Time-dependent linearisation of bottom friction for storm surge modelling in the Wadden Sea

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Abstract

The nonlinear nature of bottom friction in shallow flow complicates its analysis, particularly in idealised models. For tidal flows, Lorentz' linearisation has been widely applied, using an energy criterion to specify the friction coefficient. Here we propose an extension of this approach to storm surges, leading to a friction coefficient that may gradually vary over a storm event. The derivation is provided along with first results for a single channel.

1. Introduction

Bottom friction is important in shallow water flows, such as tides and storm surges in the marine environment. However, the nonlinear (quadratic) dependency of the bottom stress on flow speed complicates its implementation, analysis and interpretation. In his pioneering channel network model for the impact of the closure of the Zuiderzee (Fig.1a), Lorentz introduced a linearised stress parameterisation, with a coefficient r specified on the basis of equivalence of dissipated energy over a tidal period (Lorentz, 1922; Staatscommissie Zuiderzee, 1926). This coefficient is (proportional to) a velocity scale, which in the tidal context has the clear interpretation of being the tidal flow velocity amplitude of a single constituent. An iterative procedure can be applied to ensure that the velocity scale used to specify r matches the velocity scale in the model results (Reef et al., 2016). Lorentz' linearisation has been verified experimentally (Terra et al., 2005), and the effect of a dominant tidal constituent on the friction experienced by weaker components has been treated analytically (Inoue & Garrett, 2007).

For storm surges, however, such a linearisation approach is less straightforward as the velocity scale is harder to interpret. Restricting to a time-invariant approach and retaining the quadratic formulation, Lorentz algebraically analysed the equilibrium response to a representative value of the wind stress (Staatscommissie Zuiderzee, 1926). Clearly, this approach does not capture the time-varying nature of forcing (wind stress, atmospheric pressure) and response (surge).

Here we propose a linearisation of the bottom stress τ_b involving a linear friction coefficient that adjusts to the temporal development of a storm event. Assuming one-dimensional depth-averaged flow u, this can be summarised as

$$\frac{\tau_b}{\rho} = \underbrace{c_d|u|u}_{\text{quadratic}} \approx \underbrace{r(t)u}_{\text{linearised}}, \tag{1}$$

with c_d the dimensionless drag coefficient of the original quadratic parameterisation and $r(t)$ the time-dependent linear friction coefficient (in m s^{-1}). Further, ρ is the water density. The proposed linearisation enables us to simplify calculations while still capturing the (nonlinear) variation in bottom stress over the various stages of a storm event (Fig.1b). In particular, $r(t)$ will be large whenever flow is strong and small when it is weak.

To specify the linear friction coefficient, we adopt an energy criterion analogous to that of Lorentz, but now in an instantaneous rather than tide-averaged sense. Specifically, $r(t)$ must be such that the instantaneous energy dissipation by bottom friction, i.e. the *power* $\tau_b u$, averaged over the channel is identical for both parameterisations in Eq.(1):

$$\frac{1}{\ell}\int_0^\ell c_d|u|u^2 \mathrm{d}x = \frac{1}{\ell}\int_0^\ell r(t)u^2 \mathrm{d}x \qquad \Rightarrow \qquad r(t) = \frac{c_d \int_0^\ell |u|u^2 \mathrm{d}x}{\int_0^\ell u^2 \mathrm{d}x}. \tag{2}$$

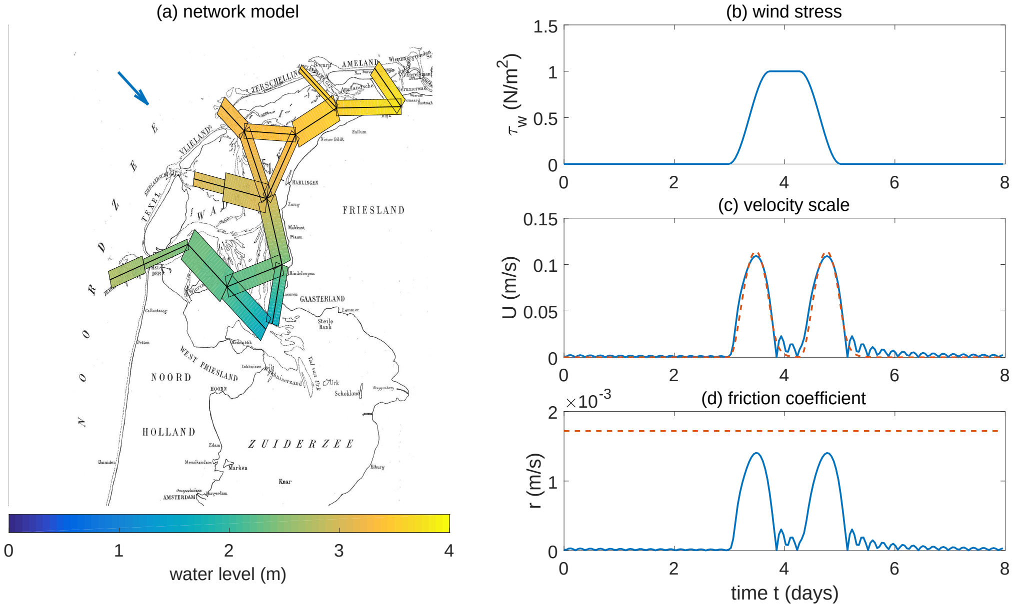

Figure 1. (a) Example simulation with network model applied by Lorentz and Staatscommissie Zuiderzee (1926), as reproduced by Reef et al. (2016). Then, for an artificial storm event, as a function of time (solid line): (b) wind stress $\tau_{\rm w}(t)$, (c) channel-averaged velocity scale, defined as $U(t) = \frac{1}{\ell}\int_0^\ell |u|{\rm d}x$, (d) time-varying friction coefficient $r(t)$. The dashed lines show an example with constant friction r_0, for the same wind event.

Here we have considered depth-averaged flow $u(x,t)$ in a channel ranging from $x = 0$ to $x = \ell$. Two remarks are in order:

- Just like the tidal case, specifying $r(t)$ requires knowledge of the flow solution $u(x,t)$, which, in turn, depends on $r(t)$ again. To deal with this cyclic dependency, the iterative procedure referred to above should be extended, now converging to a time-varying friction coefficient (rather than a scalar).
- The friction coefficient $r(t)$ depends on t but not on x. It is thus meant to represent the channel as a whole, which remains an important simplification compared to the original quadratic parameterisation in Eq.(1).

In the remainder of this extended abstract, we present an idealised one-dimensional storm surge model (§2.1) along with its solution procedure (§2.2), analyse and discuss some first results (§3), before summarising the conclusions (§4).

2. Methods

2.1 Model formulation

Consider a one-dimensional basin of length ℓ and uniform depth h with closed boundaries, subject to time-dependent wind stress. The free surface elevation is denoted by $\zeta(x,t)$, the depth-averaged flow velocity by $u(x,t)$. Assuming that $|\zeta| \ll h$, conservation of mass and momentum is expressed in linearised form according to

$$\frac{\partial \zeta}{\partial t} + h\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial u}{\partial t} + \frac{r(t)u}{h} = -g\frac{\partial \zeta}{\partial x} + \frac{\tau_{\rm w}(t)}{\rho h}, \qquad u(0,t) = 0, \qquad u(\ell,t) = 0, \tag{3}$$

where also the boundary conditions are shown. In Eq.(3), we have incorporated the linear stress parameterisation with coefficient $r(t)$ already presented in Eq.(1). The wind stress $\tau_{\rm w}(t)$ serves as the forcing of the system, assumed spatially uniform yet time-dependent (Fig.1b). Both $r(t)$ and $\tau_{\rm w}(t)$ are expressed as a discrete Fourier series according to

$$r(t) = \sum_{m=-M}^{M} R_m \exp({\rm i}\omega_m t), \qquad \frac{\tau_{\rm w}(t)}{\rho} = \sum_{m=-M}^{M} T_m \exp({\rm i}\omega_m t), \qquad \omega_m = \frac{2m\pi}{T_{\rm recur}}, \tag{4}$$

with truncation number M and complex coefficients R_m and T_m (satisfying $R_{-m} = \overline{R_m}$ and $T_{-m} = \overline{T_m}$ because r and $\tau_{\rm w}$ are real-valued, with an overbar denoting complex conjugation). Furthermore, we have introduced the angular frequency ω_m

and fictitious recurrence period T_{recur}, over which the storm event recurs in our model (due to the discrete Fourier series). Initial conditions can be safely ignored by choosing T_{recur} sufficiently large (e.g., in the order of 10 days) compared to the frictional decay time scale T_{fric} (typically less than a day). The dynamic equilibrium solution obtained will then display still water conditions at the beginning of the storm event (Chen, 2015; Reef et al., 2016).

The above model equations and boundary conditions are supplemented with the energy criterion in Eq.(2), which — through the flow field — specifies $r(t)$ as a function of time. This dependency of $r(t)$ on the solution is in fact the only nonlinear element in an otherwise linear model.

2.2 Solution procedure

Our solution method consists of a calculation of $\zeta(x,t)$ and $u(x,t)$, nested in an iterative procedure to find $r(t)$.

First, the model is expressed in terms of the free surface elevation amplitude $Z_m(x)$ only. Then, analogous to Eq.(4), the free surface elevation is written as a discrete Fourier series, introducing complex Fourier coefficients $Z_m(x)$ as a function of space. This gives the following coupled Helmholtz-type of boundary value problem for $Z_m(x)$:

$$\frac{\mathrm{d}^2 Z_m}{\mathrm{d}x^2} - \frac{1}{gh^2}\left[\sum_n R_{m-n}\mathrm{i}\omega_n Z_n\right] + \frac{\omega_m^2}{gh} Z_m = 0, \qquad \frac{\mathrm{d}Z_m}{\mathrm{d}x}(0) = \frac{\mathrm{d}Z_m}{\mathrm{d}x}(\ell) = \frac{T_m}{gh}, \qquad (\text{for } m = 0, \pm 1, \cdots, \pm M). \tag{5}$$

The convolution sum between square brackets reflects the coupling among Fourier modes due to the time-dependency of $r(t)$. Assuming given R_m-values, Eq.(5) is analytically solved for the eigenvectors of the corresponding $(2M+1)\times(2M+1)$-matrix. This provides $Z_m(x)$ and, hence, $\zeta(x,t)$. We finally use the continuity equation in Eq.(3) to find $u(x,t)$, as well.

The iterative procedure starts with a constant friction coefficient, say $r(t) = r_0$, as first guess. For this $r(t)$, we determine the solution $\zeta(x,t)$ and $u(x,t)$ as outlined above. With the aid of a Fast Fourier Transform, Eq.(2) is then applied to update the R_m-values in Eq.(4), used as input for the next iteration. This procedure is repeated until the R_m-values and consequently also $r(t)$ have converged, by which the energy criterion in Eq.(2) is indeed satisfied.

3. Results and discussion

First model results for a synthetic storm event (Fig.1b) demonstrate that our new model approach works, displaying convergence to a time-dependent friction coefficient (Fig.1d), which indeed follows the evolution of the channel-averaged flow velocity (Fig.1c). These results have been obtained with a channel of length $\ell = 100$ km and undisturbed water depth $h = 4$ m, subject to a storm event with a maximum wind stress of 1 N m^{-2}, with $T_{\mathrm{event}} = 1$ day and a recurrence period of $T_{\mathrm{recur}} = 8$ days, using $M = 96$. The drag coefficient for bottom friction has been set at a value $c_{\mathrm{d}} = 1\times 10^{-2}$. The sloshing observed in this new simulation with time-varying $r(t)$ (solid line in Fig.1cd), disappears in the example with a constant friction coefficient r_0 (dashed line), which overestimates bottom friction when flow is weak.

Importantly, our new approach takes away the possible arbitrariness associated with choosing such a constant value r_0. Finally, we remark that our new method can also be applied to include tides, e.g. for tide-surge interactions or for a tidal signal with several constituents.

4. Conclusions

We have explored a novel linearisation of the bottom friction formulation, appropriate for idealised storm surge models. The associated friction coefficient $r(t)$, which adjusts to the temporal development of storm events, follows from an energy criterion that extends Lorentz' (1922) approach. First results, obtained for an idealised single channel model in which $r(t)$ is determined iteratively, show the expected qualitative behaviour. Ongoing research focuses on quantitative sensitivities, the role of tides as well as the extension to a channel network representing the Wadden Sea.

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