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Synchronization in the presence of unknown, nonuniform and arbitrarily large communication delay

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ABSTRACT

This paper studies both output and state synchronization problems for multi-agent systems with agents that are identical and coupled through a network with unknown, nonuniform and arbitrarily large communication delay. We assume that agents are non-introspective (i.e. agents have no access to any of their own states) in the output synchronization problem. The network can be either undirected or directed. In the case of undirected network, exact knowledge of the network is not required and only a specific lower bound is needed. The objective is to design a decentralized protocol such that the multi-agent system achieves output synchronization or state synchronization for any unknown, nonuniform and arbitrarily large communication delay.

1. Introduction

In the past few decades, synchronization problems for multi-agent systems have received substantial attention, where the objective is to achieve asymptotic agreement on a common state (*state synchronization*) or output trajectory (*output synchronization*) among agents of the network through decentralized control protocols. Some early results can be found in, e.g., [4,9,11,15], for state synchronization problems of homogeneous networks (i.e. agents are identical), and in, e.g., [1,3,18,20,21], for output synchronization problems for heterogeneous networks.

Recently, synchronization in a network with time delay has attracted a great deal of interest. As clarified in [2], we can identify two kinds of delay. Firstly there is *communication delay*, which results from limitations on the communication between agents. Secondly we have *input delay* which is due to computational limitations of an individual agent. Many works have focused on dealing with input delay, progressing from single- and double-integrator agent dynamics (see e.g. [10,13,14] to more general agent dynamics (see e.g. [5,12,17,24]). Their objective is often to investigate how much input delay can be present in the system while still being able to guarantee synchronization. The amount of input delay that can still be handled will depend on the agent dynamics and the network properties.

In the case of communication delay, only for a constant synchronization trajectory do we preserve the diffusive nature of the network. This diffusive nature is an intrinsic part of the currently available design techniques and hence only this case has been studied. [13] and [19] consider single-integrator dynamics in the network and it is demonstrated that the communication delay does not affect the synchronizability of the network. [7] and [8] give the consensus conditions for networks with higher-order but SISO dynamics. In [6], second-order dynamics are investigated, but the communication delays are assumed known.

However, the above works on communication delay only consider simple dynamics. In this paper, we deal with general higher-order agent dynamics. That is, we investigate the regulated output/state synchronization problem for directed or undirected, weighted networks composed of general higher-order agent dynamics and with unknown, nonuniform and asymmetric communication delays. Regarding the output synchronization problem, we utilize the low-gain theory to do the controller design. Thus, we impose a constraint that the agent dynamics have all poles in the closed-left half complex plane. The results in this paper confirm the conclusion of [2] that it is possible to achieve consensus under a network with arbitrarily large communication delay. It is also shown that for some general agent dynamics, not all synchronization trajectories can be achieved. An explicit set of trajectories are identified and purely decentralized controllers are designed.

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1.1. Notations and definitions

Given a matrix $A \in \mathbb{C}^{m\times n}$, A' denotes its conjugate transpose while $\|A\|$ denotes the induced 2-norm. We denote by $\mathrm{diag}\{a_1,\dots,a_N\}$, a diagonal matrix with $a_1,\dots,a_N$ as the diagonal elements, and by $\mathrm{col}\{x_1,\dots,x_N\}$, a column vector with $x_1,\dots,x_N$ stacked together. Let $\mathbf{j}$ indicate $\sqrt{-1}$. Moreover, $A\otimes B$ indicates the Kronecker product between A and B.

A *weighted graph* $\mathcal{G}$ is defined by a triple $(\mathcal{V},\mathcal{E},\mathcal{A})$, where $\mathcal{V}=\{1,\dots,N\}$ is a node set, $\mathcal{E}\subseteq V\times V$ is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$ is the weighting matrix, with $a_{ij}>0$ iff $(i,j)\in\mathcal{E}$ and $a_{ii}=0$. If $a_{ij}=a_{ji}$ for all $(i,j)\in\mathcal{E}$, the graph is called *undirected*; otherwise *directed*. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1,\dots,i_k\}$ such that $(i_j,i_{j+1})\in\mathcal{E}$ for $j=1,\dots,k-1$. A graph is *connected* if there exists a path between every pair of nodes. A directed graph is *balanced* if $\sum_{j=1}^N a_{ij}=\sum_{j=1}^N a_{ji}$ for all $i=1,\dots,N$. A *directed tree* is a subgraph (subset of nodes and edges) in which every node has exactly one parent node except for one node, called the *root*, which has no parent node. In this case, the root has a directed path to every other node in the tree. A *directed spanning tree* is a subgraph which is a directed tree containing all the nodes of the original graph. An agent is called a *root agent* if it is the root of some directed spanning tree of the associated graph. For a weighted graph $\mathcal{G}$, the matrix $L=[\ell_{ij}]$ with diagonal elements $\ell_{ii}=\sum_{j=1}^N a_{ij}$ and non-diagonal elements $\ell_{ij}=-a_{ij}$ is called the *Laplacian matrix* associated with the graph $\mathcal{G}$. All eigenvalues of L are located in the closed right-half complex plane with at least one eigenvalue at zero which is associated with right eigenvector $\mathbf{1}$. In case the graph is strongly connected then the multiplicity of the eigenvalue at zero is 1 and all other eigenvalues are in the open right-half complex plane. When $\mathcal{G}$ is undirected, L is symmetric. If $\mathcal{G}$ is balanced then $x'Lx\geq 0$ for all x.

Definition 1. A linear time-invariant dynamics (A, B, C, D) is right-invertible if, given a smooth reference output y_r, there exists an initial condition $x(0)$ and an input u that ensures $y(t)=y_r(t)$ for all $t\geq 0$. For single-input-single-output system, a system is right-invertible if and only if its transfer function is nonzero.

Definition 2. The invariant zeros of a linear dynamics (A, B, C, D) are those points $\lambda\in\mathbb{C}$ for which

$$\mathrm{rank}\begin{pmatrix}\lambda I-A & -B\\ C & D\end{pmatrix} < \mathrm{normrank}\begin{pmatrix}sI-A & -B\\ C & D\end{pmatrix},$$

where by “normrank” we mean the rank of a matrix with entries in the field of rational functions.

2. Multi-agent systems

The multi-agent system (MAS) we will consider in this paper is composed of N identical general agents, which are denoted by Σ_i with $i\in\{1,\dots,N\}$,

$$\Sigma_i:\begin{cases}\dot{x}_i(t)=Ax_i(t)+Bu_i(t),\\ y_i(t)=Cx_i(t),\\ \zeta_i(t)=\displaystyle\sum_{j=1}^N a_{ij}(y_i(t)-y_j(t-\tau_{ij})),\end{cases} \tag{1}$$

where $x_i\in\mathbb{R}^n$, $u_i\in\mathbb{R}^m$ and $y_i\in\mathbb{R}^p$ are the state, input and output of agent i, and $\tau_{ij}\in\mathbb{R}^+$ $(i\neq j)$ represents an unknown constant communication delays from agent j to agent i. The communication delay implies that it takes τ_{ij} seconds for agent j to transfer its state information to agent i. In the above $a_{ij}\geq 0$ and $a_{ii}=0$. The communication presented in (1) can be represented by a weighted graph $\mathcal{G}$ with each node indicating an agent in the network and the weight of an edge is given by the coefficient a_{ij}. ζ_i is a delayed collection of relative outputs of agent i and its neighboring agents. This is referred to as *partial-state coupling*. In the special case of $C=I$, we have $y_i=x_i$ and this case is referred to as *full-state coupling*.

We need the following assumptions:

Assumption 1. (A, B) is stabilizable and (A, C) is detectable.

Assumption 2. All eigenvalues of A are in the closed left-half complex plane.

Our goal is to achieve output synchronization while the synchronous trajectory should be equal to an, a priori given, constant trajectory, denoted by $y_r\in\mathbb{R}^p$. We assume that there is a set of agents $\Pi_{\mathcal{G}}$ which have access to the constant trajectory information. Agents have access to the following information:

$$\bar{\zeta}_i(t)=\zeta_i(t)+\iota_i(y_i(t)-y_r(t)), \tag{2}$$

for $i=1,\dots,N$. If agent i is part of the set $\Pi_{\mathcal{G}}$ then $\iota_i=1$ and the agent has access to this relative information regarding y_r. For the other agents, we have $\iota_i=0$ and hence the agent does not have access to this relative information regarding y_r.

Define $\bar{\ell}_{ii}=\sum_{j=1}^N a_{ij}+\iota_i$ and $\bar{\ell}_{ij}=-a_{ij}$ for $i\neq j$. Then, the matrix $\bar{L}=[\bar{\ell}_{ij}]$ is referred to as the *expanded Laplacian matrix*.

In the following sections, we will study output and state synchronization for both undirected graphs and directed graphs. If the graph is undirected, we do not need precise information of a network communication topology, but only some rough characterization of the network. In this case, we only need a lower bound on the smallest eigenvalue of the expanded Laplacian

Definition 3. For given real number $\beta>0$, the set $\mathbb{G}_{\beta,N}$ consists of all weighted graphs composed of N nodes satisfying the following properties:

- Each agent is part of a directed tree with a root which belongs to the set $\Pi_{\mathcal{G}}$.
- The eigenvalues of the expanded Laplacian matrix $\bar{L}$, denoted by $\lambda_1,\dots,\lambda_N$ satisfy $\mathrm{Re}\,\lambda_i>\beta$.

Remark 1. If each agent is part of a directed tree with a root which belongs to the set $\Pi_{\mathcal{G}}$ then all eigenvalues of $\bar{L}$ have positive real part (see for instance [5]). Hence the graph is in $\mathbb{G}_{\beta,N}$ for some sufficiently small $\beta>0$. Our protocol design will rely only on β for undirected graphs and will be independent of other information about the network.

3. Output synchronization for agents with partial-state coupling

In this section, we will consider output synchronization problem for networks with unknown, nonuniform and arbitrarily large communication delay. We will first study undirected graphs, and then extend the result to directed graphs.

3.1. Undirected graphs

The output synchronization problem for undirected graphs can be formulated as follows.

Problem 1. Let β be a given positive real number. Consider a multi-agent system described by (1) associated with a graph $\mathcal{G}\in\mathbb{G}_{\beta,N}$. Given a constant trajectory $y_r\in R^p$ and let it be available to at least one agent. The *output synchronization* problem for networks with unknown, nonuniform and arbitrarily large communication delay is to find a distributed linear time-invariant dynamic controller of the form

$$\begin{cases} \dot{x}_{i,c} = A_c x_{i,c} + B_c \bar{\zeta}_i, \\ u_i = C_c x_{i,c}, \end{cases} \quad (i = 1, \dots, N) \tag{3}$$

for each agent such that, for any graph $\mathcal{G} \in \mathbb{G}_{\beta,N}$ and for any communication delay $\tau_{ij} \in \mathbb{R}^+$, the output of each agent converges to the given constant trajectory, i.e.,

$$\lim_{t \to \infty} (y_i(t) - y_r) = 0, \tag{4}$$

for all $i \in \{1, \dots, N\}$.

The main result will be presented in two theorems. The first theorem deals with the case when the system is right-invertible and has no invariant zeros at the origin. The second theorem deals with the general case.

Theorem 1. *Let β be a given positive real number. Consider a multi-agent system with agents described by (1) and let Assumptions 1 and 2 hold. Given any constant trajectory $y_r \in \mathbb{R}^p$ and let the relative information regarding y_r be available to at least one agent which is a root agent. Assume the above multi-agent system is associated with an undirected graph $\mathcal{G} \in \mathbb{G}_{\beta,N}$. Then, Problem 1 is solvable if the system presented by (A, B, C) is right-invertible and has no invariant zero at the origin. More specifically, there exists a linear dynamic controller of the type (3) such that output synchronization is achieved for any undirected graph $\mathcal{G} \in \mathbb{G}_{\beta,N}$, for any communication delay $\tau_{ij} \in \mathbb{R}^+$, and for any $y_r \in \mathbb{R}^p$.*

In the general case, we have to restrict our choice of y_r. Let

$$\begin{aligned} \mathscr{C}_y &= \left\{ y \in \mathbb{R}^p \,\middle|\, \begin{pmatrix} 0 \\ y \end{pmatrix} \in \operatorname{Im} \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \right\} \\ &= \{ y \in \mathbb{R}^p \mid \exists x \in \mathbb{R}^n, u \in \mathbb{R}^m : Ax + Bu = 0, Cx = y \}. \end{aligned} \tag{5}$$

Note that $\mathscr{C}_y = \mathbb{R}^p$ if (A, B, C) is right-invertible and without invariant zeros in the origin.

Theorem 2. *Let β be a given positive real number. Consider a multi-agent system with agents described by (1) and let Assumptions 1 and 2 hold. Given any constant trajectory $y_r \in \mathbb{R}^p$ and let the relative information regarding y_r be available to at least one agent which is a root agent. Assume the above multi-agent system is associated with an undirected graph $\mathcal{G} \in \mathbb{G}_{\beta,N}$. Then, Problem 1 is solvable if and only if $y_r \in \mathscr{C}_y$. More specifically, given $y_r \in \mathscr{C}_y$, there exists a linear dynamic controller of the type (3) such that output synchronization is achieved for any undirected graph $\mathcal{G} \in \mathbb{G}_{\beta,N}$ and for any communication delay $\tau_{ij} \in \mathbb{R}^+$.*

In order to prove the above theorems we need the following lemma.

Lemma 1. *Let β be a lower bound for the eigenvalues of $\bar{L}$. Then, for all communication delays $\tau_{ij} \in \mathbb{R}^+$ $(i, j = 1, \dots, N)$ and all $\omega \in \mathbb{R}$, the real part of all eigenvalues of $\bar{L}_{\mathbf{j}\omega}(\tau)$ will be larger than or equal to β, where*

$$\bar{L}_s(\tau) = \begin{pmatrix} \bar{\ell}_{11} & \bar{\ell}_{12} e^{-\tau_{12} s} & \cdots & \bar{\ell}_{1N} e^{-\tau_{1N} s} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\ell}_{k1} e^{-\tau_{k1} s} & \bar{\ell}_{kk} & \cdots & \bar{\ell}_{kN} e^{-\tau_{kN} s} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\ell}_{N1} e^{-\tau_{N1} s} & \bar{\ell}_{N2} e^{-\tau_{N2} s} & \cdots & \bar{\ell}_{NN} \end{pmatrix}$$

is the expanded Laplacian matrix in the frequency domain and τ denotes a vector consisting of all τ_{ij} $(i \neq j)$ with $i, j \in \{1, \dots, N\}$.

Proof. All eigenvalues of $\bar{L}_{\mathbf{j}\omega}(\tau)$ are in the set

$$\left\{ \nu' \bar{L}_{\mathbf{j}\omega}(\tau) \nu \mid \nu \in \mathbb{C}^N, \|\nu\| = 1 \right\},$$

and therefore it is sufficient to establish that all elements in this set have a real part larger than or equal to β.

Since $\bar{L}$ is symmetric and $\mathcal{G} \in \mathbb{G}_{\beta,N}$, $\nu' \bar{L} \nu$ is real and larger than or equal to β, provided $\|\nu\| = 1$.

Next, consider an arbitrary vector $\nu \in \mathbb{C}^N$. We have

$$\nu' \bar{L}_{\mathbf{j}\omega}(\tau) \nu = \sum_{i=1}^N |\nu_i|^2 \bar{\ell}_{ii} + \sum_{i=1}^N \sum_{j, j \neq i}^N \nu_i' \nu_j \bar{\ell}_{ij} e^{-\tau_{ij} \mathbf{j}\omega}.$$

Since $\bar{\ell}_{ij}$ $(i \neq j)$ is negative or equal to zero, we get

$$\begin{aligned} \operatorname{Re}\left(\nu' \bar{L}_{\mathbf{j}\omega}(\tau) \nu\right) &\geq \sum_{i=1}^N |\nu_i|^2 \bar{\ell}_{ii} + \sum_{i=1}^N \sum_{j, j \neq i}^N |\nu_i' \nu_j| \bar{\ell}_{ij} \\ &= \begin{pmatrix} |\nu_1| \\ \vdots \\ |\nu_N| \end{pmatrix}' \bar{L} \begin{pmatrix} |\nu_1| \\ \vdots \\ |\nu_N| \end{pmatrix} \geq \beta, \end{aligned}$$

which completes the proof. □

Proof of Theorem 1. The design consists of two steps. In the first step we will design a precompensator for each agent. In the second step, we will design a dynamic protocol for compensated MAS to achieve synchronization.

Step 1: Since (A, B, C) is right-invertible and has no invariant zeros at the origin, the matrix

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

has full row-rank. By detectability of (C, A) the first n columns of this matrix are linearly independent. This implies that there exists an injective matrix V such that

$$\begin{pmatrix} A & BV \\ C & 0 \end{pmatrix} \tag{6}$$

is square and invertible. Next consider the so-called regulator equations

$$\begin{pmatrix} A & BV \\ C & 0 \end{pmatrix} \begin{pmatrix} \Pi \\ \Gamma \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix}.$$

Invertibility of (6) trivially implies this equation has a unique solution. Next note that

$$\operatorname{rank} \begin{pmatrix} A & BV\Gamma \\ C & 0 \end{pmatrix} = n + \operatorname{rank} \Gamma$$

by the invertibility of (6). We design a precompensator for each agent of our multi-agent system

$$\begin{aligned} \dot{p}_i &= \begin{pmatrix} I & 0 \end{pmatrix} v_i, \qquad p_i(t) \in \mathbb{R}^\nu \\ u_i &= \Gamma_1 p_i + \begin{pmatrix} 0 & \Gamma_2 \end{pmatrix} v_i \end{aligned} \tag{7}$$

where v_i is the new input, Γ_1 is injective and such that $\operatorname{Im} V\Gamma = \operatorname{Im} \Gamma_1$ while $\nu = \operatorname{rank} \Gamma$. On the other hand, Γ_2 is chosen such that

$$\begin{pmatrix} \Gamma_1 & \Gamma_2 \end{pmatrix} \tag{8}$$

is square and invertible.

The interconnection of (1) and (7) (compensated MAS) is of the form

$$\begin{cases} \dot{\tilde{x}}_i(t) = \tilde{A} \tilde{x}_i(t) + \tilde{B} v_i(t), \\ y_i(t) = \tilde{C} \tilde{x}_i(t), \\ \zeta_i(t) = \sum_{j=1}^N a_{ij} (y_i(t) - y_j(t - \tau_{ij})), \end{cases} \tag{9}$$

where

$$\tilde{x}_i = \begin{pmatrix} x_i \\ p_i \end{pmatrix}, \ \tilde{A} = \begin{pmatrix} A & B\Gamma_1 \\ 0 & 0 \end{pmatrix}, \ \tilde{B} = \begin{pmatrix} 0 & B\Gamma_2 \\ I & 0 \end{pmatrix}, \ \tilde{C} = \begin{pmatrix} C & 0 \end{pmatrix}.$$

We need to verify a number of properties of this system. First note that stabilizability follows immediately from (8) and the stabilizability of (A, B). Next, we show detectability. We need to verify that

$$\operatorname{rank} \begin{pmatrix} sI - A & -B\Gamma_1 \\ 0 & sI \\ C & 0 \end{pmatrix} = n + v = n + \operatorname{rank} \Gamma_1$$

for all s in the closed right-half complex plane. For $s \neq 0$, this immediately follows from the detectability of (C, A). For $s = 0$, we have:

$$\operatorname{rank} \begin{pmatrix} -A & -B\Gamma_1 \\ 0 & 0 \\ C & 0 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} -A & -BV\Gamma \\ C & 0 \end{pmatrix} = n + \operatorname{rank} \Gamma_1.$$

Step 2: The controller for multi-agent system (9) is designed as

$$\begin{cases} \dot{\chi}_i = (\tilde{A} + K\tilde{C})\chi_i - K\bar{\zeta}_i, \\ v_i = -\alpha \tilde{B}' P_\varepsilon \chi_i, \end{cases} \tag{10}$$

where K is such that $\tilde{A} + K\tilde{C}$ is Hurwitz stable and P_ε is the unique solution of the algebraic Riccati equation

$$\tilde{A}' P_\varepsilon + P_\varepsilon \tilde{A} - P_\varepsilon \tilde{B}\tilde{B}' P_\varepsilon + \varepsilon I = 0,$$

while α and ε are design parameters to be chosen later.

Next, we will prove that with the above controllers, the output of each agent converges to the constant trajectory y_r. First we need to show that there exists a $\tilde{\Pi}$ such that $\tilde{A}\tilde{\Pi} = 0$ and $\tilde{C}\tilde{\Pi} = I$. Let W be such that $\Gamma_1 W = V\Gamma$. In that case it is easy to verify that we can choose

$$\tilde{\Pi} = \begin{pmatrix} \Pi \\ W \end{pmatrix}.$$

For $i = 1, \ldots, N$, define $\bar{x}_i = \tilde{x}_i - \tilde{\Pi} y_r$, and the output synchronization error $e_i = y_i - y_r$. Then, we get the error dynamics

$$\begin{cases} \dot{\bar{x}}_i = \tilde{A}\bar{x}_i + \tilde{B} v_i, \\ e_i = \tilde{C}\bar{x}_i. \end{cases} \tag{11}$$

Moreover, $\bar{\zeta}_i(t)$ can be rewritten as

$$\bar{\zeta}_i(t) = \sum_{j=1}^{N} a_{ij} \tilde{C}(\bar{x}_i(t) - \bar{x}_j(t - \tau_{ij})) + \iota_i \tilde{C}\bar{x}_i(t).$$

Let $\bar{x} = \operatorname{col}\{\bar{x}_1, \ldots, \bar{x}_N\}$ and $\chi = \operatorname{col}\{\chi_1, \ldots, \chi_N\}$. Then, the full closed-loop system can be written in the frequency domain as

$$\begin{pmatrix} s\bar{x} \\ s\chi \end{pmatrix} = \begin{pmatrix} I_N \otimes \tilde{A} & -\alpha I_N \otimes \tilde{B}\tilde{B}' P_\varepsilon \\ -\bar{L}_s(\tau) \otimes K\tilde{C} & I_N \otimes (\tilde{A} + K\tilde{C}) \end{pmatrix} \begin{pmatrix} \bar{x} \\ \chi \end{pmatrix}, \tag{12}$$

where $\bar{L}_s(\tau)$ is defined in Lemma 1.

Next, we will prove that (12) is asymptotically stable for all communication delay $\tau_{ij} \in \mathbb{R}^+$. We will first prove stability without communication delay and then prove stability for the case including communication delay.

When there is no communication delay in the network, according to [22], the stability of system (12) is equivalent to asymptotic stability of the matrix

$$\begin{pmatrix} \tilde{A} & -\alpha \tilde{B}\tilde{B}' P_\varepsilon \\ -\lambda_i K\tilde{C} & \tilde{A} + K\tilde{C} \end{pmatrix}$$

for all $i \in \{1, \ldots, N\}$, where λ_i is the eigenvalue of the expanded Laplacian matrix $\bar{L}$ with its lower bound of β. Note that

$$\begin{pmatrix} \lambda_i & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \tilde{A} & -\alpha \tilde{B}\tilde{B}' P_\varepsilon \\ -\lambda_i K\tilde{C} & \tilde{A} + K\tilde{C} \end{pmatrix} \begin{pmatrix} \lambda_i^{-1} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \tilde{A} & -\alpha \lambda_i \tilde{B}\tilde{B}' P_\varepsilon \\ -K\tilde{C} & \tilde{A} + K\tilde{C} \end{pmatrix}.$$

By choosing

$$\alpha > \frac{1}{\beta}, \tag{13}$$

and using the result of Lemma 4 in the appendix, we find that there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$ system (12) is asymptotically stable without any communication delay.

In the case of communication delay, according to Lemma 3 in the appendix, the closed-loop system (12) is asymptotically stable for any communication delay $\tau_{ij} \in \mathbb{R}^+$, if

$$\det \left[\mathbf{j}\omega I - \begin{pmatrix} I_N \otimes \tilde{A} & -\alpha I_N \otimes \tilde{B}\tilde{B}' P_\varepsilon \\ -\bar{L}_{\mathbf{j}\omega}(\tau) \otimes K\tilde{C} & I_N \otimes (\tilde{A} + K\tilde{C}) \end{pmatrix} \right] \neq 0 \tag{14}$$

for all $\omega \in \mathbb{R}$ and any communication delay $\tau_{ij} \in \mathbb{R}^+$. Condition (14) is satisfied if the matrix

$$\begin{pmatrix} I_N \otimes \tilde{A} & -\alpha I_N \otimes \tilde{B}\tilde{B}' P_\varepsilon \\ -\bar{L}_{\mathbf{j}\omega}(\tau) \otimes K\tilde{C} & I_N \otimes (\tilde{A} + K\tilde{C}) \end{pmatrix} \tag{15}$$

has no eigenvalues on the imaginary axis for all $\omega \in \mathbb{R}$ and all possible communication delays $\tau_{ij} \in \mathbb{R}^+$.

According to Lemma 1, all the eigenvalues of $\bar{L}_{\mathbf{j}\omega}(\tau)$ will be larger than or equal to β. Hence, when α satisfies (13), by Lemma 4 in the appendix, there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the matrix (15) has no eigenvalues on the imaginary axis. As noted before, this implies that the closed-loop system (12) is asymptotically stable for any communication delay $\tau_{ij} \in \mathbb{R}^+$.

Finally, by combining the precompensator (7) and controller (10), we get a linear dynamic controller of the type (3) with

$$A_c = \begin{pmatrix} \tilde{A} + K\tilde{C} & 0 \\ -\alpha \begin{pmatrix} I & 0 \end{pmatrix} \tilde{B}' P_\varepsilon & 0 \end{pmatrix}, \quad B_c = \begin{pmatrix} -K \\ 0 \end{pmatrix},$$

$$C_c = \begin{pmatrix} -\alpha \begin{pmatrix} 0 & \Gamma_2 \end{pmatrix} \tilde{B}' P_\varepsilon & \Gamma_1 \end{pmatrix}. \tag{*}$$

□

Proof of Theorem 2. We first show the necessity of the condition $y_r \in \mathscr{C}_y$. For any individual agent to be able to track a constant reference signal y_r, there must exist x_0 and u_0 such that

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = \begin{pmatrix} 0 \\ y_r \end{pmatrix}. \tag{16}$$

Clearly, such x_0 and u_0 exist only if y_r is in the set $\mathscr{C}_y$ which is therefore a necessary condition for the solvability of our problem.

Next, we show that the condition $y_r \in \mathscr{C}_y$ guarantees the existence of a controller which achieves output synchronization. Let R be an injective matrix such that $\mathscr{C}_y = \operatorname{Im} R$. In that case we can find Π and Γ such that:

$$\begin{pmatrix} 0 \\ R \end{pmatrix} = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} \Pi \\ \Gamma \end{pmatrix} \tag{17}$$

and

$$\operatorname{rank} \begin{pmatrix} A & B\Gamma \\ C & 0 \end{pmatrix} = n + \operatorname{rank} \Gamma. \tag{18}$$

To see that we can impose the above rank condition, we note that (C, A) detectable implies that the first n columns are linearly independent. If the above the rank condition is not satisfied, then there exist x and v such that

$$\begin{pmatrix} A & B\Gamma \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} = 0$$

with $B\Gamma v \neq 0$ and $v'v = 1$. But then

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} \Pi - xv' \\ \Gamma(I - vv') \end{pmatrix} = 0$$

which shows that $\bar{\Pi} = \Pi - x\nu'$ and $\bar{\Gamma} = \Gamma(I - \nu\nu')$ also satisfy the above equation but with rank $\bar{\Gamma} <$ rank Γ. Recursively, we can find a solution of (17) which also satisfies the extra rank condition (18).

We design a precompensator

$$\begin{aligned} \dot{p}_i &= \begin{pmatrix} I & 0 \end{pmatrix} v_i, \qquad p_i(t) \in \mathbb{R} \\ u_i &= \Gamma_1 p_i + \begin{pmatrix} 0 & \Gamma_2 \end{pmatrix} v_i \end{aligned} \tag{19}$$

where Γ_1 and Γ_2 are chosen such that $\operatorname{Im}\Gamma_1 = \operatorname{Im}\Gamma$ and

$$\begin{pmatrix} \Gamma_1 & \Gamma_2 \end{pmatrix} \tag{20}$$

is square and invertible.

The interconnection of (1) and (19) is of the form

$$\begin{cases} \dot{\tilde{x}}_i(t) = \tilde{A}\tilde{x}_i(t) + \tilde{B}v_i(t), \\ y_i(t) = \tilde{C}\tilde{x}_i(t), \\ \zeta_i(t) = \sum_{j=1}^{N} a_{ij}(y_i(t) - y_j(t - \tau_{ij})), \end{cases} \tag{21}$$

where

$$\tilde{x}_i = \begin{pmatrix} x_i \\ p_i \end{pmatrix}, \ \tilde{A} = \begin{pmatrix} A & B\Gamma_1 \\ 0 & 0 \end{pmatrix}, \ \tilde{B} = \begin{pmatrix} 0 & B\Gamma_2 \\ 1 & 0 \end{pmatrix}, \ \tilde{C} = \begin{pmatrix} C & 0 \end{pmatrix}.$$

We need to verify that the system remains stabilizable and detectable if we use a precompensator. The stabilizability of this system follows immediately from (20) and the stabilizability of (A, B).

To show the detectability of (21), we need to verify that

$$\operatorname{rank} \begin{pmatrix} sI - A & -B\Gamma_1 \\ 0 & sI \\ C & 0 \end{pmatrix} = n + \upsilon$$

with υ such that $\Gamma_1 \in \mathbb{R}^{n\times\upsilon}$ for all s in the closed right-half complex plane. For $s \neq 0$, this is achieved immediately from the detectability of (C, A). For $s = 0$, we have

$$\operatorname{rank} \begin{pmatrix} -A & -B\Gamma_1 \\ 0 & 0 \\ C & 0 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} -A & -B\Gamma \\ C & 0 \end{pmatrix} = n + \operatorname{rank}\Gamma_1.$$

We obtain the required detectability when we note that rank $\Gamma =$ rank Γ_1 and rank $\Gamma_1 = \upsilon$ (since Γ_1 is injective). The protocol for the multi-agent system, obtained after applying the precompensators, will be designed exactly as (10) and the remaining proof is similar to the proof of Theorem 1 except for the choice of $\tilde{\Pi}$ and $\bar{x}$. In this case, we choose $\bar{x}_i = \tilde{x}_i - \tilde{\Pi}z$ where z is such that $y_r = Rz$. Moreover, we set

$$\tilde{\Pi} = \begin{pmatrix} \Pi \\ W \end{pmatrix},$$

where W is such that $\Gamma_1 W = \Gamma$. It is then easily seen that $\tilde{A}\tilde{\Pi} = 0$ and $\tilde{C}\tilde{\Pi} = R$. □

3.2. Directed graphs

In this section, we will study a variation of Problem 1 for a multi-agent system with a directed graph $\mathcal{G}$. The associated Laplacian matrix L is then in general nonsymmetric, and the expanded Laplacian matrix $\bar{L}$, as defined in Section 2, will then be nonsymmetric as well.

We note that if our directed graph is balanced then the results of Theorem 1 still hold if we define β as the smallest eigenvalue of $\bar{L} + \bar{L}'$. However, if the graph is not balanced the derivation presented before might not be valid. In that case, we design a protocol for one individual graph (instead of for a set), i.e. we assume that the graph $\mathcal{G}$ is given.

We formulate the output synchronization problem for a given directed graphs as follows.

Problem 2. Consider a multi-agent system described by (1) associated with a given graph $\mathcal{G}$ which has a directed spanning tree. Given a constant trajectory $y_r \in R^p$ and let it be available to at least one agent which is a root agent. The *output synchronization* problem for networks with unknown, nonuniform and arbitrarily large communication delay is to find a distributed linear dynamic controller of the form (3) for each agent such that, for the given directed graph $\mathcal{G}$ and for any communication delay $\tau_{ij} \in \mathbb{R}^+$, the output of each agent converges to the constant trajectory, i.e.,

$$\lim_{t\to\infty} (y_i(t) - y_r) = 0, \tag{22}$$

for all $i \in \{1, \dots, N\}$.

Similar to the case of undirected graphs, the main result in this section is also presented in two theorems. The first theorem deals with the case when the system is right-invertible and has no invariant zeros at the origin. The second theorem deals with the general case.

Theorem 3. *Consider a multi-agent system described by (1) associated with a given directed graph $\mathcal{G}$ which has a directed spanning tree. Let Assumptions 1 and 2 hold. Given any constant trajectory $y_r \in \mathbb{R}^p$ and let the relative information regarding y_r be available to at least one agent which is a root agent. Then, Problem 2 is solvable if the system presented by (A, B, C) is right-invertible and has no invariant zero at the origin. Specifically, given a directed graph $\mathcal{G}$, there exists a linear dynamic controller of the type (3) such that output synchronization is achieved for any communication delay $\tau_{ij} \in \mathbb{R}^+$ and for any $y_r \in \mathbb{R}^p$.*

In the general case (not right-invertible or with zeros in the origin) the choice of the constant trajectory y_r needs to be restricted to the set given by (5).

Theorem 4. *Consider a multi-agent system described by (1) associated with a given directed graph $\mathcal{G}$ which has a directed spanning tree. Let Assumptions 1 and 2 hold. Given any constant trajectory $y_r \in \mathbb{R}^p$ and let the relative information regarding y_r be available to at least one agent which is a root agent Then, Problem 2 is solvable if and only if $y_r \in \mathscr{C}_y$. Specifically, given a directed graph $\mathcal{G}$, there exists a linear dynamic controller of the type (3) such that output synchronization is achieved for any communication delay $\tau_{ij} \in \mathbb{R}^+$.*

Remark 2. In the previous section for undirected graphs, we only used limited information about the network to design our distributed protocol. For directed graphs, we will make explicit use of our knowledge of the network to design our protocol (more specifically, to find a lower bound for our design parameter α). If we have a finite set of possible graphs, then we can still find a protocol that works for every graph in this finite set (use as a lower bound for α, the maximum of the lower bounds for each individual graph in the set).

Proof of Theorem 3. The proof follows the proof of Theorem 1, except for the choice of the parameter α given in (13).

Here, the parameter α should be designed in such a way that the eigenvalues of

$$\alpha \bar{L}_{\mathbf{j}\omega}(\tau) \tag{23}$$

have a real part larger than 1 for any communication delay $\tau_{ij} \in \mathbb{R}^+$ and for all $\omega \in \mathbb{R}$.

Since k corresponds to a root of a spanning tree, we know the expanded Laplacian matrix $\bar{L}$ is invertible and has its eigenvalues in the open right-half complex plane. Given that $\bar{L}$ is an invertible M-matrix, there exists a diagonal positive matrix $D = \operatorname{diag}\{d_1, \dots, d_N\}$ such that

$$D\bar{L} + \bar{L}'D > 0 \tag{24}$$

(in the case of an undirected or balanced graph we can simply choose $D = I$). Since this matrix (24) is positive definite, we find that

$$\mathrm{Re}(v'D\bar{L}v) = \frac{1}{2}v'(D\bar{L} + \bar{L}'D)v$$

is larger or equal to some positive constant β for all v with $\|v\| = 1$. Following the same arguments as in the proof of Lemma 1 we then obtain that

$$\mathrm{Re}\left(v'D\bar{L}_{\mathbf{j}\omega}(\tau)v\right) \geq \sum_{i=1}^{N}|v_i|^2 d_i\bar{\ell}_{ii} + \sum_{i=1}^{N}\sum_{j,j\neq i}^{N}|v_i'v_j|d_i\bar{\ell}_{ij}$$
$$= \begin{pmatrix}|v_1|\\ \vdots \\ |v_N|\end{pmatrix}' D\bar{L}\begin{pmatrix}|v_1|\\ \vdots \\ |v_N|\end{pmatrix} \geq \beta,$$

which implies:

$$\mathrm{Re}(v'D\bar{L}_{\mathbf{j}\omega}(\tau)v) \geq \beta. \tag{25}$$

Let λ be an eigenvalue of $\bar{L}_{\mathbf{j}\omega}(\tau)$ with eigenvector v. In other words, $\bar{L}_{\mathbf{j}\omega}(\tau)v = \lambda v$. Combining with inequality (25), we get

$$\mathrm{Re}(\lambda(\max d_i)v'v) \geq \mathrm{Re}(\lambda v'Dv) = \mathrm{Re}\, v'D\bar{L}_{\mathbf{j}\omega}(\tau)v \geq \beta \Rightarrow$$
$$\mathrm{Re}(\lambda) \geq \frac{\beta}{\max d_i}.$$

Thus, when choosing

$$\alpha > \frac{\max d_i}{\beta},$$

there exist an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, condition (23) is satisfied for any communication delay $\tau_{ij} \in \mathbb{R}^+$. □

Proof of Theorem 4. The proof follows the proof of Theorem 2, but with the choice of parameter α given in Theorem 3. □

4. State synchronization for agents with full-state coupling

State synchronization for multi-agent systems is a special case of the previous section when setting $C = I$. In this special case, agents are not right-invertible and we can obtain results from Theorems 2 and 4 for undirected and directed graphs, respectively.

We find in the state synchronization case that, when the agents are allowed to be introspective, we can achieve several major improvements over the previous results:

- We can use a static protocol instead of a dynamic protocol
- We do not need the assumption that all eigenvalues of matrix A are in the closed left-half complex plane.

These results will be derived in the subsections below.

4.1. Undirected graphs

The state synchronization problem for undirected graphs can be formulated as follows.

Problem 3. Let β be a given positive real number. Consider a multi-agent system described by

$$\Sigma_i : \begin{cases} \dot{x}_i(t) = Ax_i(t) + Bu_i(t), \\ \zeta_i(t) = \sum_{j=1}^{N} a_{ij}(x_i(t) - x_j(t - \tau_{ij})), \end{cases} \tag{26}$$

associated with an undirected graph $\mathcal{G} \in \mathbb{G}_{\beta,N}$ and assume that (A, B) is stabilizable. Let a constant trajectory $x_r \in R^n$ be given and let it be available to at least one agent. The *state synchronization* problem for networks with unknown, nonuniform, arbitrarily large communication delay is to find a distributed controller of the type

$$u_i = Fx_i + H\bar{\zeta}_i \tag{27}$$

for each agent such that, for any undirected graph $\mathcal{G} \in \mathbb{G}_{\beta,N}$ and for any communication delay $\tau_{ij} \in \mathbb{R}^+$, the state of each agent converges to the constant trajectory, i.e.,

$$\lim_{t\to\infty}(x_i(t) - x_r) = 0, \tag{28}$$

for all $i \in \{1, \dots, N\}$.

Before giving our result, we need to define a set

$$\mathscr{C} = \left\{x \in \mathbb{R}^n \mid Ax \in \mathrm{Im}\, B\right\}$$
$$= \left\{x \in \mathbb{R}^n \mid \exists\, u \in \mathbb{R}^m \text{ such that } Ax + Bu = 0\right\}. \tag{29}$$

Note that $\mathscr{C} = \mathscr{C}_y$ for the case $C = I$.

The main result in this section is presented in the following theorem.

Theorem 5. *Let β be a given positive real number. Consider a multi-agent system described by (26) with (A, B) stabilizable. Given a constant trajectory $x_r \in \mathbb{R}^n$ and let it be available to any agent, say agent k. Assume that the above multi-agent system is associated with an undirected graph $\mathcal{G} \in \mathbb{G}_{\beta,N}$. Then, Problem 3 is solvable if and only if the constant trajectory $x_r \in \mathscr{C}$. More specifically, there exists a distributed protocol of the type (27) for each agent such that state synchronization is achieved for any undirected graph $\mathcal{G} \in \mathbb{G}_{\beta,N}$ and for any communication delay $\tau_{ij} \in \mathbb{R}^+$.*

Proof. We first note that for an agent to be able to track a reference signal x_r, we need the existence of a u_0 such that

$$0 = Ax_r + Bu_0. \tag{30}$$

This implies that x_r must be in the set $\mathscr{C}$.

For each agent $i \in \{1, \dots, N\}$, a preliminary state feedback law

$$u_i = Fx_i + v_i, \tag{31}$$

is used such that

$$\ker(A + BF) = \mathscr{C}. \tag{32}$$

For the construction of such a matrix F, we note that there exists a state transformation T such that

$$TAT^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad TB = \begin{pmatrix} 0 \\ B_1 \end{pmatrix},$$

where B_1 has full column rank. Then, we can choose $F = -B_1^{\ell}\begin{pmatrix} A_{21} & A_{22}\end{pmatrix}T$ where B_1^{ℓ} denotes a left-inverse of B_1. Note that u_0 in (30) is then given by $u_0 = Fx_r$.

Combining each agent dynamics (26) and the state feedback law (31), the agent dynamics can be written as

$$\dot{x}_i = \bar{A}x_i + Bv_i, \tag{33}$$

where $\bar{A} = A + BF$. For the resulting multi-agent system after applying these local state feedbacks, we develop a distributed local controller

$$v_i = -\alpha B'P\bar{\zeta}_i, \tag{34}$$

where α is a design parameter that will be chosen later and P is the positive definite solution of the algebraic Riccati equation,

$$\bar{A}'P + P\bar{A} - PBB'P + I = 0. \tag{35}$$

It is worth noting that low gain is not needed here and hence no assumption needs to be made on the location of the eigenvalues of A.

If x_r is in the set $\mathscr{C}$, we will prove that the state of each agent converges to the constant trajectory x_r. Define $\bar{x}_i = x_i - x_r$ for every $i \in \{1, \ldots, N\}$. Then we have

$$\dot{\bar{x}}_i = \bar{A}\bar{x}_i + Bv_i,$$

and

$$\bar{\zeta}_i(t) = \sum_{j=1}^{N} a_{ij}(\bar{x}_i(t) - \bar{x}_j(t-\tau_{ij}) + \iota_i \bar{x}_i(t),$$

where ι_i is defined as the above.

Let $\bar{x} = \text{col}\{\bar{x}_1, \ldots, \bar{x}_N\}$. Then, the interconnection of the agents and their distributed protocols can be written in the frequency domain as

$$s\bar{x} = (I_N \otimes \bar{A})\bar{x} - \alpha(\bar{L}_s(\tau) \otimes BB'P)\bar{x}, \tag{36}$$

where $\bar{L}_s(\tau)$ is the expanded Laplacian matrix in the frequency domain as defined in (12). To prove our result, we need to prove that this system is asymptotically stable for any communication delay $\tau_{ij} \in \mathbb{R}^+$. The remaining proof will be done in two steps.

Step 1: In this step, we will first prove that the system without any communication delay is asymptotically stable. This is equivalent to showing that the matrix

$$(I_N \otimes \bar{A}) - \alpha(\bar{L} \otimes BB'P) \tag{37}$$

is Hurwitz stable. As in Definition 3, λ_i $(i = 1, \ldots, N)$ denote the eigenvalues of $\bar{L}$. Then, from [22], the stability of (37) is equivalent to the Hurwitz stability of

$$\bar{A} - \alpha\lambda_i BB'P$$

for all $i = 1, \ldots, N$. From Lemma 2 in the appendix, we conclude that these matrices are Hurwitz stable provided

$$\alpha > \frac{1}{2\lambda_i}$$

for all i. Since $\mathcal{G} \in \mathbb{G}_{\beta,N}$, it is sufficient to choose α such that $\alpha > \frac{1}{2\beta}$.

Step 2: In this step, we need to prove that the closed-loop system is asymptotically stable in the presence of communication delays. Since the system without delays is asymptotically stable, according to Lemma 3 in the appendix, the closed-loop system is asymptotically stable for any communication delay $\tau_{ij} \in \mathbb{R}^+$, if

$$\det[\mathbf{j}\omega I - (I_N \otimes \bar{A}) + \alpha\bar{L}_{\mathbf{j}\omega}(\tau) \otimes BB'P] \neq 0 \tag{38}$$

for all $\omega \in \mathbb{R}$ and any communication delay $\tau_{ij} \in \mathbb{R}^+$.

For Condition (38), it is clearly sufficient to show that

$$(I_N \otimes \bar{A}) - \alpha\bar{L}_{\mathbf{j}\omega}(\tau) \otimes BB'P \tag{39}$$

does not have eigenvalues on the imaginary axis for all $\omega \in \mathbb{R}$ and for any communication delay $\tau_{ij} \in \mathbb{R}^+$.

From Lemma 1, we know that all eigenvalues of $\bar{L}_{\mathbf{j}\omega}(\tau)$ have real part larger than or equal to β. This implies that for $\alpha > \frac{1}{2\beta}$, we know that all eigenvalues of

$$\alpha\bar{L}_{\mathbf{j}\omega}(\tau)$$

have a real part greater than $\frac{1}{2}$. According to Lemma 2 in the appendix, this implies that (39) is Hurwitz stable for any communication delay $\tau_{ij} \in \mathbb{R}^+$. As noted before, this implies that the condition (38) is satisfied. Hence, the closed-loop system is asymptotically stable for any communication delay $\tau_{ij} \in \mathbb{R}^+$. □

4.2. Directed graphs

Next, we will modify Problem 3 to allow for multi-agent systems with a given *directed* graph $\mathcal{G}$ which has a spanning tree.

We formulate the state synchronization problem for directed graphs as follows.

Problem 4. Consider a multi-agent system described by (26) associated with a directed graph $\mathcal{G}$ which has a directed spanning tree. Given a constant trajectory $x_r \in R^n$ and let it be available to at least one agent. Then the *state synchronization* problem for networks with unknown, nonuniform, and arbitrarily large communication delay is to find a distributed controller of the type (27) for each agent such that, for the given directed graph $\mathcal{G}$ and for any communication delay $\tau_{ij} \in \mathbb{R}^+$, the state of each agent converges to the constant trajectory, i.e.,

$$\lim_{t\to\infty}(x_i(t) - x_r) = 0, \tag{40}$$

for all $i \in \{1, \ldots, N\}$.

We present the result in the following theorem.

Theorem 6. *Consider a multi-agent system described by (26) with (A, B) stabilizable and associated with a given directed graph $\mathcal{G}$ which has a directed spanning tree. Let a constant trajectory $x_r \in \mathbb{R}^n$ be given and let it be available to at least one agent which is a root agent. Then, Problem 3 for a multi-agent system with a given directed graph $\mathcal{G}$ is solvable if and only if the constant trajectory x_r is in the set $\mathscr{C}$. More specifically, given a directed graph $\mathcal{G}$, there exists a distributed protocol of the type (27) for each agent such that state synchronization is achieved for any communication delay $\tau_{ij} \in \mathbb{R}^+$.*

Proof. For each $i = 1, \ldots, N$, a distributed protocol will be designed of the form

$$u_i = Fx_i - \alpha B'P\bar{\zeta}_i, \tag{41}$$

where F and P are chosen exactly as in Section 4.1, while α is a design parameter which we will choose differently in this section. Following the proof of Theorem 5, we need to design parameter α such that the eigenvalues of

$$\alpha\bar{L}_{\mathbf{j}\omega}(\tau) \tag{42}$$

have a real part larger than $\frac{1}{2}$ for any communication delay $\tau_{ij} \in \mathbb{R}^+$, $(i, j = 1, \ldots, N)$.

Similar to the proof of Theorem 3, there exists a diagonal positive matrix $D = \text{diag}\{d_1, \ldots, d_N\}$ such that

$$D\bar{L} + \bar{L}'D > 0,$$

and

$$\text{Re}(\lambda) \geq \frac{\beta}{\max d_i},$$

where λ is an eigenvalue of $\bar{L}_{\mathbf{j}\omega}(\tau)$.

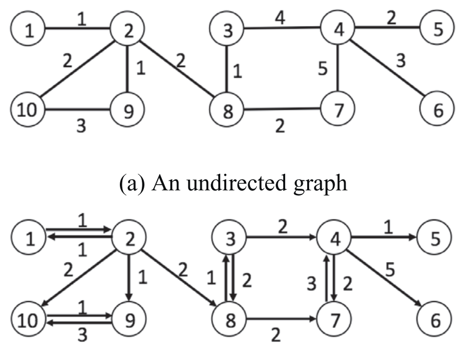
Thus, when choosing

$$\alpha > \frac{\max d_i}{2\beta},$$

condition (42) is satisfied for any communication delay $\tau_{ij} \in \mathbb{R}^+$, $(i, j = 1, \ldots, N)$. Hence, our result is proved. □

5. Examples

In this section, we will give two examples to illustrate our results on output synchronization for a MAS with $N = 10$ agents with partial-state coupling. One is for an undirected network; and the other for a directed network, illustrated in Fig. 1a and b, respectively. Agents are chosen to be right-invertible and have no invariant zeros at the origin.



(a) An undirected graph

(b) A directed graph

Fig. 1. Network topologies of $N = 10$ agents.

Undirected graph The Laplacian matrix corresponding to the undirected graph shown in Fig. 1a is written as

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 6 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & -2 \\ 0 & 0 & 5 & -4 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -4 & 14 & -2 & -3 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 & 0 & 7 & -2 & 0 & 0 \\ 0 & -2 & -1 & 0 & 0 & 0 & -2 & 5 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -3 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 5 \end{pmatrix}.$$

We allow any nonuniform arbitrarily large communication delays in the network communication. The linear agent model is described by the matrices

$$A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}.$$

Our target is to regulate all agents' outputs to a constant trajectory given by $y_r = 1$ which is available to agent 2. Then, the lower-bound on the real part of all eigenvalues of the expanded Laplacian matrix $\bar{L}$ is 0.07. Select $\beta = 0.06$. Then, the undirected graph Fig. 1a is included in a set of graphs $\mathcal{G}_{\beta,N}$.

According to Theorem 1, the precompensator for each agent is designed as,

$$\begin{aligned} \dot{p}_i &= v_i, \\ u_i &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} p_i. \end{aligned} \tag{43}$$

Now choose $\alpha = 17$, which satisfies $\alpha > \dfrac{1}{\beta} = 16.67$. By selecting

$$K = \begin{pmatrix} -20 & -40 & 15 & -1.5 \end{pmatrix}'$$

and the low-gain $\varepsilon = 10^{-7}$, the protocol is designed as

$$\begin{aligned} \dot{\chi}_i &= \begin{pmatrix} -20 & 1 & -18 & 0 \\ -41 & 0 & -40 & 1 \\ 15 & 0 & 14 & 1 \\ -1.5 & 0 & -1.5 & 0 \end{pmatrix} \chi_i - \begin{pmatrix} -20 \\ -40 \\ 15 \\ -1.5 \end{pmatrix} \bar{\zeta}_i, \\ v_i &= \begin{pmatrix} 0 & -0.0067 & -0.0067 & -0.0192 \end{pmatrix} \chi_i. \end{aligned}$$

Our simulation shows that the above protocol can tolerate any nonuniform and asymmetric communication delay. When setting $\tau_{21} = \tau_{64} = 2$, $\tau_{28} = \tau_{29} = \tau_{34} = \tau_{38} = 3$, $\tau_{2,10} = 5$, $\tau_{43} = \tau_{45} = \tau_{46} = \tau_{47} = 3$, $\tau_{54} = 5$ and other $\tau_{ij} = 1$ if it exists, we obtain Fig. 2, which shows that the outputs of all agents converge to the constant trajectory $y_r = 1$ asymptotically.

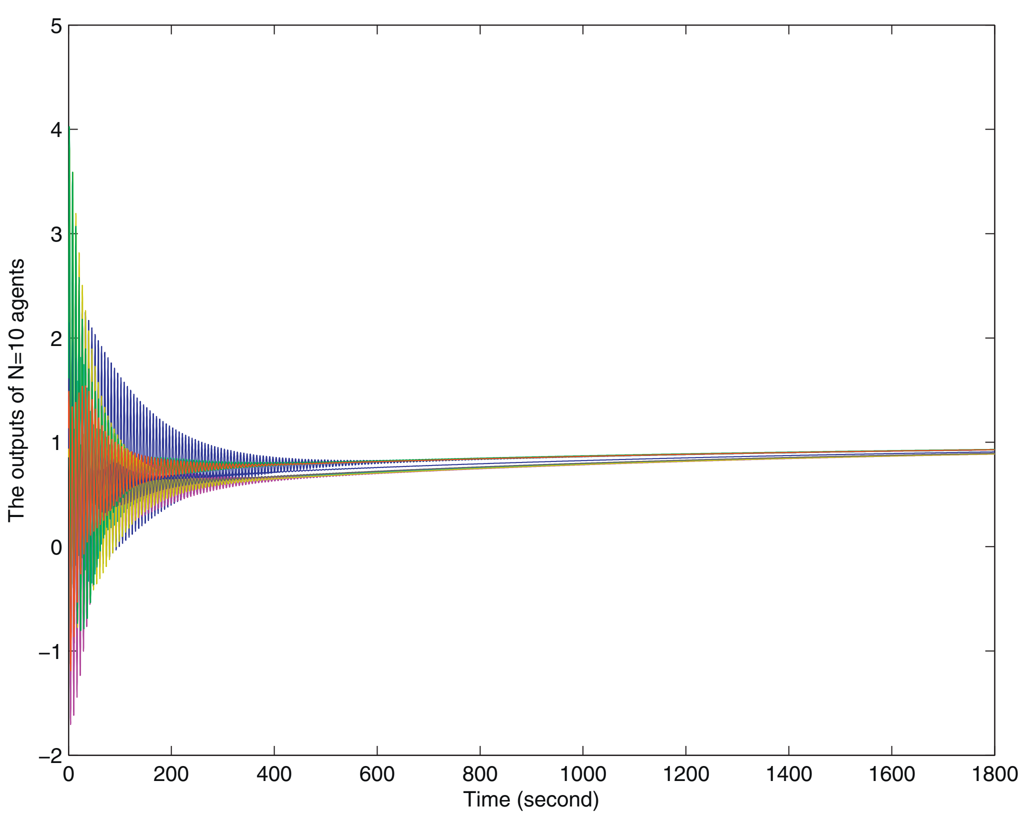


Fig. 2. The output trajectories of $N = 10$ agents under an undirected graph.

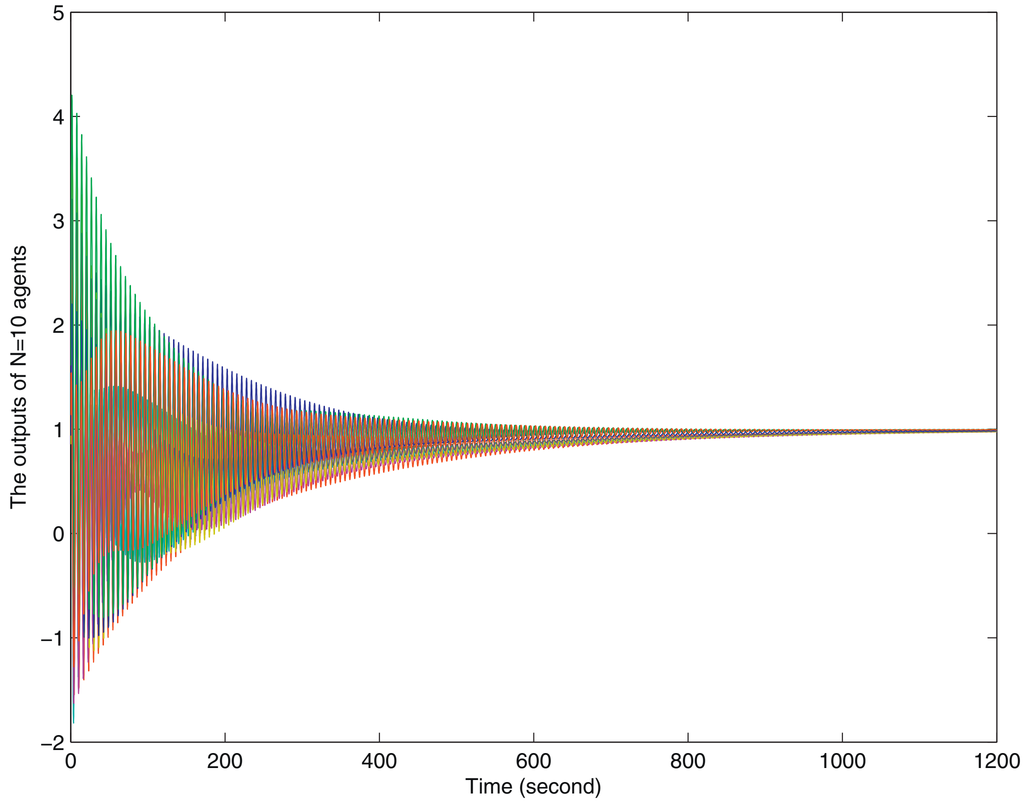


Fig. 3. The output trajectories of $N = 10$ agents under a directed graph.

Directed graph The Laplacian matrix corresponding to the directed graph shown in Fig. 1b is written as

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 5 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 4 & -2 & 0 & 0 \\ 0 & -2 & -2 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 5 \end{pmatrix}.$$

We use the same linear agent model and reference trajectory as in the case of undirected graph. Hence we have the same precompensator, that is given in (43).

In this case, we also make the constant trajectory $y_r = 1$ available to root agent 2. Now we choose the diagonal matrix

$$D = \text{diag}\{1.3781, 1.4997, 1.7023, 0.4046, 0.2745, 0.2208, 0.3915, 0.6259, 0.7497, 0.3230\}$$

such that $D\bar{L} + \bar{L}'D > 0$. We find that the real part of all eigenvalues of $D\bar{L}$ is greater than 0.53. Therefore, setting $\beta = 0.55$, we can choose $\alpha = 10$ which satisfies $\alpha > \frac{\max d_i}{\beta} = \frac{1.7023}{0.55} = 3.09$. By selecting the same K and the low-gain ε, the protocol is designed as,

$$\dot{\chi}_i = \begin{pmatrix} -20 & 1 & -18 & 0 \\ -41 & 0 & -40 & 1 \\ 15 & 0 & 14 & 1 \\ -1.5 & 0 & -1.5 & 0 \end{pmatrix} \chi_i - \begin{pmatrix} -20 \\ -40 \\ 15 \\ -1.5 \end{pmatrix} \bar{\zeta}_i,$$
$$v_i = \begin{pmatrix} -0.0000 & -0.0134 & -0.0134 & -0.0385 \end{pmatrix} \chi_i.$$

With the same communication delay, we obtain Fig. 3, which shows that the output of all agents converge to the constant trajectory $y_r = 1$ asymptotically.

Appendix A. Useful lemmas

The problem of synchronization is connected to a robust stabilization problem as presented in the following lemma which can be found in [22].

Lemma 2. *Consider a linear uncertain system*

$$\dot{x} = Ax + \lambda Bu,$$

where (A, B) *is stabilizable with* $\lambda \in \mathbb{C}$ *unknown. Consider, the state feedback* $u = \alpha Fx$ *where* $F = -B'P$, *and* P *is the unique positive definite solution of the algebraic Riccati equation*

$$A'P + PA - PB'BP + I = 0.$$

Then, we have that $A - \alpha\lambda BB'P$ *is Hurwitz stable for any*

$$\lambda \in \left\{ s \in \mathbb{C} \mid \text{Re}(s) \geq \frac{1}{2\alpha} \right\}.$$

The following lemma is a useful tool to check the stability of a delay system and can be found in [23].

Lemma 3. *Consider a linear time-delay system*

$$\dot{x} = Ax + \sum_{i=1}^{N} A_{d,i} x(t - \tau_i). \tag{44}$$

Assume that

$$A_d + \sum_{i=1}^{N} A_{d,i}$$

is Hurwitz stable. In that case, the delay system (44) is globally asymptotically stable for any $\tau_1, \ldots, \tau_N \in [0, \bar{\tau}]$ *if*

$$\det\left[\mathbf{j}\omega I - A - \sum_{i=1}^{N} e^{-\mathbf{j}\omega\tau_i} A_{d,i}\right] \neq 0,$$

for all $\omega \in \mathbb{R}$ *and* $\tau_1, \ldots, \tau_N \in [0, \bar{\tau}]$.

The following lemma can be found in [16, Lemma C.2].

Lemma 4. *There exists an* $\varepsilon^* > 0$ *such that for all* $\varepsilon \in (0, \varepsilon^*]$, *we have that*

$$\begin{pmatrix} \tilde{A} & -\lambda\tilde{B}\tilde{B}'P_\varepsilon \\ -K\tilde{C} & \tilde{A} + K\tilde{C} \end{pmatrix} \tag{45}$$

is asymptotically stable for all λ *with* $\mathrm{Re}(\lambda) \geq 1$.

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