Feedback theory extended for proving generation of contraction semigroups

MIKAEL KURULA AND HANS ZWART

Abstract. Recently, the following novel method for proving the existence of solutions for certain linear time-invariant PDEs was introduced: The operator associated with a given PDE is represented by a (larger) operator with an internal loop. If the larger operator (without the internal loop) generates a contraction semigroup, the internal loop is accretive, and some non-restrictive technical assumptions are fulfilled, then the original operator generates a contraction semigroup as well. Beginning with the undamped wave equation, this general idea can be applied to show that the heat equation and wave equations with damping are well-posed. In the present paper we show how this approach can benefit from feedback techniques and recent developments in well-posed systems theory, at the same time generalizing the previously known results. Among others, we show how well-posedness of degenerate parabolic equations can be proved.

1. Introduction

It is now a very standard technique to use semigroup theory for showing existence and uniqueness of (linear) partial differential equations (PDEs). The general results available in semigroup theory enable us to conclude existence of solutions for many PDEs once this has been proved for one PDE. For instance, if the operator A associated with a given PDE generates a C_0-semigroup, then we immediately have that for every bounded Q, also $A + Q$ generates a C_0-semigroup. Hence, the PDE associated with $A + Q$ has a unique solution given an initial condition. Even hyperbolic and parabolic PDEs are linked in the semigroup setting, since A^2 generates an (analytic) semigroup whenever A generates a C_0-group, see [4, pp. 106–107]. For contraction semigroups on Hilbert spaces, this latter result was complemented in [25].

In [25] it is shown that the existence of solutions of the heat equation,

$$\begin{cases} \dfrac{\partial x}{\partial t}(\xi, t) = \operatorname{div}\big(\alpha(\xi) \operatorname{grad} x(\xi, t)\big), & \xi \in \Omega, \ t \geq 0, \\ x(\xi, 0) = x_0(\xi), & \xi \in \Omega, \\ x(\xi, t) = 0, & \xi \in \partial\Omega, \ t \geq 0, \end{cases} \tag{1.1}$$

can be directly linked to the existence of solutions of the undamped wave equation

$$\begin{cases} \dfrac{\partial}{\partial t}\begin{bmatrix} x(\xi,t) \\ e(\xi,t) \end{bmatrix} = \begin{bmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{bmatrix}\begin{bmatrix} x(\xi,t) \\ e(\xi,t) \end{bmatrix}, & \xi \in \Omega, \ t \geq 0, \\[12pt] \begin{bmatrix} x(\xi,0) \\ e(\xi,0) \end{bmatrix} = \begin{bmatrix} x_0(\xi) \\ e_0(\xi) \end{bmatrix}, & \xi \in \Omega, \\[12pt] \qquad x(\xi,t) = 0, & \xi \in \partial\Omega, \ t \geq 0. \end{cases} \quad (1.2)$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with boundary $\partial\Omega$, div is the divergence operator div $w = \partial w_1/\partial \xi_1 + \cdots + \partial w_n/\partial \xi_n$, grad is the gradient operator grad $x = (\partial x/\partial \xi_1, \ldots, \partial x/\partial \xi_n)^\top$, and $\alpha(\xi)$ is the (strictly positive) thermal diffusivity at the point $\xi \in \Omega$.

The key to this link (more details below) is the next theorem which is taken from [25, Thm 2.6]. In the theorem, we assume that two (in general unbounded) operators are given: $A_1 : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \supset \text{dom}(A_1) \to X_1$ and $A_{21} : X_1 \supset \text{dom}(A_{21}) \to X_2$. Then we define an operator A_{ext} as

$$A_{ext} := \begin{bmatrix} A_1 \\ [A_{21}\ 0] \end{bmatrix},$$
$$\text{dom}(A_{ext}) := \left\{ \begin{bmatrix} x \\ e \end{bmatrix} \in \text{dom}(A_1) \ \Big|\ x \in \text{dom}(A_{21}) \right\}. \quad (1.3)$$

THEOREM 1.1. *Assume that A_{ext} in (1.3) generates a contraction semigroup on the pair $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ of Hilbert spaces and that S is a bounded operator on X_2 that satisfies* Re $\langle Sx, x \rangle \geq \delta \|x\|^2$ *for some $\delta > 0$ and all $x \in X_2$.*

Then the operator A_S defined using A_{ext} and S as

$$A_S x := A_1 \begin{bmatrix} x \\ SA_{21}x \end{bmatrix},$$
$$\text{dom}(A_S) := \left\{ x \in \text{dom}(A_{21}) \ \Big|\ \begin{bmatrix} x \\ SA_{21}x \end{bmatrix} \in \text{dom}(A_1) \right\} \quad (1.4)$$

generates a contraction semigroup on X_1.

In order to show how this semigroup-theoretic result links the PDEs (1.1) and (1.2), we have to identify the spaces and operators of Theorem 1.1. As Hilbert spaces X_1 and X_2 we choose $L^2(\Omega)$ and $L^2(\Omega)^n$, respectively. The operator A_{ext} is given by

$$A_{ext} = \begin{bmatrix} A_1 \\ [A_{21}\ 0] \end{bmatrix} := \begin{bmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{bmatrix}, \quad \text{dom}(A_{ext}) = \begin{bmatrix} H_0^1(\Omega) \\ H^{\text{div}}(\Omega) \end{bmatrix}, \quad (1.5)$$

where $H^1(\Omega)$ is the standard Sobolev space of functions that together with all their first-order partial derivatives lie in $L^2(\Omega)$, $H_0^1(\Omega)$ is the subspace of functions in $H^1(\Omega)$ that vanish on the boundary $\partial\Omega$ of Ω, and

$$H^{\text{div}}(\Omega) := \left\{ w \in L^2(\Omega)^n \ \Big|\ \text{div}\, w \in L^2(\Omega) \right\}.$$

It is clear that (1.2) is associated with the operator A_{ext}. Since A_{ext} is skew-adjoint on $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, see, e.g., [10], it generates a contraction semigroup. Choosing S to be the multiplication operator $(Sf)(\xi) = \alpha(\xi)f(\xi)$, $f \in X_2$, $\xi \in \Omega$, it is straightforward to see that A_S in (1.4) is the operator associated with (1.1). Hence if the thermal diffusivity α satisfies the (physically natural) condition $0 < mI \le \alpha(\xi) \le MI$, $\xi \in \Omega$ with m and M independent of ξ, then we can use Theorem 1.1 to link the two PDEs.

Theorem 1.1 was proved as [25, Thm 2.6] using a perturbation argument, and the result and its proof are also included in [26]. In the present article we give a new proof method which also allows us to generalize this theorem. In order to formulate our result, we have to introduce some notation and terminology; the precise definitions are given later in the paper.

As in Theorem 1.1, A_{ext} is assumed to generate a contraction semigroup on $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$. However, we do not assume that the lower right corner is zero. This influences the definition of A_S which now becomes $A_S x := z$ where $\begin{bmatrix} z \\ f \end{bmatrix} = A_{ext} \begin{bmatrix} x \\ Sf \end{bmatrix}$ for some f; see Definition 2.1. The *external Cayley system transform* of A_{ext} is the mapping from $\begin{bmatrix} x \\ u \end{bmatrix}$ to $\begin{bmatrix} z \\ y \end{bmatrix}$, where $\begin{bmatrix} z \\ f \end{bmatrix} := A_{ext} \begin{bmatrix} x \\ e \end{bmatrix}$ and $u := \frac{e-f}{\sqrt{2}}$, $y := \frac{e+f}{\sqrt{2}}$. A *system node* is the natural generalization to infinite-dimensional systems of the matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in the continuous-time finite-dimensional system $\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$; see Definition 2.4. A system node is *scattering passive* if and only if the following energy inequality holds:

$$2\mathrm{Re} \langle z, x \rangle_X \le \|u\|_U^2 - \|y\|_Y^2, \quad \text{with} \quad \begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}.$$

THEOREM 1.2. *Let A_{ext} generate a contraction semigroup on the pair $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ of Hilbert spaces and let $-S$ generate a contraction semigroup on X_2. Then the following claims are true:*

1. *The external Cayley system transform $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ of A_{ext} is a scattering-passive system node.*
2. *If the Cayley transform $K = (S-I)(S+I)^{-1}$ of S is an admissible static output feedback operator for $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$, then the relation A_S defined via*

$$A_S x := z, \quad \begin{bmatrix} z \\ f \end{bmatrix} = A_{ext} \begin{bmatrix} x \\ Sf \end{bmatrix} \quad \text{for some } f \in X_2,$$

   is the (single-valued) generator of a contraction semigroup on X_1.
3. *If S is bounded and $\mathrm{Re} \langle Sx, x \rangle \ge \delta \|x\|^2$ for some $\delta > 0$ and all $x \in X_2$, then K is admissible.*

The converse of assertion two in Theorem 1.2 is false; the operator A_S may generate a contraction semigroup even though K is not admissible; see Example 4.3.

REMARK 1.3. Intuitively, assertion 2 of Theorem 1.2 says that if the closed-loop system $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ with static output feedback K is a meaningful control system, i.e., a

system node, then the main operator of this system generates a contraction semigroup. For more details, see Definition 4.1 and Proposition 3.7 below.

Claim 1 of the previous theorem follows from [19, Theorem 5.2]. If S in item 2 or 3 is the identity, then $K = 0$ and we have no feedback. Moreover, in this case A_S equals the main operator A in item 1; see Theorem 3.1. It is often convenient to make the canonical choice $S = I$ [which corresponds to $\alpha \equiv I$ in the heat equation in (1.1)], but in many cases this is not preferable, or even possible. For instance, the wave equation can be transformed into the viscous Schrödinger equation [5] by choosing $S = iI + \varepsilon$. Although the solutions of the heat and Schrödinger equations have completely different properties, the existence of solutions can in both cases be proved by applying Theorem 1.2 to the wave equation (1.2). The examples in Sects. 5 and 6 have $S \neq I$. In this paper we focus on the case described in item 3; hence in all examples the operators S are bounded.

In the case of item 3, the closed-loop system $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right]$ with static output feedback K is even well-posed in the sense of Definition 2.5. This will be pointed out later in the discussion.

We do not expect that Theorem 1.2 can yield existence of solutions for a PDE for which no direct solution method exists. Rather, our point is that feedback theory can quickly solve the problem of existence of solutions, once the problem is solved for a simpler PDE; see Sect. 5. Furthermore, it follows from our method that not only homogeneous PDEs are well-posed, but also the well-posedness of some inhomogeneous PDEs is obtained; see Example 3.3. In a companion paper [11], we have shown how to easily characterize the boundary conditions which give rise to a contraction semigroup for many hyperbolic PDEs, especially those similar to the wave equation. Controllability and observability of the heat equation have previously been successfully studied using the corresponding properties of the associated wave equation in [7]; see [6,13,26] for more recent developments in this area.

The full abstract setting of the paper is described in detail in Sect. 2, together with a minimal background on continuous-time infinite-dimensional systems theory. In Sect. 3, we transform the maximal dissipative operator A_{ext} into a scattering-passive system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right]$, using a recent result on the external Cayley system transformation by Staffans and Weiss; see [23, Thm 4.6]. Then we proceed to represent A_S in terms of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right]$. The main contribution of the paper is Sect. 4, where we prove Theorem 1.2 using feedback techniques. We apply the results of Theorem 1.2 in Sect. 5, where two examples of damped wave equations are provided, one with viscous damping and one with structural damping. We end the paper with an application of Theorem 1.2 to degenerate parabolic PDEs, in Sect. 6.

Theorem 1.1 was generalized to the Banach-space setting by Schwenninger in [16]. The work [20,21,24] of Tucsnak and Weiss on "conservative systems from thin air", and that of Staffans and Weiss [19,23] on Maxwell's equations, are also very closely related to the present paper. However, it is not straightforward to translate the results from one setting to the other, and neither approach seems to be a special case of the other one. In the present paper we make extensive use of well-posed systems theory

[18], and useful connections can also be made to linear port-Hamiltonian systems [9,12,22]. Finally, it should be mentioned that Desoer and Vidyasagar used similar methods with finite-dimensional, but nonlinear, systems in [3, Sect. VI.5].

2. The abstract setting

The operator A on a Hilbert space X is *dissipative* if $\text{Re} \langle Ax, x \rangle \leq 0$ for all $x \in \text{dom} (A)$, and we say that A is *maximal dissipative* if A has no proper extension which is still a dissipative operator on X. The operator S is *(maximal) accretive* if $-S$ is (maximal) dissipative. The following definition generalizes (1.4); see Fig. 1 for an illustration:

DEFINITION 2.1. Let X_1 and X_2 be Hilbert spaces, let $A_{ext} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \supset \text{dom} (A_{ext}) \rightarrow \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ be a closed and maximal dissipative linear operator, and let S be a closed and maximal accretive linear operator on X_2.

The in general unbounded mapping A_S from $\text{dom} (A_S) \subset X_1$ into X_1 defined by

$$
\text{dom} (A_S) := \left\{ x \in X_1 \mid \exists f \in \text{dom} (S), e \in X_2 : \right.
$$
$$
\left. \begin{bmatrix} x \\ e \end{bmatrix} \in \text{dom} (A_{ext}), \ f = A_2 \begin{bmatrix} x \\ e \end{bmatrix}, \ e = Sf \right\}, \qquad (2.1)
$$
$$
A_S x := z, \quad \begin{bmatrix} z \\ f \end{bmatrix} = A_{ext} \begin{bmatrix} x \\ e \end{bmatrix}, \ e = Sf,
$$

is called *the mapping A_{ext} with internal loop through S.*

If A_{ext} is of the form $\begin{bmatrix} A_1 \\ [A_{21} \ 0] \end{bmatrix}$, then (2.1) reduces to (1.4). Moreover, it is straightforward to verify that A_S is linear, but A_S can in general be multi-valued, even when both A_{ext} and S are single-valued. For example, take $X_1 = X_2 = \mathbb{C}$, $A_{ext} = \begin{bmatrix} 0 & i \\ i & -i \end{bmatrix}$, and $S = i$. Then $\text{dom} (A_S) = \{0\}$ and the multi-valued part of A_S is $\mathbb{C}$. Fortunately, in the combinations of A_{ext} and S that we consider in the present paper A_S is always single-valued.
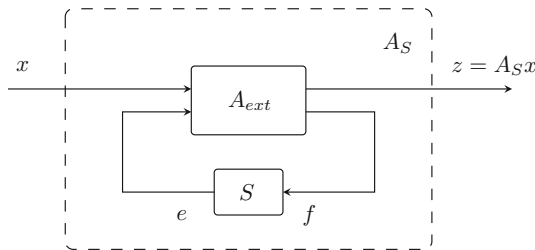

Figure 1. Representing A_S using A_{ext} and S

REMARK 2.2. Figure 1 is strongly reminiscent of feedback, but we want to emphasize that we are at this point *not working with standard feedback*. In the ODE (1.2) associated with A_{ext}, both variables x and e are state variables of a system that has no inputs or outputs. On the other hand, if we want to interpret Fig. 1 as feedback, then e would have the interpretation of input signal, x would be the state variable, $z = \dot{x}$ the (time) derivative of the state, and f would be the output signal. Sometimes, but certainly not always, it is possible to obtain useful results by making such a reinterpretation of the variables. For instance, if we in (1.2) replace $\dot{e}$ by an arbitrary variable f, and thus drop the assumption that $f = \dot{e}$, then we no longer have a meaningful system in the sense that the resulting mapping from $\left[\begin{smallmatrix} x \\ e \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} z \\ f \end{smallmatrix}\right]$ is not a system node. See Definition 2.4 and Example 3.3 below for more details.

The operator A_S is always dissipative if A_{ext} is dissipative and S is accretive. Indeed, due to (2.1), we can for all $x \in \text{dom}(A_S)$ find $z \in X_1$ $f, e \in X_2$ such that $\left[\begin{smallmatrix} z \\ f \end{smallmatrix}\right] = A_{ext} \left[\begin{smallmatrix} x \\ e \end{smallmatrix}\right]$ and $e = Sf$. Then $z = A_S x$ and it follows that

$$
\begin{aligned}
\text{Re} \langle A_S x, x \rangle = \text{Re} \langle z, x \rangle &= \text{Re} \left\langle \begin{bmatrix} z \\ f \end{bmatrix}, \begin{bmatrix} x \\ e \end{bmatrix} \right\rangle - \text{Re} \langle f, e \rangle \\
&= \text{Re} \left\langle A_{ext} \begin{bmatrix} x \\ e \end{bmatrix}, \begin{bmatrix} x \\ e \end{bmatrix} \right\rangle - \text{Re} \langle f, Sf \rangle \le 0.
\end{aligned}
\tag{2.2}
$$

According to the following famous theorem, A_S generates a contraction semigroup if and only if A_S is closed and *maximal* dissipative:

THEOREM 2.3. (Lumer-Phillips) *For a linear operator A on a Hilbert space X, the following conditions are equivalent:*

1. *A generates a contraction semigroup on X.*
2. *A is closed and maximal dissipative, i.e., dissipative with no dissipative proper extension.*
3. *A is densely defined, closed, and dissipative, and A^* is also dissipative.*
4. *A is dissipative and there exists at least one $\alpha \in \mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \text{Re } \lambda > 0\}$ such that $\text{ran}(\alpha I - A) = X$.*
5. *A is dissipative and $\alpha I - A$ has a bounded inverse on X for every $\alpha \in \mathbb{C}_+$.*

The standard definition of a contraction semigroup and additional background can be found in most books on semigroup theory. Here we assume that the reader is familiar with this theory, and we refer to Chapter 3 of [18] for more details. For a proof of Theorem 2.3, see in particular [18, Thms 3.4.8 and 3.4.9], noting that $\alpha I - A$ is always injective when $\alpha \in \mathbb{C}_+$ and A is dissipative. The importance of the assumption that A is closed in item two of the Lumer–Phillips Theorem was investigated in [15, §I.1.1].

Let X be a Hilbert space and A a linear operator defined on some subset of X. Defining the *resolvent set* of A to be the set $\rho(A)$ of all $\lambda \in \mathbb{C}$ for which $\lambda I - A$ is both injective and surjective, we can state assertion 4 of Theorem 2.3 equivalently as "A is dissipative and $\mathbb{C}_+ \cap \rho(A) \neq \emptyset$." Similarly, assertion 5 is equivalent to "A is dissipative and $\mathbb{C}_+ \subset \rho(A)$," due to the closed-graph Theorem.

Next we introduce the concept of a *system node*. It is helpful to think about a system node $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ as a generalization to infinite dimensions of the matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in the standard finite-dimensional linear system with input signal $u(\cdot)$, state trajectory $x(\cdot)$, and output signal $y(\cdot)$:

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad x(0) = x_0. \tag{2.3}$$

The associated semigroup is the mapping $t \mapsto e^{At}$, $t \geq 0$, which for zero input $u(t) = 0$, $t \geq 0$, sends the initial state x_0 into the state $x(t)$ at time $t \geq 0$.

The following definition of a system node is slightly different from the standard definition [18, Def. 4.7.2] that uses rigged Hilbert spaces, but the definitions are seen to be equivalent by combining [18, Lem. 4.7.7] with the fact that every generator of a C_0-semigroup has a non-empty resolvent set; see [18, Thm 3.2.9].

DEFINITION 2.4. By a *system node* with *input space* U, *state space* X, and *output space* Y, all Hilbert spaces, we mean an in general unbounded linear operator

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} : \begin{bmatrix} X \\ U \end{bmatrix} \supset \mathrm{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \right) \to \begin{bmatrix} X \\ Y \end{bmatrix}$$

with the following properties:

1. The operator $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ is closed.
2. The operator $A\&B$ is closed, where $A\&B$ is the projection of $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ onto $\begin{bmatrix} X \\ \{0\} \end{bmatrix}$ along $\begin{bmatrix} \{0\} \\ Y \end{bmatrix}$.
3. the *main operator* $A \colon \mathrm{dom}(A) \to X$, which is defined by

$$Ax = A\&B \begin{bmatrix} x \\ 0 \end{bmatrix}, \quad \mathrm{dom}(A) = \left\{ x \in X \ \Big| \ \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathrm{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \right) \right\}, \tag{2.4}$$

    is the generator of a C_0-semigroup on X.
4. The domain of $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ satisfies the condition

$$\forall u \in U \ \exists x \in X : \quad \begin{bmatrix} x \\ u \end{bmatrix} \in \mathrm{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \right).$$

By a *classical trajectory* of the system node $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ we mean a triple (u, x, y) where $u \in C(\mathbb{R}_+; U)$, $x \in C^1(\mathbb{R}_+; X)$, $y \in C(\mathbb{R}_+; Y)$, $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathrm{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \right)$ for all $t \geq 0$, and

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \tag{2.5}$$

using the derivative from the right at 0.

When we in the sequel use the notation $A\&B$, we mean that the operators A and B can in general no longer be separated from each other (without extending the co-domain and the domain).

Let $\pi_{[0,T]}$ denote the linear operator which first restricts a function to the interval $[0, T]$ and then extends the restricted function by zero on $\mathbb{R} \setminus [0, T]$, and introduce the Sobolev space

$$H_0^1(\mathbb{R}_+; U) := \left\{ u \in L^2(\mathbb{R}_+; U) \mid \frac{du}{d\xi} \in L^2(\mathbb{R}_+; U), \ u(0) = 0 \right\}.$$

Let $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ be a system node. Then there for every $u \in H_0^1(\mathbb{R}_+; U)$ exist $x \in C^1(\mathbb{R}_+; X)$ and $y \in C(\mathbb{R}_+; Y)$, such that (u, x, y) is a classical trajectory of $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ with $x(0) = 0$; see [18, Lemma 4.7.8]. Thus, for every $u_0 \in \pi_{[0,T]}H_0^1(\mathbb{R}_+; U)$ there exists a classical trajectory (u, x, y) of $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ with $x(0) = 0$ and $\pi_{[0,T]}u = u_0$. It is well known that $\pi_{[0,T]}H_0^1(\mathbb{R}_+; U)$ is dense in $L^2([0, T]; U)$, and therefore for all $T > 0$,

$$\mathcal{U}_0^T := \left\{ \pi_{[0,T]}u \mid (u, x, y) \text{ classical trajectory of } \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \wedge x(0) = 0 \right\} \quad (2.6)$$

is a dense subspace of $L^2([0, T]; U)$.

DEFINITION 2.5. A system node $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ is *(L^2-)well-posed* if there for every $T \geq 0$ exists a corresponding constant $M_T \geq 0$ such that all classical trajectories (u, x, y) of $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ satisfy

$$\|x(T)\|_X^2 + \int_0^T \|y(t)\|_Y^2 \, dt \leq M_T \left(\|x(0)\|_X^2 + \int_0^T \|u(t)\|_U^2 \, dt \right). \quad (2.7)$$

The system node is *(scattering) passive* if (2.7) holds with $M_T = 1$ for all $T \geq 0$.

A system node is well-posed (passive) if and only if there exist one $T > 0$, such that the inequality in (2.7) holds with some $M_T \geq 0$ (with $M_T = 1$). Often M_T grows with growing T in the non-passive well-posed case. By [18, Thm 11.1.5], a system node $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ is passive if and only if it for all $\begin{bmatrix} x \\ u \end{bmatrix} \in \text{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \right)$ holds that

$$2\text{Re} \ \langle z, x \rangle_X \leq \|u\|_U^2 - \|y\|_Y^2, \quad \text{with} \quad \begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}. \quad (2.8)$$

Let (u, x, y) be a classical trajectory with $x(0) = 0$ of a well-posed system node $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ and fix $T > 0$ arbitrarily. The mapping $\mathfrak{D}_0^T$ from $\pi_{[0,T]}u$ into $\pi_{[0,T]}y$ is a linear operator defined on $\text{dom}\left(\mathfrak{D}_0^T \right) = \mathcal{U}_0^T$ with values in $L^2([0, T]; Y)$. The domain $\mathcal{U}_0^T$ of $\mathfrak{D}_0^T$ is dense in $L^2([0, T]; U)$, and as an operator from $L^2([0, T]; U)$ into $L^2([0, T]; Y)$, the operator $\mathfrak{D}_0^T$ is bounded by M_T in (2.7).

DEFINITION 2.6. Let $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ be a well-posed system node and $T > 0$ be arbitrary. We call the unique extension of $\mathfrak{D}_0^T$ into a bounded operator $\mathfrak{D}_0^T : L^2([0, T]; U) \rightarrow L^2([0, T]; Y)$ the *T-input/output map* of $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$, and we also denote this extension by $\mathfrak{D}_0^T$.

For a passive system node, $\mathfrak{D}_0^T$ is a contraction, again by (2.7).

REMARK 2.7. Combining Definition 2.2.7 and Theorem 4.6.11 in [18] with our derivation of $\mathfrak{D}_0^T$, we see that our operator $\mathfrak{D}_0^T$ coincides with the operator represented by the same notation in [18]. Indeed, $\mathfrak{D}_0^T$ maps an input signal $u \in L^2([0, T]; U)$ into the corresponding output signal $y \in L^2([0, T]; Y)$ of a mild trajectory (u, x, y) of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right]$ with $x(0) = 0$; see also [18, Sect. 2.1].

Compared to our derivation of $\mathfrak{D}_0^T$, Staffans [18] proceeds in the opposite direction. More precisely, he considers an extension $\mathfrak{D}$ of the operator $\mathfrak{D}_0^T$ to the space of functions in $L_{loc}^2(\mathbb{R}; U)$, with support bounded from the left, to be part of the definition of a well-posed system. Using the operator $\mathfrak{D}$, he defines $\mathfrak{D}_0^T$ by $\mathfrak{D}_0^T := \pi_{[0,T]} \mathfrak{D} \pi_{[0,T]}$ in Definition 2.2.6, and only later he defines the system node and classical trajectories.

We end the section with a result that is useful when working on examples. The simple proof, which uses causality and the identity $\mathfrak{D}_0^\tau := \pi_{[0,\tau]} \mathfrak{D} \pi_{[0,\tau]}$, $\tau > 0$, is omitted.

PROPOSITION 2.8. *For a well-posed system* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right]$ *with input space U and output space Y, the norm of* $\mathfrak{D}_0^T$, $T > 0$, *as an operator from* $L^2([0, T]; U)$ *to* $L^2([0, T]; Y)$, *is a non-decreasing function of T.*

3. Representing A_S using a passive system node

We provide sufficient conditions for A_S to generate a contraction semigroup by using the following theorem, which is a reformulation of Theorem 5.2 in [19]. We give a new elementary and self-contained proof, where we show directly that the conditions of Definition 2.4 and the inequality (2.8) are satisfied.

THEOREM 3.1. *If* $A_{ext} = \left[\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix} \right]$ *is a closed and maximal dissipative operator on the pair* $\left[\begin{smallmatrix} X_1 \\ X_2 \end{smallmatrix} \right]$ *of Hilbert spaces, then the* external Cayley system transform

$$
\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} : \begin{bmatrix} x \\ (e - f)/\sqrt{2} \end{bmatrix} \mapsto \begin{bmatrix} z \\ (e + f)/\sqrt{2} \end{bmatrix}, \quad \begin{bmatrix} z \\ f \end{bmatrix} = A_{ext} \begin{bmatrix} x \\ e \end{bmatrix},
$$
$$
\mathrm{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \right) = \left\{ \begin{bmatrix} x \\ (e - f)/\sqrt{2} \end{bmatrix} \mid \begin{bmatrix} x \\ e \end{bmatrix} \in \mathrm{dom}\,(A_{ext}),\ f = A_2 \begin{bmatrix} x \\ e \end{bmatrix} \right\},
\tag{3.1}
$$

of A_{ext} *is a passive (in particular well-posed) system node, with state space* X_1, *and input and output space* X_2.

The main operator A equals A_S *of Definition 2.1 for S = I.*

Proof. The following useful equivalence is straightforward to verify:

$$
u = \frac{e - f}{\sqrt{2}} \quad \text{and} \quad y = \frac{e + f}{\sqrt{2}} \quad \Longleftrightarrow \quad f = \frac{y - u}{\sqrt{2}} \quad \text{and} \quad e = \frac{y + u}{\sqrt{2}}.
\tag{3.2}
$$

In order to prove (2.8), we let $\left[\begin{smallmatrix} x \\ u \end{smallmatrix} \right] \in \mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right] \right)$ be arbitrary, and we set $\left[\begin{smallmatrix} z \\ y \end{smallmatrix} \right] :=$ $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right] \left[\begin{smallmatrix} x \\ u \end{smallmatrix} \right]$, $e := (y + u)/\sqrt{2}$, and $f := (y - u)/\sqrt{2}$. Then $\left[\begin{smallmatrix} x \\ e \end{smallmatrix} \right] \in \mathrm{dom}\,(A_{ext})$ and $\left[\begin{smallmatrix} z \\ f \end{smallmatrix} \right] = A_{ext} \left[\begin{smallmatrix} x \\ e \end{smallmatrix} \right]$ by (3.1) and (3.2), and the dissipativity of A_{ext} yields

$$0 \geq 2\mathrm{Re}\left\langle \begin{bmatrix} z \\ f \end{bmatrix}, \begin{bmatrix} x \\ e \end{bmatrix} \right\rangle = 2\mathrm{Re}\,\langle z, x \rangle + 2\mathrm{Re}\left\langle \tfrac{y-u}{\sqrt{2}}, \tfrac{y+u}{\sqrt{2}} \right\rangle$$

$$\geq 2\mathrm{Re}\,\langle z, x \rangle + \|y\|^2 - \|u\|^2. \tag{3.3}$$

We have proved (2.8), and setting $u = 0$, we obtain for all $x \in \mathrm{dom}\,(A)$ that $z = Ax$ and $2\mathrm{Re}\,\langle Ax, x \rangle \leq -\|y\|^2 \leq 0$, where A is the main operator of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right]$; see (2.4). Hence, A is dissipative.

As A_{ext} is *maximal* dissipative, $1 \in \rho\,(A_{ext})$ by the Lumer–Phillips Theorem 2.3, which implies that the operator $I - A_{ext}$ has range $\left[\begin{smallmatrix} X_1 \\ X_2 \end{smallmatrix} \right]$. Therefore, for arbitrary $\tilde{x} \in X_1$ and $\tilde{u} \in X_2$ there exists an $\left[\begin{smallmatrix} x \\ e \end{smallmatrix} \right] \in \mathrm{dom}\,(A_{ext})$ such that

$$\begin{bmatrix} \tilde{x} \\ \sqrt{2}\,\tilde{u} \end{bmatrix} = \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - A_{ext} \right) \begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} x - z \\ e - f \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} z \\ f \end{bmatrix} = A_{ext} \begin{bmatrix} x \\ e \end{bmatrix}.$$

Comparing this to (3.1), we see that condition 4 of Definition 2.4 is met. Moreover, setting $\tilde{u} = 0$, we see that $I - A$ is surjective, and since we already know that A is dissipative, we can conclude that A is maximal dissipative, hence the generator of a contraction semigroup. Thus condition 3 of Definition 2.4 is also met.

Next we prove that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right]$ inherits closedness from A_{ext}. Indeed, let $\left[\begin{smallmatrix} x_n \\ u_n \end{smallmatrix} \right] \in \mathrm{dom}\,(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right])$, $\left[\begin{smallmatrix} z_n \\ y_n \end{smallmatrix} \right] = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right]\left[\begin{smallmatrix} x_n \\ u_n \end{smallmatrix} \right]$, $x_n \to x$ and $z_n \to z$ in X_1, and $u_n \to u$ and $y_n \to y$ in X_2. Then

$$e_n := \frac{y_n + u_n}{\sqrt{2}} \to \frac{y + u}{\sqrt{2}} =: e \quad \text{and} \quad f_n := \frac{y_n - u_n}{\sqrt{2}} \to \frac{y - u}{\sqrt{2}} =: f$$

in X_2, and moreover $\left[\begin{smallmatrix} x_n \\ e_n \end{smallmatrix} \right] \in \mathrm{dom}\,(A_{ext})$ with $\left[\begin{smallmatrix} z_n \\ f_n \end{smallmatrix} \right] = A_{ext}\left[\begin{smallmatrix} x_n \\ e_n \end{smallmatrix} \right]$ due to (3.1) and (3.2). By the closedness of A_{ext}, $\left[\begin{smallmatrix} x \\ e \end{smallmatrix} \right] \in \mathrm{dom}\,(A_{ext})$ and $\left[\begin{smallmatrix} z \\ f \end{smallmatrix} \right] = A_{ext}\left[\begin{smallmatrix} x \\ e \end{smallmatrix} \right]$, and since $u = (e - f)/\sqrt{2}$ and $y = (e + f)/\sqrt{2}$ by (3.2), we obtain from (3.1) that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix} \right] \in \mathrm{dom}\,(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right])$ and $\left[\begin{smallmatrix} z \\ y \end{smallmatrix} \right] = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right]\left[\begin{smallmatrix} x \\ u \end{smallmatrix} \right]$. We have proved that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right]$ is closed, and so condition 1 of Definition 2.4 is met.

Finally we need to show that condition 2 of Definition 2.4 is satisfied. Therefore, we assume that $\left[\begin{smallmatrix} x_n \\ u_n \end{smallmatrix} \right] \in \mathrm{dom}\,(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right])$, $\left[\begin{smallmatrix} z_n \\ y_n \end{smallmatrix} \right] = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right]\left[\begin{smallmatrix} x_n \\ u_n \end{smallmatrix} \right]$, $x_n \to x$ and $z_n \to z$ in X_1, and $u_n \to u$ in X_2. Then x_n, z_n, and u_n are all Cauchy sequences such that $\left[\begin{smallmatrix} z_n - z_m \\ y_n - y_m \end{smallmatrix} \right] = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right]\left[\begin{smallmatrix} x_n - x_m \\ u_n - u_m \end{smallmatrix} \right]$, and combining (3.3) with the Cauchy–Schwartz inequality, we obtain that

$$\|y_n - y_m\|^2 \leq \|u_n - u_m\|^2 - 2\mathrm{Re}\,\langle z_n - z_m, x_n - x_m \rangle$$

$$\leq \|u_n - u_m\|^2 + 2\|z_n - z_m\|\|x_n - x_m\|.$$

This implies that y_n is also a Cauchy sequence in X_2. Hence y_n also converges to some $y \in X_2$ and by the closedness of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right]$, we have that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix} \right] \in \mathrm{dom}\,(A\&B)$ and $A\&B\left[\begin{smallmatrix} x \\ u \end{smallmatrix} \right] = z$. We conclude that $A\&B$ is closed.

By equation (2.4), x is mapped to $z = Ax$ whenever $u = 0$. However, $u = 0$ corresponds to $e = f$, see (3.2). Hence $e = If$, and so (2.1) gives $z = A_I x$. $\qquad\square$

In [17] it was shown that there exist maximal scattering dissipative operators which are not closed, and so the closedness assumption in Theorem 3.1 is essential.

The following alternative representation of the operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ in (3.1) is useful in computations; see also [23]:

PROPOSITION 3.2. *Let* $A_{ext} = \left[\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix}\right]$ *be a dissipative operator on the pair* $\left[\begin{smallmatrix} X_1 \\ X_2 \end{smallmatrix}\right]$ *of Hilbert spaces and define* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *by* (3.1). *Then the operator* $\left[\begin{smallmatrix} \sqrt{2}\,I & 0 \\ 0 & I \end{smallmatrix}\right] - \left[\begin{smallmatrix} 0 \\ A_2 \end{smallmatrix}\right]$ *maps* dom (A_{ext}) *one to one onto* dom $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ *and*

$$
\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} = \begin{bmatrix} \sqrt{2}\,A_1 \\ A_2 + \begin{bmatrix} 0 & I \end{bmatrix} \end{bmatrix} \left(\begin{bmatrix} \sqrt{2}\,I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 \\ A_2 \end{bmatrix} \right)^{-1} \quad \text{with}
$$
$$
\text{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right) = \left(\begin{bmatrix} \sqrt{2}\,I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 \\ A_2 \end{bmatrix} \right) \text{dom}\,(A_{ext}).
$$

$$(3.4)$$

In particular, if there exist linear operators A_{12} *and* A_{21}, *such that* $A_1\left[\begin{smallmatrix} x \\ e \end{smallmatrix}\right] = A_{12}\,e$ *and* $A_2\left[\begin{smallmatrix} x \\ e \end{smallmatrix}\right] = A_{21}x$ *for all* $\left[\begin{smallmatrix} x \\ e \end{smallmatrix}\right] \in$ dom (A_{ext}),[1] *then*

$$
\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} = \begin{bmatrix} A_{S=I}\ \&\ (\sqrt{2}\,A_{12}) \\ (\sqrt{2}\,A_{21})\ \&\ I \end{bmatrix},
$$
$$
\text{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right) = \begin{bmatrix} \sqrt{2}\,I & 0 \\ -A_{21} & I \end{bmatrix} \text{dom}\,(A_{ext}),
$$

$$(3.5)$$

where $A_{S=I} := A_{12}A_{21}$, *cf.* (1.4).

The notation in the second part of the proposition is analogous to that in Theorem 1.1.

Proof. Fix $\left[\begin{smallmatrix} x \\ e \end{smallmatrix}\right] \in$ dom (A_{ext}) arbitrarily and set $\left[\begin{smallmatrix} z \\ f \end{smallmatrix}\right] := A_{ext}\left[\begin{smallmatrix} x \\ e \end{smallmatrix}\right]$, $u := (e-f)/\sqrt{2}$, and $y := (e+f)/\sqrt{2}$. It then follows that $e - A_2\left[\begin{smallmatrix} x \\ e \end{smallmatrix}\right] = \sqrt{2}u$ and by (3.2) also $y = \sqrt{2}A_2\left[\begin{smallmatrix} x \\ e \end{smallmatrix}\right] + u$. Hence

$$
\left(\begin{bmatrix} \sqrt{2}\,I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 \\ A_2 \end{bmatrix} \right) \begin{bmatrix} x \\ e \end{bmatrix} = \sqrt{2} \begin{bmatrix} x \\ u \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} A_1 \\ \sqrt{2}\,A_2 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}.
$$

$$(3.6)$$

We next prove that the operator $\left[\begin{smallmatrix} \sqrt{2}\,I & 0 \\ 0 & I \end{smallmatrix}\right] - \left[\begin{smallmatrix} 0 \\ A_2 \end{smallmatrix}\right]$ is injective and therefore assume that $\left(\left[\begin{smallmatrix} \sqrt{2}\,I & 0 \\ 0 & I \end{smallmatrix}\right] - \left[\begin{smallmatrix} 0 \\ A_2 \end{smallmatrix}\right]\right)\left[\begin{smallmatrix} x \\ e \end{smallmatrix}\right] = 0$. Then $x = 0$ and $f = A_2\left[\begin{smallmatrix} 0 \\ e \end{smallmatrix}\right] = e$, which implies that $u = \frac{e-f}{\sqrt{2}} = 0$, and it follows from (3.3) that $y = 0$, which in turn implies that $e = \frac{y+u}{\sqrt{2}} = 0$. This shows that $\left[\begin{smallmatrix} \sqrt{2}\,I & 0 \\ 0 & I \end{smallmatrix}\right] - \left[\begin{smallmatrix} 0 \\ A_2 \end{smallmatrix}\right]$ is injective. The domain of this operator is clearly dom (A_{ext}), and by (3.1) its range is dom $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$.

---

[1] By writing that $A_2\left[\begin{smallmatrix} x \\ e \end{smallmatrix}\right] = A_{21}x$ for all $\left[\begin{smallmatrix} x \\ e \end{smallmatrix}\right] \in$ dom (A_{ext}), we mean that the given operator A_2 has the property that $A_2\left[\begin{smallmatrix} x \\ e_1 \end{smallmatrix}\right] = A_2\left[\begin{smallmatrix} x \\ e_2 \end{smallmatrix}\right]$ whenever $\left[\begin{smallmatrix} x \\ e_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} x \\ e_2 \end{smallmatrix}\right] \in$ dom (A_{ext}). Then we set dom $(A_{21}) := \{x \mid \left[\begin{smallmatrix} x \\ e \end{smallmatrix}\right] \in$ dom $(A_{ext})\}$ and $A_{21}x := A_2\left[\begin{smallmatrix} x \\ e \end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} x \\ e \end{smallmatrix}\right] \in$ dom (A_{ext}). The same is meant for A_1.

Moreover, (3.6) yields that

$$
\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} A_1 \\ \sqrt{2}\,A_2 \end{bmatrix} \left(\begin{bmatrix} \sqrt{2}\,I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 \\ A_2 \end{bmatrix} \right)^{-1} \sqrt{2} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}
$$

$$
= \left(\begin{bmatrix} \sqrt{2}\,A_1 \\ 2A_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \left(\begin{bmatrix} \sqrt{2}\,I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 \\ A_2 \end{bmatrix} \right) \right)
$$

$$
\times \left(\begin{bmatrix} \sqrt{2}\,I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 \\ A_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} x \\ u \end{bmatrix}
$$

$$
= \begin{bmatrix} \sqrt{2}\,A_1 \\ A_2 + \begin{bmatrix} 0 & I \end{bmatrix} \end{bmatrix} \left(\begin{bmatrix} \sqrt{2}\,I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 \\ A_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} x \\ u \end{bmatrix}
$$

for all $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathrm{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \right)$, and the first assertion is proved. From here (3.5) follows easily.   □

If one assumes more structure of A_{ext} in the preceding proposition, essentially that A_{ext} is a system node, then one can alternatively obtain the result by flow inversion, using [18, Thm 6.3.9].

We now continue the example in the introduction, where A_{ext} in (1.5) is a skew-adjoint operator that is not a system node.

EXAMPLE 3.3. The operator $A_{ext} = \begin{bmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & 0 \end{bmatrix}$ with $\mathrm{dom}\,(A_{ext}) := \begin{bmatrix} H_0^1(\Omega) \\ H^{\mathrm{div}}(\Omega) \end{bmatrix}$ is *not* a system node with input space $L^2(\Omega)^n$, because

$$
\left\{ u \in L^2(\Omega)^n \mid \exists x \in L^2(\Omega) : \begin{bmatrix} x \\ u \end{bmatrix} \in \mathrm{dom}\,(A_{ext}) \right\} = H^{\mathrm{div}}(\Omega),
$$

which is a proper subspace of $L^2(\Omega)^n$, and so condition 4 of Definition 2.4 is violated. Moreover, the "main operator" of A_{ext} is zero:

$$
x \mapsto \begin{bmatrix} 0 & \mathrm{div} \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = 0, \quad \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathrm{dom}\,(A_{ext}), \text{ i.e., } x \in H_0^1(\Omega),
$$

and the "control operator" div is unbounded from $L^2(\Omega)^n$ into $L^2(\Omega)$, and so A_{ext} also fails the standard test that the main operator should be the most unbounded operator of the system node.

Although A_{ext} is not a system node, it is maximal dissipative and closed (even self-adjoint; see [11, Cor. 3.4]), and hence the extended Cayley system transform $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ of A_{ext} is a system node; see Theorem 3.1. The state space of $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ is $X = L^2(\Omega)$, the input and output spaces are $U = Y = L^2(\Omega)^n$, and according to Proposition 3.2, the system node itself is given by:

$$
\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} = \begin{bmatrix} \Delta\ \& \ \sqrt{2}\,\mathrm{div} \\ \sqrt{2}\,\mathrm{grad}\ \& \ I \end{bmatrix} \Bigg|_{\mathrm{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \right)}, \tag{3.7}
$$

where

$$\text{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right) = \left\{ \begin{bmatrix} \sqrt{2}\,x \\ e - \text{grad}\,x \end{bmatrix} \in \begin{bmatrix} L^2(\Omega) \\ L^2(\Omega)^n \end{bmatrix} \,\middle|\, \begin{bmatrix} x \\ e \end{bmatrix} \in \begin{bmatrix} H_0^1(\Omega) \\ H^{\text{div}}(\Omega) \end{bmatrix} \right\}. \tag{3.8}$$

Here the main operator A equals the *Laplacian* $\Delta x := \text{div}\,(\text{grad}\,x)$ defined on

$$\text{dom}\,(\Delta) = \left\{ x \,\middle|\, \begin{bmatrix} x \\ 0 \end{bmatrix} \in \text{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right) \right\} = \left\{ x \in H_0^1(\Omega) \mid \text{grad}\,x \in H^{\text{div}}(\Omega) \right\}.$$

We can confirm that A of $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ is the most unbounded operator of $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$.

The PDE associated with the operator $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ in (3.7)–(3.8) is

$$\begin{cases} \dfrac{\partial x}{\partial t}(\xi, t) = \Delta x(\xi, t) + \sqrt{2}\,\text{div}\,u(\xi, t) \\[2mm] y(\xi, t) = \sqrt{2}\,\text{grad}\,x(\xi, t) + u(\xi, t), \quad \text{a.e. } \xi \in \Omega, \ t \geq 0, \\[2mm] x(\xi, 0) = x_0(\xi), \quad \text{a.e. } \xi \in \Omega, \\[2mm] x(\xi, t) = 0, \quad \text{a.e. } \xi \in \partial\Omega, \ t \geq 0. \end{cases} \tag{3.9}$$

Thus, the external Cayley system transformation of the wave equation is the heat equation with constant thermal conductivity $\alpha(\cdot) = I$ and control and observation along all of the spatial domain.

In the definition (2.1) of A_S, we expressed A_S in terms of A_{ext}, and we now proceed to express A_S in terms of the transform $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$. Combining (2.1) and (3.1), we see that $x \in \text{dom}\,(A_S)$ and $z = A_S x$ if and only if

$$\exists f \in \text{dom}\,(S), e \in X_2 : \quad \begin{bmatrix} x \\ e \end{bmatrix} \in \text{dom}\,(A_{ext}), \ \begin{bmatrix} z \\ f \end{bmatrix} = A_{ext} \begin{bmatrix} x \\ e \end{bmatrix}, \ e = Sf$$

$$\Longleftrightarrow \quad \exists f \in \text{dom}\,(S), e \in X_2 : \quad \begin{bmatrix} x \\ (e-f)/\sqrt{2} \end{bmatrix} \in \text{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right),$$

$$\text{and} \ \begin{bmatrix} z \\ (e+f)/\sqrt{2} \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x \\ (e-f)/\sqrt{2} \end{bmatrix}, \ e = Sf \tag{3.10}$$

$$\Longleftrightarrow \quad \exists u, y \in X_2 : \quad y - u \in \text{dom}\,(S), \ \begin{bmatrix} x \\ u \end{bmatrix} \in \text{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right),$$

$$\text{and} \ \begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \ \frac{y+u}{\sqrt{2}} = S\frac{y-u}{\sqrt{2}}.$$

Since $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ is a well-posed system node, contrary to A_{ext}, it now makes sense to write the equation $y + u = S(y - u)$ in the form $u = Ky$ and interpret K as an output feedback operator for $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$. We next show that $y - u \in \text{dom}\,(S)$ and $y + u = S(y - u)$ if and only if $u = Ky$, where

$$K := (S - I)(S + I)^{-1}. \tag{3.11}$$

We call this K the *operator Cayley transform* of the maximal accretive operator S.

It is important to pay attention to the condition $\delta \geq 0$ versus the condition $\delta > 0$ in (3.13) below. If $\delta = 0$ then S is only accretive, whereas $\delta > 0$ implies that S is *uniformly* accretive. Neither of these conditions alone implies any kind of maximality; see the second assertion in the following lemma.

LEMMA 3.4. *The following claims are true:*

1. *Let S be a closed and maximal accretive operator on X_2. Then $S + I$ has a bounded inverse and the operator K in (3.11) is an everywhere-defined contraction on X_2, i.e., $\|K\| \leq 1$.*

   The contraction K has the additional property that $I - K$ is injective with range dense in X_2, and S can be recovered from K using the formula

   $$S = (I + K)(I - K)^{-1} \quad \text{with} \quad \text{dom}(S) = \text{ran}(I - K). \qquad (3.12)$$

2. *If S is a closed and accretive and everywhere-defined operator on X_2, then S is bounded and maximal accretive.*

3. *If S is closed, defined on all of X_2, and uniformly accretive, i.e., there exists a $\delta > 0$ such that*

   $$\text{Re} \langle Sf, f \rangle \geq \delta \|f\|^2, \quad f \in X_2, \qquad (3.13)$$

   then K in (3.11) is a strict contraction:

   $$\|K\| \leq \varepsilon < 1 \quad \text{where} \quad \varepsilon := \sqrt{1 - \frac{4\delta}{\|S + I\|^2}}.$$

Proof. Assertion 2 holds because S is accretive and bounded (by the closed graph theorem), and clearly S has no proper extension to an operator on X_2.

Now assume that S is an arbitrary closed and maximal accretive operator on X_2. Then $-S$ is closed and maximal dissipative, and hence $-1 \in \rho(S)$ by the Lumer–Phillips Theorem 2.3, and so $S + I$ is boundedly invertible. Moreover, K is a contraction because the accretivity of S implies that for all $y \in \text{ran}(S + I) = X_2$:

$$\|Ky\|^2 - \|y\|^2 = \left\langle (S - I)(S + I)^{-1}y, (S - I)(S + I)^{-1}y \right\rangle$$
$$- \left\langle (S + I)(S + I)^{-1}y, (S + I)(S + I)^{-1}y \right\rangle \qquad (3.14)$$
$$= -4\text{Re} \left\langle S(S + I)^{-1}y, (S + I)^{-1}y \right\rangle \leq 0.$$

It follows directly from $K = (S - I)(S + I)^{-1}$ that $I + K = 2S(S + I)^{-1}$ and $I - K = 2(S + I)^{-1}$, so that $I - K$ is injective with $\text{ran}(I - K) = \text{dom}(S)$ and $(I + K)(I - K)^{-1} = S$. According to Theorem 2.3, $\text{ran}(I - K) = \text{dom}(S)$ is dense in X_2, and this finishes the proof of assertion one.

Now assume that S is bounded with $\text{dom}(S) = X_2$ and $\text{Re} \langle Sf, f \rangle \geq \delta \|f\|^2$ for some $\delta > 0$ and all $f \in X_2$. Then it holds for all $f \in X_2$ that

$$\text{Re} \langle Sf, f \rangle \geq \delta \|f\|^2 \geq \frac{\delta}{\|S + I\|^2} \|S + I\|^2 \|f\|^2 \geq \frac{\delta}{\|S + I\|^2} \|(S + I)f\|^2,$$

and choosing $f := (S + I)^{-1}y$ for an arbitrary $y \in X_2$, we obtain that

$$\frac{\delta}{\|S + I\|^2}\|y\|^2 \leq \mathrm{Re}\,\left\langle S(S + I)^{-1}y, (S + I)^{-1}y\right\rangle \quad \forall y \in X_2.$$

Thus we can sharpen (3.14) into

$$\frac{\|Ky\|^2}{\|y\|^2} = \frac{\|y\|^2 - 4\mathrm{Re}\,\left\langle S(S + I)^{-1}y, (S + I)^{-1}y\right\rangle}{\|y\|^2} \leq 1 - \frac{4\delta}{\|S + I\|^2},$$

and therefore $\|K\| \leq \sqrt{1 - 4\delta/\|S + I\|^2} < 1$, as claimed in assertion 3. $\qquad\square$

The following lemma gives a converse to the preceding result:

LEMMA 3.5. *Assume that K is an everywhere-defined contraction with $I - K$ injective. Then S defined by (3.12) is a maximal accretive, in general unbounded but densely defined and closed, operator on X_2.*

The operator $S + I$ has a bounded inverse defined on all of X_2, and K can be recovered from S using (3.11). Moreover, (3.13) holds with

$$\delta := \frac{1 - \|K\|^2}{\|I - K\|^2}. \tag{3.15}$$

In particular, if $\|K\| < 1$, then $I - K$ has a bounded inverse and $\delta > 0$ in (3.15). In this case S is also bounded: $\|S\| \leq (1 + \|K\|)/(1 - \|K\|)$.

Proof. Assume that K is an arbitrary contraction such that $I - K$ is injective. It follows from (3.12) that $S + I = 2(I - K)^{-1}$, and $S - I = 2K(I - K)^{-1}$. Hence $\mathrm{ran}\,(S + I) = \mathrm{dom}\,(I - K) = X_2$ and (3.11) holds. From (3.11) it follows that (3.14) holds, and from (3.14) it in turn follows that for all $f \in \mathrm{dom}\,(S)$:

$$\begin{aligned}
\mathrm{Re}\,\langle Sf, f\rangle &= \frac{\|(S + I)f\|^2 - \|K(S + I)f\|^2}{4} \\
&\geq \frac{\|(S + I)f\|^2 - \|K\|^2\|(S + I)f\|^2}{4} \\
&\geq \frac{1 - \|K\|^2}{4}\|(S + I)f\|^2 \geq \frac{1 - \|K\|^2}{4}\|2(I - K)^{-1}f\|^2 \\
&\geq \frac{1 - \|K\|^2}{\|I - K\|^2}\|I - K\|^2\|(I - K)^{-1}f\|^2 \geq \frac{1 - \|K\|^2}{\|I - K\|^2}\|f\|^2 \geq 0.
\end{aligned}$$

Thus (3.13) holds with δ in (3.15), and we have showed that S is accretive with the property $\mathrm{ran}\,(S + I) = X_2$. By the Lumer–Phillips Theorem 2.3, S is maximal accretive, densely defined, and closed.

Finally assume that $\|K\| < 1$, then $I - K$ is boundedly invertible, or more precisely, $\|(I - K)^{-1}\| \leq 1/(1 - \|K\|)$, as can easily be seen using Neumann series. Thus

$$\|S\| = \|(I + K)(I - K)^{-1}\| \leq \|I + K\|\|(I - K)^{-1}\| \leq \frac{1 + \|K\|}{1 - \|K\|}.$$

$\qquad\square$

The following simple observation turns out to be useful:

COROLLARY 3.6. *Let the operators S and K be related by (3.11)–(3.12). Then $u = Ky$ if and only if $y - u \in \text{dom}(S)$ and $y + u = S(y - u)$.*

Proof. Assume that $y - u \in \text{dom}(S)$ and $y + u = S(y - u)$. Then $(S + I)(y - u) = 2y$ and $(S - I)(y - u) = 2u$, which implies that $2u = (S - I)(S + I)^{-1}2y = 2Ky$. Conversely, if $u = Ky$, then it follows from (3.12) that $y - u = (I - K)y \in \text{dom}(S)$ and $y + u = S(y - u)$. $\square$

The main findings of this section are now collected in the following proposition:

PROPOSITION 3.7. *Let A_{ext} be a closed and maximal dissipative operator on the pair $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ of Hilbert spaces, and let S be a closed and maximal accretive operator on X_2. Define $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ by (3.1) and K by (3.11). Then the following claims are true:*

1. *The operator $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ is a passive system node with state space X_1 and input/output space X_2, and K is a contraction on X_2. The operator K is a strict contraction if and only if S is bounded and uniformly accretive.*
2. *The operator A_S defined in (2.1) has the alternative representation*

$$\text{dom}(A_S) = \left\{ x \in X_1 \mid \exists u \in X_2 : \begin{bmatrix} x \\ u \end{bmatrix} \in \text{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right), \right.$$

$$\left. y = K[C\&D]\begin{bmatrix} x \\ u \end{bmatrix}, \text{ and } u = Ky \right\}, \qquad (3.16)$$

$$A_S x = z, \quad \text{where } \begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} \text{ and } u = Ky.$$

Proof. Item 1 follows from Theorem 3.1 together with assertions 1 and 3 of Lemma 3.4 and Lemma 3.5. The second item holds because the last line of (3.10) and (3.16) are equivalent by Corollary 3.6. $\square$

In the next section we give some sufficient conditions for A_S to be maximal dissipative by considering K as a static output feedback operator for $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$; see (3.16).

4. Proof of Theorem 1.2 using feedback theory

We first recall some background on feedback in infinite-dimensional systems. We start with a system node $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ and a bounded static output feedback operator K. We then create a feedback loop from the output y of $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ to the input of K, and the output of K is fed back into the input u of $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$. To the input u of $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ we also add another external input v, and if the resulting mapping $\begin{bmatrix} A^f\&B^f \\ C^f\&D^f \end{bmatrix}$ from $\begin{bmatrix} x \\ v \end{bmatrix}$ to $\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}$ is again a system node, then we say that K is an admissible static feedback operator for $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$. The superscript f stands for "feedback"; see Fig. 2 for an illustration of $\begin{bmatrix} A^f\&B^f \\ C^f\&D^f \end{bmatrix}$. Definition 4.1 gives the precise definition of the concept which is referred to as *system node admissibility* in [18, Def. 7.4.2].
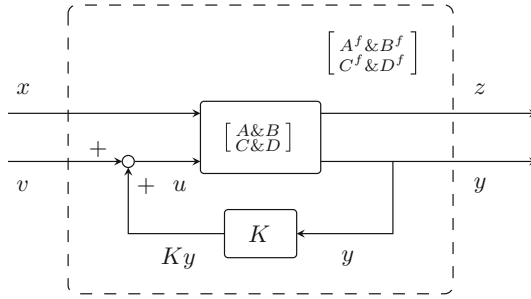
Figure 2. A standard feedback connection illustrating the closed-loop system node in Definition 4.1

DEFINITION 4.1. Let $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node with input space U and output space Y. The bounded linear operator K from Y into U is an *admissible static output feedback operator* for $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ if there exists another system node $\left[\begin{smallmatrix} A^f\&B^f \\ C^f\&D^f \end{smallmatrix}\right]$ with the same input, state, and output spaces as $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, such that the following conditions all hold:

1. The operator

$$M := \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} [0\ 0] \\ K[C\&D] \end{bmatrix} \tag{4.1}$$

    maps dom $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ continuously into dom $\left(\left[\begin{smallmatrix} A^f\&B^f \\ C^f\&D^f \end{smallmatrix}\right]\right)$.

2. M is invertible and the inverse satisfies

$$M^{-1} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} [0\ 0] \\ K[C^f\&D^f] \end{bmatrix}.$$

3. The two system nodes are related by

$$\begin{bmatrix} A^f\&B^f \\ C^f\&D^f \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} M^{-1}. \tag{4.2}$$

We refer to $\left[\begin{smallmatrix} A^f\&B^f \\ C^f\&D^f \end{smallmatrix}\right]$ in the above result as the *closed-loop* system node corresponding to the coupling of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ and K. Note that the operator M^{-1} in Definition 4.1 corresponds to the mapping from $\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ in Fig. 2. The T-input/output map of Definition 2.6 plays a key role in determining if a given operator K is an admissible static input/output feedback operator:

LEMMA 4.2. *Fix $T > 0$ arbitrarily and let $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a* passive *system node with input space U, output space Y, and T-input/output map $\mathfrak{D}_0^T$. Let K be a bounded operator from Y into U. Then the following claims are true:*

1. *The operator K is an admissible static output feedback operator for $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ if $I - K\mathfrak{D}_0^T$ has a bounded inverse in $L^2([0, T]; U)$, where K is applied point wise to a function in $L^2([0, T]; Y)$.*

2. *If $\|K\mathfrak{D}_0^T\| < 1$ as an operator on $L^2([0, T]; U)$, then K is admissible.*

Proof. Since K is applied point wise, we have that

$$\pi_{[0,T]}K\mathfrak{D}\pi_{[0,T]} = K\pi_{[0,T]}\mathfrak{D}\pi_{[0,T]} = K\mathfrak{D}_0^T.$$

By Remark 2.7 combined with [18, Thm 7.1.8(ii)], K is admissible even in the well-posed sense described in [18, Def. 7.1.1] if $I - K\mathfrak{D}_0^T$ has a bounded inverse in $L^2([0, T]; U)$. By [18, Thm 7.4.1], K is then admissible also in the sense of Definition 4.1, and this proves item one. Item two is [18, Cor. 7.1.9(i)]. □

The preceding proof together with Lemma 3.4.3 proves the last claim in Remark 1.3.

We now focus on the sufficient condition 2 in Lemma 4.2. First recall that $\|\mathfrak{D}_0^T\| \le 1$ for a passive system node by the construction of $\mathfrak{D}_0^T$ and that $\|K\| \le 1$ if S is maximal accretive and closed. Hence, if A_{ext} is maximal dissipative and S is maximal accretive, both being closed, then $\|K\mathfrak{D}_0^T\| \le \min\{\|K\|, \|\mathfrak{D}_0^T\|\}$, which is strictly less than one if $\|K\| < 1$ or $\|\mathfrak{D}_0^T\| < 1$. We can now prove the main result of the paper, Theorem 1.2.

Proof of Theorem 1.2. We assume that A_{ext} is maximal dissipative and closed on $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, that S is maximal accretive and closed on X_2, and that K is an admissible static feedback operator for $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ defined in (3.1). By Theorem 3.1, $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ is a scattering-passive system node, and the operator $\begin{bmatrix} A^f\&B^f \\ C^f\&D^f \end{bmatrix}$ in Definition 4.1 is also a system node due to the assumption on K. We next compute the main operator A^f of the latter, showing that $A^f = A_S$.

By (2.4) and Definition 4.1, $x \in \mathrm{dom}\left(A^f\right)$ and $A^f x = z$ if and only if

$$\begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathrm{dom}\left(\begin{bmatrix} A^f\&B^f \\ C^f\&D^f \end{bmatrix}\right) = \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 \\ K[C\&D] \end{bmatrix}\right)\mathrm{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right)$$

$$\text{and} \quad z = A\&B\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 \\ K[C\&D] \end{bmatrix}\right)^{-1}\begin{bmatrix} x \\ 0 \end{bmatrix},$$

which holds if and only if there exist $\begin{bmatrix} \tilde{x} \\ u \end{bmatrix} \in \mathrm{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right)$, such that

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 \\ K[C\&D] \end{bmatrix}\right)\begin{bmatrix} \tilde{x} \\ u \end{bmatrix} \quad \text{and} \quad z = A\&B\begin{bmatrix} \tilde{x} \\ u \end{bmatrix}. \tag{4.3}$$

The equations (4.3) clearly hold if and only if

$$\tilde{x} = x \quad \text{and} \quad \begin{bmatrix} z \\ u \end{bmatrix} = \begin{bmatrix} A\&B \\ K[C\&D] \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix},$$

and summarizing, we find that $x \in \mathrm{dom}\left(A^f\right)$ and $A^f x = z$ if and only if

$$\exists u \in X_2 : \quad \begin{bmatrix} x \\ u \end{bmatrix} \in \mathrm{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right), \ u = K[C\&D]\begin{bmatrix} x \\ u \end{bmatrix}, \ z = A\&B\begin{bmatrix} x \\ u \end{bmatrix}. \tag{4.4}$$

By (3.16), (4.4) is equivalent to $x \in \mathrm{dom}\,(A_S)$ and $z = A_S x$. Hence $A^f = A_S$.

Now we prove that A_S generates a contraction semigroup on X_1. According to Definitions 2.4 and 4.1, the operator $A^f = A_S$ generates a C_0-semigroup. By the Hille–Yosida Theorem [2, Thm 2.1.12], there exists some $\omega \in \mathbb{C}_+ \cap \rho\left(A^f\right)$, and since A_S is dissipative by (2.2), we have that A_S generates a contraction semigroup by the Lumer–Phillips theorem 2.3.

It now only remains to point out that K is admissible if S is bounded and uniformly accretive, and this follows from Proposition 3.7.1, Lemma 4.2, and $\|K\mathfrak{D}_0^T\| \leq \|K\| < 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following simple example shows that admissibility of K is not necessary for A_S to generate a contraction semigroup:

EXAMPLE 4.3. Take $X_1 = X_2 = \mathbb{C}$, $A_{ext} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & i \end{smallmatrix}\right]$, and $S = i$. Then $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 0 \\ 0 & -i \end{smallmatrix}\right]$ and $K = i$, so that $M = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ which is not injective. Hence, K is not admissible, but by (1.4) we have $A_S = 0$ which nevertheless generates the constant semigroup on $\mathbb{C}$.

In the introduction we proved that the heat equation (1.1) is associated with a contraction semigroup using the knowledge that the wave equation (1.2) is associated with a contraction semigroup. In the case where the thermal diffusivity $\alpha(\cdot)$ is constantly I, we obtain $S = I$ which gives $K = 0$. In the notation of Definition 4.1, we thus have that $M^{-1} = \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right]$ and hence $\left[\begin{smallmatrix} A^f \& B^f \\ C^f \& D^f \end{smallmatrix}\right] = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$. Comparing (1.1) to (3.9), we can confirm that in this example indeed $A_S = A^f = A = \Delta$.

In the next section we study two more examples that fall under Theorem 1.2. Now we present a list of sufficient conditions on $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ for $\|\mathfrak{D}_0^T\| < 1$ to hold.

PROPOSITION 4.4. *Assume that A_{ext} is maximal dissipative and closed. Define $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ by (3.1). If at least one of the following conditions is satisfied for some $T > 0$, then $\|\mathfrak{D}_0^T\| < 1$:*

1. *There exist $T > 0$ and $N_T < 1$, such that it for all classical trajectories with initial state $x(0) = 0$, input signal $u(\cdot)$, and output signal $y(\cdot)$ holds that*

$$\int_0^T \|y(t)\|_Y^2 \, \mathrm{d}t \leq N_T \int_0^T \|u(t)\|_U^2 \, \mathrm{d}t. \tag{4.5}$$

2. *For some $T > 0$, some $\varepsilon > 0$, and all classical trajectories with input signal $u(\cdot)$ and state trajectory $x(\cdot)$ satisfying $x(0) = 0$, it holds that*

$$\|x(T)\|_X^2 \geq \varepsilon \int_0^T \|u(t)\|_U^2 \, \mathrm{d}t. \tag{4.6}$$

3. *The system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ has a delay $\tau > 0$ from input to output, i.e., all classical trajectories (u, x, y) with initial state $x(0) = 0$ satisfy $\pi_{[0,\tau)} y = 0$.*

In fact, assumptions 2 and 3 both imply that assumption 1 is satisfied, with $N_T = 1 - \varepsilon$, and $T := \tau$, $N_\tau = 0$, respectively.

Proof. Combining (4.5) with the denseness of $\mathcal{U}_0^T$ in $L^2([0, T]; U)$, see (2.6), we obtain that $\|\mathfrak{D}_0^T\| \leq N_T < 1$. If (4.6) holds, then (4.5) holds with $N_T := 1 - \varepsilon$, according to (2.7). Finally, if assumption 3 holds, then $\int_0^\tau \|u(t)\| \, dt = 0$ for all classical trajectories with $x(0) = 0$, so (4.5) holds with $T := \tau$ and $N_T := 0$. $\qquad\square$

By Proposition 2.8, it is enough to check the conditions in Proposition 4.4 for small T. The condition (4.6) implies that the input-to-state map $u \mapsto x(T)$, $x(0) = 0$, is injective. This condition seems quite rare; it does not hold for any finite-dimensional system, since the input-to-state map maps the dense subspace $\mathcal{U}_0^T$ of $L^2([0, T]; U)$ into the finite-dimensional state space. The condition (4.6) does, however, hold with $\varepsilon = 1$ if A generates the outgoing shift on the right half-line with input u at the boundary $\xi = 0$.

PROPOSITION 4.5. *Let A_{ext} and $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right]$ be as in Proposition 4.4, and assume that condition 1 in that proposition holds. Then A_{ext} is in fact a well-posed system node which is in addition* impedance passive*, i.e.,*

$$\operatorname{Re} \langle z, x \rangle_{X_1} \leq \operatorname{Re} \langle f, e \rangle_{X_2}, \qquad \begin{bmatrix} x \\ e \end{bmatrix} \in \operatorname{dom}(A_{ext}), \quad \begin{bmatrix} z \\ f \end{bmatrix} = A_{ext} \begin{bmatrix} x \\ e \end{bmatrix}. \qquad (4.7)$$

Proof. The operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right]$ is a well-posed system node by Theorem 3.1. By Proposition 4.4 it holds that $\|\mathfrak{D}_0^T\| < 1$ and by Lemma 4.2, $-I$ is then a well-posed admissible static feedback operator [18, Def. 7.1.1] of

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} = \begin{bmatrix} \sqrt{2}\,A_1 \\ A_2 + \begin{bmatrix} 0 & I \end{bmatrix} \end{bmatrix} \left(\begin{bmatrix} \sqrt{2}\,I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 \\ A_2 \end{bmatrix} \right)^{-1}; \qquad (4.8)$$

see Proposition 3.2 (here $A_{ext} = \left[\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix} \right]$). Using Definition 4.1, we calculate the corresponding well-posed closed-loop system node by inserting (4.8) into (4.1):

$$M = \begin{bmatrix} \sqrt{2}I & 0 \\ 0 & 2I \end{bmatrix} \left(\begin{bmatrix} \sqrt{2}I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 \\ A_2 \end{bmatrix} \right)^{-1}.$$

Using this and (4.8) in (4.2), one then obtains

$$\begin{bmatrix} A^f & B^f \\ C^f & D^f \end{bmatrix} = \begin{bmatrix} A_1 \\ \frac{1}{\sqrt{2}}\left(A_2 + \begin{bmatrix} 0 & I \end{bmatrix} \right) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \frac{1}{\sqrt{2}}I \end{bmatrix}. \qquad (4.9)$$

It is now established that $\left[\begin{smallmatrix} A^f & B^f \\ C^f & D^f \end{smallmatrix} \right]$ satisfies the conditions in Definition (2.4) and that for any fixed $T > 0$ there exists an $M_T \geq 0$, such that (2.7) holds for all trajectories of $\left[\begin{smallmatrix} A^f & B^f \\ C^f & D^f \end{smallmatrix} \right]$. We leave it for the reader to verify that this implies that $\left[\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix} \right]$ also satisfies the conditions in Definition (2.4) and that for the same T and all trajectories of $\left[\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix} \right]$, the inequality (2.7) holds with $4M_T$ instead of M_T.

The inequality (4.7) is obtained by substituting $u = (e - f)/\sqrt{2}$ and $y = (e + f)/\sqrt{2}$ into (2.8), and this completes the proof. $\qquad\square$

The preceding result was kindly pointed out to us by the anonymous referee. It says that Proposition 4.4 is only applicable to well-posed systems. Here is furthermore an example showing that Proposition 4.4 fails to cover the (well-posed) wave equation:

EXAMPLE 4.6. Unfortunately, the external Cayley system transform (3.7)–(3.8) of the wave equation (1.2) does not satisfy (4.5) for any $N_T < 1$, because $\|\mathfrak{D}_0^T\| = 1$.

Indeed, since Ω is a *bounded* Lipschitz domain, we can choose a nonzero constant input signal $u(\xi, t) := u_0 \in \mathbb{R}^n$ for all $t \geq 0$ and almost every $\xi \in \Omega$. With this input signal and $x_0 = 0$ in (3.9), we obtain that $\partial x(\xi, t)/\partial t = 0$ for every $t \geq 0$ and almost every $\xi \in \Omega$, and so the state stays at zero: $x(\cdot, t) = 0$ in $L^2(\Omega)$ for all $t \geq 0$. Hence the corresponding output is $y(\xi, t) = u(\xi, t) = u_0$ for all $t \geq 0$ and almost every $\xi \in \Omega$. This implies that

$$\int_0^T \|y(t)\|_{L^2(\Omega)^n}^2 \, dt = \int_0^T \|u(t)\|_{L^2(\Omega)^n}^2 \, dt = T \operatorname{vol} \Omega \, \|u_0\|_{\mathbb{R}^n}^2 > 0$$

for all $T > 0$, and so $N_T = 1$ is the smallest possible choice in (4.5) for all $T > 0$.

5. Wave equations with damping along the spatial domain

In this section we use the approach outlined in the introduction to show that the wave equation with viscous damping and the wave equation with structural damping, both with the damping along the spatial domain, are also associated with contraction semigroups. We shall make use of the following operators A_{ext}.

PROPOSITION 5.1. *For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, the following operators are skew-adjoint (and closed) on $L^2(\Omega)^{2n+1}$ and $L^2(\Omega)^{n+2}$, respectively:*

$$A_{ext,s} := \begin{bmatrix} 0 & \operatorname{div} \begin{bmatrix} I & I \end{bmatrix} \\ \begin{bmatrix} I \\ I \end{bmatrix} \operatorname{grad} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \quad \text{with}$$

$$\operatorname{dom}\left(A_{ext,s}\right) := \left\{ \begin{bmatrix} x_1 \\ x_2 \\ e \end{bmatrix} \in \begin{bmatrix} H_0^1(\Omega) \\ L^2(\Omega)^n \\ L^2(\Omega)^n \end{bmatrix} \mid x_2 + e \in H^{\operatorname{div}}(\Omega) \right\}, \quad \text{and} \tag{5.1}$$

$$A_{ext,v} := \begin{bmatrix} 0 & \operatorname{div} & I \\ \operatorname{grad} & 0 & 0 \\ -I & 0 & 0 \end{bmatrix} \quad \text{with} \quad \operatorname{dom}\left(A_{ext,v}\right) := \begin{bmatrix} H_0^1(\Omega) \\ H^{\operatorname{div}}(\Omega) \\ L^2(\Omega) \end{bmatrix}. \tag{5.2}$$

Proof. By Theorem 6.2 in [10], $\operatorname{grad}|_{H_0^1(\Omega)}^* = -\operatorname{div}|_{H^{\operatorname{div}}(\Omega)}$. Combining this with Lemma A.1 below, we obtain that

$$A^*_{ext,s} = \begin{bmatrix} 0 & \mathrm{div}\begin{bmatrix} I & I \end{bmatrix} \\ \begin{bmatrix} I \\ I \end{bmatrix} \mathrm{grad}|_{H^1_0(\Omega)} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}^*$$

$$= \begin{bmatrix} 0 & \left(\begin{bmatrix} I \\ I \end{bmatrix} \mathrm{grad}|_{H^1_0(\Omega)} \right)^* \\ \left(\mathrm{div}\begin{bmatrix} I & I \end{bmatrix} \right)^* & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\mathrm{div}\begin{bmatrix} I & I \end{bmatrix} \\ \begin{bmatrix} I \\ I \end{bmatrix}(-\mathrm{grad}|_{H^1_0(\Omega)}) & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} = -A_{ext,s},$$

where we used that the diagonal blocks are zero operators and that the domain of $A_{ext,s}$ decomposes into the product of dom $\left(\begin{bmatrix} I \\ I \end{bmatrix} \mathrm{grad}|_{H^1_0(\Omega)} \right)$ and dom $\left(\mathrm{div}\begin{bmatrix} I & I \end{bmatrix} \right)$.

We also have that $(Q + R)^* = Q^* + R^*$ if R is bounded and everywhere defined. From this it immediately follows that

$$A_{ext,v} = \begin{bmatrix} 0 & \mathrm{div} & 0 \\ \mathrm{grad}|_{H^1_0(\Omega)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ -I & 0 & 0 \end{bmatrix}$$

is skew-adjoint.   □

We remark that [10, Thm 6.2] allows a wide range of boundary conditions in addition to those used above for $A_{ext,v}$ and $A_{ext,s}$.

5.1. Wave equations with viscous damping

We first consider the wave equation with *viscous damping* on a bounded Lipschitz domain Ω:

$$\begin{cases} \rho(\xi)\dfrac{\partial^2 x}{\partial t^2}(\xi, t) = \mathrm{div}\left(T(\xi)\,\mathrm{grad}\,x(\xi, t)\right) - k_v(\xi)\dfrac{\partial x}{\partial t}(\xi, t), & \xi \in \Omega,\ t \geq 0, \\[2mm] x(\xi, 0) = x_0(\xi), \quad \dfrac{\partial x(\xi, 0)}{\partial t} = z_0(\xi), & \xi \in \Omega, \\[2mm] \dfrac{\partial x(\xi, t)}{\partial t} = 0, \quad \xi \in \partial\Omega,\ t \geq 0, \end{cases}$$

$$\tag{5.3}$$

where $x(\xi, t)$ is the deflection at point ξ and time t, $\rho(\cdot)$ is the mass density, $T(\cdot)$ is Young's modulus, and $k_v(\cdot)$ is the scalar viscous damping coefficient. For physical reasons $\rho(\cdot), k_v(\cdot) \in L^\infty(\Omega)$ take real values and $T(\cdot) \in L^\infty(\Omega)^{n \times n}$ with $T(\xi)^* = T(\xi)$ for almost all $\xi \in \Omega$. We make the additional assumption that $\rho(\cdot)$, $T(\cdot)$, and $k_v(\cdot)$ are bounded away from zero, i.e., that there exists a $\delta > 0$, such that $\rho(\xi) \geq \delta$, $k_v(\xi) \geq \delta$, and $T(\xi) \geq \delta I$ for almost all $\xi \in \Omega$. This implies that the operators of multiplication by $\rho(\cdot)$, $T(\cdot)$, and $k_v(\cdot)$ are self-adjoint, bounded, and uniformly accretive on $L^2(\Omega)$, $L^2(\Omega)^{n \times n}$, and $L^2(\Omega)$, respectively.

The following multiplication operator is also bounded, everywhere defined, self-adjoint, and uniformly accretive on $X_1 := \begin{bmatrix} L^2(\Omega) \\ L^2(\Omega)^n \end{bmatrix}$:

$$\mathcal{H}x := \xi \mapsto \begin{bmatrix} 1/\rho(\xi) & 0 \\ 0 & T(\xi) \end{bmatrix} x(\xi), \quad \xi \in \Omega, \ x \in X_1. \tag{5.4}$$

This operator defines an alternative, but equivalent, inner product on X_1 through $\langle z_1, z_2 \rangle_{\mathcal{H}} := \langle \mathcal{H}z_1, z_2 \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\begin{bmatrix} L^2(\Omega) \\ L^2(\Omega)^n \end{bmatrix}$. We denote X_1 equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ by $X_{\mathcal{H}}$, and by X_1 we mean X_1 equipped with the standard $L^2(\Omega)^{n+1}$-inner product.

We can write (5.3) in the first-order abstract ODE form

$$\begin{cases} \dfrac{d}{dt} \begin{bmatrix} \rho(\cdot)\frac{dx(t)}{dt} \\ \operatorname{grad} x(t) \end{bmatrix} = \begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{bmatrix} \mathcal{H} \begin{bmatrix} \rho(\cdot)\frac{dx(t)}{dt} \\ \operatorname{grad} x(t) \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} e(t), \\[4mm] \qquad e(t) = k_v(\cdot) \begin{bmatrix} -I & 0 \end{bmatrix} \mathcal{H} \begin{bmatrix} \rho(\cdot)\frac{dx(t)}{dt} \\ \operatorname{grad} x(t) \end{bmatrix}, \quad t \geq 0, \end{cases} \tag{5.5}$$

whose state is $\begin{bmatrix} \rho(\cdot)\frac{dx(t)}{dt} \\ \operatorname{grad} x(t) \end{bmatrix}$. The natural state space is $X_{\mathcal{H}} := \begin{bmatrix} L^2(\Omega) \\ L^2(\Omega)^n \end{bmatrix}$ [with the $\mathcal{H}$-inner product induced by $\mathcal{H}$ in (5.4)].

Following Sect. 2 in [26], we define $X_2 := L^2(\Omega)$, and we choose S_v to be the bounded and uniformly accretive multiplication operator

$$S_v x := \xi \mapsto k_v(\xi)\, x(\xi) \quad \text{on} \quad X_2.$$

This allows us to rewrite (5.5) as

$$\frac{d}{dt} \begin{bmatrix} \rho(\cdot)\frac{dx(t)}{dt} \\ \operatorname{grad} x(t) \end{bmatrix} = A_{S,v} \mathcal{H} \begin{bmatrix} \rho(\cdot)\frac{dx(t)}{dt} \\ \operatorname{grad} x(t) \end{bmatrix}, \quad t \geq 0, \tag{5.6}$$

where, using (2.1),

$$A_{S,v} = \begin{bmatrix} -S_v & \operatorname{div} \\ \operatorname{grad} & 0 \end{bmatrix} \quad \text{with} \quad \operatorname{dom}(A_{S,v}) = \begin{bmatrix} H_0^1(\Omega) \\ H^{\operatorname{div}}(\Omega) \end{bmatrix}.$$

By the following result (see [9, Lem. 7.2.3]), (5.6) is associated with a contraction semigroup on $X_{\mathcal{H}}$ if and only if $A_{S,v}$ is maximal dissipative on X_1 (with the standard $L^2(\Omega)^{n+1}$-inner product):

LEMMA 5.2. *Let $\mathcal{H}$ be a bounded, self-adjoint, and uniformly accretive operator on a Hilbert space X_1. Then a linear operator A generates a contraction semigroup (a unitary group) on X_1 if and only if the operator $A\mathcal{H}$ with domain $\operatorname{dom}(A\mathcal{H}) = \{x \in X_1 \mid \mathcal{H}x \in \operatorname{dom}(A)\}$ generates a contraction semigroup (unitary group) on $X_{\mathcal{H}}$.*

Since S_v is bounded and uniformly accretive and $A_{ext,v}^* = -A_{ext,v}$ by Proposition 5.1, $A_{S,v}$ is maximal dissipative on X_1; see Theorem 1.2. Therefore (5.3) is governed

by a contraction semigroup on $X_{\mathcal{H}}$ in the following sense: The PDE (5.3) has a unique solution x for every initial condition, and for this solution the family of mappings

$$\begin{bmatrix} \rho(\cdot)z_0(\cdot) \\ \text{grad } x_0(\cdot) \end{bmatrix} \mapsto \begin{bmatrix} \rho(\cdot)\frac{\partial x}{\partial t}(\cdot, t) \\ \text{grad } x(\cdot, t) \end{bmatrix}, \quad t \geq 0,$$

is a contraction semigroup on $X_{\mathcal{H}}$, cf. (5.6).

It follows from Proposition 3.2 that the external Cayley system transform of $A_{ext,v}$ is

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_v := \left[\begin{array}{c} \begin{bmatrix} -I & \text{div} \\ \text{grad} & 0 \end{bmatrix} \& \begin{bmatrix} \sqrt{2}\,I \\ 0 \end{bmatrix} \\ \begin{bmatrix} -\sqrt{2}\,I & 0 \end{bmatrix} \& I \end{array} \right] \quad \text{with}$$

$$\text{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_v\right) := \left\{ \begin{bmatrix} \sqrt{2}\,x_1 \\ \sqrt{2}\,x_2 \\ e - \text{grad } x_1 \end{bmatrix} \in \begin{bmatrix} H_0^1(\Omega) \\ H^{\text{div}}(\Omega) \\ L^2(\Omega) \end{bmatrix} \,\middle|\, e \in L^2(\Omega) \right\}. \tag{5.7}$$

It is a consequence of the following result that $\|\mathfrak{D}_0^T\| = 1$ for the system node (5.7), and hence Proposition 4.4 is not applicable to the wave equation with viscous damping:

PROPOSITION 5.3. *For a well-posed system* $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ *with input space U and output space Y, the following claims are true:*

1. *Let $D : U \to Y$ be bounded and let Λ_D^T denote the bounded operator from $L^2([0, T]; U)$ to $L^2([0, T]; Y)$ of point-wise multiplication by D. If $\lim_{T \to 0+} \|\Lambda_D^T - \mathfrak{D}_0^T\| = 0$, where $\|\cdot\|$ denotes the norm of bounded linear operators from $L^2([0, T]; U)$ to $L^2([0, T]; Y)$, then $\|\mathfrak{D}_0^T\| \geq \|D\|$ for all $T > 0$.*
2. *Denote the state space of $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ by X and assume that there exist bounded operators $B : U \to X$, $C : X \to Y$, and $D : U \to Y$, such that $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\big|_{\text{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right)}$. Then there for every $T_0 > 0$ exists a constant $k_0 \geq 1$, such that $\|\mathfrak{D}_0^T - \Lambda_D^T\| \leq k_0 T$ for all $0 < T \leq T_0$. In particular, Assertion (1) applies, so that $\|\mathfrak{D}_0^T\| \geq \|D\|$.*

One uses the triangle inequality to establish the first assertion and the second assertion is proved by using a standard convolution estimate on the variation of constants formula.

5.2. Structural damping

Using exactly the same argument as in Sect. 5.1, we can prove that the wave equation with *structural damping*,

$$\begin{cases} \rho(\xi)\dfrac{\partial^2 x}{\partial t^2}(\xi, t) = \text{div}\left(T(\xi) \text{ grad } x(\xi, t)\right) + \text{div}\left(k_s(\xi) \text{ grad } \dfrac{\partial x}{\partial t}(\xi, t)\right), \\[2mm] \qquad x(\xi, 0) = x_0(\xi), \quad \dfrac{\partial x(\xi, 0)}{\partial t} = z_0(\xi), \quad \xi \in \Omega, \\[2mm] \dfrac{\partial x(\xi, t)}{\partial t} = 0, \quad \xi \in \partial\Omega, \ t \geq 0, \end{cases} \tag{5.8}$$

is also associated with a contraction semigroup on $X_{\mathcal{H}}$. We make the same assumptions on $\rho(\cdot)$ and $T(\cdot)$ as in (5.3), so that $\mathcal{H}$ in (5.4) again defines the inner product of a Hilbert space $X_{\mathcal{H}}$. Moreover, we assume that $k_s(\cdot) \in L^\infty(\Omega)^{n \times n}$ satisfies $k_s(\xi) + k_s(\xi)^* \geq \delta I$ for some $\delta > 0$ and almost every $\xi \in \Omega$, so that the multiplication operator

$$S_s x := \xi \mapsto k_s(\xi)\, x(\xi) \quad \text{on} \quad X_2 := L^2(\Omega)^n$$

is bounded, everywhere defined, and uniformly accretive. As extended operator we use $A_{ext,s}$ in (5.1), and we can use Theorem 1.2 and Lemma 5.2 to conclude that (5.8) is governed by a contraction semigroup on $X_{\mathcal{H}}$. The corresponding operator A_S is

$$A_{S,s} = \begin{bmatrix} \operatorname{div} \begin{bmatrix} S_s\,\operatorname{grad} & I \end{bmatrix} \\ \begin{bmatrix} \operatorname{grad} & 0 \end{bmatrix} \end{bmatrix},$$

$$\operatorname{dom}\left(A_{S,s}\right) = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \begin{bmatrix} H_0^1(\Omega) \\ L^2(\Omega)^n \end{bmatrix} \;\middle|\; S_s\,\operatorname{grad} x_1 + x_2 \in H^{\operatorname{div}}(\Omega) \right\}.$$

By Proposition 3.2, the external Cayley system transform of $A_{ext,s}$ is

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_s := \begin{bmatrix} \begin{bmatrix} \Delta & \operatorname{div} \\ \operatorname{grad} & 0 \end{bmatrix} & \& \begin{bmatrix} \sqrt{2}\,\operatorname{div} \\ 0 \end{bmatrix} \\ \begin{bmatrix} \sqrt{2}\,\operatorname{grad} & 0 \end{bmatrix} & \& \; I \end{bmatrix} \quad \text{with}$$

$$\operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_s\right) := \left\{ \begin{bmatrix} \sqrt{2}\,x_1 \\ \sqrt{2}\,x_2 \\ e - \operatorname{grad} x_1 \end{bmatrix} \in \begin{bmatrix} H_0^1(\Omega) \\ L^2(\Omega) \\ L^2(\Omega) \end{bmatrix} \;\middle|\; \right.$$

$$\left. e \in L^2(\Omega),\; x_2 + e \in H^{\operatorname{div}}(\Omega) \right\}.$$

Hence the main operator A is given by (see (2.4))

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \operatorname{div}(\operatorname{grad} x_1 + x_2) \\ \operatorname{grad} x_1 \end{bmatrix},$$

$$\operatorname{dom}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \begin{bmatrix} H_0^1(\Omega) \\ L^2(\Omega) \end{bmatrix} \;\middle|\; \operatorname{grad} x_1 + x_2 \in H^{\operatorname{div}}(\Omega) \right\}.$$

Here the control and observation operators are unbounded, so Proposition 5.3 is not applicable. However, the technique in Example 4.6 can easily be adapted to show that $\|\mathfrak{D}_0^T\| = 1$ also in this case, so application of Proposition 4.4 is excluded.

One can also treat wave equations with both viscous and structural damping. Indeed, from the proof of Proposition 5.1 it follows that the operator

$$A_{ext,vs} := \begin{bmatrix} 0 & \mathrm{div}\begin{bmatrix} I & I \end{bmatrix} & I \\ \begin{bmatrix} I \\ I \end{bmatrix}\mathrm{grad} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ -I & \begin{bmatrix} 0 & 0 \end{bmatrix} & 0 \end{bmatrix},$$

$$\mathrm{dom}\left(A_{ext,vs}\right) := \left\{ \begin{bmatrix} x_1 \\ x_2 \\ e_1 \\ e_2 \end{bmatrix} \in \begin{bmatrix} H_0^1(\Omega) \\ L^2(\Omega)^n \\ L^2(\Omega)^n \\ L^2(\Omega) \end{bmatrix} \mid x_2 + e_1 \in H^{\mathrm{div}}(\Omega) \right\},$$

$$(5.9)$$

is skew-adjoint (in particular closed) on $L^2(\Omega)^{2n+2}$. This operator can be associated with a wave equation with both viscous and structural damping by defining S_{vs} to be the operator of multiplication by $\begin{bmatrix} k_s(\cdot) & 0 \\ 0 & k_v(\cdot) \end{bmatrix}$ on $\begin{bmatrix} L^2(\Omega)^n \\ L^2(\Omega) \end{bmatrix}$. From here we can, however, not immediately deduce that the PDEs (5.3) and (5.8) are associated with contraction semigroups by setting $k_v(\cdot) := 0$ or $k_s(\cdot) := 0$, because S_{vs} is no longer uniformly accretive in that case.

6. Degenerate parabolic equations

In [25] it is shown how well-posedness of the heat equation (1.1) can be obtained from the well-posedness of the associated wave equation (1.2) by means of Theorem 1.1. In this section we show that Theorem 1.2 allows this same approach to be extended to degenerate parabolic PDEs, see e.g. [1,4,14]. In a degenerate parabolic equation, the physical parameter, such as α in equation (1.1), may become zero at the boundary of the spatial domain.

Let $H_0^{\mathrm{div}}(\Omega)$ denote the closure in $H^{\mathrm{div}}(\Omega)$ of the set of all functions in $C^\infty(\Omega)^n$ with support contained in the open set Ω. This equals the set of all functions in $H^{\mathrm{div}}(\Omega)$ for which the normal trace map is zero; see [10, Thm 5.4.2] or [8, Thm I.2.6]. Let K be a linear operator which maps $H_0^{\mathrm{div}}(\Omega)$ boundedly into U, where U is any Hilbert space. In addition assume that the operator $\begin{bmatrix} \mathrm{div} \\ -K \end{bmatrix}$ with domain $H_0^{\mathrm{div}}(\Omega)$ is closed as an unbounded operator $L^2(\Omega)^n \to \begin{bmatrix} L^2(\Omega) \\ U \end{bmatrix}$.

Now set $H := L^2(\Omega)$, $E := L^2(\Omega)^n$, $E_0 := H_0^{\mathrm{div}}(\Omega)$, $L := -\mathrm{div}|_{E_0}$, $G := 0$. Denoting the dual of E_0 with pivot space $L^2(\Omega)^n$ by E_0', we obtain that $L^* = \mathrm{grad} : L^2(\Omega) \to E_0'$ is bounded. It follows from [19, Thm 1.1] and Definition 2.4 that the following operator generates a contraction semigroup on $\begin{bmatrix} L^2(\Omega) \\ L^2(\Omega)^n \end{bmatrix}$:

$$A_{ext} = \begin{bmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & -K^*K \end{bmatrix} \quad \text{with domain}$$

$$\mathrm{dom}\left(A_{ext}\right) = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \begin{bmatrix} L^2(\Omega) \\ H_0^{\mathrm{div}}(\Omega) \end{bmatrix} \mid \mathrm{grad}\, x_1 - K^*K\, x_2 \in L^2(\Omega)^n \right\}.$$

$$(6.1)$$

Next we apply Theorem 1.2 with S bounded on $E = L^2(\Omega)^n$ (satisfying the conditions of item 3) to A_{ext}. We find that A_S generates a contraction semigroup on $L^2(\Omega)$,

where A_S is the mapping from x_1 to z_1 in

$$
\begin{cases}
z_1 = \operatorname{div} x_2 \\
z_2 = \operatorname{grad} x_1 - K^* K x_2 \\
x_2 = S z_2
\end{cases}
\iff
\begin{cases}
z_1 = \operatorname{div} x_2 \\
(S^{-1} + K^* K) x_2 = \operatorname{grad} x_1 \\
z_2 = S^{-1} x_2
\end{cases}.
\tag{6.2}
$$

Since E_0' is the dual of E_0 with pivot space E, we can regard S^{-1} as a bounded mapping from E_0 into E_0' in (6.2). Furthermore, for $x_2 \in E$ we have by item 3 of Theorem 1.2 that, with $\tilde{x}_2 = S^{-1} x_2$,

$$
\operatorname{Re} \left\langle S^{-1} x_2, x_2 \right\rangle_E = \operatorname{Re} \langle \tilde{x}_2, S \tilde{x}_2 \rangle_E \geq \delta \|\tilde{x}_2\|_E^2 \geq \tilde{\delta} \|x_2\|_E^2.
$$

Thus in particular, the operator $S^{-1} + K^* K$ is injective. Hence, (6.2) is solvable, i.e., $x_1 \in \operatorname{dom}(A_S)$ and $z_1 = A_S x_1$, if and only if

$$
x_1 \in L^2(\Omega), \quad \operatorname{grad} x_1 \in (S^{-1} + K^* K) H_0^{\operatorname{div}}(\Omega), \quad \text{and}
$$

$$
z_1 = \operatorname{div}\left(\left(S^{-1} + K^* K \right)^{-1} \operatorname{grad} x_1 \right);
$$

indeed then also x_1 and

$$
x_2 = \left(S^{-1} + K^* K \right)^{-1} \operatorname{grad} x_1 \in H_0^{\operatorname{div}}(\Omega)
$$

satisfy $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix} \right] \in \operatorname{dom}(A_{ext})$, since

$$
\operatorname{grad} x_1 - K^* K x_2 = z_2 = S^{-1} x_2 \in S^{-1} H_0^{\operatorname{div}}(\Omega) \subset L^2(\Omega).
$$

We conclude by Theorem 1.2 that

$$
A_S = \operatorname{div}\left(S^{-1} + K^* K \right)^{-1} \operatorname{grad}
\tag{6.3}
$$

with domain

$$
\operatorname{dom}(A_S) = \left\{ x \in L^2(\Omega) \mid \operatorname{grad} x \in (S^{-1} + K^* K) H_0^{\operatorname{div}}(\Omega) \right\}
\tag{6.4}
$$

generates a contraction semigroup on $L^2(\Omega)$. Here the multiplication by α in (1.1) has been replaced by the operator $(S^{-1} + K^* K)^{-1}$. This makes it possible to treat the degenerate case, as we make explicit in the next example.

The boundary condition on the operator A_S in equation (6.3) and (6.4) is that the normal trace of $\left(S^{-1} + K^* K \right)^{-1} \operatorname{grad} x$ should be zero along all of the boundary, and this case is technically rather simple to deal with. To illustrate how more challenging boundary conditions (where different parts of the boundary are coupled) can be handled, we take a one-dimensional spatial domain.

We set $\beta(\xi) := \xi^{-\alpha}$, $\xi \in (0, 1)$, with $\alpha \in (0, 1)$. Then the corresponding multiplication operator $K = M_\beta$ maps $E_0 := \{x \in H^1(0, 1) \mid x(0) = 0\}$ with the $H^1(0, 1)$ norm into $L^2(0, 1)$, because

$$|\beta(\xi)x(\xi)| = \beta(\xi)\left|\int_0^\xi 1 \cdot x'(\tau)\,d\tau\right| \le \beta(\xi)\sqrt{\xi}\,\|x'\|_{L^2(0,1)} \le \beta(\xi)\sqrt{\xi}\,\|x\|_{E_0}$$

by Cauchy–Schwartz, and $\int_0^1 \left(\beta(\xi)\sqrt{\xi}\right)^2 d\xi = \frac{1}{2-2\alpha}$. Hence the norm of M_β is bounded by $\frac{1}{\sqrt{2-2\alpha}}$, and M_β^* is multiplication by $\bar{\beta} = \beta$, mapping $L^2(0, 1)$ continuously into the dual E_0' of E_0 with pivot space $L^2(0, 1)$.

Take $\kappa > 0$ arbitrarily and observe that $x_1' + \beta e \in L^2(0, 1)$ and $\beta|_{(\frac{1}{2}, 1)}$ bounded implies that $x_1' = (z - \beta e)|_{(\frac{1}{2}, 1)} \in L^2(\frac{1}{2}, 1)$. Hence $x_1|_{(\frac{1}{2}, 1)} \in H^1(\frac{1}{2}, 1)$ and $x_1(1)$ is well-defined. We leave it to the reader to verify that the (unbounded) adjoint of the operator

$$A_{ext,0} = \begin{bmatrix} 0 & \frac{\partial}{\partial \xi} & 0 \\ \frac{\partial}{\partial \xi} & 0 & M_\beta^* \\ 0 & -M_\beta & 0 \end{bmatrix} \tag{6.5}$$

with domain

$$\mathrm{dom}\,(A_{ext,0}) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ e \end{bmatrix} \in L^2(0, 1)^3 \;\middle|\; x_2 \in H^1(0, 1),\; x_1' + \beta e \in L^2(0, 1), \right.$$

$$\left. x_2(0) = 0,\; x_1(1) = -\kappa x_2(1) \right\}$$

is

$$A_{ext,0}^* = -\begin{bmatrix} 0 & \frac{\partial}{\partial \xi} & 0 \\ \frac{\partial}{\partial \xi} & 0 & M_\beta^* \\ 0 & -M_\beta & 0 \end{bmatrix} \quad \text{with domain}$$

$$\mathrm{dom}\,(A_{ext,0}^*) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ e \end{bmatrix} \in L^2(0, 1)^3 \;\middle|\; x_2 \in H^1(0, 1),\; x_1' + \beta e \in L^2(0, 1), \right.$$

$$\left. x_2(0) = 0,\; x_1(1) = \kappa x_2(1) \right\}.$$

A main step in this verification is showing that $z_1|_{[a,1]} \in H^1(a, 1)$ for all $a \in (0, 1)$ whenever $(z_1, z_2, h) \in \mathrm{dom}\,(A_{ext,0}^*)$, which again follows from the boundedness of β on every interval $[a, 1]$, $a \in (0, 1)$. Since both $A_{ext,0}$ and $A_{ext,0}^*$ are closed and dissipative, $A_{ext,0}$ is the generator of a contraction semigroup on $L^2(\Omega)^3$.

Applying Theorem 1.1 to the operator in (6.5) with $S = I$, we obtain

$$A_{S,0} = \begin{bmatrix} 0 & \frac{\partial}{\partial \xi} & 0 \\ \frac{\partial}{\partial \xi} & 0 & M_\beta^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & -M_\beta \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \xi} & -M_\beta^* M_\beta \end{bmatrix}, \quad (6.6)$$

with domain

$$\mathrm{dom}\,(A_{S,0}) = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \begin{bmatrix} L^2(0,1) \\ H^1(0,1) \end{bmatrix} \,\middle|\, x_1' - \beta^2 x_2 \in L^2(0,1), \right.$$

$$\left. x_2(0) = 0, \ x_1(1) = -\kappa x_2(1) \right\}.$$

Note that the operator (6.6) is of a similar form as the operator in (6.1), but now the boundary conditions on x_1 and x_2 are coupled at $\xi = 1$. By Theorem 1.1, $A_{S,0}$ generates a contraction semigroup on $L^2(\Omega)^2$.

We next apply Theorem 1.2 to the operator $A_{ext} := A_{S,0}$ with $S = M_s$, i.e., multiplication by the function s. The calculations here are the same as in the n-D case above, and the result is

$$A_{S,1} x = \qquad \frac{\partial}{\partial \xi} \left(\frac{1}{s^{-1}(\xi) + \beta(\xi)^2} \frac{\partial x}{\partial \xi} \right) \quad \text{with domain}$$

$$\mathrm{dom}\,(A_{S,1}) = \qquad \left\{ x \in L^2(0,1) \,\middle|\, \frac{1}{s^{-1} + \beta^2} x' \in H^1(0,1), \right.$$

$$\left. \left(\frac{1}{s^{-1} + \beta^2} x' \right)(0) = 0, \ x(1) = -\kappa \left(\frac{1}{s^{-1} + \beta^2} x' \right)(1) \right\}. \quad (6.7)$$

Using the expression $\beta(\xi) = \xi^{-\alpha}$, this becomes

$$A_{S,1} x = \frac{\partial}{\partial \xi} \left(\frac{s(\xi)\xi^{2\alpha}}{1 + s(\xi)\xi^{2\alpha}} x'(\xi) \right) \qquad \text{with domain}$$

$$\mathrm{dom}\,(A_{S,1}) = \left\{ x \in L^2(0,1) \,\middle|\, \frac{s(\xi)\xi^{2\alpha}}{1 + s(\xi)\xi^{2\alpha}} x'(\xi) \in H^1(0,1), \right.$$

$$\left. \left(s(\xi)\xi^{2\alpha} x'(\xi) \right)(0) = 0, \ x(1) = -\kappa \frac{s(1)}{1 + s(1)} x'(1) \right\}.$$

Here the thermal diffusivity $s(\xi)\xi^{2\alpha}(1 + s(\xi)\xi^{2\alpha})^{-1}$ becomes zero at $\xi = 0$.

This way any thermal diffusivity that can be written as $\tilde{s}\beta^{-2}$ with $\tilde{s}$ positive, bounded and bounded away from zero can be captured. We leave it for future work to extend the situation with mixed boundary conditions to the n-D case.

Acknowledgements

Appendix A. A lemma on unbounded adjoints

The following result must be well-known in the literature, but we could not find a suitably formulated reference:

LEMMA A.1. *Let H, K, and L be Hilbert spaces, and let $Q : K \to L$ and $R : H \to K$ be possibly unbounded operators. If Q is bounded, or if R is bounded and surjective, then $(QR)^* = R^*Q^*$.*

Proof. The proof for the case where Q is bounded is trivial. Moreover, the inclusion $R^*Q^* \subset (QR)^*$ always holds for linear operators Q and R, as one easily shows. We finish the proof by showing that if R is bounded and surjective, then the converse inclusion also holds.

Assume that there exists a w such that $\langle QRx, z \rangle = \langle x, w \rangle$ for all $x \in \mathrm{dom}\,(QR)$. Then in particular $0 = \langle x, w \rangle$ for all $x \in \ker\,(R)$, so that $w \in \ker\,(R)^\perp = \mathrm{ran}\,(R^*)$, since R^* has closed range by the Closed Range Theorem. Writing $w = R^*v$, we thus obtain that $\langle QRx, z \rangle = \langle x, R^*v \rangle = \langle Rx, v \rangle$ for all $Rx \in \mathrm{dom}\,(Q)$, again using the boundedness and surjectivity of R. Therefore $z \in \mathrm{dom}\,(Q^*)$ and $Q^*z = v$.

Hence $z \in \mathrm{dom}\,((QR)^*)$ and $w = (QR)^*z$ imply $z \in \mathrm{dom}\,(R^*Q^*)$ and $w = R^*Q^*z$, i.e., that $(QR)^* \subset R^*Q^*$. $\qquad\square$

REFERENCES

[1] M. Campiti, G. Metafune, and D. Pallara, *Degenerate self-adjoint evolution equations on the unit interval*, Semigroup Forum **57** (1998), no. 1, 1–36.

[2] Ruth F. Curtain and Hans Zwart, *An introduction to infinite-dimensional linear systems theory*, Springer-Verlag, New York, 1995.

[3] Charles A. Desoer and Mathukumalli Vidyasagar, *Feedback systems: Input-output properties*, Electrical Science Series, Academic Press, New York San Francisco London, 1975.

[4] Klaus-Jochen Engel and Rainer Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000.

[5] Sylvain Ervedoza and Enrique Zuazua, *Uniform exponential decay for viscous damped systems*, Advances in Phase Space Analysis of Partial Differential Equations, vol. 78, 95–112, Birkhäuser Boston, Inc., Boston, MA, 2009.

[6] Sylvain Ervedoza and Enrique Zuazua, *Sharp observability estimates for heat equations*, Arch. Ration. Mech. Anal. **202** (2011), no. 3, 975–1017.

[7] H. O. Fattorini and D. L. Russell, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. **43** (1971), 272–292.

[8] Vivette Girault and Pierre-Arnaud Raviart, *Finite element methods for Navier-Stokes equations: Theory and algorithms*, Springer Series in Computational Mathematics, vol. 5, Springer-Verlag, Berlin, 1986.

[9] Birgit Jacob and Hans Zwart, *Linear port-hamiltonian systems on infinite-dimensional spaces*, Operator Theory: Advances and Applications, vol. 223, Birkhäuser-Verlag, 2012.

[10] Mikael Kurula and Hans Zwart, *The duality between the gradient and divergence operators on bounded Lipschitz domains*, Memorandum 1994, University of Twente, 2012, available at http://doc.utwente.nl/84362/.

[11] Mikael Kurula and Hans Zwart, *Linear wave systems on n-D spatial domains*, Internat. J. Control **88** (2015), 1063–1077.

[12] Mikael Kurula, Hans Zwart, Arjan van der Schaft, and Jussi Behrndt, *Dirac structures and their composition on Hilbert spaces*, J. Math. Anal. Appl. **372** (2010), 402–422.

[13] Luc Miller, *The control transmutation method and the cost of fast controls*, SIAM J. Control Optim. **45** (2006), no. 2, 762–772 (electronic).

[14] Amnon Pazy, *Semi-groups of linear operators and applications to partial differential equations*, Springer-Verlag, Berlin, 1983.

[15] Ralph S. Phillips, *Dissipative operators and hyperbolic systems of partial differential equations*, Trans. Amer. Math. Soc. **90** (1959), 193–254.

[16] Felix Schwenninger and Hans Zwart, *Generators with a closure relation*, Operators and Matrices **8** (2014), no. 1.

[17] Olof J. Staffans, *On scattering passive system nodes and maximal scattering dissipative operators*, Proc. Amer. Math. Soc. **141** (2013), no. 4, 1377–1383.

[18] Olof J. Staffans, *Well-posed linear systems*, Cambridge University Press, Cambridge and New York, 2005.

[19] Olof J. Staffans and George Weiss, *A physically motivated class of scattering passive linear systems*, SIAM J. Control Optim. **50(5)** (2012), 3083–3112.

[20] Marius Tucsnak and George Weiss, *How to get a conservative well-posed linear system out of thin air. Part II. Controllability and stability*, SIAM J. Control Optim. **42** (2003), 907–935.

[21] Marius Tucsnak and George Weiss, *Observation and control for operator semigroups*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 2009, (electronic version).

[22] Javier Andres Villegas, *A port-Hamiltonian approach to distributed parameter systems*, Ph.D. thesis, 2007, Available at http://doc.utwente.nl/57842/.

[23] George Weiss and Olof J. Staffans, *Maxwell's equations as a scattering passive linear system*, SIAM J. Control Optim. **51** (2013), no. 5, 3722–3756.

[24] George Weiss and Marius Tucsnak, *How to get a conservative well-posed linear system out of thin air. Part I. Well-posedness and energy balance*, ESAIM. Control, Optim. Calc. Var. **9** (2003), 247–274.

[25] Hans Zwart, Yann Le Gorrec, and Bernhard Maschke, *Linking hyperbolic and parabolic p.d.e.'s.*, Proceedings of the 50th IEEE Conference on Decision and Control (CDC), 2011.

[26] Hans Zwart, Yann Le Gorrec, Bernhard Maschke, and Javier Villegas, *Building parabolic and hyperbolic partial differential equations from simple hyperbolic ones*, submitted, 2015.

M. Kurula
Åbo Akademi Mathematics,
Fänriksgatan 3B,
20500 Åbo,
Finland
E-mail: mkurula@abo.fi

H. Zwart
Department of Applied Mathematics,
University of Twente,
7500 AE Enschede,
The Netherlands
E-mail: h.j.zwart@utwente.nl