The viscous modulation of Lamb's dipole vortex

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A description of the adiabatic decay of the Lamb dipolar vortex is motivated by a variational characterization of the dipole. The parameters in the description are the values of the entropy and linear momentum integrals, which change in time due to the dissipation. It is observed that the dipole dilates during the decay process [radius $R\sim(\nu t)^{1/2}$], while the amplitude of the vortex and its translation speed diminish in time proportional to $(\nu t)^{-3/2}$ and $(\nu t)^{-1}$. © *1996 American Institute of Physics.* [S1070-6631(96)00206-7]

In this Brief Communication we follow Swaters[1] in a description of the adiabatic frictional dissipation of the Lamb (or Batchelor) dipolar vortex.[2,3] In Swaters[1] this is done by means of an asymptotic expansion for a large Reynolds number, or say small kinematic viscosity. The solvability conditions in the perturbation analysis are translated towards the end of his paper to transport or decay equations for the energy and entropy integrals. Here it is really the decay rates of the integrals that motivated us to give a description of the adiabatic decay.

The point of view that only the decay of a few of the infinitely many integrals seem to be important in the description of the dissipative dynamics is motivated by the self-organization hypothesis (see Hasegawa[4] and references therein). This hypothesis specifies the idea of selective dissipation of integrals and the relation with the asymptotic behavior of the system. We view the Navier–Stokes equations as a Poisson system (or generalized Hamiltonian system) given by the Euler equations, with a non-Hamiltonian perturbation if ν, the kinematic viscosity, is small. Due to the dissipation term the Poisson structure of the dynamical system is destroyed and the integrals of the conservative, Hamiltonian, system are no longer preserved.

The adiabatic decay is an approximation of the evolution of solutions of the dissipative system by means of a projection on a set of solutions of the conservative system. For the Lamb dipole this is no more than a projection on a manifold of relative equilibria, given suitable dynamics on this manifold. In van Groesen[5,6] this adiabatic decay is shaped into a dynamical self-organization principle. The dynamics introduced on the manifold of relative equilibria is found from the dissipative evolution of the integrals which characterize the manifold. This gives the adiabatic decay by means of dissipating constrained minimizers. In van Groesen[5,6] this is shown for the decay of Taylor vortices. In van de Fliert and van Groesen[7] the principle is applied to the adiabatic decay of monopolar vortices.

For a short recap of this dynamic principle, let $\Omega(\gamma_1,\ldots,\gamma_n)$ be a family of relative equilibria, corresponding to exact solutions of the (conservative) evolution equation and found variationally from

$$\operatorname*{crit}_{\omega}\{H(\omega)|I_1(\omega)=\gamma_1;\ldots;I_n(\omega)=\gamma_n\}. \tag{1}$$

Starting with a function Ω of this form, consider the evolution $t\to\Omega(\gamma_1(t),\ldots,\gamma_n(t))$, where the parameters $\gamma_i(t)$ are determined from the initial data and the evolution equations for the integrals, such that

$$\frac{d}{dt}\gamma_i(t)=\langle\delta I_i(\Omega),\Omega_t\rangle=\langle\delta I_i(\Omega),\nu\Delta\Omega\rangle. \tag{2}$$

In this Brief Communication we apply the principle to the Lamb dipole, which will show that the vortex pair dilates during the (adiabatic) decay process, contrary to what was observed by Swaters.[1] This difference is explained by the decay rate of the linear momentum integral which is an important parameter in the characterization of the dipole.

We first write Lamb's dipole as a relative equilibrium solution, i.e. a critical point of the energy under specific constraints. With this characterization we apply the dynamic self-organization principle to find the correct decay rates not only for energy and entrophy, but also for the linear momentum integral.

The evolution of vorticity for an incompressible planar fluid is described by the Navier–Stokes equation with kinematic viscosity coefficient ν. Denoting the scalar vorticity by ω and the streamfunction by ψ, we write

$$\partial_t\omega=-\nabla\omega\cdot J\nabla\psi+\nu\Delta\omega,$$
$$\omega=-\Delta\psi,\qquad \text{with } J=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{3}$$

The spatial flow domain is the plane $\mathbb{R}^2$ and conditions at infinity are that the vorticity vanishes, while the streamfunction ψ satisfies

$$\psi\in H^2(\mathbb{R}^2),\qquad \text{with } \psi(x)\sim-\frac{\Gamma(\omega)}{2\pi}\ln|x|,$$
$$\text{as } |x|\to\infty.$$

Here Γ denotes the total vorticity, see (5). In case $\nu=0$, this dynamical system has a Poisson structure, with the Hamiltonian H given by the kinetic energy,

$$H(\omega)=\frac{1}{2}\int\psi\omega\,dx\,dy. \tag{4}$$

Other invariant integrals are the Casimir integrals $C(\omega)=\int f(\omega)dx\,dy$, for an arbitrary function f of ω, with special cases given by the circulation and the entrophy:

$$\Gamma(\omega)=\int \omega dxdy, \qquad W(\omega)=\frac{1}{2}\int \omega^2 dxdy, \tag{5}$$

and furthermore the linear and angular momentum integrals,

$$L_x(\omega)=\int x\omega dxdy, \qquad L_y(\omega)=\int y\omega dxdy,$$
$$P(\omega)=\frac{1}{2}\int (x^2+y^2)\omega dxdy. \tag{6}$$

Relative equilibria of the Euler equations [i.e., of (3) with $\nu=0$] are found from variational principles like (1) using as constraints a choice of the Casimir and/or momentum integrals. The non-differentiable integrals for positive and negative circulations can be used separately to find relative equilibria with a compact support (see van de Fliert and van Groesen),[7,8]

$$\Gamma_+(\omega)=\int (\omega)_+ dxdy, \qquad \Gamma_-(\omega)=\int (\omega)_- dxdy, \tag{7}$$

with $(\omega)_+=\max(\omega,0)$ and $(\omega)_-=\min(\omega,0)$.

From the literature the best known dipolar vortex is the Lamb- or Batchelor-dipole with distributed vorticity.[2,3] As pointed out in van de Fliert,[8] this dipolar vortex is a relative equilibrium characterized by the variational principle:

$$\underset{\gamma_+}{\text{crit}}\ \underset{\omega}{\text{crit}}\{H(\omega)|L_x(\omega)=0,\ L_y(\omega)=l,\ P(\omega)=0,$$
$$W(\omega)=w, \quad \Gamma_+(\omega)=\gamma_+, \quad \Gamma_-(\omega)=-\gamma_+\}. \tag{8}$$

It is in fact not difficult to show that for a solution of a variational principle of the form

$$\underset{\omega}{\text{crit}}\{H(\omega)|L_x(\omega)=0, \quad L_y(\omega)=l,$$
$$P(\omega)=0, \quad C(\omega)=c\},$$

with parameter value $l\in\mathbb{R}\backslash\{0\}$, the flow of the angular momentum is the identity and the Hamiltonian flow is a translation in the x-direction. So for Lamb's dipole we have a steady translation in the x-direction, with the speed given by the Lagrange multiplier corresponding to the linear momentum constraint $L_y(\omega)=l$.

Due to the symmetry in positive and negative vorticity and the optimization over γ_+, the multipliers for Γ_+ and Γ_- are equal to zero. Assuming that the free boundary problem for the vorticity is solved by a circular support, it holds on $r\leqslant R$ that $\psi=\mu\omega+\lambda y$ and the vorticity satisfies in this circle

$$\omega(r,\theta)=\frac{-2k\lambda}{J_0(\rho_1)}J_1(kr)\sin\theta, \tag{9}$$

with, in terms of the specified parameters (ρ_1 is the first non-trivial zero of J_1):

$$R^2=\frac{l\rho_1}{2\sqrt{\pi w}}, \qquad \mu=\frac{1}{k^2}=\frac{R^2}{\rho_1^2}, \qquad \lambda=\frac{l}{2\pi R^2}. \tag{10}$$

We remark again that the multiplier λ is the translation speed of the dipole.

The dissipative equations for the integrals can be found directly from the Navier–Stokes equations; for smooth vorticity solutions ω of the Navier–Stokes equations it holds that

$$\dot H(\omega)=-2\nu W(\omega), \quad \dot W(\omega)=-2\nu D(\omega),$$
$$\dot\Gamma(\omega)=0, \quad \dot P(\omega)=2\nu\Gamma(\omega), \tag{11}$$
$$\dot L_x(\omega)=0, \quad \dot L_y(\omega)=0. \tag{12}$$

Here $D(\omega)=\frac{1}{2}\int(\nabla\omega)^2$ and the dot denotes differentiation with respect to time. We approximate the dissipative dynamics of the Navier–Stokes equations with initial condition given by the steadily translating Lamb dipole, by projection on the manifold of relative equilibria. In this adiabatic approximation we are led by the dissipative equations for entrophy and linear momentum: $\dot W=-2\nu D, \dot L_y=0$, to consider the dynamics:

$$\Omega(w(t),l(t)), \quad \text{with } \dot w=-2\nu d, \quad \dot l=0. \tag{13}$$

Here $d=D(\Omega)$. Since Lamb's dipole $\Omega(w,l)$ as given by (9) satisfies the Laplace eigenvalue problem $\Omega=-\mu\Delta\Omega$, we simply find the relation

$$d=D(\Omega)=\frac{W(\Omega)}{\mu}=\frac{w}{\mu},$$

and we obtain compatibility for the decay of the energy $h=H(\Omega)$:

$$\dot h=\frac{\partial h}{\partial w}\dot w+\frac{\partial h}{\partial l}\dot l=\mu\dot w=-2\nu w. \tag{14}$$

With $\mu=l/2\rho_1\sqrt{w\pi}$ it follows that $\dot w=-4\nu\rho_1\sqrt{w^3\pi}/l$, such that we can conclude:

$$w\sim(\nu t)^{-2}, \qquad h\sim(\nu t)^{-1}, \qquad R^2\sim\nu t,$$
$$\mu=\frac{1}{k^2}\sim\nu t, \qquad \lambda\sim(\nu t)^{-1}. \tag{15}$$

We compare these decay rates with Swaters,[1] who describes the viscous adiabatic decay of the Lamb dipole vortex using the transport equations (i.e., the dissipative equations) for the energy and entrophy: $\dot H=-2\nu W, \quad \dot W=-2\nu D$, in a singular perturbation theory with small parameter ν. Taking into account the decay rates of energy and entrophy, together with the "dispersion" relation $J_1(kR)=0$, a solution is constructed with $\dot k = 0$, $\dot R=0$ and $\dot\lambda=-\nu\rho_1^2\lambda/R^2$, such that

$$w\sim(\nu t)^{-2}, \qquad h\sim(\nu t)^{-1}, \qquad R^2\sim 1,$$
$$\mu=\frac{1}{k^2}\sim 1, \qquad \lambda\sim\exp(-k^2\nu t). \tag{16}$$

In comparison with the adiabatic decay described above, these dynamics are not adjusted to the decay of the linear momentum integral. An important consequence of including the decay of the linear momentum besides that of entrophy and energy is the dilation of the dipolar vortex, with R^2 growing linearly in time. This is similar to the dissipative behavior of monopolar vortices, where we observe a dilation

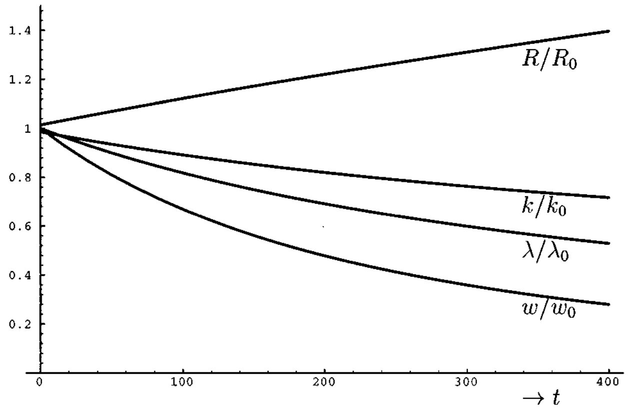

FIG. 1. The evolution in time of the parameters which characterize the Lamb dipole. For the initial parameter setting we chose $R_0=8$ cm and $\lambda_0=0.31$ cm s^{-1}, which are typical values as in the experiments by Flór *et al.*[9,10] Then we have $k_0=0.48$ cm^{-1}, $l_0=129$ cm^3 s^{-1} and $w_0=4.5$ cm^2 s^{-2}. With $\nu=0.01$ cm^2 s^{-1}, the Reynolds number for this flow is $Re=2R_0\lambda_0/\nu=500$.

of the vortex domain while the amplitude diminishes. (For an elliptically shaped vortex support the expansion is accompanied by a relaxation towards axisymmetry.)[7]

In Fig. 1 the evolution of the parameter values is shown, starting with a typical dipole velocity and radius of the support. For comparison with experimental and numerical results on the decay of dipolar vortices with a linear ψ-ω relation, we refer to Flór, van Heijst and Delfos,[9] Flór and van Heijst[10] and Cavazza, van Heijst and Orlandi.[11] In the numerical simulations of the latter paper it was observed that the dipole may eject patches of vorticity, while the remaining dipole still satisfies a linear ψ-ω relationship. Similar to the generation of filaments in the evolution of monopolar vortices (as was observed by for instance Melander, McWilliams and Zabusky),[12] this ejection of patches can not be described by the decay through the quasi-stationary solutions given by the relative equilibria.

In this respect it also needs to be remarked that although the relative equilibria have compact support, a solution of the Navier–Stokes equation will not be compact, since the diffusion term will smooth out the support instantaneously. Usually in numerical and laboratory experiments the vortex domain is defined by a (constant) threshold for the vorticity, i.e., an isolated eddy or vortex is defined by that region where $|\omega(x)|\geqslant\omega_{th}$. In this way the vortex support is well defined and the radius of the dipole as given by the adiabatic approximation should be viewed as such.

Concluding we can say that, led by the variational formulation of the Lamb dipole, the adiabatic decay is described using the decay rates for the entrophy and the linear momentum. The energy then automatically satisfies the balance equation as found from the Navier–Stokes equations, such that the orthogonality conditions in the asymptotic expansion are met as well. In terms of the homogeneous adjoint problem formulated by Swaters[1] we find that the variational formulation gives a motivated choice for the orthogonality conditions of the inhomogeneity.

Finally we remark that the decay rates for the Lamb dipole seem special, as the energy decays as $(\nu t)^{-1}$ and not as $\log \nu t$, as would be expected. This can be related to the special, optimal choice of the constraint value γ_+. For another choice of this value we find that the Lagrange multiplier for the circulation is not equal to zero, in which case $\mu\sim 1/w$ and it follows that $\dot{w}=-2\nu d\sim -2\nu w^2$ such that

$$w\sim(\nu t)^{-1}, \qquad h\sim\log\ \nu t, \qquad R^2\sim\nu t,$$
$$\mu=\frac{1}{k^2}\sim\nu t, \qquad \lambda\sim(\nu t)^{-1}. \tag{17}$$

These decay rates hold for instance for the Chaplygin vortex, which satisfies (8) for another value of γ_+ than the optimal value; see van de Fliert.[8] These decay rates also correspond to the rates found for monopolar vortices[7] and for Oseen's diffusing vortex, which is known to be an exact but unconfined solution to the Navier–Stokes equations (see for instance van de Fliert and van Groesen).[7]

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