Consistent Preferences with Fixed Point Updates*

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Abstract

We propose a normative framework for complete preference orderings, in which updating is based on a straightforward fixed point principle. Its scope goes far beyond the Sure Thing Principle, and covers Bayesian updating as a special case, without reference to probabilities. The induced preference reversals are justified on the basis of an interpretation that reconciles consequentialism with adequate forms of choice consistency, and absence of arbitrage. We emphasize the distinction between dynamic choices with or without rights in bygone states, and between choosing if it comes to obtaining versus offering. The framework accommodates the preferences in the Allais and Ellsberg paradoxes, which indicates that the gap between descriptive and normative models is narrower than generally believed.

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1 Introduction

The challenge of developing a normative framework for decision making without the Sure Thing Principle is to reconcile forward ex-ante optimization and backward inductive implications of updating. According to many, this seems impossible, and the quest for a suitable updating principle outside the STP has not reached consensus (Wakker, 2010; Machina and Viscusi, 2013).

As usual for such a long-lasting debate, the heart of the problem does not lie in the mathematics itself, but rather in its mapping to reality. In our opinion, there are two semantic aspects that deserve more attention.

Firstly, what does it take to actually have a choice? We emphasize that an opportunity to choose in a future state E requires some form of preparatory agreement *before* E obtains, whenever the choice set involves rights or obligations outside E. We call these *embedded* choices, to discern them from *fresh* choices in E. Only for the latter, arranging the choice set can wait till E has actually obtained. In our framework, updates only apply to fresh choices, precisely as initial preference orderings do not refer to rights or obligations outside the overall state space. This opens the way to combine dynamic consistency with uncomplicated, consequentialist updating.

Secondly, what does it mean to prefer one act, or lottery, over the other? We make a sharp distinction between choices to obtain and to offer. The corresponding two preference relations can be derived from each other, by identifying obtaining an act with offering its negative. We call them *twins* - they share many features but should not be confused. This explains why absence of arbitrage is easily achieved without the STP.

These are two distinctive features behind the normative interpretation of preferences without the STP, which we explain in more detail by an application to the Ellsberg and Allais paradoxes in the next section.

The description of our axiomatic framework can be divided into three parts. In the first part, Sections 3 and 4, we develop a framework with existence and uniqueness of updates, for a general class of complete preference orderings. We also show how this materializes for its concave and convex subclasses, in terms of their representations. For the sake of the argument, we strive for mathematical simplicity, and restrict attention to acts with monetary outcomes with finite range, on a finite outcome space. The update rule we propose is also known as Pires' rule (Pires 2002), but we found it independently, as explained in Section 8.1. However, its potential as universal update principle has not yet been recognized in economic literature, probably due to treating embedded and fresh choices at the same footing. Our first main result is to establish fixed point updating as a cornerstone of normative modeling that generalizes the STP.

In the second part, Sections 5–7, we address absence of arbitrage, and choice consistency. Our second main result is a comfortingly general criterion guaranteeing that any ex-ante strictly optimal plan remains compatible with updates. We define two slightly different notions that combine our consistency requirements, Monetary Consistency and Twin Consistency.

The third part is devoted to discussion of related literature and conclusions. Proofs are in the appendix.

2 Illustration by the paradoxes

We illustrate our approach by the Ellsberg paradox, and then briefly address how the Allais paradox can be treated in a largely similar way.

2.1 The Ellsberg paradox

In the Ellsberg paradox (Ellsberg, 1961), a ball is drawn from an urn with 90 balls, 30 of which are red, and 60 black or yellow, in unknown proportion. A decision maker (DM) considers choices between betting on red or black, combined or not with a bet on yellow as well. A winning bet yields prize \$100. In obvious notation, with acts (r, b, y) on state space $\{R, B, Y\}$ in units of \$100, the two pairs of bets are

choice pair I: $\{(1,0,0),(0,1,0)\}$

choice pair II: $\{(1,0,1),(0,1,1)\}$.

An ambiguity averse DM chooses for the bet on red in the first pair, and on 'black or yellow' in the second, thus avoiding ambiguity, but violating the Sure Thing Principle.

To become concrete, we assume that the DM has preference ordering $\preceq$ represented by

$$V(r, b, y) = \min\{r/3 + 2b/9 + 4y/9, r/3 + 4b/9 + 2y/9\}. \tag{2.1}$$

We interpret V as the (maximum) price that the DM is willing to pay for a bet. Note that $V(c, c, c) = c$, which makes sense, assuming that side aspects as credit risk and interest rates are irrelevant, or at least can be ignored.

The fixed point update rule, for acts (r, b) in the state $E := \{R, B\}$, takes the form

$$V_E(r, b) = V(r, b, y) \text{ with } y \text{ such that } V(r, b, y) = y. \tag{2.2}$$

The intuition is that if the DM is willing to pay y for act (r, b, y), he gets his payment back in case the ball is yellow, as if he only bets when E obtains. In other words, the ‘fresh’ act (r, b) in E is treated like that ex-ante act (r, b, y) for which neither the DM nor the betting agency needs an ex-ante agreement to guarantee rights outside E. It follows that

$$V_E(r, b) = \min\{3r/5 + 2b/5, 3r/7 + 4b/7\}, \tag{2.3}$$

so the DM prefers a fresh bet in E on red (valued at 3/7) over black (valued at 2/5), again in line with being ambiguity averse. It may be noted here that fixed point updating is a generalization of full Bayesian updating, without reference to probabilities. We propose it as a serious candidate for the unifying update principle outside the STP.

Once updating is settled, the classic tension between ex-ante optimization and backward induction arises in choice pair II, when the DM is given the option to revise his choice in state E. The DM first chooses black, and is determined not to exercise the option in E, so as to lock in $(0, 1, 1)$. Will he stick to his plan when E actually obtains, despite the updated preference for betting on red over black?

Yes. Firstly, the ex-post decision is an embedded choice the DM only can have on the basis of an ex-ante agreement that secures his right on the prize when Y would have obtained. Even when state Y is bygone, he still knows that sticking to the plan completes an unambiguous bet, while switching to red would not (a standard argument). There is a concrete agreement that confirms the relevance of the bygone outcome for the ex-post decision. We assume that embedded choices are directly governed by the stick-to-the-plan principle, and do not further underpin them by updates. The relevance of non-consequentialist updates for embedded choices is discussed Section 8.2.

Secondly, updates only govern fresh choices. The update implies that the DM would have preferred red over black in E if ex ante he would not have had rights or obligations outside E. But he had.

Thirdly, the update is compatible with plan consistency in the following sense:

$$V_E(0,0) > V_E(1,-1). \tag{2.4}$$

The incremental effect of the decision to stick to his plan, as fresh choice in E, is zero, while switching involves giving up the bet on black for one on red. Note that also $V_E(0,0) > V_E(-1,1)$, so that also for choice pair I choice consistency is not undermined by the update. The DM sticks to his plans, in both choice pairs, because plans are sticky, so to speak. This form of compatibility is guaranteed, for any choice set, when V satisfies the implication

$$V(f) > V(g) \Rightarrow 0 > V(g-f). \tag{2.5}$$

So, analogous to the idea of consistent planning (Strotz, 1955; Siniscalchi, 2009), we impose a restriction on initial preferences to avoid conflicts between their given updates and ex-ante optimal plans. However, a much simpler and weaker one, since, in the terminology of Gilboa (2015), sticking to initially optimal plans only has to be weakly rational in view of the updates, not strongly.

Finally, making book against the DM has no chance. Whereas $V(f)$ is his price for obtaining f, he sets a higher price $V^*(f)$ for offering f, with 'twin' value V^* defined by the reflection principle $V^*(f) := -V(-f)$. In words, we identify the opposite of obtaining with obtaining the negative, which is straightforward in a monetary setting in terms of incremental wealth. Twins are both updated by the fixed point principle, which commutes with the reflection priciple. It follows that V^* and V^*_E are given by (2.1) and (2.3) with min replaced by max. For instance, $V^*_E(1,0) = 3/5$, and we say that the DM assigns a 'thick' value $[2/5, 3/5]$ to a fresh bet on red in E. The DM hence trades at prices, ex ante as well as ex post, that are at least as favorable to him as taking expected value, which is arbitrage-free. This is a standard argument in bid-ask price modelling.

These normative considerations are combined in our axiomatic definition of Monetary Choice Consistency. A variant, called Twin Consistency, avoids the subtraction of acts in (2.4), replacing it by

$$V_E^*(0,1) > V_E(1,0). \tag{2.6}$$

It is left to the reader to verify that the DM is consistent in both senses. We conclude that the Ellsberg preferences can be given a normative interpretation.

2.2 The Allais paradox

We briefly sketch how the same line of reasoning applies to the Allais paradox (Allais, 1953). The Allais lotteries can be represented in our setting as acts (a, b, c) on an outcome space $\{\omega_1, \omega_2, \omega_3\}$, with probabilities resp. 0.10, 0.01, and 0.89. Preferences only take the probability distributions of acts into account. Again, there are two choice pairs, one with rights and with obligations in the third state,

choice pair A: {(1,1,1),(5,0,1)}
choice pair B: {(5,0,0),(1,1,0)}.

It has been well documented that many people prefer the first lottery in both pairs, contrary to the Independence Axiom, and hence to the Sure Thing Principle. The anomaly arises for updating in $E := \{\omega_1, \omega_2\}$.

A particularly simple idea to accomodate the Allais preferences in our framework is to consider worst expected values, with the given probabilities up- or downscaled by at most a factor ten, but there are many more possible. The same explanation in terms of fresh and embedded choices applies. Updates are consequentialist and only apply to fresh choices. For the ex-post decision in E on exercising the option to revise the choice for $(5, 0, 0)$ over $(1, 1, 0)$, the outcome 0 in the bygone state still matters - it is about completing the initial distribution.

This last argument may be less pronounced than in the Ellsberg setting with complementary ambiguity. However, the recognition of a twin provides another argument to justify the decline of backward induction[1], in both paradoxes. Intuitively speaking, the willingness to pay today does not depend alone on the willingness tomorrow to pay again, as also argued in RS16 in the context of bid and ask prices. The same is true for the willingness tomorrow to stick to a plan.

In summary, our framework supports the normative interpretation of the delicate aspects of decision making revealed by the Allais and Ellsberg experiments.

3 Setting and notation

3.1 Acts and class of preference orderings

We consider acts of the form $f : \Omega \to X$, with Ω a finite outcome space, and X a finite interval $[w, b] \subseteq \mathbb{R}$ of monetary outcomes. The set of all acts is denoted as $\mathcal{A}$, and $\mathcal{A}^*$ is the set $-\mathcal{A}$, with outcome range $[-b, -w] =: X^*$. The case with $w = -b$, and hence $\mathcal{A} = \mathcal{A}^*$, will play a special role. The interval $[\min f, \max f]$ is denoted as range(f). If an act f has $f(\omega) = c \in X$ on Ω, it is called a constant (act), and then we use the symbol c also for f. The (pointwise) mixture $\lambda f + (1 - \lambda)g$ is the act with final outcome $\lambda f(\omega) + (1 - \lambda)g(\omega)$ in ω. An act is also called a lottery when an externally given probability measure on Ω is specified.

Our scope is the class $\mathcal{P}$ of preference orderings satisfying the usual basic axioms.

Definition 3.1 $\mathcal{P}$ is the class of preference orderings $\preceq \subset \mathcal{A} \times \mathcal{A}$ that satisfy

A1 *(Weak order)* $\preceq$ is complete and transitive.

[1]This states that the ceq of a sub-act in E must be equal to its replacement value r, i.e., the constant by which the sub-act can be replaced without changing the value of the whole.

A2 *(Monotonicity in final outcomes)* If $f(\omega) \leq g(\omega)$ on Ω, then $f \preceq g$.

A3 *(Strict monotonicity for constants)* For $c, d \in X$: $c < d$ implies $c \prec d$.

A4 *(Continuity)* For all $f \in \mathcal{A}$, the upper set $\{g \in \mathcal{A} \,|\, g \succeq f\}$ and the lower set $\{g \in \mathcal{A} \,|\, g \preceq f\}$ are closed.

Orderings in $\mathcal{P}$ are called *regular*. The equivalence $f \sim c$ if and only if $V(f) = c$ defines a one-to-one correspondence between $\mathcal{P}$ and the class of value functions $V : A \to X$ that are continuous, monotone, and normalized, i.e., have $V(c) = c$. This V is called the (normalized) *value function* of $\preceq$, or the *certainty equivalence function* of $\preceq$. Proofs of these elementary facts are left to the reader. We will often abbreviate the term certainty equivalent to ceq.

3.2 State space and sub-acts

To streamline the exposition, updates are defined with respect to a state space S that corresponds to one degree of information under consideration. Formally, S is a partition of Ω. The state space S may be externally specified, or just hypothesized as a thought experiment in the decision making process.[2] How the case with several degrees of information can be reduced to one-shot updating with respect to a state space, is indicated by a reference to a compatibility result in Section 4.1.

The sub-act of an act $f \in \mathcal{A}$ in $s \in S$ is denoted as f_s, and $\mathcal{A}_s$ denotes the set of all sub-acts in state s. A (state) update of $\preceq$ in s is a preference ordering on $\mathcal{A}_s$, commonly denoted as $\preceq_s$. For the vectors $(\mathcal{A}_s)_{s \in S}$ and $(\preceq_s)_{s \in S}$, we use the notation $\mathcal{A}_1$ and $\preceq_1$, but $(f_s)_{s \in S}$ is simply identified with f. The (vector of) preference ordering(s) $\preceq_1$ is referred to as a (vector) update of $\preceq$. The definition

[2]For horse-roulette lotteries, S can be taken the end-states of the horse-lotteries (indeed commonly denoted as S), but also a partition of these end-states.

of regularity extends to updates in the obvious way. We write $f_s h$ for the result of pasting sub-act f_s in state s in act h.

The axioms with reference to the state space S that we consider give rise to several subclasses of regular preference orderings in $\mathcal{P}$. In the first part we define the class $\mathcal{S}$ with well-defined sequentially consistent updates, the concave class $\mathcal{C}$, and the class $\mathcal{R}$ with dynamic risk aversion. In the second part we eventually define the classes $\mathcal{M}$ and $\mathcal{T}$ of resp. monetary and twin consistent preferences.

3.3 Notation for probability measures

Although the core of our axiomatic setup does not refer to probability measures they are instrumental in the representation of concave preferences. Δ is the space of probability measures on Ω, $\Delta^+ = \{Q \in \Delta \,|\, Q(s) > 0 \text{ on } S\}$, and $\Delta^{s+} = \{Q \in \Delta \,|\, Q(s) > 0\}$. $E^Q f$ is the expected value of f under Q, Δ_s and E_s^Q are the corresponding conditional versions in state s, and $E_1 f$ is $(E_s f)_{s \in S}$. Further, TQ_s is the probability measure obtained by pasting Q conditioned on s in T, formally defined by $E^{TQ_s} f = E^T(1_s E_s^Q f + (1 - 1_s) E_1^T f)$. In the context of risk aversion, we use P for the reference measure.

All notation generalizes from states s to events $E \subseteq \Omega$ in the obvious way. The usual double role of the symbol E should not cause confusion.

4 Updating without the STP

We formulate an axiomatic framework for unambiguous updating, based on the notion of sequential consistency, and the corresponding Equal Level Principle as generalization of the Sure Thing Principle. The general result is made concrete for the concave and convex subclasses, in terms of their representations. The update formulas are simplified under additional assumptions on risk aversion.

4.1 Main axioms and update rule

The following axioms form the cornerstone of our framework. They apply to $f \in \mathcal{A}$.

S1 *(Sequential Consistency)* If $f_s \sim_s c$ on S, then $f \sim c$.

S2 *(Equal Level Principle)* If $f_s c \sim c \in \text{range}(f_s)$ on S, then $f \sim c$.

S3 *(c-Sensitivity)* If $f_s c \sim c$, then $f_s d \succ d$ for $d < c$ and $f_s d \prec d$ for $d > c$, for all $s \in S$ and $c, d \in \text{range}\, f_s$.

Definition 4.1 $\mathcal{S}$ is the subclass of preferences in $\mathcal{P}$ that satisfy axioms S2 and S3.

The first axiom, *Sequential Consistency*, is our key notion for consistent updating, replacing the common notion of monotonicity,

$$f \preceq_s g \text{ on } S \quad \Rightarrow \quad f \preceq g \quad (f, g \in \mathcal{A}), \tag{4.1}$$

The notion of sequential consistency has been developed and analyzed in a long-standing research line on risk measures and valuations, see Section 8.1. It is equivalent, in $\mathcal{P}$, to the condition

$$c \preceq_1 f \preceq_1 d \quad \Rightarrow \quad c \preceq f \preceq d. \tag{4.2}$$

In other words, *values should be in the range of their sequential updates.* The *Equal Level Principle* (axiom S2) is the corresponding weakening of the STP that characterizes existence of consistent updates, under the sensitivity condition of axiom S3 that guarantees their uniqueness.

As shown in the theorem below, these axioms lead to the following update mechanism, which we call *fixed point updating (fpu)*:

$$f_s \sim_s c \quad :\Leftrightarrow \quad f_s c \sim c \text{ with } c \in \text{range}(f_s). \tag{4.3}$$

Recall that in Section 2.1 we justified this rule by interpreting $f_s c$ as the act that gives no right nor obligations outside s. We call $\preceq_s$ a *fixed point update* of $\preceq$ (in state s) if it satisfies the forward implication in (4.3); it satisfies (4.3) if and only if it is the unique one.

Theorem 1 *A preference ordering $\preceq$ in $\mathcal{P}$ has unique fixed point updates $\preceq_s$ on S if and only if $\preceq$ satisfies axiom S3, and then $\preceq_s$ is given by (4.3), and regular. The (vector) update $(\preceq_s)_{s\in S} =: \preceq_1$ is then sequentially consistent (axiom S1) if and only if $\preceq$ also satisfies axiom S2, otherwise $\preceq$ has no regular sequentially consistent update.*

So $\mathcal{S}$ is the subclass of $\mathcal{P}$ for which the fixed point update (4.3) is well-defined and produces the sequentially consistent update.

It may be illuminating to compare the implications of axiom S2 and the STP for a strictly monotone preference ordering $\preceq$ on acts with three outcomes, (x, y, z), and two states (s, s'), corresponding to resp. the first two outcomes and the third. The STP requires that $(x, y, z) \sim (c, c, z)$ either for all z or none. Axiom S2 amounts to the implication that if $(x, y, c) \sim c$ and $(c, c, z) \sim c$, then $(x, y, z) \sim c$, which is void, since $z = c$ when $(c, c, z) \sim c$. The knowledgeable reader will immediately recognize what this means for the Allais paradox. Note that axiom S3 is satisfied, for instance, when the induced value function $V(x, y, z)$ has both the third partial derivative strictly bounded by 1, as well as the sum of the first two.

We conclude this subsection by a remark on compatibility of updates. The notation and preceding results generalize from states $s \in S$ to events E in a partition of S in the obvious way. In particular, under the analogues of axiom S2 and S3, the consistent update $\preceq_E$ is then determined by (4.3), with s replaced by E. This satisfies a compatibility property, called commutativity in Gilboa and Schmeidler (1989), which requires that $\preceq_s$ can also be obtained as the update of $\preceq_E$ with $s \in E$. Compatibility is addressed in (RS13, Prop. 4.6) and (RS16, Prop. 6.7) in technically more advanced settings.

4.2 The concave subclass and its representation

Concave preferences are those that satisfy

C1 *(Concavity)* If $f, g \succeq c$, then $\lambda f + (1-\lambda) g \succeq c$ for all $\lambda \in [0,1]$.

In other words, their upper contour sets,

$$\mathcal{A}^c := \{f \in \mathcal{A} \mid f \succeq c\}, \tag{4.4}$$

are convex, for all $c \in X$. Concavity is strongly related to risk- and ambiguity aversion, also when it is defined in terms of probability mixtures. The corresponding property for value functions is quasi-concavity:

$$V(\lambda f + (1-\lambda) g) \geq \min\{V(f), V(g)\}.$$

In the representation lemma below, which is a simplified yet slightly different version of the results in Cerreia-Vioglio et al. (2011b), we use the following regularity condition.

C2 *(Sensitivity for constants)* $f + d \succ f$ for $d > 0$, $f + d \in \mathcal{A}$.

Lemma 4.2 *A preference ordering $\preceq$ is concave (axiom C1) and regular if it can be represented as*

$$f \sim c \quad \Leftrightarrow \quad \min_{Q \in \Delta} E^Q f + \theta(c, Q) = c, \tag{4.5}$$

for some function $\theta : X \times \Delta \to \mathbb{R}^+$ that is (i) continuous in the first argument, (ii) satisfies $\min_{Q\in\Delta} \theta(c, Q) = 0$, for all $c \in X$, and (iii) has the property that $\theta(c, Q) - c$ is equi-strictly decreasing.[3] *Under axiom C2, $\preceq$ is concave and regular only if*

$$\theta(c, Q) = \max\{c - E^Q f \mid V(f) = c\} \tag{4.6}$$

is such a representation, and this representation is minimal.

[3]This means that for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $Q \in \Delta$, $|d - c| \geq \varepsilon$ implies $|\theta(d, Q) - d - (\theta(c, Q) - c)| \geq \delta$.

Concave preference orderings can also be represented by R-representations,

$$V(f) = \min_{Q \in \Delta} R(E^Q f, Q) \tag{4.7}$$

with $R : X \times \Delta \to X$. These can be obtained from θ-representations, and vice versa, by the equivalence

$$R(m, Q) = c \quad \Leftrightarrow \quad \theta(c, Q) = c - m. \tag{4.8}$$

It follows that the minimal R-representation of $\preceq$ in $\mathcal{C}$ is given by

$$R(m, Q) = \max\{V(f) \mid E^Q f = m\}. \tag{4.9}$$

The proof of Lemma 4.2 is in terms of R-representations, and gives the properties of R analogous to (i)-(iii) in the lemma.

4.3 The fixed point update for concave preferences

We now come to the translation of the fixed point update rule in terms of representations of concave preferences. In addition to axiom C2, we impose another regularity condition to guarantee the relevance of each state.

C3 *(Sensitivity per state)* $b_s c \succ c$ for $c \in [w, b)$ and $s \in S$.

Definition 4.3 $\mathcal{C}$ is the class of preference orderings in $\mathcal{P}$ that satisfy axioms C1-3.

Lemma 4.4 *For a preference ordering $\preceq$ in $\mathcal{C}$, represented by θ according to (4.5), the condition $f_s c \sim c$ implies that $c \in \operatorname{range} f_s$, and is characterized by*

$$f_s c \sim c \quad \Leftrightarrow \quad \min_{Q \in \Delta^{s+}} E_s^Q f + \hat{\theta}_s(c, Q) = c \tag{4.10}$$

with

$$\hat{\theta}_s(c, Q) := \min_{T \in \Delta^{s+}} \frac{\theta(c, TQ_s)}{T(s)}. \tag{4.11}$$

For θ minimal, $\hat{\theta}_s(c, Q) = \min\{c - E_s^Q f_s \mid f_s \in \mathcal{A}_s^c\}$ with $\mathcal{A}_s^c := \{f_s \in \mathcal{A}_s \mid f_s c \succeq c\}$.

In terms of the R-representation, the rule (4.11) reads

$$\hat{R}_s(m, Q) := \min_{T \in \Delta^{s+}} R(T(s)m + (1 - T(s))c, TQ_s).$$

The specification of Theorem 1 for $\mathcal{C}$ can now be formulated as follows.

Theorem 2 *A preference ordering $\preceq$ in $\mathcal{C}$ has a unique fixed point update (4.3) in state s, if it has a θ-representation so that*

$$\text{the mapping } c \mapsto \hat{\theta}_s(c, Q) - c \text{ is strictly decreasing, for all } Q \in \Delta, \tag{4.12}$$

with $\hat{\theta}_s$ defined by (4.11). Then this unique fixed point update is regular, and represented by $\hat{\theta}_s$. Furthermore, then $\preceq_1 := (\preceq_s)_{s \in S}$ is sequentially consistent if

$$\min\{\theta(c, TQ_1) - E^T \theta_1(c, Q) \,|\, T \in \Delta\} = 0 \text{ for all } Q \in \Delta_1. \tag{4.13}$$

The conditions (4.12) and (4.13) are not only sufficient, but also necessary, for the minimal θ-representation (4.6) of $\preceq$.

So for minimal θ-representations of $\preceq$ in $\mathcal{C}$, (4.12) is equivalent to axiom S3, and then axiom S2 amounts to (4.13).

4.4 Updating under Dynamic Risk Aversion

More explicit characterizations can be obtained under additional assumptions in terms of dynamic risk aversion. This is a somewhat stronger version of the notion of consistent risk aversion, introduced in RS16. Starting point is a given reference measure $P \in \Delta^+$ with respect to which risk aversion is defined. The idea is to impose, in addition to straightforward risk aversion with respect to P, also some dynamic properties in the same spirit. The vector $V_1(f)$ of certainty equivalents (ceqs) of f conditioned on S, is identified with an act in $\mathcal{A}$. As before, f stands for an arbitrary act in $\mathcal{A}$.

R1 *(Risk Aversion)* $f \preceq E^P f$.

R2 *(Consistent Risk Aversion, to S)* $f \preceq E^P V_1(f)$.

R3 *(Consistent Risk Aversion, from S)* $f \preceq E_1^P f$.

R4 *(Super-recursiveness)* $f \succeq V_1(f)$.

Definition 4.5 The class $\mathcal{R}$ consists of preference orderings in $\mathcal{S}$ that satisfy axioms R1-4.

Axiom R1 requires that risk premiums $V - E^P$ are nonnegative. It is easily verified that sequentially consistent updates then have the corresponding property,

$$f \preceq_1 E_1^P f \tag{4.14}$$

The axioms R2 and R3 reflect the intuition that ignoring risk premiums leads to higher values, over resp. the 'stage' towards S, and the stage from S. Axiom R3 is not used in the results below, but added for symmetry. Axiom R4 expresses that the overall risk premium should not exceed the aggregation of those towards and from S, whereas the monotonicity axiom would require equality here.[4] Notice that sequential consistency (axiom S1) directly follows from axiom R2 and R4, so axiom S2 and S3 are in fact redundant in the definition of $\mathcal{R}$.

Theorem 3 *Consider a preference ordering $\preceq$ in $\mathcal{C}$ that satisfies axiom R1, R2 and R4. It has a unique fixed point update in state s if and only it has a θ-representation (4.5) with*

$$c \mapsto \theta(c, PQ_s) - cP(s) \text{ strictly decreasing, for all } Q \in \Delta.$$

[4] In RS16, Chapter 7 it is described how this generalization amounts to the extension from recursive valuation per separate risk aversion level to a joint recursion in a range of levels.

If so, the fpu $\preceq_s$ is regular, and represented by

$$\hat{\theta}_s(c,Q) := \frac{\theta(c,PQ_s)}{P(s)}.$$

The corresponding vector-update $\preceq_1$ is then sequentially consistent.

The proof in the appendix also contains a characterization of the θ-representations for each of the axioms R1–4. In terms of the R-representations,

$$\hat{R}_s(m,Q) := R(P(s)m + (1-P(s))c, PQ_s),$$

when $R(m+h,PQ_s) - R(m,PQ_s) \le h/(1-P(s))$.

4.5 The convex class and the reflection principle

Preference orderings $\preceq$ induce a counterpart $\preceq^*$ by the reflection principle

$$f \preceq^* g \ \Leftrightarrow \ -g \preceq -f. \tag{4.15}$$

This is a preference ordering on $\mathcal{A}^*$, which is equal to $-\mathcal{A}$. Of particular interest is the case with symmetric outcome range $X = [-b,b]$, since then also $\preceq^*$ is a preference ordering on $\mathcal{A}$. As $\preceq^*$ shares many properties with $\preceq$, we call them twin preferences. Proofs of the list of claims below are left to the reader.

All axioms A1-4, S1-3 and C2 hold for $\preceq^*$ precisely when they hold for $\preceq$. The concavity axiom C1 itself reflects into convexity, and axiom C3 yields $w_s c \prec^* c$ for constants $c \in (w,b]$. In a similar way, the inequalities in the anti-symmetric axioms R1-4 reverse for $\preceq^*$. The representation (4.5) transforms into

$$f \sim^* c \quad \Leftrightarrow \quad \max_{Q\in\Delta} E^Q f - \theta(-c,Q) = c.$$

The reflection principle commutes with fixed point updating, i.e., $(\preceq^*)_s = (\preceq_s)^*$, and we can simply write $\preceq_s^*$ in the context of sequentially consistent updating.

5 Consistency considerations without the STP

We now shift the focus from syntax to semantics. We first describe some preliminary considerations regarding the interpretation of twin preferences, and no-arbitrage conditions. Then we propose two notions of consistency: Monetary Consistency, tuned to a truly monetary setting, and Twin Consistency, with a broader scope.

5.1 On the consistency of twin preferences

It is critical to our normative claim that our interpretation of twin preferences is accepted as consistent. We therefore start with an elementary example.

Consider the elementary lottery e with equal probability on an outcome +1 and -1, in dollars say, and a DM with preference ordering $\preceq$ corresponding to expected utility with a strictly increasing concave utility function u. Then $e \sim -d$ for some constant $d > 0$. This is commonly phrased as "to the DM, e is indifferent to $-d$."

However, then his willingness to offer e is not also $-d$, but d. The monetary effect of writing a contract that gives someone else the right on e, is exactly the same as obtaining such a contract. For both he asks compensation d. This 'gap' between willingness to pay ($-d$) and to accept (d) is, in our framework, the opposite of an inconsistency: it is the same risk aversion that has opposite effects in opposite directions of trade. Intuitively spoken, the DM assigns a 'thick' value $[-d, d]$ to e, and 'thin' values c to constants c. Consequently, "$e \sim -d$" denotes only indifference "*if it comes to obtaining*", $e \sim^* d$ only if it comes to offering.

More generally, we identify obtaining an act f with offering $-f$, so that $V(f)$ can be interpreted as the willingness to obtain, and $V^*(f)$ as the willingness to keep, or the attractiveness to have.[5] One is the implication of the other, but they are not

[5] We avoid the standard term 'willingness to accept' (WTA) here, since (i) 'accept' is easily confused with obtaining, and (ii) in fact, it measures *un*willingness, since a higher outcome means

the same.

The crucial observations here are that the syntax of complete preference orderings does not impose that values are thin, and that thick values allow a consistent interpretation in a monetary setting.

5.2 Absence of arbitrage

Absence of arbitrage is easy to obtain by a standard condition in bid-ask price modeling, cf. Section 8.3. With $V(f)$ the DM's price he is willing to pay for f, Axiom A2 already rules out the most direct form of arbitrage: the DM paying a positive amount for an act with only non-positive outcomes. Depending on the context, also series of acts $f_1, \ldots, f_K$ have to be excluded that have the same net effect, i.e., with non-positive sum yet positive sum of values. For $K = 2$ this amounts to excluding *round trip arbitrage*, with $f_2 = -f_1$. The following lemma formulates a few standard results that we use. We assume a symmetric outcome range $[-b, b]$.

Lemma 5.1 *A preference ordering $\preceq$ in $\mathcal{P}$ is arbitrage free when it satisfies axiom R1. It is free from round trip arbitrage iff $V \leq V^*$, i.e., for all $f \in \mathcal{A}$, if $f \sim c$, then $-f \preceq -c$. For concave preferences in $\mathcal{C}$ this is the case if and only if for all $c \in [-b, b]$, there exists $P^c \in \Delta^+$ such that $\theta(c, P^c) = 0 = \theta(-c, P^c)$.*

It is essential for our interpretation that $V \leq V^*$, since it would be absurd to be willing to pay an amount c for f, and at the same time to offer it for less. This allows us to refer to V and V^* as resp. the upper and lower value induced by $\preceq$, following the terminology in e.g. Walley (1991). Updates inherit this ordering, i.e., $V_1 \leq V_1^*$, when they are sequentially consistent.

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less willingness to offer. The so-called WTP-WTA-bias is discussed in Section 8.3

6 Monetary Consistency

The notion of Monetary Consistency that we propose below, combines several consistency requirements, tuned to a monetary interpretation, in which replacing act f for g can simply be resembled by its monetary net effect $g - f$. It relies on the following static axiom for choice consistency.

MCC *(Monetary Choice Consistency)* $f \succ g \Rightarrow 0 \succ g - f \quad (f, g, g - f \in \mathcal{A})$.

The axiom requires that if one prefers to obtain f rather than g, it cannot be that at the same time one has in mind that it would be attractive to exchange f for g.

Definition 6.1 The class $\mathcal{M}$ of *monetarily consistent preferences* is the set of preferences in $\mathcal{S}$ that satisfy MCC, and axiom R1 for some $P \in \Delta^+$.

The axioms S1-3 guarantee that updates in $\mathcal{M}$ are well defined, consequentialist, and have the property that values $V(f)$ are in the range of the vector-update $V_1(f)$. Axiom R1 rules out arbitrage opportunities, and makes sure that value has nonnegative thickness $V^* - V$.

Concerning MCC, we first derive a two dynamic implications it has under fixed point updating, called Forward and Backward MCC:

$$\text{(FMCC) } f \succ g \quad \Rightarrow \quad 0 \succ_s g_s - f_s \text{ for some } s \in S \quad (f, g, g - f \in \mathcal{A}) \tag{6.1}$$

$$\text{(BMCC) } f \succ_1 g \quad \Rightarrow \quad 0 \succ g - f \quad (f, g, g - f \in \mathcal{A}). \tag{6.2}$$

Theorem 4 *If $\preceq$ in $\mathcal{P}$ is monetarily choice consistent (axiom MCC), and has sequentially consistent update $\preceq_1$, then also the forward and backward versions (6.1) and (6.2) are satisfied.*

As we have indicated in the Ellsberg example, FMCC guarantees that sticking to an ex-ante (strictly) optimal plan remains justified in view of updates. In particular, in case of a reversal, with $f \succ g$, g only differing from f in s, yet $g_s \succ f_s$, FMCC implies that $g_s - f_s \prec 0$, and hence that a 'fresh', ex-post opportunity to replace f by g will be declined.

The subtle point here concerns an option to switch in E that is already given ex ante. By assumption, $g = g_s f$. First observe that f, and hence g, must resemble rights or obligations outside s. Otherwise, $f = f_s c$ and $g = g_s c$ for $c := V(f) = V(g)$, but then, by definition of fixed point updates, no reversal can arise. Hence the option to switch in E must have been accompanied by some ex-ante agreement that guaranteed the outcomes when s would not have obtained. For a decision on such an embedded choice we rely on the idea that the inital plan was well-thought, and hence is maintained. It simply falls outside the scope of the consequentialist update, since the agreement provides a still relevant context outside s.

Note that if we legally would commit to ex-ante choices, the DM effectively already owns f ex ante, which further underpins the 0 in the compatibility condition in FMCC. We do not change the reference point from f to 0, it is 0 *de facto*.

So preference reversals do not lead to choice inconsistencies, when axiom MCC is imposed. Its forward aspect, FMCC, guarantees that carrying out an ex-ante strictly optimal plan remains weakly rational in view of updates along the way. We leave an interpretation of BMCC to the reader.

We believe that this justifies to deem $\mathcal{M}$ a class of consistent preferences. At the same time it accommodates the Allais and Ellsberg preferences, as shown in Section 2.

6.1 Monetary Consistency in the concave class

For concave preferences, axiom MCC can be characterized as follows. We call $\theta(c, Q)$ strictly binding if there is an f in the interior of $\mathcal{A}$ such that $E^Q f + \theta(c, Q) = c$. The lemma does not rely on symmetry of outcome ranges.

Lemma 6.2 *A concave preference in $\mathcal{C}$ with θ-representation (4.5) satisfies axiom MCC if it has a θ-representation with $\theta(0,Q) = 0$ for all Q with $\theta(c,Q)$ strictly binding for some $c \in X$, and only if its minimal representation has this property.*

We remark that the MCC criterion has quite strong implications for $\theta(0,Q)$: it is either zero or ineffective. A milder criterion is obtained by restricting the MCC condition to pairs f, g with either f or g a constant. It can be shown that this is equivalent to the condition that for all Q, $\theta(c,Q)$ achieves its minimum in $c = 0$. We will not elaborate on this.

7 Twin Consistency

The criterion for choice consistency, axiom MCC, relies on a truly monetary setting in which the expression $g - f$ makes sense. This may hinder extensions to other settings, for instance with nonnegative outcomes, or in units of utility (utils).

We therefore propose a variant of MCC that avoids the expression $g - f$, simply called Twin Consistency.

TC *(Twin Consistency)* $V(V_1(\cdot)) \leq V(\cdot) \leq V^*(V_1(\cdot)) \leq V^*(\cdot) \leq V^*(V_1^*(\cdot))$.

It has the following forward and backward aspects.

Theorem 5 *A preference ordering $\preceq$ in $\mathcal{S}$ satisfies axiom TC if and only if it satisfies both*

FCC (Forward Choice Consistency) $f \succ g \Rightarrow \quad V_1^*(f) \not\leq V_1(g)$.

BCC (Backward Choice Consistency) $f \succeq_1 g \Rightarrow \quad V^*(f) \geq V(g)$.

So, where in the formalization of stick-to-your plan the standard criterion compares $V_1(f)$ and $V_1(g)$, and FMCC compares $V_1(g - f)$ and 0, the idea of FCC is to compare $V_1^*(f)$ and $V_1(g)$. A similar comparison applies to BMCC and BCC, apart from the non-strictness in BCC.

Definition 7.1 The class $\mathcal{T}$ of *twin consistent preferences* is the set of preferences in $\mathcal{S}$ that satisfy axiom TC.

It is easily verified that axioms S1 and S2 are redundant here, that axiom R4 holds true in $\mathcal{T}$, and that $\mathcal{T}$ includes $\mathcal{R}$ (even when axiom R3 is relaxed). Furthermore, roundtrip arbitrage is excluded in TC, since it implies $V \leq V^*$, cf. Lemma 5.1. Of course, complete absence of arbitrage could also be imposed, as in $\mathcal{M}$, but we view it less compelling in non-monetary contexts.

We conclude that the axioms for $\mathcal{T}$ provide sufficient basis for a normative interpretation. Further comparison between axioms TC and MCC, and classes $\mathcal{M}$ and $\mathcal{T}$, is left as a topic of future research.

7.1 On setups with nonnegative outcomes only

Our normative interpretation, in particular for $\mathcal{T}$, extends to setups which take starting point in nonnegative outcomes only, $X = [0, b]$ in our notation. The point we distilled from the elementary example in Section 5.1 is less straightforward in this case, but perhaps therefore even more important. The twin preference $\preceq^*$ then only applies to non-positive acts, and this domain $-\mathcal{A}$ has no overlap with that of $\preceq$, other than the constant 0. So $\preceq^*$ on $-\mathcal{A}$ no longer derives from $\preceq$ on $\mathcal{A}$, but depends on the way $\preceq$ is extended to acts with symmetric outcome range $[-b, b]$. Then it is mathematically quite straightforward to insist on thin values for all acts, by taking $V(-f) = -V(f)$, so that $V^*(f) = V(f)$ on $\mathcal{A}$ (and $-\mathcal{A}$). In view of the elementary example, however, there is no reason to adopt this as a normative principle *per se*. In the contrary, our analysis suggests that a specification of the extension, be it in the form $\preceq$ on $-\mathcal{A}$, or $\preceq^*$ on $\mathcal{A}$, is an indispensable element in its normative interpretation. Value can have thickness, $V^* - V$, whether one is aware of it or not, and this justifies to decouple consequentialist update values from non-consequentialist replacement values. We view this a fundamental aspect of complete preference orderings, also in non-monetary settings, as indicated in Section 8.4.

Summarizing, we claim that regular preferences $\preceq$ in $\mathcal{P}$, combined with a specification of $\preceq^*$ in as far it does not derive from $\preceq$ by reflection, admit a normative interpretation when axioms S3 and TC are satisfied, despite the reversals they induce.

8 Related literature

We first explain the background of this paper. Then we discuss related literature on two central topics: updating, and the distinction between buying and selling prices that we based on the reflection principle. We conclude by discussing some additional topics.

8.1 Background

The central notion in our framework, sequential consistency, has been introduced in Roorda and Schumacher (2007) (henceforth RS07) and further developed in mathematically more advanced settings in RS13; RS16.[6] These papers describe axiomatic frameworks with sequential consistency as basis for unique updating, applicable to both a pricing context, with $V(f)$ the bid price of f, as well as a regulatory context,

[6]A common assumption in these papers is that value functions V are *translation invariant*, i.e., satisfy $V(f + c) = V(f) + c$. Sequential consistency then amounts to the criterion in axiom S1 restricted to $c = 0$.

with $E^P(f) - V(f)$ a required capital buffer against extreme losses.[7] The fixed point update rule (4.3) closely relates to the conditionally consistent updating rule in RS07 and the refinement update introduced in RS13. It turned out that the fixed point update rule already was present in the literature on preference orderings, as discussed below.

For a general introduction to sequential consistency, and other forms of non-recursive, so-called *weak time consistency concepts* we refer to the aforementioned papers and the references therein. That recursiveness is problematic in a regulatory context has been signaled in RS07, Ex. 8.8. Similar, yet less pronounced, concerns about recursive pricing are indicated in RS13, Ex. 3.9. The observed preference reversals described in these examples gave rise to further investigation at the level of complete preference orderings, of which the current paper is a reflection. As compared to the aforementioned papers, the new elements in this paper are the relaxation of translation invariance, the more extensive disussion of a normative interpretation, the notion of fresh choice, the results on choice consistency, and the connection with the Allais and Ellsberg paradox. Also, the updating formulas for quasi-concave value functions extend those in RS16 for ordinary concave valuations. The insights in Cerreia-Vioglio et al. (2011b) and Frittelli and Maggis (2011) provided the motivation and the tools for this extension.

[7] There is a continuous spectrum from pricing with small risk premiums (close to expected values) to so-called risk measures (much closer to worst-case), which explains that axiomatic frameworks in both domains are strikingly similar. For instance, the seminal paper Artzner et al. (1999) advocates coherent risk measures, which corresponds to MEU with trivial utility $u(x) = x$. Convex risk measures correspond to (ordinary) concave, translation invariant, monotone V, whereas convexity is not imposed for monetary risk measures, see Föllmer and Schied (2011).

8.2 On the fixed point update rule

The fixed point update rule (4.3) (fpu) is not new. It is essentially the same as the notion of *conditional ceq consistency* in Eichberger et al. (2007), building on Pires (2002, Axiom 9), which is the forward implication in (4.3). In Siniscalchi et al. (2001) it appears as constant-act dynamic consistency, and is interpreted as fixed point criterion. Its close connection with the Generalized Bayesian Rule in Walley (1991) and the Full Bayesian Updating Rule in Jaffray (1994) is well understood for the Gilboa-Schmeidler framework with Multiple Priors, also called Maxmin Expected Utility (MEU), see Pires (2002). In Eichberger et al. (2007) this connection is addressed at the level of capacities, and applied to the class of Choquet Expected Utility (CEU), also where it lies outside MEU (see also Horie (2007) for a correction).

Our main contribution is to establish this rule as a fundamental principle of updating outside the STP. As already indicated in Section 2.1, there are basically two remedies proposed in the literature, to cope with the preference reversals that arise, i.e., pairs f, g with $f \succ g$ yet $f \preceq_1 g$. Both rule out such reversals, either taking starting point in ex-ante optimality $f \succ g$, or in the backward induced implication $f \preceq_1 g$. We first put our results in this perspective.

Hanany and Klibanoff (2007, 2009) propose a non-consequentialist updating principle that by definition supports ex-ante optimal plans for a given choice set, building on the findings in Machina (1989). Such updates support plan consistency, not as an argument why, which would be circular, but as a description how outcomes in bygone states still influence embedded choices. We do not incorporate such updates in our framework, but respect their consequences by letting embedded choices be directly governed by the stick-to-the-plan principle, and in that sense avoid dynamic inconsistency as defined in these references. Compared to their approach, the choice consistency criteria MCC and TC hence should be viewed as static requirements on ex-ante preference orderings that could be added to their framework.

The alternative to cope with preference reversals is to 'fold back' backwardly induced conditional choices to initial choices, according to the principle of *consistent planning* (Strotz, 1955; Siniscalchi, 2009) and *behavioral consistency* (Karni and Safra, 1990). However, this leads to complicated requirements on ex-ante preference orderings to rule out reversals under a given update rule, which are hard to interpret. The reason that our static requirement (essentially TC, or MCC combined with Axiom S2) is much simpler, lies in the fact that we do not require that updates reproduce conditional choices, but only that they do not undermine them. As we said before, sticking to initially optimal plans only has to be weakly rational in view of the updates, not strongly. We obey consistent planning in this sense.

In view of the existing alternatives, doubts have been expressed that a universal update principle can exist outside expected utility models (Wakker, 2010; Machina and Viscusi, 2013). However, we think we found a third way out, that of accepting reversals by rethinking the very definition of choice consistency, without loosing the good elements in both alternatives.

In addition, we provided further underpinning of the fpu, by linking it to fresh choices, and deriving it as the only candidate for a sequentially consistent update (axiom S1). These key notions have received little attention in the literature so far.

Concerning the scope of the fixed point update rule, we see the following points of contribution, as compared to aforementioned references. From axiom S1, we arrive at the Equal Level Principle (axiom S2) as the fundamental criterion to ensure that the rule indeed produces a consistent update (while uniqueness is characterized by a sensitivity condition, axiom S3). So we identified Axiom S2 as the essential criterion that determines the domain of consistent updating, all by one and the same fpu rule. This normative domain for updating extends far outside the STP's, even the Comonotonic STP (Wakker, 2010), since only a condition on acts with the same value in each state S is imposed. We refer to RS16 for concrete descriptions of the extra freedom, allowing for jointly recursive evaluation at a range of risk aversion levels. How this relates to the empirical findings on violations of the STP is a topic of future research.

Finally, note that, under just the sensitivity condition axiom S3, the fpu produces an update for any regular preference ordering in $\mathcal{P}$, and axiom S1 rules out any alternative. It means that, under sequential consistency, the fpu provides the only candidate for consistent updating for virtually all known classes of complete preference orderings. This includes the quite general class of Uncertainty Averse Preferences (Cerreia-Vioglio et al., 2011a), which covers all known models that are extensions of MEU, and also Smooth Ambiguity Preferences (Klibanoff et al 2005). For instance, Variational Preferences (Maccheroni et al 2006) are those with θ-representations (4.5) independent of c. The specification of Thm. 2 to the case with $\theta(c, Q) \in \{0, \infty\}$ and independent of c, provides the criterion for MEU, and hence for convex capacities in CEU.[8] Moreover, utility functions with convex and concave segments are included, so that the fpu also applies to Cumulative Prospect Theory (CPT) (Tversky and Kahneman, 1992). For all these models, fpu is therefore the adequate update principle if sequential consistency is adopted.

[8]This criterion has been interpreted as a junctedness condition in RS07, which imposes that each set of conditional probabilities $P^i(\cdot|E_i)_{i\in\mathbb{N}}$ with $\theta(P^i) = 0$ that occurs, also occurs as the set of conditional probabilities of one 'junct' P^* with $\theta(P^*) = 0$ (see also RS16, Section 4 for the concave case). It shows the difference with the rectangularity condition implied by the STP for Π, as in Epstein and Schneider (2003).

8.3 On the reflection principle and bid-ask prices

One of the cornerstones of our framework is the observation that acts have not just one price, but (at least) two. This is well recognized in the literature, in several ways, under the heading of e.g. first order risk aversion, endowment effect, the WTP-WTA bias, and the law of two prices. Many aspects of our framework are already treated in these branches of literature, at a deeper level, and in a much more advanced mathematical and economic setting. An overview is far beyond our scope, and we only discuss a few representative examples, with a focus on the given interpretation of bid-ask spreads.

The reflection principle is a standard way to relate bid and ask prices, in particular in monetary settings. It is used for instance in *conic finance*, introduced in Madan and Cherny (2010) as a new way to model markets with bid-ask spreads.[9] Also the way in which loss aversion and probability weighting induce bid-ask spreads, as described in (Wakker, 2010, Ex. 6.6.1 and 9.3.2),[10] is in line with the reflection principle. For instance, in RDU, the reflection principle amounts to reversed rank order for ask prices. In fact, the findings in Birnbaum and Stegner (1979) already point in this direction. It links bid-ask spreads to so-called *configural weighting*, in the context of estimating used car prices. They find that "Judges instructed to take the buyer's point of view gave greater weight to the lower estimate, whereas judges who identified with the seller placed a greater weight on the high estimate," and they emphasize the point that it is the same cautiousness that results in different prices for opposite directions of trade.

To discern two prices for the same thing is hence by no means new, nor to

[9]The term conic refers to cones as the acceptance set of so-called coherent risk measures, which have been introduced by Artzner et al. (1999).

[10]Loss aversion in CPT is a form of first-order risk aversion, which corresponds to utility functions with a kink in a reference point, commonly at zero, cf. (Segal and Spivak, 1990).

base it on reflection, but we think that the consequences for interpretation and rationalization have not yet been fully recognized. For instance, in conic finance, recursiveness is still commonly imposed in bid and ask prices separately (Madan, 2016), so that the induced (dynamic) preference reversals are avoided. However, as we have indicated in RS13, Ex. 3.9, this may lead to a market in which round trip costs can always be avoided. An exception is Bielecki et al. (2013), which applies a notion of weak time consistency (their definition D7) to conic finance that corresponds to sequential consistency; the special role of the fpu is however not addressed, nor the idea of a joint recursion involving bid and ask prices. An interesting point of future research is to test for such a joint recursion on the basis of market prices.

Empirical research has led to extensive descriptions and interpretations of the discrepancy between selling and buying prices, or closely related notions. Common terms in this literature, like *WTP-WTA bias* (Machina and Viscusi, 2013, Chapter 4), *endowment effect* (Kahneman et al., 1990), (static) *preference reversal* (Karni and Safra, 1987), *failure of procedure invariance* (Tversky and Thaler, 1990) indicate that two values for the same thing is primarily viewed as irrational by nature. Our results give reason to rethink the normative content of these findings, and provides a general framework for further developing normative bid-ask models. Issues observed in eliciting certainty equivalents, as described in (Machina and Viscusi, 2013, Chapter 4), may be clarified by a careful distinction between replacement values and certainty equivalents, both for obtaining and offering. The notions of sequential consistency and choice consistency we proposed, suggest new empirical questions on dynamic choice.

Intuitively speaking, our analysis emphasizes the importance of keeping 'looking through' sub-acts, rather than perceiving them as represented by just one 'thin' value and then concentrate on the weight of that value. We do agree that such a weight is configural, but it need not be a function of the ceqs of sub-acts alone.

8.4 Other topics

It may be illuminating to compare our framework to the standard monotonicity axiom in Anscombe-Aumann (AA)-settings, with sub-lotteries identified with their probability distribution.[11] This is often accepted as a basic rationality axiom, see e.g. Gilboa (2015). In our approach, the fpu rule (4.3) assigns to each roulette lottery its certainty equivalent, $V_s(f_s)$, in a consequentialist way, but its replacement value in the horse lottery (r such that $V(f) = V(r_s f)$) may depend on the roulette lotteries in other states. We only require, by axiom S2, that if all probability distributions have the same ceq c (if it comes to obtaining), then c is the ceq of the horse lottery as a whole (if it comes to obtaining).

The STP is largely equivalent to the monotonicity axiom, and implies backward induction (ceqs are replacement values, i.e., $r \in X$ such that $r_s f = f$). Normative objections against backward induction in the literature concentrate on versions with ambiguous states (see Trautmann and Wakker (2015) and references therein). The objections confirm our viewpoint that replacement values are non-consequentialist by nature, but our line of reasoning is different, and does not rely on the distinction between risk and ambiguity, cf. Section 2.2. The reference to twin preferences may bring a new element to these considerations.

We have proposed two notions of choice consistency, MCC in Section 6, and TC in Section 7. To our knowledge, the interpretation of MCC as static requirement to ensure the dynamic forward and backward versions, is new. The criterion for MCC in Lemma 6.2 exhibits similarities with first-order risk aversion, that deserve further

[11]The AA-setting can be approximated in our setting by taking Ω sufficiently large. State independence can be obtained by taking $\Omega = S \times \Omega'$. Monotonicity in final outcomes (axiom A2) amounts to first-order stochastic dominance under law-invariance.

investigation. We did not find a reference that relates the elementary conditions in the second notion, TC, to choice consistency.

There is an interesting link with how Bewley's incomplete preferences are used in Gilboa (2015) to model objective rationality. A given preference $\preceq$ induces an incomplete preference ordering $\preceq^i$, with $f \preceq^i g$ if and only if $V^*(f) \leq V(g)$. Forward choice consistency (axiom FCC) requires that when $f \preceq_1^i g$, it cannot be that $f \succ g$. If we interpret $\preceq_1^i$ as the 'objectively rational' part of $\preceq_1$, axiom FCC can be phrased as "a preference ordering should respect the objectively rational consequences of its update".

We relied on the reflection principle (4.15) to justify the disconnection of ceqs and replacement values. Although we believe that it is rather compelling in a monetary setting, it may require modification in other contexts, especially when the negative of outcomes is not well defined. For instance, one can first express outcomes in utils, and then identify offering x utils with obtaining $-x$ utils. When the utiltity function u is continuous and strictly monotone, the axioms for $\mathcal{S}$ each amount to the same for $\preceq$ and the transformed ordering $\tilde{\preceq}$ on acts $u \circ f$. This indicates that essence of our interpretation is also relevant in non-monetary settings.

9 Conclusions

We described a normative framework for preferences without the STP, in which consequentialist updating is combined with choice consistency. A key aspect of our interpretation is the recognition of a twin side inherent in preference orderings, giving value 'thickness'. We identified the fixed point rule as the fundamental unifying update principle, which can be so straightforward because it only pertains to fresh choices. In this way, we extended the scope of consistent updating from the STP to the much weaker Equal Level Principle (axiom S2). The proposed notions of choice consistency are tuned to these viewpoints, and pull out the sting of the Allais and Ellsberg paradoxes.

This extension brings an extra dimension to the many concepts and techniques rooted in the STP, such as risk neutral valuation, dynamic programming, sub-game perfectness, and exponential discounting. It invites less monotone ways to break time and value into pieces, accepting preference reversals without a sigh.

A Proofs

A.1 Proof of Theorem 1

We first prove that (4.3) defines a unique update $\preceq_s$ under S3, for each $s \in S$. Let V denote the (normalized) value function of $\preceq$. Consider, for given $f_s \in \mathcal{A}_s$, the mapping $\rho : c \mapsto V(f_s c)$ on the domain range$(f_s) =: [l, r]$. Since V is continuous and monotone, ρ is continuous, $\rho(l) \geq l$ and $\rho(r) \leq r$. So ρ has a fixed point c' on this domain, i.e., there exists c' satisfying the right-hand side (rhs) of (4.3). Axiom S3 guarantees that such c' is unique, and hence that $\preceq_s$ is uniquely determined by (4.3). This means that $\preceq_1$ is indeed unambiguously defined by (4.3).

This proves the if-part of the first claim of the theorem. The only if-part is obvious from the formulation of S3.

Regularity of $\preceq_s$, under axiom S3, follows straightforwardly from regularity of $\preceq$. In particular, $\preceq_s$ is continuous, because for a series $f_s^k \to f_s$ in $\mathcal{A}_s$, with c_k the unique solution of the rhs of (4.3) for f_s^k, any converging subseries $(c_k)_{k \in \mathcal{I} \subset \mathbb{N}} \to c'$ yields $V(f_s c') = c' \in$ range(f_s), by continuity of V; so c' must be the unique solution of the rhs in (4.3), and hence the full series $(c_k)_{k \in \mathbb{N}}$ is converging to c'.

To see that $\preceq_1$ defined by (4.3) is sequentially consistent if $\preceq$ satisfies axiom S2, consider $f \in \mathcal{A}$ with $f \sim_1 c$. Then (4.3) implies that for all $s \in S$, $f_s c \sim c$ with $c \in$ range(f_s), and by axiom S2, $f \sim c$, so that axiom S1 follows.

It remains to show, under axiom S3, that if $\preceq$ has a regular sequentially consistent update $\preceq_1$, then $\preceq$ must satisfy axiom S2. Let an act $f \in \mathcal{A}$ be given with $f_s c \sim c$ and $c \in \text{range}(f_s)$ for all $s \in S$. We have to prove that $f \sim c$. Consider an $s \in S$. As $\preceq_1$ is regular, there exists $c' \in \text{range}(f_s)$ such that $f_s \sim_s c'$, and hence $f_s c' \sim_1 c'$. But then $f_s c' \sim c'$ by axiom S1, while also $f_s c \sim c$ by assumption, and axiom S3 implies that $c' = c$. Since $s \in S$ was arbitrary, $f_s \sim_s c$ for all $s \in S$, and, again by axiom S1, indeed $f \sim c$.

A.2 Proof of Lemma. 4.2

We prove the lemma in terms of R-representations (4.7), in line with Cerreia-Vioglio et al. (2011b), using the 1-to-1 correspondence (4.8), which is easily verified. Note that $m \mapsto R(m, Q)$ is the inverse of the strictly increasing mapping $c \mapsto c - \theta(c, Q)$. The properties (i)-(iii) translate for R into (i′) R is strictly increasing in m, (ii′) $\min_Q R(m, Q) = m$, and (iii′) R is equi-continuous in m, i.e., for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $Q \in \Delta$, $|m' - m| < \delta$ implies $|R(m', Q) - R(m, Q)| < \varepsilon$.

For the if-claim, we have to prove that V in (4.7) is monotone, normalized, continuous, and quasi-concave if R satisfies (i′)-(iii′). Monotonicity is obvious, and (ii′) implies that V is normalized, i.e., that $V(c) = c$. That V is quasi-concave follows from (i′) and the inequality $E^Q(\lambda f + (1 - \lambda)g) \geq \min\{E^Q f, E^Q g\}$. It remains to show that V is continuous.

Consider a series $f^n \to \bar{f}$ and define $c_n := V(f^n)$ and $c := V()$. We have to prove that $c_n \to c$. From (4.7) it follows that for all n, $R(E^Q f^n, Q) \geq c_n$ for all Q, and that equality is reached for some $Q^n \in \Delta$. From continuity of $R(m, Q)$ in m it follows that $R(E^Q \bar{f}, Q) \geq \limsup c_n =: \hat{c}$, for all $Q \in \Delta$, and hence $V(\bar{f}) \geq \hat{c}$. On the other hand, $\liminf R(E^{Q^n} \bar{f}, Q^n) \leq \liminf c_n =: \bar{c}$, and by equi-continuity of R hence $V(\bar{f}) \leq \bar{c}$. It follows that $V(\bar{f}) = \hat{c} = \bar{c} = \lim c_n = c$.

For the only-if claim under axiom C2, consider a concave preference ordering in

$\mathcal{P}$. Let R be given by (4.9), which corresponds to θ in (4.6), and define $\bar{V}(f) := \min_{Q\in\Delta} R(E^Q f, Q)$. We have to prove that $\bar{V} = V$, and that R satisfies (i′)-(iii′). It is obvious that R is minimal.

That $\bar{V} = V$ is a standard duality result: $\bar{V} \geq V$ by construction, for any $\preceq$, since for f with $V(f) = c$, $R(E^Q f, Q) \geq c$ on Δ, and hence $\bar{V}(f) \geq c$. By a standard separating hyperplane argument, which relies on the convexity of the upper contour sets $\mathcal{A}^c$ (4.4), equality follows when $\preceq$ is concave.

To derive (i′), notice that $R(m, Q)$ is non-decreasing in m, since V is monotone. If it were not strictly increasing, say $R(m, Q) = R(m + d, Q)$ for some m and Q, then there would exist an f with $V(f) = R(m, Q)$, by (4.9), and hence with $V(f + d) \leq R(m + d, Q) = V(f)$, which violates axiom C2.

Property (ii′) follows from $V(c) = c$. This implies that $R(c, Q) \geq c$ on Δ, and also that $V(c) = R(E^Q c, Q) = R(c, Q) = c$ for some $Q \in \Delta$, by (4.7) for $f = c$. So $\min_{Q\in\Delta} R(c, Q) = c$, which is (ii′).

Finally, (iii′) follows from continuity of V. By continuity of V, for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|V(f) - V(g)| < \varepsilon$ for all $|f - g| < \delta$, with $|\cdot|$ the sup-norm. Then also $R(m + d, Q) - R(m, Q) < \varepsilon$ for all $Q, m, d < \delta$, since

$$\begin{aligned} R(m + d, Q) &= \max\{V(f) \,|\, E^Q f \leq m + d\} \\ &= \max\{V(g + h) \,|\, E^Q g \leq m, |h| \leq d\} \\ &\leq \max\{V(g) \,|\, E^Q g \leq m\} + \varepsilon = R(m, Q) + \varepsilon \end{aligned}$$

Note that indeed ε is independent of m and Q.

A.3 Proof of Lemma 4.4

The preference ordering $\preceq$ has a θ-representation (4.5), since it satisfies axiom C1 and C2 by assumption. Axiom C3, restricted to a given state $s \in S$, can be characterized in terms of (any) θ-representation of $\preceq$ by the criterion

$$\theta(c, Q) = 0 \text{ with } c < b \Rightarrow Q(s) > 0. \tag{A.1}$$

This can be seen as follows. If the criterion does not hold true, for some c, Q, s,there exists a Q, $c < b$ with $Q(s) = 0$ and $\theta(c, Q) = 0$, and then $E^Q b_s c + \theta(c, Q) = c$ and hence $b_s c \sim c$. Conversely, if $b_s c \sim c$ for some $c < b$, there exists a $Q \in \Delta$ that achieves the minimum in (4.5) for $f = b_s c$, which is $E^Q b_s c + \theta(c, Q) = c$. This Q must have $Q(s) = 0$ and $\theta(c, Q) = 0$, and hence (A.1) does not hold true.

Hence, for (any) θ that represents $\preceq$, we have, with $\Delta^{s+} := \{Q \in \Delta \,|\, Q(s) > 0\}$,

$$\begin{aligned}
f_s c \sim c &\Leftrightarrow \min_{Q \in \Delta} E^Q f_s c + \theta(c, Q) = c \\
&\Leftrightarrow \min_{Q \in \Delta^{s+}} E^Q f_s c + \theta(c, Q) = c \\
&\Leftrightarrow \min_{Q \in \Delta^{s+}} E^Q_s f + \frac{\theta(c, Q)}{Q(s)} = c \\
&\Leftrightarrow \min_{T, Q \in \Delta^{s+}} E^Q_s f + \frac{\theta(c, TQ_s)}{T(s)} = c \\
&\Leftrightarrow \min_{Q \in \Delta^{s+}} E^Q_s f + \hat{\theta}_s(c, Q) = c
\end{aligned}$$

That $f_s c \sim c$ only admits solutions $c \in \text{range}(f_s)$, is obvious from the third line above, combined with the fact that $\min\{\theta(c, Q) \,|\, Q \in \Delta^+\} = 0$, by (A.1).

Note that $\hat{\theta}_s(c, RQ_s)$ is independent of $R \in \Delta^{s+}$, so that we can write $\hat{\theta}_s(c, Q_s)$ with Q_s the conditional probability of $Q \in \Delta^{s+}$ in s. We show that when θ is minimal, $Q_s \mapsto \hat{\theta}_s(c, Q_s)$ is lower semi-continuous (l.s.c.) and convex in Q_s, for all $c \in X$. By a standard duality result, then $\hat{\theta}_s(c, \cdot)$ must be the minimal representation of $\mathcal{A}^c_s$, which is closed and convex, so that the last claim follows.

To establish l.s.c., consider a series $Q^n_s \to \bar{Q}_s$, and take T^n such that $\hat{\theta}_s(c, Q^n_s) = \theta(c, T^n Q^n_s)/T^n(s)$. Axiom C3 implies that $T^n(s)$ is bounded away from zero, as follows. Note that $\hat{\theta}_s(c, \cdot) \leq c - w$, hence bounded, since in (4.11) we can take T with $T(s) = 1$, and $\theta(c, \cdot) \leq c - w$ for minimal θ, by (4.6). So if a sub-series has

$T^n(s) \to 0$, then it also must have $\theta(c, T^n Q^n_s) \to 0$, and hence, because θ is l.s.c., have a limit point T' with $\theta(c, T') = 0$ and $T'(s) = 0$, precisely what is excluded by axiom C3, see (A.1). So $T^n(s)$ must be bounded away from zero. Consequently, there must exists $\bar{T} \in \Delta^{s+}$ such that $\liminf \theta(c, T^n Q^n_s) = \theta(c, \bar{T}\bar{Q}_s)$, and hence

$$\liminf \hat{\theta}_s(c, Q^n_s) = \liminf \theta(c, T^n Q^n_s)/T^n(s) \geq \theta(c, \bar{T}\bar{Q}_s)/\bar{T}(s) \geq \hat{\theta}_s(c, \bar{Q}_s),$$

where the first inequality follows from l.s.c. of θ, and the second by definition of $\hat{\theta}_s$.

To derive convexity, consider $\tilde{Q} = \lambda Q + (1-\lambda)Q'$ for some $\lambda \in (0, 1)$. We have to show that $\hat{\theta}_s(c, \tilde{Q}) \leq \lambda \hat{\theta}_s(c, Q) + (1-\lambda)\hat{\theta}_s(c, Q')$. By definition of $\hat{\theta}_s$, the convex combination is equal to

$$\lambda \frac{\theta(c, TQ_s)}{T(s)} + (1-\lambda)\frac{\theta(c, T'Q\prime_s)}{T'(s)}$$

for some $T, T' \in \Delta^{s+}$. This can be rewritten as

$$\frac{\theta(c, \tilde{T}\tilde{Q}_s)}{\tilde{T}(s)} \text{ with } \tilde{T} = \mu T + (1-\mu)T', \mu = \frac{\lambda T'(s)}{\lambda T'(s) + (1-\lambda)T(s)}.$$

By definition of $\hat{\theta}_s$, it follows that this dominates $\hat{\theta}_s(c, \tilde{Q})$.

A.4 Proof of Theorem 2

To derive that the criterion (4.12) is sufficient for uniqueness of fixed point updates (and hence implies axiom S3), suppose $f_s c \sim c$ and $f_s d \sim d$ for some $c < d$. From the first indifference, it follows by (4.10) that there exists Q^* with $E^{Q^*}_s f + \hat{\theta}_s(c, Q^*) - c = 0$. But then, since also $f_s d \sim d$, $E^{Q^*}_s f + \hat{\theta}_s(d, Q^*) - d \geq 0$, and hence, contrary to (4.12),

$$\hat{\theta}_s(c, Q^*) - c \leq \hat{\theta}_s(d, Q^*) - d. \tag{A.2}$$

To prove the necessity of (4.12) when θ is minimal, suppose (A.2) holds true. By Lemma 4.4, there exists f_s with $f_s d \sim d = E^{Q^*}_s f_s + \hat{\theta}_s(d, Q^*)$, while, by (A.2), $f_s c \preceq c$. So axiom S3 is not satisfied, and hence fixed point updates are not unique.

That the unique fpu is regular, has already been proved in Theorem 1, and that it is represented by $\hat{\theta}_s$ is obvious.

The criterion (4.13) is essentially the same as in RS16, Corollary 4.3, and the proof that it characterizes sequential consistency, is entirely analogous, though technically far less complicated due to the simplicity of our mathematical setting.[12] Sufficiency of (4.13) is straightforward, and its necessity is derived from a separating hyperplane argument.[13]

A.5 Proof of Theorem 3

As an additional result, we first characterize the axioms R1–4.

Lemma A.1 *Let representations θ and θ_1 be given that represent resp. $\preceq$ and $\preceq_1$, by (4.5), and let V denote the value function of $\preceq$. The pair $\preceq, \preceq_1$ satisfies*

1. *axiom R1 if* $\theta(c, P) = 0$
2. *axiom R2 if* $\theta(E^P c_1, PQ_1) \leq E^P \theta_1(c_1, Q)$
3. *axiom R3 if* $\theta(c, Q'P_1) \leq \theta(c, Q'Q_1)$
4. *axiom R4 if* $\theta(V(c_1), Q'Q_1) - V(c_1) \geq E^{Q'}(\theta_1(c_1, Q) - c_1)$

for all $c \in X$, $c_1 \in X^{\#S}$, $Q' \in \Delta$, $Q \in \Delta^+$. Conversely, each of the four conditions is implied by resp. axiom R1–4 for minimal θ and θ_1.

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[12] On the other hand, in RS16 only *translation invariant* value functions are considered, with $\mathcal{A}^c = \mathcal{A}^0 + c$, cf. (4.4), so that only the case $c = 0$ has to be addressed. In our setting, for a fixed c we can define $\tilde{\mathcal{A}}^d := \mathcal{A}^c + d - c$, which is translation invariant, and apply the proof in RS16 to $\tilde{\mathcal{A}}^0$.

[13] Applied to two disjoint convex sets, denoted as $\mathcal{Y}$ and $\mathcal{Z}$. In our setting, for given $c \in X$, we can take $\mathcal{Z} := \{(Q,t) \,|\, t \geq \theta(c, Q)\}$ and $\mathcal{Y} := \{(Q,t) \,|\, t \leq E^Q \theta_1(c, Q^*)$, where Q^* denotes a probability measure for which (4.13) is not satisfied.

Proof of the auxiliary lemma

On axiom R1. sufficiency of the criterion is obvious from (4.5), necessity from (4.6).

On axiom R2. For the sufficiency of the criterion, consider f with $V_1(f) = c_1 = E_1^{\tilde{Q}} f + \theta_1(c_1, \tilde{Q})$, for some $\tilde{Q} \in \Delta^+$. Then

$$E^P V_1(f) = E^{P\tilde{Q}_1} f + E^P \theta_1(c_1, \tilde{Q}) \geq E^{P\tilde{Q}_1} f + \theta(E^P c_1, P\tilde{Q}_1),$$

and hence $f \preceq E^P c_1$. For its necessity, suppose it does not hold true for minimal θ, i.e., for some $\tilde{c}_1, \tilde{Q}$,

$$\theta(E^P \tilde{c}_1, P\tilde{Q}_1) > E^P \theta_1(\tilde{c}_1, \tilde{Q}) \tag{A.3}$$

Define $\tilde{c} := E^P \tilde{c}_1$. Since θ is minimal, there exists $\tilde{f} \in \mathcal{A}$ with

$$V(\tilde{f}) = E^P \tilde{c}_1 = E^{P\tilde{Q}_1} \tilde{f} + \theta(\tilde{c}, P\tilde{Q}_1) = \tilde{c} \tag{A.4}$$

(note that generally not $V_1(\tilde{f}) = \tilde{c}_1$). But then, contrary to axiom R2, $E^P V_1(\tilde{f}) < \tilde{c}$, which can be derived as follows. From (A.3) and (A.4), $E^P[E_1^{\tilde{Q}} \tilde{f} + \theta_1(\tilde{c}_1, \tilde{Q})] < \tilde{c}$, so there must exist $s \in S$ with $E_s^{\tilde{Q}} \tilde{f} + \theta_s(\tilde{c}_s, \tilde{Q}) < \tilde{c}_s$. Hence, by property (ii) in Lemma 4.2, there exist $\delta > 0$ so that $E_s^{\tilde{Q}} \tilde{f} + \theta_s(\tilde{c}_s - \delta, \tilde{Q}) = \tilde{c}_s - \delta$. Define $t_1 := \tilde{c}_1 - 1_s \delta$. Using an alternative expression for V_1, namely

$$V_1(f) = \min\{E_1^Q f + \theta_1(c_1, Q) \wedge c_1 \,|\, c_1 \in X^{\#S}, Q \in \Delta\},$$

it follows that

$$E^P(V_1(\tilde{f})) \leq E^P(E_1^{\tilde{Q}} \tilde{f} + \theta_1(t_1, \tilde{Q}) \wedge t_1) < E^P \tilde{c}_1 = \tilde{c}.$$

On axiom R3. For the sufficiency of the criterion, consider f with $E_1^P f =: g \sim c$. By (4.5) there exists $\tilde{Q}'$ with $E^{\tilde{Q}'} g + \theta(c, \tilde{Q}') = c$. The criterion implies that $\theta(c, \tilde{Q}') \geq \theta(c, \tilde{Q}' P_1)$, so $V(f) \leq E^{\tilde{Q}' P_1} f + \theta(c, \tilde{Q}' P_1) \leq c$.

For its necessity, suppose the criterion does not hold true for minimal θ, i.e, there exists c, $\tilde{Q}'$ with $\theta(c, \tilde{Q}' P_1) > \theta(c, \tilde{Q}')$. Since θ is minimal, there exists $f \sim c$ with $E^{\tilde{Q}' P_1} f + \theta(c, \tilde{Q}' P_1) = c$. But then $E_1^P f \prec c$, as $E^{\tilde{Q}'}(E_1^P f) + \theta(c, \tilde{Q}') < c$.

On axiom R4. We first derive an expression for the minimal representation of $\check{\mathcal{V}} := V(V_1(\cdot))$. Define $\check{\theta}$ by $\check{\theta}(c,Q) = \max\{c - E^Q f \mid \check{\mathcal{V}}(f) \geq c\}$. Note that $\check{\mathcal{V}}$ is continuous and concave, so that (4.6) indeed applies. Rewrite, with θ_1 minimal,

$$\begin{aligned}\check{\theta}(c,Q) &= \max\{c - E^Q f \mid V(V_1(f)) \geq c\} \\ &= \max_{\{c_1 \mid V(c_1)=c\}} \max\{c - E^Q f \mid V_1(f) \geq c_1\} \\ &= \max_{\{c_1 \mid V(c_1)=c\}} \max\{c - E^Q c_1 + E^Q(c_1 - E_1^Q f) \mid V_1(f) \geq c_1\} \\ &= \max_{\{c_1 \mid V(c_1)=c\}} \{c - E^Q c_1 + E^Q \theta_1(c_1, Q)\}\end{aligned}$$

Axiom R4 amounts to the requirement that $\check{\theta} \leq \theta$, and this is exactly the criterion. *End of the proof of the auxiliary lemma*

To prove the theorem, notice that the criterion for R4, with $c_1 = c \in X$, yields $\theta(c, Q'Q_1) \geq E^{Q'}\theta_1(c,Q)$. Combined with R1, this implies $\theta_1(c,P) = 0$, and with R2 it yields $\theta(c, PQ_1) = E^P\theta_1(c,Q)$. Taking Q of the form PQ_s, i.e., only differing from P in s, gives $\theta(c, PQ_s) = E^P\theta_1(c, PQ_s) = P(s)\theta_s(c, PQ_s)$. This implies that the minimum in (4.11) is achieved for $T = P$. All claims follow.

A.6 Proof of Lemma. 5.1

For the first claim, consider $f_1, \ldots, f_K$ with $\Sigma_k f_k \leq 0$. By axiom R1, we have $V \leq E^P$, and hence $\Sigma_k V(f_k) \leq \Sigma_k E^P f_k = E^P \Sigma_k f_k \leq 0$, which we had to show.

By definition, a round trip arbitrage opportunity is the existence of $f \in \mathcal{A}$ with $V(f) + V(-f) > 0$. The second claim follows from $V(-f) = -V^*(f)$, cf. (4.15).

The condition in the last claim implies that the two convex spaces $\mathcal{A}^c$ and $-\mathcal{A}^{-c}$ (see (4.4)) are separated by $\{f \mid E^{P^c} f = c\}$ (not strictly, because c is in their intersection). This implies $V \leq V^*$. Conversely, $V \leq V^*$ implies that there is a separating hyperplane between both aforementioned spaces. This takes the form $\{f \mid E^Q f = c\}$ for some $Q \in \Delta$. By axiom C3, $Q \in \Delta^+$, cf. (A.1), so we can take Q

as P^c, and the last claim follows.

A.7 Proof of Theorem 4

The implication (6.1) follows from $f \succ g \Rightarrow 0 \succ g - f \Rightarrow 0 \not\preceq_1 g - f$, by resp. MCC and (4.2). The implication (6.2) follows similarly, from $f \succ_1 g \Rightarrow 0 \succ_1 g - f \Rightarrow 0 \succ g - f$, where for the first implication we now used the fact that a sequentially consistent update $\preceq_1$ inherits the static property MC (in each $s \in S$) from $\preceq$. To see this, assume $f_s \succ_s g_s$. Then, by axiom S3, $f_s c \succ g_s c$ for $c \sim f$, and hence $g_s 0 - f_s 0 \prec 0$, by MCC for $\preceq$, so $g_s - f_s \prec_s 0$, again by S3.

A.8 Proof of Lemma 6.2

Observe that MCC is not weakened if we impose it for f, g with $f, g, g - f$ in the interior of $\mathcal{A}$, since V is continuous. So MCC is the condition that $f + h \succeq f$ for all $f, f + h$ in the interior of $\mathcal{A}$ with $h \succeq 0$.

To derive that the criterion is necessary for MCC, suppose there exists $\tilde{Q}$ with (i) $\theta(0, \tilde{Q}) > 0$ and (ii) $\theta(c, \tilde{Q})$ strictly binding, i.e., with $E^{\tilde{Q}} \tilde{f} + \theta(c, \tilde{Q}) = c$ for some $\tilde{f} \sim c$ in the interior of $\mathcal{A}$. From (i), and minimality of θ, it follows that there exists $h \sim 0$ with $E^{\tilde{Q}} h + \theta(0, \tilde{Q}) = 0$, hence $E^{\tilde{Q}} h < 0$. By (ii), $\tilde{f} \sim c \succ \tilde{f} + \lambda h$ for all $\lambda \in (0, 1]$, and $\tilde{f} + \lambda h$ is in the interior of $\mathcal{A}$ for sufficiently small λ, contrary to MCC.

To derive sufficiency, consider $c \sim g \prec f$. We can assume that g is in the interior of $\mathcal{A}$ (otherwise add a sufficiently small constant to g). So there exists $\tilde{Q}$ with $E^{\tilde{Q}} g + \theta(c, \tilde{Q}) = c < E^{\tilde{Q}} f + \theta(c, \tilde{Q})$, hence with $E^{\tilde{Q}}(g - f) < 0$. Since $\theta(c, \tilde{Q})$ is strictly binding, the criterion imposes that $\theta(0, \tilde{Q}) = 0$, and consequently $g - f \prec 0$.

A.9 Proof of Theorem 5

If part: FCC is the implication $V_1^*(f) \leq V_1(g) \Rightarrow f \preceq g$. Indeed, by TC, $V(f) \leq V(V_1^*(f)) \leq V(V_1(g)) \leq V(g)$ when $V_1^*(f) \leq V_1(g)$. Similarly, TC implies that $V(g) \leq V^*(V_1(g)) \leq V^*(V_1(f)) \leq V^*(f)$ when $V_1(f) \geq V_1(g)$, and BCC follows.

Only-if part: From FCC with $f = V_1(g)$ and $g \in \mathcal{A}$, it follows that $V(V_1(g)) \leq V(g)$, and from BCC we have $V(g) \leq V^*(V_1(g))$. The third inequality in TC is equivalent to $V(V_1^*(f)) \geq V(f)$, which follows from FCC with $g = V_1^*(f)$. The last inequality in TC is equivalent to the first (and added for symmetry).

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