On solving the Helmholtz equation in terms of amplitude and phase.

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Abstract

Solving the Helmholtz equation for high wavenumbers is a major challenge. Since one needs at least some elements per wavelength, the computational effort in finite element or boundary element calculations increases drastically with increasing wavenumbers. However, for exterior problems in unbounded domains, the amplitude and phase are smooth and non-oscillatory functions (at least some distance away from the radiating object). Therefore, we propose to solve the Helmholtz equation in terms of amplitude and phase instead of pressure. The paramount advantage of this approach is that any common discretization remains accurate for high wavenumbers. A drawback of the method is that the equations for amplitude and phase are non-linear and hence need to be solved iteratively.
Substitution of $p = Ae^{i\phi}$ in the Helmholtz equation yields two, non-linear, coupled, real-valued, differential equations. These equations have been reported some decades ago but have only been used to trace 'rays' for given solutions of the Helmholtz equation. To eventually be able to include inhomogeneous regions and fluid-structure interaction efficiently, a finite element discretization is preferred over a boundary element discretization. Therefore, finite elements were used to discretize the governing equations. We present several solutions and, despite of its non-linearity, show the efficiency of the method for high wavenumbers.

1 Introduction

The major problem in numerically solving the Helmholtz equation ($\nabla^2 p + k^2 p = 0$, where p is the pressure perturbation and k is the wavenumber) is solving the equation at high wavenumbers. Since an accurate simulation requires at least a number of degrees of freedom per wavelength, especially for 2 and 3 dimensional problems, the model size and computational effort to solve the system of equations increase drastically with increasing wavenumber.

For exterior problems however, one can reduce the model size by recognizing that the amplitude A and phase ϕ are, at least some distance away from the radiating object, smooth and non-oscillatory. Hence, it requires much less degrees of freedom to accurately represent the amplitude and phase than to represent the (complex) pressure. Therefore, we propose to solve the Helmholtz equation in terms of amplitude and phase instead of pressure. As an illustration consider the simple plane wave solution $p = Be^{-ikx}$ of a wave propagating in the positive x-direction. The amplitude $A = |B|$ is a constant and the phase $\phi = -kx$ is a linear function of the coordinate. This implies that only a single linear finite element for both amplitude and phase suffices to capture the analytical solution for any wavenumber! Another example is the radiation of a monopole source $p = Be^{-ikr}/r$, where r denotes the radius. The amplitude $A = |B/r|$ decays smoothly and the phase $\phi = -kr$ again remains linear (smooth) for any wavenumber. Hence, a small size numerical model is already accurate.

Note that the current approach does not lead to model size reduction for interior acoustics as in that case the amplitude varies with wavelength.

A finite element discretization of the Helmholtz equation in terms of amplitude and phase, as will be given in the next section, allows one to analyze regions of varying k, i.e. $k = k(\mathbf{x})$. Also fluid structure interaction can efficiently be taken into account if the finite element is coupled to a structural finite element. Hence we use finite elements, as opposed to boundary elements, to discretize the set of equations.

2 Theory

2.1 Helmholtz in terms of amplitude and phase.

In terms of the (complex) pressure $p = p(\mathbf{x}, k) = p_r(\mathbf{x}, k) + ip_i(\mathbf{x}, k)$, where $\mathbf{x}$ denotes the position vector, $k = \omega/c_0$ is the wavenumber, ω is the angular frequency and c_0 is the speed of sound, the Helmholtz equation reads:

$$\nabla^2 p + k^2 p = 0. \tag{1}$$

On a boundary $\partial\Omega$ of the computational domain Ω, either the pressure ($p = \hat{p}$, where $\hat{}$ denotes a prescribed variable), the normal velocity ($\partial p/\partial n = -i\rho_0 c_0 k \hat{v}_n$, where ρ_0 denotes the density and $\hat{v}_n$ the prescribed normal velocity) or a specific impedance ($\hat{\zeta}\partial p/\partial n + ikp = 0$, where $\hat{\zeta} = Z/(\rho_0 c_0)$ is a prescribed characteristic impedance) is given. For exterior problems, the solution should satisfy the Sommerfeld radiation condition on the enclosing boundary:

$$\lim_{r\to\infty} r\left(\frac{\partial p}{\partial r} + ikp\right) = 0, \tag{2}$$

allowing only outgoing waves.

The real and imaginary part of Eq.(1) transform to two coupled, non-linear, equations for both the amplitude $A = A(\mathbf{x}, k)$ and phase $\phi = \phi(\mathbf{x}, k)$:

$$\nabla^2 A + \left(k^2 - \nabla\phi\cdot\nabla\phi\right) A = 0 \quad \text{and} \tag{3}$$

$$A\nabla^2\phi + 2\nabla A\cdot\nabla\phi = 0. \tag{4}$$

These equations were given in [1] and were used, as a post-processing step, to trace rays for a given solution of the Helmholtz equation.

In terms of amplitude and phase, a pressure amplitude $\hat{A}$ and phase $\hat{\phi}$ can be prescribed:

$$A = \hat{A} \quad \text{and} \tag{5}$$

$$\phi = \hat{\phi}. \tag{6}$$

The boundary condition for a prescribed normal velocity amplitude $\hat{V}$ and phase $\hat{\phi}_V$ reads:

$$\frac{\partial A}{\partial n} + \rho_0 c_0 k\hat{V} sin(\phi - \hat{\phi}_V) = 0 \quad \text{and} \tag{7}$$

$$A\frac{\partial\phi}{\partial n} + \rho_0 c_0 k\hat{V} cos(\phi - \hat{\phi}_V) = 0, \tag{8}$$

$$\tag{9}$$

where n denotes the unit normal vector on the boundary. For a prescribed complex impedance $\hat{\zeta} = \hat{\zeta}_r + i\hat{\zeta}_i$, the boundary condition is:

$$\frac{\partial A}{\partial n} + \frac{\hat{\zeta}_i}{\hat{\zeta}_r^{\,2} + \hat{\zeta}_i^{\,2}} kA = 0 \quad \text{and} \tag{10}$$

$$A\frac{\partial\phi}{\partial n} + \frac{\hat{\zeta}_r}{\hat{\zeta}_r^{\,2} + \hat{\zeta}_i^{\,2}} kA = 0. \tag{11}$$

The Sommerfeld radiation condition (actually the term between brackets in Eq.(2)) on the enclosing boundary $\partial\Omega$ is obtained from the equation above by setting the impedance ζ to the impedance of a plane wave; $\zeta_r = 1$ and $\zeta_i = 0$. This results in:

$$\frac{\partial A}{\partial n} = 0 \tag{12}$$

$$\frac{\partial \phi}{\partial n} = -k. \tag{13}$$

Note the impedance of a spherical wave on the outer boundary results in:

$$\frac{\partial A}{\partial n} + \frac{A}{r} = 0 \tag{14}$$

$$\frac{\partial \phi}{\partial n} = -k, \tag{15}$$

which seems, generally, more appropriate than the Sommerfeld radiation condition.

2.2 Finite element discretization

Using test functions v and w for, respectively, Eqs.(3) and (4) and applying the divergence theorem, a weak formulation is obtained:

$$-\int_\Omega \nabla v \cdot \nabla A \, dV + \int_\Omega v(k^2 - \nabla\phi \cdot \nabla\phi) A \, dV + \int_{\partial\Omega} v \nabla A \cdot \mathbf{n} \, dS = 0 \tag{16}$$

$$\int_\Omega w \nabla A \cdot \nabla \phi \, dV - \int_\Omega A \nabla w \cdot \nabla \phi \, dV + \int_{\partial\Omega} w A \nabla \phi \cdot \mathbf{n} \, dS = 0 \tag{17}$$

Note that the last terms on the left hand side are the natural boundary conditions $\partial A/\partial n$ and $A\partial\phi/\partial n$ as were stated above for a velocity and impedance (and Sommerfeld) boundary condition. The variables A and ϕ within each element are now interpolated between the nodal values using shape functions. Choosing test functions equal to the shape functions, a system of equations is obtained which can be solved iteratively.

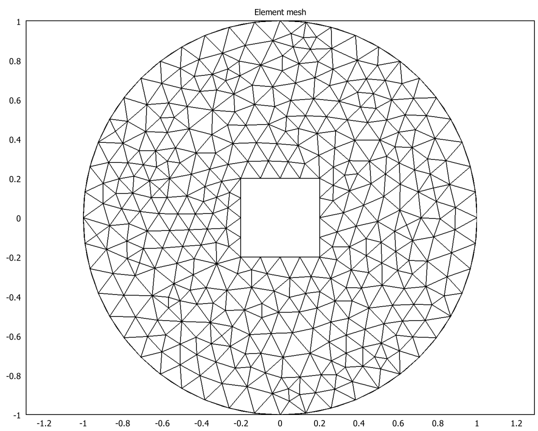

Figure 1: Finite element mesh used for solving the Helmholtz equation in terms of pressure and in terms of amplitude and phase.

3 Example

As an example we consider the radiation from a simple square radiator as given in Fig.(1). On the radiator the pressure is set to unity, i.e. $p = 1$. In terms of amplitude and phase, the boundary conditions are thus $A = 1$ and $\phi = 0$. A circular outer boundary of the computational domain, as shown in Fig.(1), is defined at a radius $r = 1m$. The Sommerfeld radiation condition is applied on this boundary.

The domain is meshed arbitrarily using quadratic elements. The mesh consists of 690 elements (4332 dof's). The size of each element is approximately $0.1x0.1m$. Hence, based on a minimum of 2 quadratic elements per wavelength, the mesh would be sufficient for frequencies upto $1715Hz$ (for standard air conditions $\rho_0 = 1.25kg/m^3$ and $c_0 = 343m/s$) when solving the Helmholtz equation in terms of pressure. Above this frequency, the element mesh is too coarse to capture the pressure variations.

To compare the solution of the Helmholtz equation in terms of pressure to the solution of the non-linear set in terms of amplitude and phase, both sets were discretized and solved. Based on the pressure solution obtained from the linear Helmholtz equation, the amplitude and phase can simply be calculated. These results will be denoted by $A_p = \sqrt{p_r^2 + p_i^2}$ and $\phi_p = atan(p_i/p_r)$ to discriminate them between the amplitude and phase which have been calculated directly from the non-linear set of equations.

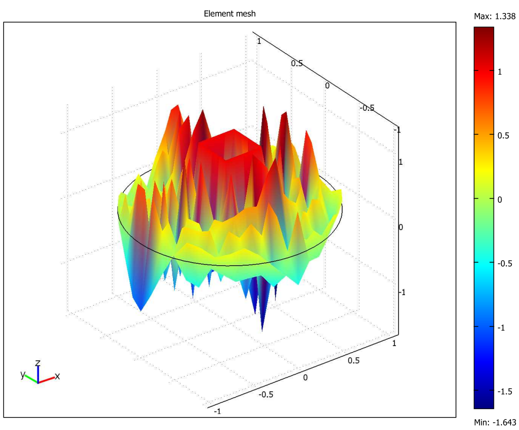

Figure 2: Pressure (real part, p_r) based on the Helmholtz equation in terms of pressure ($2kHz$).

In Fig.(2) the real part of the pressure is shown at a frequency of $2kHz$ ($k = 36.6$) based on solving the linear Helmholtz equation in terms of pressure. As can be seen, the mesh starts to become too coarse and the discretization error is becoming large.

Based on the pressure, the amplitude A_p and phase ϕ_p are shown in Fig.(3). The amplitude and phase directly calculated from the Helmholtz equation are shown in Fig.(4). As can already be seen, the larger discretization error results in less accurate results for the pressure based quantities. The discretization error for the directly calculated amplitude and phase is much less, specifically far away from the radiating object. In addition, it is noted that the current mesh is irregular. A regular mesh would result in smoother results. Note that the directly calculated phase is non-oscillatory.

The major advantage of the new approach is illustrated in Fig.(5), showing the amplitude A_p and A at $10kHz$ ($k = 183.2$). At this frequency and mesh there is only 1 element per 3 wavelengths! and the A_p results are obviously erroneous. However, the results for A are still accurate despite the non-structured mesh. Note that the left figure in Fig.(4) and the left figure in Fig.(5) are nearly the same. This is similar to the amplitude for a spherical wave; for high frequencies the amplitude becomes independent of frequency ($A \sim 1/r$). It is noted that even less elements per wavelength can be applied whenever possible as long as the discretization error is low, e.g. further away from the radiating object.

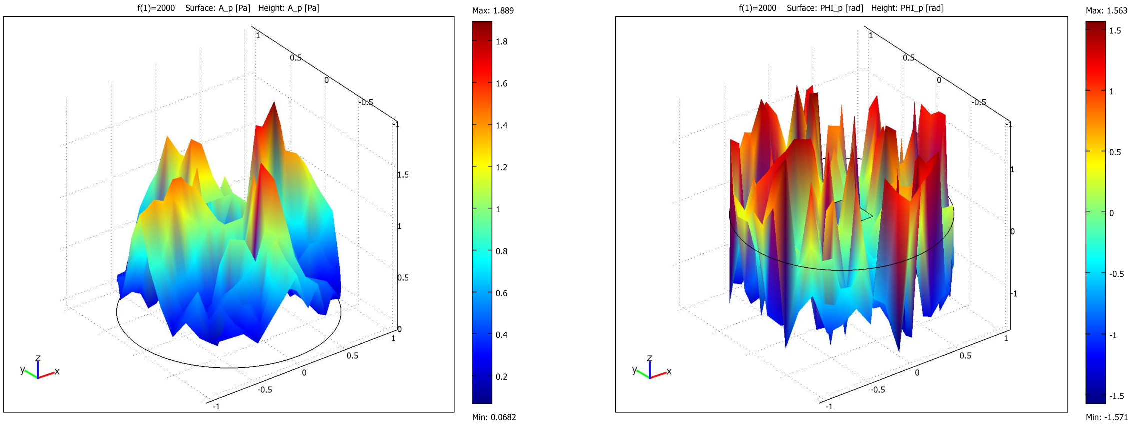

Figure 3: Amplitude $A_p = \sqrt{p_r^2 + p_i^2}$ (left) and phase $\phi_p = atan(p_i/p_r)$ (right) based on the Helmholtz equation in terms of pressure $(2kHz)$.

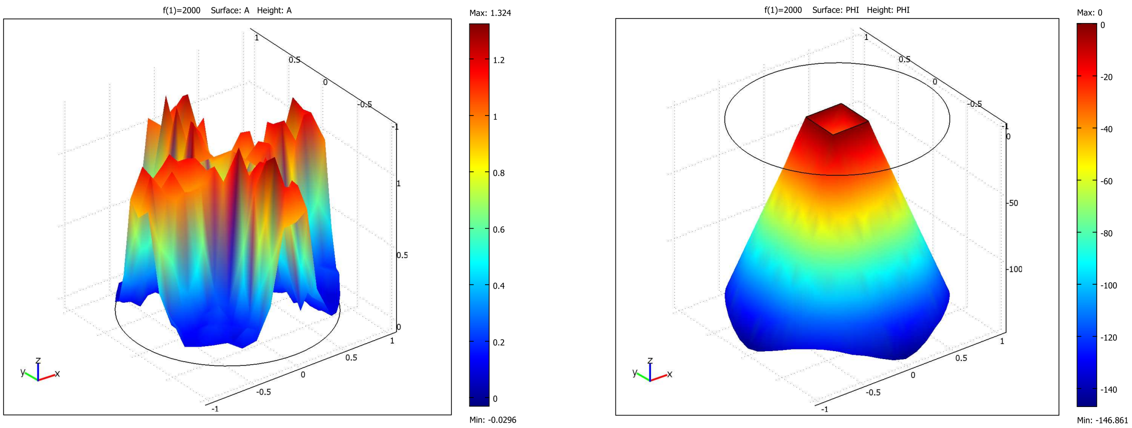

Figure 4: Amplitude A and phase ϕ based on the Helmholtz equation in terms of amplitude and phase $(2kHz)$.

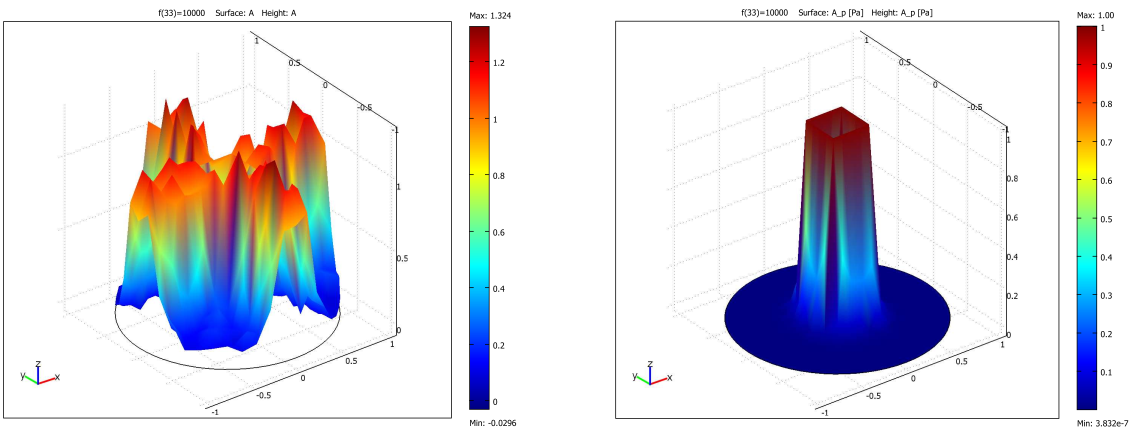

Figure 5: Amplitude A based on the Helmholtz equation in terms of amplitude and phase (left) and $A_p = \sqrt{p_r^2 + p_i^2}$ based on the Helmholtz equation in terms of pressure (right) $(10kHz)$.

4 Conclusions

For exterior problems, we presented a method to solve the Helmholtz equation for high wave numbers efficiently. This has been accomplished by solving the Helmholtz equation in terms of amplitude and phase instead of pressure. The transformation from complex pressure to pressure amplitude and phase yields two non-linear, non-complex, coupled equations. This transformation leads to a large model size reduction and, despite the non-linearity, can be used to predict sound radiation efficiently.

References

[1] T.L. Foreman, *An exact ray theoretical formulation of the Helmholtz equation*, J. Acoustic. Soc. Am., Vol. 86, (1989), p. 234.