# On Stability of the Hamiltonian Index Under Contractions and Closures

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**Abstract:** The hamiltonian index of a graph $G$ is the smallest integer $k$ such that the $k$-th iterated line graph of $G$ is hamiltonian. We first show that, with one exceptional case, adding an edge to a graph cannot increase its hamiltonian index. We use this result to prove that neither the contraction of an $A_G(F)$-contractible subgraph $F$ of a graph $G$ nor the closure operation performed on $G$ (if $G$ is claw-free) affects the value of the hamiltonian index of a graph $G$. AMS Subject Classification (2000): 05C45, 05C35.


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## 1. INTRODUCTION

In this paper, we consider only finite undirected loopless graphs $G = (V(G), E(G))$. However, except for Section 4, we admit $G$ to have multiple edges. We generally follow the most common graph-theoretical notation and terminology and for concepts and notations not defined here we refer the reader to [1].

A *dominating closed trail* (abbreviated DCT) in a graph $G$ is a closed trail (or, equivalently, an eulerian subgraph) $T$ in $G$ such that every edge of $G$ has at least one vertex on $T$. The following result by Harary and Nash-Williams relates the existence of a DCT in a graph $G$ and the existence of a hamiltonian cycle in its line graph $L(G)$. Here the line graph of a graph $G$, denoted by $L(G)$, is the graph with vertex set $E(G)$ and with two vertices adjacent in $L(G)$ if and only if the corresponding edges of $G$ have a vertex in common.

**Theorem A** [5]. *Let G be a graph with at least three edges. Then L(G) is hamiltonian if and only if G has a DCT.*

If $P = x_1, \ldots, x_k$ is a path in a graph $G$ and $S, T \subset G$ are subgraphs of $G$, then we say that $P$ is an $(S,T)$-*path* if $x_1 \in V(S)$ and $x_k \in V(T)$. The *distance* of two subgraphs $S, T \subset G$ (denoted $\mathrm{dist}_G(S,T)$) is the minimum length of an $(S,T)$-path. For any integer $i \geq 0$ set $V_i(G) = \{v \in V(G) : d_G(v) = i\}$ (here $d_G(v)$ denotes the degree of a vertex $v$ in $G$) and $W(G) = V(G) \setminus V_2(G)$. *A branch* in $G$ is a nontrivial path with endvertices in $W(G)$ and with internal vertices, if any, of degree 2 in $G$ (and thus not in $W(G)$). If a branch has length 1, then it has no internal vertex. Let $B(G)$ denote the set of branches of $G$, and let $B_1(G)$ be the subset of $B(G)$ in which every branch has an end in $V_1(G)$. For any subgraph $H$ of $G$ let $B_H(G)$ be the set of those branches of $G$ which have all edges in $H$.

If $G$ is a graph and $k \geq 2$ an integer, then $EU_k(G)$ denotes the set of all subgraphs $H$ of $G$ that satisfy the following conditions:

I. $d_H(x) \equiv 0 \pmod 2$ for every $x \in V(H)$;
II. $V_0(H) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$;
III. $\mathrm{dist}_G(H_1, H - H_1) \leq k - 1$ for every subgraph $H_1$ of $H$;
IV. $|E(b)| \leq k + 1$ for every branch $b \in B(G) \setminus B_H(G)$;
V. $|E(b)| \leq k$ for every branch $b \in B_1(G)$.

The following theorem, which can be considered as an analog of Theorem A for the $k$-th iterated line graph $L^k(G)$ of a graph $G$, shows the importance of subgraphs from $EU_k(G)$. Here $L^k(G)$ is defined recursively by $L^0(G) = G$, $L^1(G) = L(G)$ and $L^k(G) = L(L^{k-1}(G))$.

**Theorem B** [13]. *Let $G$ be a connected graph with at least three edges and let $k \geq 2$ be an integer. Then $L^k(G)$ is hamiltonian if and only if $EU_k(G) \neq \emptyset$.*

The *hamiltonian index* of a graph $G$, denoted by $h(G)$, is the smallest integer $k$ such that the $k$-th iterated line graph $L^k(G)$ of $G$ is hamiltonian. Thus, Theorem B equivalently says that for an integer $k \geq 2$ and for any graph $G$, $h(G) \leq k$ if and only if $EU_k(G) \neq \emptyset$.

If $F$ is a subgraph of a graph $G$, then a vertex $x$ is said to be a *vertex of attachment of $F$ in $G$* if $x \in V(F)$ and $x$ has a neighbor in $V(G)\backslash V(F)$. The set of all vertices of attachment of a subgraph $F$ in $G$ is denoted by $A_G(F)$.

For a subgraph $F$ of $G$, $G|_F$ denotes the graph obtained from $G$ by identifying the vertices of $F$ as a (new) vertex $v_F$, and by replacing the created loops by pendant edges (i.e., edges with one vertex of degree 1) attached to $v_F$. We say that the graph $G|_F$ is obtained from $G$ by *contracting* the subgraph $F$ (observe that $|E(G)| = |E(G|_F)|$).

If $G$ is a graph, $X \subset V(G)$ and $\mathcal{A}$ is a partition of $X$ into subsets, then $E(\mathcal{A})$ denotes the set of all edges $a_1a_2$ (not necessarily in $E(G)$) such that $a_1$, $a_2$ are in the same element of $\mathcal{A}$, and $G^{\mathcal{A}}$ denotes the graph with vertex set $V(G^{\mathcal{A}}) = V(G)$ and edge set $E(G^{\mathcal{A}}) = E(G) \cup E(\mathcal{A})$. Note that $E(G)$ and $E(\mathcal{A})$ are considered to be disjoint, i.e., if $e_1 = a_1a_2 \in E(G)$ and $e_2 = a_1a_2 \in E(\mathcal{A})$, then $e_1$, $e_2$ are parallel edges in $G^{\mathcal{A}}$.

Let $F$ be a graph and let $A \subset V(F)$. Following [11], we say that the graph $F$ is *$A$-contractible*, if for every even subset $X \subset A$ and for every partition $\mathcal{A}$ of $X$ into two-element subsets the graph $F^{\mathcal{A}}$ has a DCT containing all vertices of $A$ and all edges of $E(\mathcal{A})$. Note that this definition comprises the case where $X$ is empty and $F^{\mathcal{A}} = F$. Also, if $F$ is $A$-contractible, then $F$ is $A'$-contractible for any $A' \subset A$ (since every subset $X$ of $A'$ is a subset of $A$).

Set $d_T(G) = \max\{|S| : S \subset E(G)$ and there is a closed trail $T \subset G$ such that every edge $e \in S$ has at least one vertex on $T\}$. The following result was proved in [11].

**Theorem C** [11]. *Let $F$ be a connected graph and let $A \subset V(F)$. Then $F$ is $A$-contractible if and only if*

$$d_T(G) = d_T(G|_F)$$

*for every graph $G$ such that $F \subset G$ and $A_G(F) = A$.*

For $d_T(G) = |E(G)|$ we get the following immediate corollary.

**Corollary D** [11]. *Let $G$ be a graph and let $F \subset G$ be an $A_G(F)$-contractible subgraph of $G$. Then $G$ has a DCT if and only if $G|_F$ has a DCT.*

Note that $G|_F$ may contain multiple edges even if $G$ is a simple graph. However, it is easy to observe that a multiple edge is a contractible subgraph and hence, by a series of subsequent contractions, it is always possible to reduce $G|_F$ to a certain simple graph $G'$ with $d_T(G') = d_T(G|_F) = d_T(G)$.

We say that a graph $G$ is *claw-free* if $G$ is a simple graph that does not contain a copy of the *claw* as an induced subgraph. It is well-known that every line graph is claw-free.

Let $G$ be a claw-free graph. A vertex $x \in V(G)$ is *locally connected* if $G[N(x)]$ is a connected graph. For $x \in V(G)$, the graph $G'_x$ with vertex set $V(G'_x) = V(G)$ and edge set $E(G'_x) = E(G) \cup \{yz|\ y, z \in N(x)\}$ is called the *local completion of G at x*. It was shown in [9] that the local completion of a claw-free graph $G$ at $x$ is again claw-free, and if $x$ is a locally connected vertex, then $c(G'_x) = c(G)$, where $c(G)$ denotes the circumference of $G$, i.e., the length of a longest cycle in $G$.

The following concept was introduced in [9]. Let $G$ be a claw-free graph and let $\mathrm{cl}(G)$ be a graph obtained from $G$ by recursively performing the local completion operation at locally connected vertices with noncomplete neighborhood, as long as this is possible. The graph $\mathrm{cl}(G)$ is called the *closure* of the graph $G$. The following theorem summarizes basic properties of the closure operation.

**Theorem E** [9]. *Let G be a claw-free graph. Then*

i. $\mathrm{cl}(G)$ *is uniquely determined,*
ii. $c(\mathrm{cl}(G)) = c(G)$,
iii. $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph.*

Theorem E has the following immediate consequence.

**Corollary F** [9]. *Let G be a claw-free graph. Then G is hamiltonian if and only if* $\mathrm{cl}(G)$ *is hamiltonian.*

If $\mathcal{C}$ is a class of graphs, $\Gamma$ is a graph operation on $\mathcal{C}$ and $\mathcal{P}$ is a graph property, then $\mathcal{P}$ is said to be *stable under* $\Gamma$ if, for any $G \in \mathcal{C}$, $G$ has $\mathcal{P}$ if and only if $\Gamma(G)$ has $\mathcal{P}$. Similarly, a graph invariant $\pi$ is said to be *stable under* $\Gamma$ if for any $G \in \mathcal{C}$ we have $\pi(G) = \pi(\Gamma(G))$. In this terminology, Theorem C and Corollary D say that $d_T(G)$ and the existence of a DCT are stable under the operation of contraction of an $A_G(F)$-contractible subgraph $F$, and Theorem E and Corollary F say that the circumference and hamiltonicity are stable under the closure operation on claw-free graphs. Stability of some further graph properties and invariants under the closure operation was studied, e.g., in [2], [10], [6] or [8] (see also the survey paper [3]).

The main results of this paper, Theorems 7 and 10, show that the hamiltonian index is stable under the operation of contraction of an $A_G(F)$-contractible subgraph $F$ and under the closure operation on claw-free graphs.

## 2. THE HAMILTONIAN INDEX OF A SUBGRAPH

Our first result shows that, with one exceptional case, adding an edge to a graph cannot increase its hamiltonian index.

**Theorem 1.** *Let G be a connected graph with at least three edges that is not a path. Then for any two vertices* $a, b \in V(G)$ *with* $d_G(a) + d_G(b) \geq 3$, *either* $h(G) = 1$ *and* $h(G + ab) = 2$ *or* $h(G) \geq h(G + ab)$. *Moreover, if* $dist_G(a, b) = 2$, *then*

$$h(G) \geq h(G + ab).$$

***Proof.*** Let $G' = G + ab$. We distinguish the following cases.

**Case 1.** $h(G') = 0$. Then $h(G) \geq 0 = h(G')$.

**Case 2.** $h(G') = 1$. Then $G'$ is not hamiltonian, implying that $G$ is also not hamiltonian. Hence $h(G) \geq 1 = h(G')$.

**Case 3.** $h(G') \geq 2$.

If $h(G) = 0$, then $G$ is hamiltonian and since $V(G) = V(G')$, we have $h(G') = 0$, a contradiction.

Next, suppose $h(G) = 1$. Then, by Theorem A, $G$ has a DCT $T$. Since $h(G') \geq 2$, $T$ is not a DCT of $G'$. Hence neither $a$ nor $b$ are in $V(T)$, and necessarily all neighbors of $a$ and all neighbors of $b$ are on $T$. This implies that any hamiltonian cycle in $L(G)$ is a DCT in $L(G')$, implying that $h(G') \leq 2$. Since, by the assumption, $h(G') \geq 2$, we have $h(G) = 1$ and $h(G') = 2$.

Now, for $a, b \in V(G)$ with $\text{dist}_G(a, b) = 2$, neither $a$ nor $b$ are in $V(T)$ and hence there is a vertex $c_{ab}$ in $N_G(a) \cap N_G(b)$ with $c_{ab} \in V(T)$. Let $T'$ be a closed trail in $G'$ obtained from $T$ by adding the cycle $c_{ab}abc_{ab}$. Then $T'$ is a DCT in $G'$, implying $h(G') \leq 1$, a contradiction.

Hence, we can suppose that $h(G) \geq 2$ and $d_G(a) + d_G(b) \geq 3$. By Theorem B, there is a subgraph $H \in EU_{h(G)}(G)$. Let $H'$ be the subgraph of $G'$ with vertex set

$$V(H') = V(H) \cup \{v \in \{a, b\} : d_{G'}(v) \geq 3\}$$

and edge set

$$E(H') = E(H).$$

We will show that $H' \in EU_{h(G)}(G')$, i.e., $H'$ satisfies the conditions (I) – (V) of the definition of $EU_{h(G)}(G')$ for the graph $G'$ and $k = h(G)$. Obviously, $H'$ satisfies conditions (I) and (II).

If one of $a, b$ has degree 1 in $G$, say, $d_G(a) = 1$, then $d_G(b) \geq 2$ since $d_G(a) + d_G(b) \geq 3$. The branch $P$ of $B_1(G)$, which contains $a$, will become a new branch $P' = Pb$ in $B(G') \backslash (B_{H'}(G') \cup B_1(G'))$ of length $|E(P)| + 1 \leq h(G) + 1$. The other branches of $B(G') \backslash B_{H'}(G)$ are the same as those of $B(G) \backslash B_H(G)$ except when $d_G(b) = 2$ and $b$ is not in $V(H)$; in this exceptional case, the branch containing $b$ turns into two shorter branches in $B(G')\backslash B_{H'}(G')$. This shows that $H'$ satisfies (IV) and (V). If both $a$ and $b$ have degree at least 2 in $G$, then the branches in $B(G')\backslash B_{H'}(G')$ are the same as those in $B(G)\backslash B_H(G)$ except when $a$ or $b$ (or both) have degree exactly 2 in $G$ and they are not in $V(H)$; in this exceptional case, the branches in $B(G')\backslash B_{H'}(G')$ will be shorter than those in $B(G)\backslash B_H(G)$. This shows that $H'$ satisfies (IV) and (V).

It remains to show that $H'$ satisfies (III). Suppose there is a subgraph $H'_1$ of $H'$ such that $\text{dist}_{G'}(H'_1, H' - H'_1) \geq h(G) \geq 2$. It is easy to see that $V(H'_1) \cap V(H)$ and $V(H' - H'_1) \cap V(H)$ cannot be both empty. Suppose first that $V(H'_1) \cap V(H) = \emptyset$ and $V(H' - H'_1) \cap V(H) \neq \emptyset$ (note that the case that $V(H'_1) \cap V(H) \neq \emptyset$ and $V(H' - H'_1) \cap V(H) = \emptyset$ is symmetric). Then $V(H'_1) \subseteq \{a, b\}$. Let $x$ be a vertex of $H'_1$. Since $V(H'_1) \cap V(H) = \emptyset$, $d_G(x) \leq 2$ due to (II). But by the definition of $H'$, $d_{G'}(x) \geq 3$, hence $d_G(x) = 2$ and $x$ belongs to a branch in $B(G)\backslash B_H(G)$. Since $H$ satisfies (IV) and (V), $\text{dist}_G(\{x\}, H) \leq h(G) - 1$. Now, every shortest path from $V(H'_1)$ to $H$ in $G$ is also an $(H'_1, H' - H'_1)$-path in $G'$ which implies $\text{dist}_{G'}(H'_1, H' - H'_1) \leq \text{dist}_G(\{x\}, H) \leq h(G) - 1$, a contradiction. This implies that $H'_1$ has exactly one vertex, say, $V(H'_1) = \{a\}$. Similarly, $\text{dist}_G(\{a\}, H) \leq h(G) - 1$ and any shortest $(\{a\}, H)$-path in $G$ is an $(H'_1, H' - H'_1)$-path in $G'$, implying that $\text{dist}_{G'}(H'_1, H' - H'_1) \leq \text{dist}_G(\{a\}, H) \leq h(G) - 1$, a contradiction. Finally, suppose that both $V(H'_1) \cap V(H)$ and $V(H' - H'_1) \cap V(H)$ are nonempty, and set $H_1 = H'_1 \cap H$. Analogously as above, any shortest $(H_1, H - H_1)$-path in $G$ is also an $(H'_1, H' - H'_1)$-path in $G'$. Hence $\text{dist}_{G'}(H'_1, H' - H'_1) \leq \text{dist}_G(H_1, H - H_1) \leq h(G) - 1$, a contradiction. This shows that $H'$ satisfies (III). Thus $H' \in EU_{h(G)}(G')$, implying $h(G') \leq h(G)$.

If $\text{dist}_G(a, b) = 2$ and $d_G(a) + d_G(b) = 2$, then both $a$ and $b$ are on branches of length 1 which are all in $B_1(G)$. It is obvious that $h(G) = 1$ implies $h(G') = 1$. If $h(G) \geq 2$, then every member of $EU_{h(G)}(G)$ is also a member of $EU_{h(G)}(G')$, thus $h(G') \leq h(G)$. ■

**Example 2.** We construct an infinite family of graphs showing that the assumption $d_G(a) + d_G(b) \geq 3$ in Theorem 1 cannot be relaxed. Let $C$ be a cycle of length $|E(C)| \geq 6$ and let $x, y$ be two vertices on $C$ with maximum $\text{dist}_C(x, y)$. Take two disjoint paths $P_1, P_2$ with endvertices $x', a$ and $y', b$, respectively. Let $G$ be the graph obtained from $C$ and $P_1, P_2$ by identifying $x', x$ and $y', y$ respectively (for $|E(C)| = 6$ see Fig. 1(a)). It is easy to see that $P_1$ and $P_2$ are two branches in $B_1(G)$. If $|E(P_1)|, |E(P_2)| \leq (|E(C)| - 2)/4$, then $h(G) = \max\{|E(P_1)|, |E(P_2)|\}$ (see [12] and [13]) and $h(G + ab) = |E(P_1)| + |E(P_2)| = h(G) + \min\{|E(P_1)|, |E(P_2)|\} > h(G)$ (see [12] and [14]).

**Remark 3.** In fact, using the method of the proof of Theorem 1 (with just a slight modification of the proof of (IV) and (V)), it would be possible to show that without the assumption $d_G(a) + d_G(b) \geq 3$ one can still prove that $2h(G) \geq h(G + ab)$. The graph $G$ from Example 2 with $|E(P_1)| = |E(P_2)| \leq$

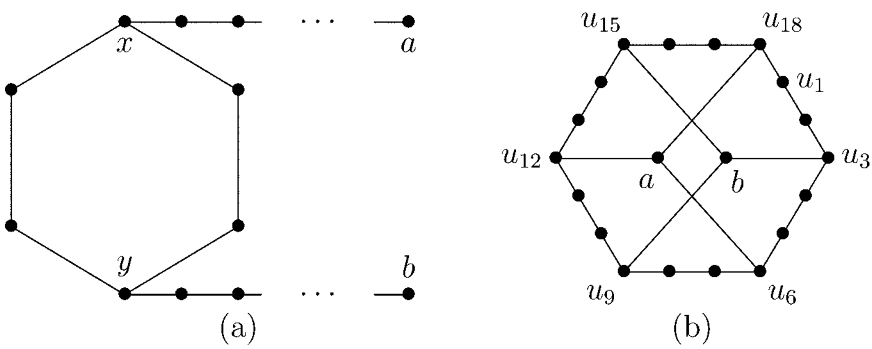


FIGURE 1.

$(|E(C)|-2)/4$ gives $2h(G) = h(G+ab)$ (see [14]), which shows that this inequality is sharp.

Using a similar modification of the proof of Theorem 1, it would also be possible to prove that $2h(G) \geq h(G')$ if $G$ is a spanning subgraph of $G'$. Details are left to the reader.

**Example 4.** Without the condition $\text{dist}_G(a,b) = 2$, we can construct a graph $G$ such that $h(G) = 1$ and $h(G+ab) = 2$ even if $d_G(a) + d_G(b) \geq 3$ is required. Let $t, s \geq 3$ be integers, let $C = u_1u_2 \cdots u_t \cdots u_{2t} \cdots u_{st}u_1$ be a cycle of length $st$ and let $a$ and $b$ be two distinct vertices that are not on $C$. The graph $G$ is obtained from $C$ and $a, b$ by adding $s$ new edges between $a, b$ and $u_t, u_{2t}, \ldots, u_{st}$ such that each of $a$, $b$ is incident to at least one and each of $u_t, u_{2t}, \ldots, u_{st}$ is incident to exactly one of the new edges (for $t = 3$, $s = 6$ and one of the possible choices of the new edges see Fig. 1(b)). By the construction, $d_G(a) + d_G(b) = s \geq 3$. It is easy to see (by Theorem A) that $h(G) = 1$ and $h(G+ab) = 2$.

The following corollary is easily obtained from Theorem 1.

**Corollary 5.** *Let $G$ be a connected graph with at least three edges that is not a path and let $G'$ be a graph obtained from $G$ by recursively adding the edges whose ends $a$ and $b$ satisfy the assumptions of the first part of Theorem 1. Then either $h(G) = 1$ and $h(G') = 2$, or $h(G) \geq h(G')$.*

## 3. THE HAMILTONIAN INDEX IS STABLE UNDER CONTRACTION

We begin this section with the following easy observation which will be used in our proof.

**Lemma 6.** *Let $G$ be a graph with $h(G) \geq 2$. For any $H \in EU_{h(G)}(G)$ and any subgraph $H_1$ of $H$, if the distance between $H_1$ and $H - H_1$ is at least 2, then the shortest path of $G$ between $H_1$ and $H - H_1$ is a branch of $G$, whose ends are not adjacent in $G$.*

***Proof.*** The lemma follows easily from the condition (II) of the definition of $EU_{h(G)}(G)$. ■

We will also need the following well-known result.

**Theorem G** [7]. *A connected graph is eulerian if and only if each minimum edge cut contains an even number of edges.*

If $G$ is a hamiltonian graph (i.e., $h(G) = 0$) and $F \subset G$ is a nontrivial subgraph of $G$, then $G|_F$ cannot be hamiltonian (since it has connectivity 1), and it is easy to observe that any hamiltonian cycle in $G$ turns into a DCT in $G|_F$. Hence, $h(G) = 0$ implies $h(G|_F) = 1$ for any nontrivial subgraph $F \subset G$. However, the following theorem shows that for $h(G) \geq 1$, i.e., for nonhamiltonian graphs, the hamiltonian index is stable under contraction of a contractible subgraph.

**Theorem 7.** *Let $G$ be a nonhamiltonian graph other than a path and $F$ be an $A_G(F)$-contractible subgraph of $G$. Then $h(G) = h(G|_F)$.*

***Proof.*** Let $G' = G|_F$. By Theorem A and Corollary D, $h(G) \leq 1$ if and only if $h(G') \leq 1$. Equivalently, $h(G) \geq 2$ if and only if $h(G') \geq 2$. It is sufficient to consider the case $h(G) \geq 2$. We first prove that $h(G') \leq h(G)$. By Theorem B and $h(G) \geq 2$, we can take a subgraph $H$ in $EU_{h(G)}(G)$. Let $H'$ be the graph obtained from $H|_F$ by deleting the new pendant edges. We shall prove that $H'$ is in $EU_{h(G)}(G')$, i.e., that $H'$ satisfies the conditions of the definition of $EU_{h(G)}(G')$ for the graph $G'$ and $k = h(G)$. It is easy to see that $H'$ satisfies the conditions (I) and (II) due to Theorem G.

The following claim is immediate from the definitions of $A_G(F)$ and $A$-contractible graph.

**Claim 1.** *Every vertex in $A_G(F)$ has degree at least 3 in $G$.*

Now Claim 1 and Lemma 6 easily imply that $H'$ satisfies also the other conditions in the definition of $EU_{h(G)}(G')$, and hence $h(G') \leq h(G)$.

We will prove that $h(G) \leq h(G')$. Since $h(G') \geq 2$, by Theorem B, we can take a subgraph $H'$ in $EU_{h(G')}(G')$. Obviously, every edge of $H'$ can be considered as an edge of $G$. Set $V_b(H') = \{x \in F : x \text{ is an endvertex of a branch of } B_{H'}(G)\}$ and let $r(x)$ denote the number of branches of $B_{H'}(G)$ which have $x$ as an endvertex. Set $V_b^j = \{x \in V_b(H') : r(x) \equiv j \pmod 2\}$. Since $H'$ satisfies (I), $\sum_{x \in V_b^1} r(x) + \sum_{x \in V_b^2} r(x) = \sum_{x \in V_b} r(x) = d_{H'}(v_F)$ is even. But $\sum_{x \in V_b^2} r(x)$ is even, hence $\sum_{x \in V_b^1} r(x)$ is also even, which implies that $|V_b^1|$ is even. Let $X = V_b^1$ and take one partition $\mathcal{A}$ of $X$ into two-element subsets. Since $F$ is $A_G(F)$-contractible, $F^{\mathcal{A}}$ has a DCT $T$ containing all vertices of $A_G(F)$ and all edges of $E(\mathcal{A})$. Now we let $H$ be the graph with vertex set

$$V(H) = V(H') \cup \left( \bigcup_{i=3}^{\Delta(G)} V_i(G) \right) \cup V(T)$$

and edge set

$$E(H) = E(H') \cup (E(T) \backslash E(\mathcal{A})).$$

We prove that $H \in EU_{h(G')}(G)$. Obviously, $H$ satisfies the conditions (I) and (II) in the definition of $EU_{h(G')}(G)$. Since $T$ is a DCT which contains all vertices of $A_G(F)$ and all edges of $E(\mathcal{A})$, by Claim 1, $H$ satisfies (IV) and (V). By Lemma 6, $H$ satisfies (III). Hence $H \in EU_{h(G')}(G)$, implying $h(G) \leq h(G')$. This completes the proof of Theorem 7. ■

**Remark 8.** Catlin [4] introduced a reduction technique based on the concept of a collapsible graph. It was shown in [11] that every collapsible graph $F$ is $V(F)$-contractible. Thus, Theorem 7 implies that the hamiltonian index is stable under contraction of a collapsible subgraph.

## 4. THE HAMILTONIAN INDEX OF A CLAW-FREE GRAPH IS STABLE UNDER THE CLOSURE

In this section we assume all graphs to be simple (i.e., without multiple edges).

**Lemma 9.** *Let $G$ be a connected claw-free graph with at least three edges which is not a path. Then*

i. $h(G) = 0$ *if and only if* $h(cl(G)) = 0$;
ii. $h(G) = 1$ *if and only if* $h(cl(G)) = 1$.

***Proof.*** By Corollary F, it is sufficient to prove that $h(G) \leq 1$ if and only if $h(\text{cl}(G)) \leq 1$. Since $V(\text{cl}(G)) = V(G)$, using Theorem 1 we obtain $h(\text{cl}(G)) \leq h(G)$. Hence $h(G) \leq 1$ implies $h(\text{cl}(G)) \leq 1$.

Conversely, suppose that $h(\text{cl}(G)) \leq 1$, i.e., by Theorem A, $\text{cl}(G)$ has a DCT. We prove that $G$ also has a DCT. It is sufficient to prove that if there is a DCT in $G' = G + xy$ for any pair of vertices $x$ and $y$ with $xy \notin E(G)$ such that they have a common neighbor $c_{xy}$ in $G$ which is a locally connected vertex of $G$, then there is also a DCT in $G$. Let $P$ be a shortest $(x, y)$-path in $G[N_G(c_{xy})]$. Since $G$ is claw-free and $P$ is chordless, $|E(P)| \leq 3$. Since $xy \notin E(G)$, $2 \leq |E(P)| \leq 3$. Let $F = G[V(P) \cup \{c_{xy}\}]$ and $F' = G'[V(P) \cup \{c_{xy}\}]$. Then $F$ is isomorphic to the graph $F_1$ or $F_2$ and $F'$ is isomorphic to the graph $F_3$ or $F_4$ of Figure 2.

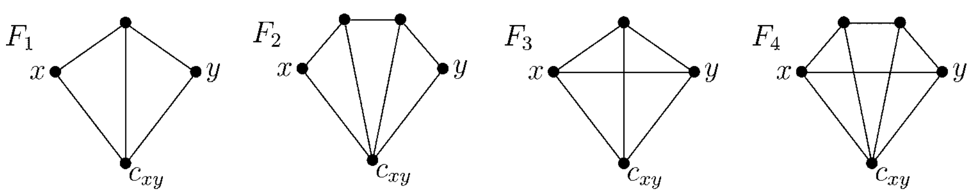


FIGURE 2.

It is easy to verify that each of the graphs $F_i$ is $V(F_i)$-contractible, $i = 1, 2, 3, 4$. Let $e$ be one of the pendant edges of $G'$ adjacent to the vertex $v_{F'}$. Since $G|_F \simeq G'|_{F'} - e$ and clearly $G'|_{F'}$ has a DCT if and only if $G'|_{F'} - e$ has a DCT, by Corollary D, $G'$ has a DCT if and only if $G$ has a DCT. Hence the lemma follows. ∎

The following result, which is the main result of this section, shows that the hamiltonian index is stable under the closure operation in claw-free graphs.

**Theorem 10.** *Let G be a connected claw-free graph with at least three edges which is not a path. Then*

$$h(G) = h(\mathrm{cl}(G)).$$

***Proof.*** By Lemma 9, we only need to prove the case when $h(G) \geq 2$. Since $G \subseteq \mathrm{cl}(G)$ and $V(G) = V(\mathrm{cl}(G))$, we have $h(G) \geq h(\mathrm{cl}(G))$ by the definition of $\mathrm{cl}(G)$ and by Theorem 1. For the reverse inequality, it is sufficient to prove that $h(G) \leq h(G + xy)$ for any pair of vertices $x$ and $y$ with $xy \notin E(G)$ such that they have a common neighbor in $G$ which is a locally connected vertex of $G$.

Let $G' = G + xy$ and let $u$ be a locally connected common neighbor of $x$ and $y$. Then there is an $(x, y)$-path $P$ in $G[N(u)]$ such that $|E(P)| \geq 2$. The following claim is immediate.

**Claim 1.** The internal vertices of $P$ have degree at least 3 in $G$.

By Lemma 9 and since $h(G) \geq 2$, we have $h(\mathrm{cl}(G)) \geq 2$. Thus, by the definition of $\mathrm{cl}(G)$ and by Theorem 1, $h(G') \geq h(\mathrm{cl}(G)) \geq 2$. By Theorem B, $EU_{h(G')}(G') \neq \emptyset$. Taking an $H \in EU_{h(G')}(G')$, we construct a subgraph $H'$ of $G$ as follows:

$$V(H') = V(H) \backslash \{v \in \{x, y\} : d_G(v) = 2 \text{ and } d_H(v) = 0\},$$

$$E(H') = \begin{cases} E(H) & \text{if } xy \notin E(H), \\ (E(H)\Delta E(P))\backslash\{xy\} & \text{if } xy \in E(H), \end{cases}$$

where $E(H)\Delta(E(P)$ denotes the symmetric difference $(E(H)\backslash E(P)) \cup (E(P)\backslash E(H))$.

We show that $H' \in EU_{h(G')}(G)$, i.e., $H'$ satisfies the conditions of the definition of $EU_{h(G')}(G)$ for the graph $G$ and $k = h(G')$. Obviously, $H'$ satisfies conditions (I) and (II). By the definition of $G + xy$ and Claim 1, all branches of length at least 2 in $G$ are the same as in $G'$ except the case when $x$ or $y$ (or both) have degree 2 in $G$; in this exceptional case, each of $x, y$ is on a branch in $B(G)\backslash B_1(G)$ with adjacent endvertices and length exactly 2. Hence by Claim 1 and Lemma 6, $H'$ satisfies the other conditions of the definition of $EU_{h(G')}(G)$, implying $H' \in EU_{h(G')}(G)$. By Theorem B, $h(G) \leq h(G')$, which proves Theorem 10. ∎

**Remark 11.** It was shown in [11] that the operation of contraction of an $A_H(F)$-contractible subgraph of a graph $H$ can be equivalently reformulated as a closure operation performed on its line graph $G = L(H)$. Combined with the closure concept for claw-free graphs this yields a powerful closure operation on claw-free graphs, called the $\mathcal{C}$-*closure* (for details we refer the reader to [11]). Theorems 7 and 10 then immediately imply that the hamiltonian index of a claw-free graph is also stable under the $\mathcal{C}$-closure operation.

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