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Logarithmic temperature profiles in the ultimate regime of thermal convection

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We report on the theory of logarithmic temperature profiles in very strongly developed thermal convection in the geometry of a Rayleigh-Bénard cell with aspect ratio (defined by cell width divided by cell height) $\Gamma = 1$, and discuss the degree of agreement with the recently measured profiles in the ultimate state of very large Rayleigh number flow. The parameters of the log-profile are calculated and compared with the measured ones. Their physical interpretation as well as their dependence on the radial position are discussed. © 2012 American Institute of Physics. [<http://dx.doi.org/10.1063/1.4767540>]

I. INTRODUCTION

The ultimate state of thermal convection in a Rayleigh-Bénard (RB) cell is generally believed to be fully turbulent, both in the bulk and in the boundary layers (BL) and with respect to both the velocity as well as to the temperature fields. As shown in Ref. 1, this implies very characteristic scaling behavior of the heat transport flux, $Nu \propto Ra^{0.38}$ as well as of the strength of the large scale flow, $Re \propto Ra^{0.50}$. Here, the Rayleigh number Ra , the Nusselt number Nu , and the Reynolds number Re are defined as usual, see, e.g., Refs. 2 and 3. In a series of recent experiments, cf. Refs. 4–8, these scaling laws of Nu and Re have been measured and reported.

In the previous work,¹ we have calculated the global scaling exponents in the ultimate state. This ultimate state theory, though derived by employing the characteristic profiles of fully developed turbulent flows, concentrated on the implications for and the interpretation of the various global scaling exponents. We did not explicitly describe and report the local profiles themselves. Meanwhile the local thermal profiles in the very large Ra -regime have been measured, cf. Ref. 9. We thus present here the corresponding local profiles, in the spirit of and in extension of our earlier theory from Ref. 1.

II. THERMAL LOG-PROFILES OF FLOW ALONG PLATES

We start from the equations of motion for the thermal field $T(\vec{x}, t)$. The involved velocity field is understood to solve the corresponding Navier-Stokes equation for an incompressible flow, $\vec{\nabla} \cdot \vec{u} = 0$, i.e., we assume that the temperature behaves like a passive scalar. The molecular properties kinematic viscosity ν and thermal diffusivity κ are taken as temperature independent fluid parameters. It is

$$\partial_t T = -\vec{u} \cdot \vec{\nabla} T + \kappa \Delta T, \quad (1)$$

together with the respective boundary conditions (bc) $T(z = 0) = 0$ and $T(z = L) = -\Delta$ and no heat flux through the sidewalls. In order to understand its turbulent solution we model the RB flow in a rather simplified way, which in the first steps allows to make some use of textbook wisdom, cf. Sec. 54 of Ref. 10, or Sec. 7.1.4 of Ref. 11. Consider a flow along an infinite plate in x -direction and ask for the profile in z -direction perpendicular to the plate. First, decompose the fields into their long time means \vec{U} , Θ and their fluctuating parts \vec{u}' , θ' , defined by $\vec{u}' = \vec{u} - \vec{U}$ and $\theta' = T - \Theta$. The time averages depend on z only and the velocity has an x -component only, chosen to be in positive

x -direction. Per definition we choose $\Theta(z=0) = 0$. If there is heat flow upwards, $J = \langle u_z \theta \rangle_{A,t} - \kappa \partial_z \langle \theta \rangle_{A,t}$ positive, Θ decreases with increasing distance z from the plate and vice versa. This describes the situation at the bottom plate of a RB-cell. At the top plate the temperature flux J is still positive, directed towards the plate, but Θ increases with growing distance from the top plate or decreasing z , starting from $z = L$ downwards.

$$0 = -\vec{U} \cdot \vec{\nabla} \Theta - \overline{u' \cdot \vec{\nabla} \theta'} + \kappa \Delta \Theta. \quad (2)$$

The first term vanishes, since only the x -derivative can contribute but Θ depends on z only. The last term for the same reason reads $\kappa \partial_z^2 \Theta(z)$. The second term has to be modeled; we use the mixing length type Ansatz for an eddy thermal diffusivity. First, $\overline{u'_i \theta'}$ as a long time mean depends on z only and is approximated by employing the temperature gradient hypothesis,

$$\overline{u'_z \theta'} \hat{=} -\kappa_{turb}(z) \partial_z \Theta(z), \quad (3)$$

and $\overline{u'_x \theta'} = 0$ and $\overline{u'_y \theta'} = 0$. The eddy diffusivity $\kappa_{turb}(z)$ is a property of the flow, not of the fluid. The physical flow properties on which $\kappa_{turb}(z)$ can depend, are the distance z from the plate and the characteristic velocity scale u_* of the turbulent fluctuations, defined by the wall stress $\nu \partial_z U_x(z=0) \equiv u_*^2$, i.e., the shear rate or drag on the wall. From dimensional reasons we write

$$\kappa_{turb}(z) \hat{=} \bar{\kappa}_\theta z u_*. \quad (4)$$

Here, the factor $\bar{\kappa}_\theta$ is the dimensionless thermal von Kármán constant, whose empirical value depends on the flow type and for Rayleigh-Bénard flow is not known.

The equation for the temperature profile then is $\partial_z [0 + \kappa_{turb}(z) \partial_z \Theta(z) + \kappa \partial_z \Theta(z)] = 0$. Integrate from $z = 0$ to some arbitrary z and use $\kappa_{turb}(z=0) = 0$ to find $(\kappa_{turb} + \kappa) \partial_z \Theta(z) - \kappa \partial_z \Theta(z)|_0 = 0$. The last term is just J or $Nu \cdot \kappa \Delta L^{-1}$, where $\Delta = T_b - T_t$ is the temperature difference between the bottom and the top plates, which causes the thermal flow. J is z -independent. This leads to the thermal profile equation

$$\partial_z \Theta(z) = \frac{-J}{\kappa + \bar{\kappa}_\theta z u_*}. \quad (5)$$

If J is positive, i.e., the heat flows upwards, the thermal gradient is negative, $\Theta(z)$ decreases, as is characteristic for the bottom plate. With the choice $\Theta(z=0) = 0$ at the bottom plate we have $\Theta(z) \equiv T(z) - T_b$ relating Θ with the physical temperature $T(z)$.

From this equation, we draw conclusions for the bottom part profile. In the immediate vicinity of the plate, it is $\kappa \gg \kappa_{turb}$, giving rise to the “linear thermal sublayer” of the profile, $\Theta(z) = -J \cdot (z/\kappa)$, $0 \leq z \lesssim z_{*,\kappa}$. This linear thermal sublayer extends until $z = z_{*,\kappa}$ for which $\kappa \hat{=} \bar{\kappa}_\theta u_* z_{*,\kappa}$, i.e., when the molecular and the eddy thermal diffusivity are of the same size. The relation to the kinetic sublayer width z_* is

$$z_{*,\kappa} \equiv \frac{\kappa}{\bar{\kappa}_\theta u_*} \rightarrow z_{*,\kappa} = Pr^{-1} \frac{\nu}{\bar{\kappa}_\theta u_*} = Pr^{-1} \frac{z_*}{\bar{\kappa}_\theta}, \quad \text{with } z_* \equiv \frac{\nu}{u_*}. \quad (6)$$

Beyond the linear thermal sublayer the profile is increasingly dominated by the eddy diffusivity, $\kappa_{turb}(z) \gg \kappa$, implying $\partial_z \Theta(z) = -J/(\bar{\kappa}_\theta u_* z)$ and thus

$$\Theta(z) = -\frac{J}{\bar{\kappa}_\theta u_*} \left(\ln \frac{z}{z_*} + f \right) \quad (7)$$

is the thermal profile in the turbulent BL. The integration constant f may depend, of course, on κ and ν , but only in the form ν/κ , since f has unit 1. Thus, $f = f(Pr)$, which is reported in Ref. 10 to empirically be about 1.5 for air, i.e., for $Pr \approx 0.7$.

To make use of this formula for insight into the thermal log-profile we need the strength of the velocity fluctuations u_* as a function of $Re(Ra)$. This has been derived in Ref. 1. But before determining u_* we make now contact to the experimentally measured profile, cf. Ref. 9. The experimental log-profile is parametrized in the form

$$\frac{T(z) - T_m}{T_b - T_t} = A \ln \frac{z}{L} + B. \quad (8)$$

Here, $T_m = (T_b + T_t)/2 = T_b - \Delta/2$ is the mean temperature in the Rayleigh-Bénard cell. Comparing Eqs. (7) and (8) leads to

$$\frac{\Theta(z)}{\Delta} = A \ln \frac{z}{L} + B - \frac{1}{2} = -\frac{J}{\bar{\kappa}_\theta u_* \Delta} \left(\ln \frac{z}{L} + \ln \frac{L}{z_*} + f \right). \quad (9)$$

We now can identify the dimensionless empirical parameters A and B

$$A \equiv -\frac{\kappa Nu}{L \bar{\kappa}_\theta u_*} \quad (10)$$

and

$$B = \frac{1}{2} - \frac{\kappa Nu}{L \bar{\kappa}_\theta u_*} \left[\ln \frac{L}{z_*} + f(Pr) \right] = A \left[\ln \frac{L}{z_*} + f(Pr) \right] + \frac{1}{2}. \quad (11)$$

Besides trivial parameters and a yet undetermined empirical parameter, the thermal von Kármán constant $\bar{\kappa}_\theta$, two physical quantities determine the measured constants A and B . This is first the strength of the heat flux $J = Nu \cdot \kappa \Delta L^{-1}$, describing the amplitude of the log-profile, and second the strength of the turbulent velocity fluctuations u_* . While Nu and its scaling behavior in the ultimate range is experimentally (and theoretically) rather well known, the velocity amplitude u_* has been calculated in Ref. 1. We make use of those results now.

III. THE LOG-PROFILE PARAMETERS A AND B

A. Parameter A

We start with the discussion of the amplitude A of the log-profile. From Eq. (10) the parameter A apparently can be written as

$$A = -\frac{\kappa Nu}{L \bar{\kappa}_\theta u_*} = -\frac{Re^{-1} Pr^{-1} Nu}{\bar{\kappa}_\theta u_* / U} \quad (12)$$

with $Re \equiv UL/\nu$. Here, we have introduced the wind amplitude U and the corresponding Reynolds number Re . For not yet too large Ra this amplitude U usually is visualized as a large scale circulation (LSC) in the RB sample. For very large Ra on the other hand, which are considered here, such LSC will probably not survive under the strong turbulent fluctuations. But its remnants locally in space and time still must have physical importance, since there apparently is enough shear in the plates' boundary layers to induce transition to turbulence, as it can be observed experimentally in the scaling exponents of the Nusselt number Nu and the Reynolds number Re versus Ra as well as the measured characteristic changes of the scaling exponents indicating this transition, see Refs. 4–8 and also Ref. 9.

To quantify this we use results from Ref. 12. In the ultimate range $Ra \gtrsim 10^{15}$, the effective Reynolds number was found to be $Re = 0.0439 \times Ra^{0.50}$, resulting in $Re(Ra = 10^{15}) = 1.39 \times 10^6$. Extrapolating Re from smaller Ra , known as the classical range of RB flow, in Ref. 12 it is found $Re = 0.407 Ra^{0.423}$, the scaling exponent being well consistent with the Grossmann-Lohse (GL) theory.¹³ According to this classical range formula the wind amplitude would be measurably smaller at $Ra = 10^{15}$, namely, $Re = 0.901 \times 10^6$.

In the cited RB experiment, it is $Nu = 5631$ and $Pr = 0.859$ at $Ra = 1.075 \times 10^{15}$. From this we can calculate the average coherence length in the turbulent bulk, which is defined as $\ell_{coh} \approx 10\eta_{Kol} = 10(\nu^3/\epsilon)^{1/4} = 10(Pr^{-2}Ra(Nu - 1))^{-1/4}L$. One gets $\ell_{coh}/L \approx 1.87 \times 10^{-4}$ or in physical units (using $L = 2.24$ m for the height of the Goettingen Uboot device) $\ell_{coh} = 0.419$ mm. This sets the lower bound of the turbulence eddy sizes in the bulk. There will be larger eddies too, whose extension is between the external scale of order L ($= D$ in the case $\Gamma = 1$) and ℓ_{coh} . These even if fluctuating temporally and in position can provide the necessary shear in the boundary layers.

To quantify this we compare the average coherence length with the thickness z_* of the linear viscous sublayer above the plate, which will be introduced and calculated later. It will be estimated as $z_*/L = 1.98 \times 10^{-5}$ for $Ra = 10^{15}$. Then $z_*/\ell_{coh} = 0.106$, meaning that even the smallest eddies of the bulk cascade are still 10 times larger than the kinetic viscous sublayer extension. That holds

all the more for the energy carrying larger eddies up to the macroscopic scale L . Thus, the bulk flow, even if strongly fluctuating, can provide sufficient shear to imply the turbulence transition at the plates (and most probably also at the side walls).

Having clarified the meaning and physical importance of the wind amplitude U , we can determine the log-profile and its parameters A and B . In Ref. 1, we have derived the following two expressions for the Nusselt number Nu and the velocity fluctuation amplitude relative to the given asymptotic flow velocity, u_*/U , in the logarithmic ultimate range, cf. Eqs. (20) and (2) of that paper, namely,

$$Nu = \frac{\frac{\bar{\kappa}_\theta}{2} \frac{u_*}{U} Re Pr}{\ln\left(Re \frac{u_*}{U} \frac{1}{b}\right) + \tilde{f}(Pr)}. \quad (13)$$

The fluctuation amplitude u_* solves the implicit equation

$$\frac{u_*}{U} = \frac{\bar{\kappa}}{\ln\left(Re \frac{u_*}{U} \frac{1}{b}\right)}. \quad (14)$$

Here $\tilde{f}(Pr) \equiv f(Pr) + \ln(b/2)$, and b is an empirical parameter of the velocity profile. It characterizes the position of the buffer range, in which the transition occurs from the linear viscous sublayer to the log-layer range in the velocity profile. Usually that logarithmic profile – known as the *law of the wall* – is written in the form

$$\frac{U_x(z)}{u_*} = \frac{1}{\bar{\kappa}} \ln\left(\frac{zu_*}{\nu}\right) + B_u. \quad (15)$$

Here, $\bar{\kappa}$ is the kinetic von Kármán constant, taken in the following as $\bar{\kappa} = 0.4$. B_u describes the velocity $U(z)$ at $z = z_*$, i.e., it characterizes the buffer range of the velocity profile by indicating how far the log-law velocity $U(z)$ is shifted in amplitude if extended to the linear viscous sublayer. There are various values of the empirical constant B_u given in the literature, such as 5.1, cf. Refs. 10 and 11. To the best of our knowledge there is not yet any information available for its value in the case of Rayleigh-Bénard flow in a closed container. Note that this RB flow neither is stream-wise infinite nor does it approach a given constant amplitude U in vertical direction, but rather becomes small again near the mid-plane and then even changes sign. Thus, B_u remains a parameter still to be measured in RB cells. We include it into the profile equation (14) in the form

$$b \equiv e^{-\bar{\kappa} B_u}. \quad (16)$$

According to Eq. (14) the numerical value of b determines the amplitude of the velocity fluctuation scale u_* (or shear stress ρu_*^2) together with the Reynolds number Re in the combination Re/b . From Eq. (14) we see that the smaller b is the smaller u_* will be. Similarly, u_* will decrease with increasing Reynolds number Re and thus with growing Rayleigh number Ra . For $Re(Ra)$ we meanwhile have experimental information, see Ref. 12. In the ultimate state, there is an effective scaling $Re = 0.0439 Ra^{0.50}$. We thus have typical Re numbers of order 10^6 or more in the ultimate range of thermal convection with $Ra \sim 10^{15}$.

The velocity profile parameter b also enters the Pr -dependence of the temperature profile via the definition $\tilde{f}(Pr) = f(Pr) + \ln(b/2)$. Using all these previous results we can express the amplitude parameter A in the form

$$A = -\frac{1}{2\left[\ln\left(\frac{u_*}{U} Re \frac{1}{b}\right) + \tilde{f}(Pr)\right]} = -\frac{1}{2\left[\bar{\kappa} \frac{U}{u_*} + \tilde{f}(Pr)\right]}. \quad (17)$$

Two remarks may be useful. In this expression for A the thermal von Kármán constant $\bar{\kappa}_\theta$ does not appear explicitly; instead the kinetic one, $\bar{\kappa}$, shows up. This comes from eliminating $Nu/\bar{\kappa}_\theta$ in the defining equation (12) for A by using Eq. (13). Second, the originally derived explicit Nu -dependence can be completely substituted and expressed in terms of the log-corrections originating from the log-profile of the velocity. The only reminder to the thermal profile is the thermal buffer range shift $\tilde{f}(Pr)$, the thermal analog of the corresponding kinetic shift B_u .

Both A and B depend on Re as well as on u_*/U , and thus on Ra . This determines the Ra -dependence of these measured fit-amplitudes A and B . Numerical values for Re and u_*/U have been given in Table I of Ref. 1 for $Ra = 10^{14}$ and $b = 1$. For any other Ra (and b) they can be computed by solving the implicit equation (14), either numerically or via its continued fraction representation given in Ref. 1. We do this below for $Ra = 10^{15}$.

Taking for the parameters f and b the values reported in textbooks for channel or pipe flow of gases, i.e., $f(Pr = 0.7) \approx 1.5$ (cf. Ref. 10) and $b = O(1)$, one finds $\tilde{f}(Pr) = O(1)$. The estimate for the magnitude of the relative fluctuations u_*/U in Ref. 1 gave values of the order of 0.05. Thus, the first term in the denominator of Eq. (17) is an order of magnitude larger, even more so for larger $Re(Ra)$. Neglecting therefore \tilde{f} we find the following approximate formula for the parameter A :

$$A \approx -\frac{1}{2\ln\left(\frac{u_*}{U} Re^{\frac{1}{b}}\right)} = -\frac{1}{2\bar{\kappa}} \frac{u_*}{U}. \quad (18)$$

For $Ra = 10^{14}$ it is $u_*/U = 0.039$ and 0.065 for the choices Re_1 and Re_2 in Table I of Ref. 1, respectively, leading to the numerical values $A = -0.0488$ and $A = -0.0813$, respectively. For $Ra = 10^{15}$ and $Re = 1.4 \times 10^6$ we calculate from Eq. (14) (and the choice $b = 1$) the value $u_*/U = 0.036$, leading to $A = -0.045$, still about half an order of magnitude larger as the measured value $A_{exp} = -0.0082$, given in Ref. 9.

The scaling of the profile parameters A and B with the Rayleigh number Ra can be determined as follows. First, transform the Re -dependence of u_*/U into a scaling Ansatz, $u_*/U \propto Re^{\alpha'}$. Here, the Reynolds number dependent (“local”) exponent $\alpha' = \alpha'(Re)$ decreases with Re and therefore with Ra . In the relevant Re -range, one calculates $\alpha' = -0.0855$. The negative scaling exponent α' means that the ratio $u_*/U = Re_*/Re$ decreases with increasing Re and correspondingly increasing Ra . But, of course, the turbulent fluctuation scale $Re_* = u_*L/\nu$ itself increases in size for stronger thermal driving Ra . It is

$$Re_* \propto Re^{1+\alpha'} = Re^{0.9145}, \text{ in the } Ra\text{-range around } 10^{15}. \quad (19)$$

Introducing now this Re -scaling of Re_* into Eq. (18) results in

$$A \propto Re^{\alpha'} \propto Ra^{0.50\alpha'} = Ra^{-0.043}, \text{ in the } Ra\text{-range around } 10^{15}. \quad (20)$$

The amplitude A thus decreases with Ra , though very slowly. Over the next decade it will be smaller by approximately a factor of 0.906, i.e., its value is expected to be about 9.4% less. This predicted slow decrease of A with Ra could be consistent with experimental observation.⁹ We emphasize that the absolute value of α' decreases even further with increasing Re , i.e., asymptotically in Ra the amplitude A will approach a constant, Ra -independent limit A_∞ .

One may wish to try coming closer to the experimental value for A in the measured range of $Re \approx 10^6$ and $Ra = 10^{15}$ (Ref. 12) by adjusting the still not well-known value of b . Instead of $b = O(1)$ one could reduce b , mimicking thus a larger Re , which leads to smaller u_*/U . To be more precise: Take as above $u_*/U = 0.036$ and $Re(Ra = 10^{15}) = 1.4 \times 10^6$ and insert this into (14). This then determines b . We obtain $\ln(1/b) = \bar{\kappa} B_u \approx 0.28$, thus $B_u \approx 0.71$, and $b \approx 0.75$. Note that for a bit smaller fluctuations $u_*/U = 0.031$ we find $b \approx 0.11$ and $B_u \approx 5.6$, very close to the von Karman values for shear flow. The measured $A_{exp} = -0.0082$ according to Eq. (18) would correspond to a very small relative fluctuation amplitude $(u_*/U)|_{exp} \hat{=} 0.00656$, implying an unusually small b and large B_u .

Obviously the model for the velocity field’s U -profile is oversimplified. An important missing feature when assuming an asymptotically constant flow amplitude U with increasing distance z from the plate is as follows. The LSC in a RB-sample does not approach a nonzero constant value when going off the plate. Instead $U(z)$ not only will decrease again, it even passes through 0 at $z/L = 0.5$, changing sign in the upper half of the RB cell. Thus, between about $0.25 \lesssim z/L \lesssim 0.5$ the average flow has a negative instead of a positive or zero z -slope. The thermal flux J keeps its positive sign. Because of this quite different $U(z)$ profile in comparison to pipe or channel flow we have to expect a deviation from the temperature log-profile in the range above $z/L \gtrsim 0.25$. Indeed such deviation can be seen in experiment, see Figure 1 of Ref. 9 – Another consequence of such deviation from the

pure log-profile is that the intimate connection between the amplitudes A and B , to be discussed in the next chapter, will lose validity.

B. Parameter B

Let us now analyze the parameter B as derived in Eq. (11), in particular the right hand part of this equation. Consider first the term $\ln \frac{z}{z_*}$. With the above given definition $z_* = \nu/u_*$ one obtains $L/z_* = Lu_*/\nu$ and thus $B = A [\ln (Re \frac{u_*}{U}) + f(Pr)] + 1/2 = A [\ln (Re \frac{u_*}{U} \frac{1}{b}) + \tilde{f}(Pr) + \ln 2] + 1/2$. Now, the ln-term plus the next one are just $-\frac{1}{2A}$, which after multiplication with A leaves $-1/2$, canceling the final term $1/2$. Thus,

$$B = A \cdot \ln 2 \text{ or } B/A = \ln 2 = 0.693\dots \quad (21)$$

Irrespective of any approximate calculation of the coefficient A , the other coefficient B is always about 70% of A , in particular is also negative. Using this the T -profile can be written as

$$\frac{T(z) - T_m}{T_b - T_t} = A \left(\ln \frac{z}{L} + \ln 2 \right) = A \ln \frac{z}{L/2}. \quad (22)$$

This is consistent with the underlying idea that $T(z = L/2) = T_m$, which has been used when relating the T -profile, expressed in terms of the thermal current J or Nu , with the driving temperature difference $\Delta = T_b - T_t$, and which also is an immediate consequence of the Ansatz (8) for fitting the data.

Let us come back to the consequences of the deviation of the (x -component of the) velocity profile from the classical case with asymptotically constant amplitude U . In a RB cell, the $U(z)$ -profile does not stay asymptotically constant with z but goes through a maximum, then decreases further and even changes sign at $z/L \approx 0.5$. This different maximum-and-beyond behavior of the wind leads to deviations in the thermal profile from a log-law. Therefore, the intimate relation between A and B , viz. $B/A = \ln 2$, typical for the log-profile, will no longer be valid. Also this statement is consistent with experimental observation, cf. Ref. 9: the ratio B/A has quite some scatter and apparently differs from $\ln 2$, though not much.

To briefly summarize, the thermal profile is characterized by – starting from the plate – (i) a tiny linear thermal sublayer of extension an order of magnitude less than the bulk coherence length, (ii) a buffer range in which the linear increase in the sublayer turns over into the (iii) log-law, observable over a broader z -range up to about a quarter of the RB cell height, then (iv) the profile changes again due to the decrease of the wind amplitude with further increasing z including its directional change, from positive to negative (or vice versa), which may be denoted as the temperature's center profile. This latter one, the center part, still has to be explored in more detail.

IV. POSITION DEPENDENCE OF PROFILE PARAMETERS $A(r)$ AND $B(r)$

The basic assumption of the just given derivation of the profile parameters A, B is a plane parallel homogeneous flow with velocity U over an infinitely extended plate. In a cylindrical Rayleigh-Bénard cell this—at best—is realized in the center range of the circular cell, i.e., at $r = 0$. We now make a crude model for the shape of the LSC. We have in mind the case of aspect ratio $\Gamma = 1$, but one can argue similarly for $\Gamma = 1/2$ (and other). Then, if one moves away from the center range at $r = 0$, the relevant flow velocity leading to the velocity shear in the BL and the perpendicular logarithmic profiles of velocity and temperature is the x -component of the LSC only. We therefore have to substitute in above formulas always $U \Rightarrow U_x = U_x(r)$. If we consider for simplicity a circular LSC, we get $U_x = U \cos \phi = U \sqrt{1 - \frac{r^2}{R^2}}$; here U still denotes the LSC amplitude, which defines $Re = UL/\nu$.

Considering expression (18) will result in an r -dependent profile coefficient

$$A(r) = \frac{A}{\sqrt{1 - (r/R)^2}}, \text{ with } A \text{ the coefficient at the center.} \quad (23)$$

Expressed in terms of the (relative) side wall distance $\xi \equiv \frac{R-r}{R}$, thus $0 \leq \xi \leq 1$ between the wall and the center, respectively, it is

$$A(\xi) = \frac{A}{\sqrt{2\xi - \xi^2}} \approx \frac{A}{\sqrt{2}} \xi^{1/2}. \quad (24)$$

The absolute value of the coefficient $A(\xi)$ decreases with increasing distance from the wall $\propto \xi^{-1/2}$. This result is consistent with the experimental finding of an $|A|$ -decrease towards the center. The data presented in Ref. 9, where $L = 4R$ in the $\Gamma = 1/2$ sample, have been taken at the position $(R - r)/L = 0.0045$, corresponding to $\xi = 0.018$. This implies $A(\frac{r}{R} = 0.982) = A(\xi = 0.018) = 5.29A$.

We note that these last formulas cease to be valid in the limit $\xi \rightarrow 0$ or $r \rightarrow R$, because for sufficiently small $U_x(r)$ the BL locally is no longer turbulent, finally there is only upward flow and one is in the BL of the side wall. But note further that with the full, non-approximate expression (17) for the profile coefficient A the additional term in the denominator weakens the r -dependence, all the more the larger r becomes. Then even the limit $r \rightarrow R$ exists, leading to

$$A(r \rightarrow R) = -\frac{1}{2\tilde{f}(Pr)}. \quad (25)$$

One can draw various conclusions from the r -dependence of $A(r)$ on the shape of the LSC as follows. First, we assume that the magnitude of the LSC wind (U or in dimensionless form Re) is approximately the same along its closed trajectory, thus ignoring geometrical effects. The direction of the wind varies along the LSC orbit. Its local unit vector be called $\vec{t}(s)$, which is the local tangential unit vector along the (closed) LSC curve. s denotes the arc length along the LSC curve. Then the velocity relevant for $A(r)$ is given by $U_x = \vec{e}_x \cdot \vec{t}U$, the x -component of the local LSC velocity vector $U\vec{t}$. Given a model for the LSC-curve, e.g., a circle or an ellipse, this can be described in terms of a suitable parameter ϕ by $\vec{x}(\phi)$. Expressing the (arbitrarily given) parameter ϕ in terms of the arc length s , the tangential vector then is the derivative of the curve, i.e., $\vec{t} = \frac{d\vec{x}(s)}{ds}$. Measuring on the other hand $U_x(r)$ via $A(r)$ allows to re-construct the LSC curve.

Some qualitative features of the r -dependence of $A(r)$ are the following: (i) For small r the amplitude $A(r)$ always increases as $A(r) = A(1 + \text{const}(\frac{r}{R})^2)$; there is no term linear in r because of analyticity reasons. (For a strictly circular flow it is $\text{const} = 0.5$.) (ii) For r near the side wall, i.e., for small $\xi = (R - r)/R$, the amplitude $A(\xi) \sim \xi^{-n}$ decreases with increasing distance from the wall with an exponent $n = 1/2$ for a circular LSC; for an elliptically shaped wind curve it will be steeper, thus n is larger. In the particular case of aspect ratio $\Gamma = 1/2$ there may be two circular rolls above each other, which again would lead to $n = 1/2$. In case there is only one single roll this will be elliptically shaped and thus n is larger. In case both LSC shapes are present part of the time, the exponent n will be somewhere in between, i.e., in any case one would find a steeper decrease of A with the wall distance as compared to the circular case for $\Gamma = 1$. This is consistent with the measurement of $n \approx 2/3$ in the case $\Gamma = 0.5$ (Goettingen Uboot team, private communication). (iii) If Γ is a little below 1, n will be a bit larger than $1/2$; if Γ is a little above 1, n is expected to be somewhat smaller than $1/2$; etc.

V. CONCLUDING REMARKS

We have calculated the profile parameters A and B , defined in Eq. (8), of the experimentally measured logarithmic temperature profiles in the ultimate state of Rayleigh-Bénard convection as realized for very large Ra . In the case of a pure thermal log-law as well as reflection symmetry with respect to the middle plane $z = L/2 = R$ of the $\Gamma = 1$ cylindrical sample, which we expect and which should hold for approximately temperature independent material properties, the coefficient B equals A up to a factor $\ln 2$. Since the real wind profile is different from the standard case of approaching a constant when going off the plate, in the RB cell instead going down in magnitude towards the center, even followed by a directional inversion of the wind direction, one finds deviations from this value.

The amplitude A physically measures the strength of the turbulent velocity fluctuation scale u_* relative to the LSC velocity U (together with the von Kármán constant $\bar{\kappa}$); if one is sufficiently near

to the side wall, it also measures the Prandtl number dependent temperature profile shift constant $\tilde{f}(Pr)$. We explain the r -dependence of $A(r)$ by the decreasing magnitude of the local flow velocity $U_x(r)$ parallel to the bottom plate with increasing distance r from the center, which by (18) or (17) leads to an increase of the A -amplitude with increasing r .

Finally, we reiterate that there is very interesting information on the velocity ratio u_*/U in the measured T -profile parameters $A(r)$ and $B(r)$. But, as usual in Rayleigh-Bénard flow, to complete, check, and confirm theoretical interpretation and explanation will need information also about the velocity profile and its expected considerable deviations from the model of classical channel or pipe flow as is used here for a lack of better.

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- ¹ S. Grossmann and D. Lohse, "Multiple scaling in the ultimate range of thermal convection," *Phys. Fluids* **23**, 045108 (2011).
- ² G. Ahlers, S. Grossmann, and D. Lohse, "Heat transfer and large scale dynamics in turbulent Rayleigh-Bénard convection," *Rev. Mod. Phys.* **81**, 503 (2009).
- ³ D. Lohse and K.-Q. Xia, "Small-scale properties of turbulent Rayleigh-Bénard convection," *Annu. Rev. Fluid Mech.* **42**, 335 (2010).
- ⁴ D. Funfschilling, E. Bodenschatz, and G. Ahlers, "Search for the ultimate state in turbulent Rayleigh-Bénard convection," *Phys. Rev. Lett.* **103**, 014503 (2009).
- ⁵ G. Ahlers, D. Funfschilling, and E. Bodenschatz, "Transitions in heat transport by turbulent convection at Rayleigh numbers up to 10^{15} ," *New J. Phys.* **11**, 123001 (2009).
- ⁶ G. Ahlers, D. Funfschilling, and E. Bodenschatz, "Addendum to transitions in heat transport by turbulent convection at Rayleigh numbers up to 10^{15} ," *New J. Phys.* **13**, 049401 (2011).
- ⁷ X. He, D. Funfschilling, E. Bodenschatz, and G. Ahlers, "Heat transport by turbulent Rayleigh-Bénard convection for $Pr \simeq 0.8$ and $4 \times 10^{11} \lesssim Ra \lesssim 2 \times 10^{14}$: Ultimate-state transition for aspect ratio $\Gamma = 1.00$," *New J. Phys.* **14**, 063030 (2012).
- ⁸ G. Ahlers, X. He, D. Funfschilling, and E. Bodenschatz, "Heat transport by turbulent Rayleigh-Bénard convection for $Pr \simeq 0.8$ and $3 \times 10^{12} \lesssim Ra \lesssim 10^{15}$: Aspect ratio $\Gamma = 0.50$," *New J. Phys.* **14**, 103012 (2012).
- ⁹ G. Ahlers, E. Bodenschatz, D. Funfschilling, S. Grossmann, X.-Z. He, D. Lohse, R. J. A. M. Stevens, and R. Verzicco, "Logarithmic temperature profiles in turbulent Rayleigh-Bénard convection," *Phys. Rev. Lett.* **108**, 114501 (2012).
- ¹⁰ L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1987).
- ¹¹ S. B. Pope, *Turbulent Flows* (Cambridge University Press, 2000).
- ¹² X.-Z. He, D. Funfschilling, H. Nobach, E. Bodenschatz, and G. Ahlers, "Transition to the ultimate state of turbulent Rayleigh-Bénard convection," *Phys. Rev. Lett.* **108**, 024502 (2012).
- ¹³ S. Grossmann and D. Lohse, "Scaling in thermal convection: A unifying view," *J. Fluid. Mech.* **407**, 27 (2000); "Thermal convection for large Prandtl number," *Phys. Rev. Lett.* **86**, 3316 (2001); "Prandtl and Rayleigh number dependence of the Reynolds number in turbulent thermal convection," *Phys. Rev. E* **66**, 016305 (2002); "Fluctuations in turbulent Rayleigh-Bénard convection: The role of plumes," *Phys. Fluids* **16**, 4462 (2004).