Di et al., 2016). Algorithms were presented in Di et al. (2016) and Lou et al. (2010). We underline that the considered toll setting is optimal from a designer's perspective (i.e. with respect to the Best/Worst-case BRUE). But if any other BRUE (different from the Best/Worst-case BRUE) realizes after the toll setting, a different toll might have been optimal. For a recent study on the robust NDP with congestion-free links under bounded rationality we refer to Sun et al. (2017).

6. Illustrative example

We showed that the feasible set and optimal value function of problem $\mathcal{Q}(\varepsilon)$ are continuous in ε, but that the optimal solution set is not necessarily continuous. In this section, we illustrate that the mentioned properties are natural, i.e. we give two simple examples of the Best-case BRUE in which the optimal value function $v(\varepsilon)$ is continuous in ε but that the optimal solution set $\mathcal{S}(\varepsilon)$ is not. In addition, we show that $\mathcal{F}(\varepsilon)$ is not necessarily convex for fixed ε.

Since for small networks we can easily enumerate all possible paths in the network of Fig. 1, we use the branch approach (4) to solve $\mathcal{Q}(\varepsilon)$ for a fixed ε (See Section 3.3).

6.1. Example with affine linear cost functions

Fig. 1 shows the network we consider. There are 6 links in the network with cost vector function $l(x) = x + 1$. There are two OD pairs, namely (O^1, D) and (O^2, D). Demand for the commodities is 5 and 8 respectively. OD pair (O^1, D) is connected

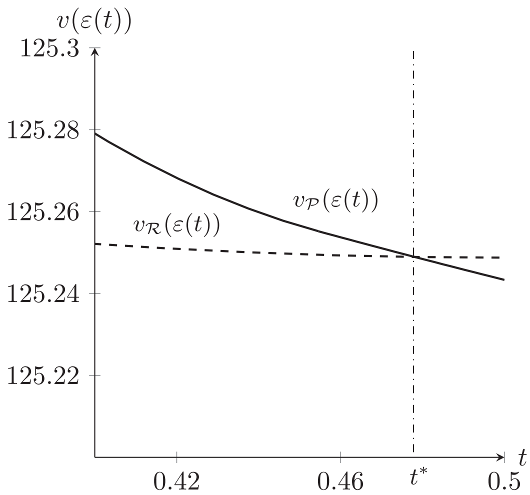

Fig. 2. Best-case BRUE optimal value function corresponding to the network in Fig. 1 with affine linear link cost functions.

by the paths

$$p_1^1 = \{a, c, e\}, \qquad \text{and} \qquad p_2^1 = \{b, e\},$$

while the paths p_1^2, p_2^2 connect OD pair (O^2, D):

$$p_1^2 = \{d, c, e\}, \qquad \text{and} \qquad p_2^2 = \{f\}.$$

We solve the Best-case BRUE using the Branch approach (see Section 3.1) and choose the two relevant path sets, namely

$$\mathcal{R} = \{p_1^1, p_2^1, p_2^2\}, \qquad \text{and} \qquad \mathcal{P} = \{p_1^1, p_2^1, p_1^2, p_2^2\}. \tag{13}$$

Note that the solution of $\mathcal{Q}_{\mathcal{R}}(0)$ is the PRUE flow distribution (f^{PRUE}, x^{PRUE}), while the solution of $\mathcal{Q}_{\mathcal{P}}(\varepsilon^{so})$, ε^{so} sufficiently large, is the system-optimal flow (f^{so}, x^{so}).

We define $\varepsilon(t) := t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $t \in [0, 1/2]$ (for the sake of clarity).

$\mathcal{Q}_{\mathcal{R}}(\varepsilon(t))$ is solvable for $t \in [0, 1/2]$ and $\mathcal{Q}_{\mathcal{P}}(\varepsilon(t))$ has a solution for $t \in [1/4, 1/2]$, i.e., $\mathcal{F}_{\mathcal{P}}(\varepsilon(t)) = \emptyset$ for $0 \le t < 1/4$. We solve both (strictly convex) quadratic programs as function of t (see Bank et al., 1983) and find the (unique for each t) optimal solution vectors $x_{\mathcal{R}}(\varepsilon(t))$, $x_{\mathcal{P}}(\varepsilon(t))$ respectively, i.e.

$$x_{\mathcal{R}}(\varepsilon(t)) = \left(\frac{4+t}{3}, \frac{11-t}{3}, \frac{4+t}{3}, 0, 5, 8\right), \qquad \text{for } t \in [0, \frac{1}{2}],$$

$$x_{\mathcal{P}}(\varepsilon(t)) = \begin{cases} (1+t, 4-t, 1+t, 0, 5, 8) & \text{if } t \in [\frac{1}{4}, \frac{5}{11}]; \\ \left(\frac{16}{11}, \frac{39}{11}, \frac{59+11t}{44}, \frac{11t-5}{44}, \frac{215+11t}{44}, \frac{357-11t}{44}\right) & \text{if } t \in [\frac{5}{11}, \frac{1}{2}]. \end{cases}$$

The corresponding value functions are

$$v_{\mathcal{R}}(\varepsilon(t)) = \frac{1}{3}(t^2 - t + 376), \qquad \text{for } t \in \left[0, \frac{1}{2}\right],$$

$$v_{\mathcal{P}}(\varepsilon(t)) = \begin{cases} 3t^2 - 3t + 126 & \text{if } t \in [\frac{1}{4}, \frac{5}{11}]; \\ \frac{t^2}{4} - \frac{t}{2} + \frac{5519}{44} & \text{if } t \in [\frac{5}{11}, \frac{1}{2}]. \end{cases}$$

It directly follows that the optimal value function behaves continuously in $t \in [0, 1/2]$. We find

$$v(\varepsilon(t)) = \begin{cases} v_{\mathcal{R}}(\varepsilon(t)) & \text{if } t \in [0, 2\sqrt{\frac{6}{11}} - 1]; \\ v_{\mathcal{P}}(\varepsilon(t)) & \text{if } t \in [2\sqrt{\frac{6}{11}} - 1, \frac{1}{2}]. \end{cases}$$

Fig. 2 illustrates the continuous behavior of $v(\varepsilon(t))$, $t \in [2/5, 1/2]$. In other words, a gradual change in ε for a given measure has only small impact on the total travel time in the Best-case BRUE.

To the contrary, the optimal solution set is not continuous in ε. For edge a we find the following optimal solution as function of t:

$$x_a(\varepsilon(t)) = \begin{cases} \frac{4+t}{3} & \text{if } t \in [0, 2\sqrt{\frac{6}{11}} - 1); \\ \{\frac{4+t}{3}, \frac{16}{11}\} & \text{if } t = 2\sqrt{\frac{6}{11}} - 1; \\ \frac{16}{11} & \text{if } t \in (2\sqrt{\frac{6}{11}} - 1, \frac{1}{2}]. \end{cases}$$

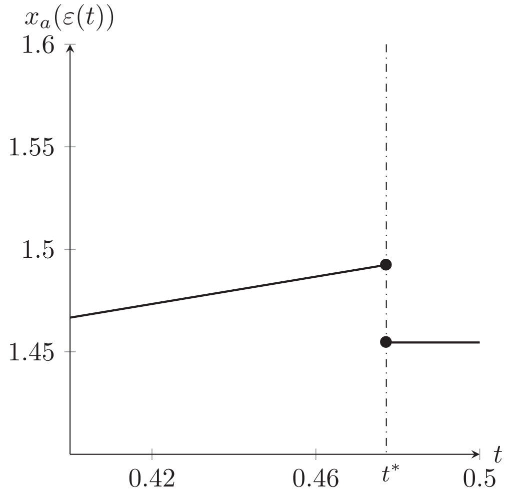

Fig. 3. Best-case BRUE optimal solution for link a as function of t corresponding to the network in Fig. 1 with affine linear link cost functions.

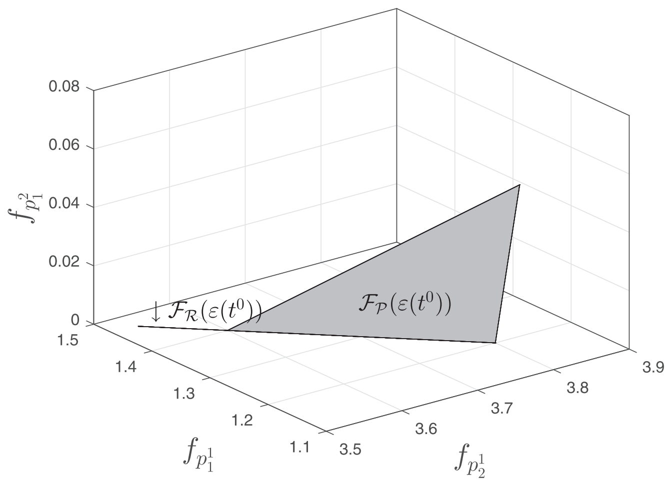

Fig. 4. Non-convex feasible set $\mathcal{F}^f(\varepsilon(t^0))$ for $t^0 = 0.4$ corresponding to the network in Fig. 1 with affine linear link cost functions.

Fig. 3 illustrates that $\mathcal{S}^x(\varepsilon(t))$ is usc but not lsc at $t^* = 2\sqrt{6/11} - 1$. Indeed, for any sequence $t^l \to t^*$ the sequence $\mathcal{S}^x(\varepsilon(t^l))$ does not converge to all possible solutions in $\mathcal{S}^x(\varepsilon(t^*))$.

We now show that for fixed $\varepsilon(t^0)$ the feasible set $\mathcal{F}(\varepsilon(t^0))$ is not convex (see also Section 3). Fig. 4 shows the feasible set $\mathcal{F}^f(\varepsilon(t^0))$ (the projection of $\mathcal{F}(\varepsilon(t^0)) = \mathcal{F}_{\mathcal{P}}(\varepsilon(t^0)) \cup \mathcal{F}_{\mathcal{R}}(\varepsilon(t^0))$ onto the f-space) for $t^0 = 0.4$. Obviously, we cannot draw a line between any two feasible points and remain feasible.

The example showed that even in a simple network with linear latencies the Best-case BRUE optimal solution set is not necessarily continuous in ε.

6.2. Example with nonlinear cost functions

We present a numerical example for the Best-case BRUE with nonlinear (quadratic) cost functions under Assumption 1. We consider the same network (Fig. 1) and demand vector as in previous example (Section 6.1). The cost vector function is $l(x) = \frac{1}{2}x^2 + b$, where

$$b = (1, 1, 1, 20, 1, 1).$$

We applied the branch approach (Section 3.1) to all possible subsets of the path set. In this example, $\mathcal{R}$ and $\mathcal{P}$ (see (13)) are again the two relevant path sets. In fact, subproblem $\mathcal{Q}_{\mathcal{R}}(\varepsilon(t))$ is equivalent to solving $\mathcal{Q}(\varepsilon(t))$ for $t \in [0, 3.34]$, while subproblem $\mathcal{Q}_{\mathcal{P}}(\varepsilon(t))$ finds the minimizer of $\mathcal{Q}(\varepsilon(t))$ for $t \in [3.34, 5]$ (note that these intervals are evaluated numerically).

We see similar behavior as in our previous example. Fig. 5 shows that the optimal value function $v(\varepsilon(t))$ is continuous with respect to $t \in [0, 5]$. On the other hand, Fig. 6 shows that the Best-case BRUE optimal solution $x_f(\varepsilon(t))$ for link f is not continuous everywhere on $t \in [0, 5]$. For this example, we used the MATLAB environment.

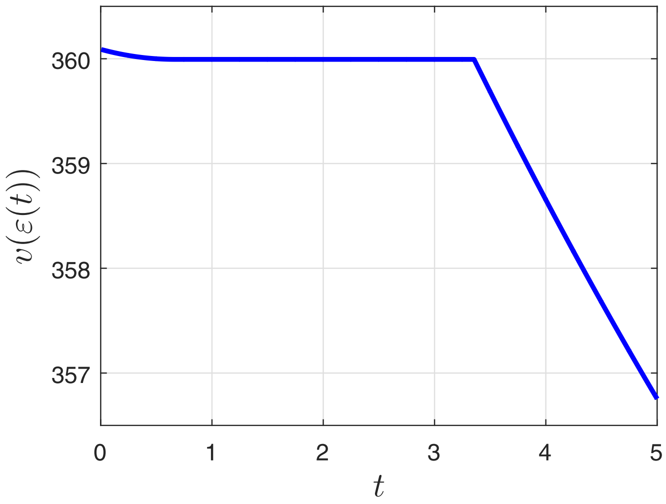

Fig. 5. Best-case BRUE optimal value function corresponding to the network in Fig. 1 with quadratic link cost functions.

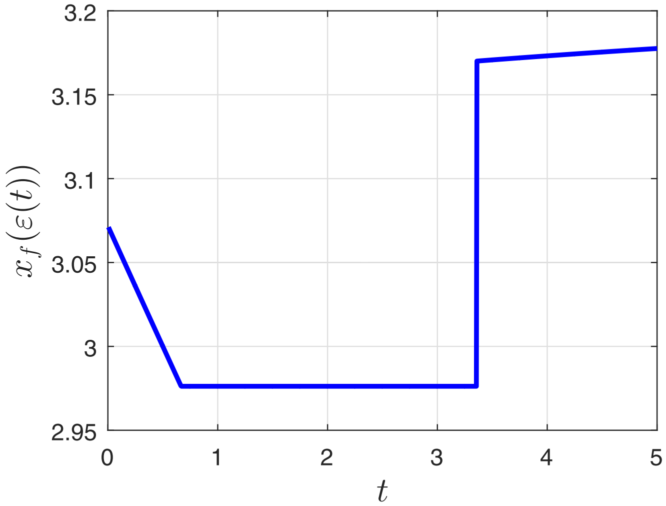

Fig. 6. Best-case BRUE optimal solution for link f as function of t corresponding to the network in Fig. 1 with quadratic link cost functions.

7. Conclusion

This study incorporates boundedly rational travel behavior in the static traffic assignment with the aim to increase validity of ex-ante evaluation. Earlier studies indicated that boundedly rational travel behavior leads to uncertainty in flow distributions and therefore we propose to evaluate a policy measure with respect to its best and worst-performing BRUE flow.

We introduce a bilevel formulation of the Best/Worst-case BRUE problem. This bilevel program is generally difficult to solve since the feasible set is not convex, a regularity condition does not necessarily hold and many local minimizers may appear. However, this bilevel structure eases parametric analysis. We applied parametric analysis to study the behavior of the program under perturbations in the indifference band parameter ε to account for, e.g., calibration difficulties. We show that the feasible link flow set and optimal value function are continuous in ε for general latency functions. The optimal solution set is however only guaranteed to be usc. The claims are illustrated by two simple examples. The continuous optimal value function implies that the optimal network settings from a designer's perspective are likely to remain (nearly) optimal if ε lightly changes. Furthermore, we extend the setting to a particular application of the NDP in which the Best/Worst-case BRUE serves as subproblem. We use our techniques to prove existence of a risk-prone and risk-averse second-best toll pricing scheme under boundedly rational route choice.

We have shown that the feasible set and value function of the Best/Worst-case BRUE program changes continuously with ε. However, different local minimizers can coexist. Together with the complicated non-convex structure of the feasible set makes the Best/Worst-case BRUE problem difficult to solve and a global solution of the problem for large networks is to our opinion out of reach.

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